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ENDOGENOUS-HORIZON RANDOMLY FURCATING DIFFERENTIAL GAMES

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1 Introduction

In many game situations, horizon endogeniety is common. Important cases include the extraction of exhaustible resource from a common pool, competition for market shares under given demand, sharing of CPU time by competing computer users, allocations under fixed funding, claims on frontier land by homesteaders, and economic development under a global environmental constraint. These situations are essentially differential games, in which playing ends when a subset of state variables reaches certain target levels at terminal time. A special feature of such games is that the horizon is not fixed at the outset: Instead, it is determined endogenously by the actions of the players. Due to the fact that the horizon is related to the initial state, only open-loop solutions have been considered for such games (see for examples, Dasgupta and Heal (1979), Kemp and Long (1980), Khalatbari (1977) and Sinn (1984). Yeung (2000) provides a theorem to characterize feedback solutions of differential games with endogenous horizon games.

This paper widens the scope of endogenous-horizon differential games by introducing uncertainty into the model along the line of the work of Yeung (2001 and 2003). In particular, we propose a new class of games - designated as *endogenous-horizon randomly furcating stochastic differential games.* The analysis allows random shocks to appear in the players' future payoffs. Since these payoffs are uncertain, the term "randomly furcating" is introduced to emphasize that a useful way to analyze such situations is to assume that payoffs change at any instant of time according to known probability distributions defined in terms of multiple-branching stochastic processes.

A particularly fruitful way of approaching complicated games with endogenous horizons follows. The application of differential game theory is widened to problems where future environments are not known with certainty. For the first time, theorems characterizing feedback Nash equilibrium solutions become possible for this kind of game problems. In addition, an illustration is provided.

The paper is organized as follows. Section 2 establishes the basic formulation of endogenous-horizon randomly furcating stochastic differential games. Section 3 develops a theorem characterizing a feedback Nash equilibrium solution of the game. Section 4 presents a two-state endogenous-horizon differential game with uncertain payoffs as an example of this class of games, and applies it to illustrate the application of the solution theorem of Section 3. Section 5 concludes.

2 Game formulation

Consider a class of N-person differential games in which player i, for $i \in N$, maximizes:

$$E_{t_0}\left\{\int_{t_0}^{t_1} g^{i(0)}\left[s, x(s), u^1(s), u^2(s), \ldots, u^N(s)\right] ds \right.$$
$$+\sum_{h=1}^{\tau-1}\int_{t_h}^{t_{h+1}} g^{i(h)}\left[\theta^h, s, x(s), u^1(s), u^2(s), \ldots, u^N(s)\right] ds$$
$$\left. +\int_{t_\tau}^{T} g^{i(\tau)}\left[\theta^\tau, s, x(s), u^1(s), u^2(s), \ldots, u^N(s)\right] ds + q^{i(T)}\left[\theta^T, T, x(T)\right]\right\}, \quad (2.1)$$

where $x(s) = [x_1(s), x_2(s), \ldots, x_m(s)] \in R^m$ is a vector of state variable at time s, $u_i(s) \in U_i \subset R^{m_i}$ is of player i's control, and $t_{\tau+1} = T$ is time when the game terminates. The terms θ^k, for $k = 1, 2, \ldots, \tau$, is a random variable stemming from the branching process.
$\theta^1 = \left\{\theta^1_1, \theta^1_2, \ldots, \theta^1_{\eta_1}\right\}$ with corresponding probabilities $\left\{\lambda^1_1, \lambda^1_2, \ldots, \lambda^1_{\eta_1}\right\}$.
Given that $\theta^1_{a_1}$ is realized in time interval $[t_1, t_2)$, for $a_1 = 1, 2, \ldots, \eta_1$,

$\theta^2 = \left\{\theta_1^{2[(1,a_1)]}, \theta_2^{2[(1,a_1)]}, \ldots, \theta_{\eta_{2[(1,a_1)]}}^{2[(1,a_1)]}\right\}$ would be realized with the corresponding probabilities $\left\{\lambda_1^{2[(1,a_1)]}, \lambda_2^{2[(1,a_1)]}, \ldots, \lambda_{\eta_{2[(1,a_1)]}}^{2[(1,a_1)]}\right\}$.

Given that $\theta_{a_1}^1$ is realized in time interval $[t_1, t_2)$ and $\theta_{a_2}^{2[(1,a_1)]}$ is realized in time interval $[t_2, t_3)$, for $a_1 = 1, 2, \ldots, \eta_1$ and $a_1 = 1, 2, \ldots, \eta_{2[(1,a_1)]}$,

$\theta^3 = \left\{\theta_1^{3[(1,a_1)(2,a_2)]}, \theta_2^{3[(1,a_1)(2,a_2)]}, \ldots, \theta_{\eta_{3[(1,a_1)(2,a_2)]}}^{3[(1,a_1)(2,a_2)]}\right\}$ would be realized with the corresponding probabilities

$$\left\{\lambda_1^{3[(1,a_1)(2,a_2)]}, \lambda_2^{3[(1,a_1)(2,a_2)]}, \ldots, \lambda_{\eta_{3[(1,a_1)(2,a_2)]}}^{3[(1,a_1)(2,a_2)]}\right\}.$$

In general, given that $\theta_{a_1}^1$ is realized in time interval $[t_1, t_2)$, $\theta_{a_2}^{2[(1,a_1)]}$ is realized in time interval $[t_2, t_3), \ldots,$ and $\theta_{a_{k-1}}^{k-1[(1,a_1)(2,a_2)\ldots(k-2,a_{k-2})]}$ is realized in time interval $[t_{k-1}, t_k)$, for $a_1 \in [1, 2, \ldots, \eta_1], a_2 \in \left[1, 2, \ldots, \eta_{2[(1,a_1)]}\right], \ldots, a_{k-1} \in \left[1, 2, \ldots, \eta_{k-1[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}\right]$,

$$\theta^k = \left\{\theta_1^{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}, \theta_2^{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}, \ldots, \theta_{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}^{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}\right\}$$

would be realized with the corresponding probabilities

$$\left\{\lambda_1^{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}, \lambda_2^{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}, \ldots, \lambda_{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}^{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}\right\}$$

for $k = 1, 2, \cdots, \tau$.

The payoffs of all players will remain unchanged within the time interval $[t_{k-1}, t_k)$, and to be common knowledge to all players within that time. Moreover, at time t_0, the terminal payoffs when the game terminates are known to be:

$$q^{i(T)}\left[\theta^T, T, x(T)\right], \qquad \text{for } i \in N,$$

where θ^T is a random variable with range $\left\{\theta_1^T, \theta_2^T, \ldots, \theta_{\eta_T}^T\right\}$ and corresponding probabilities $\left\{\lambda_1^T, \lambda_2^T, \ldots, \lambda_{\eta_T}^T\right\}$.

The state of the game evolves according to the dynamical equation:

$$\dot{x}(s) = -f\left[s, x(s), u_1(s), u_2(s), \ldots, u_N(s)\right], \qquad \text{and} \qquad x(t_0) = x^0, \tag{2.2}$$

where $f = [f_1, f_2, \ldots, f_m]$ is a vector valued function. There are integral constraints on ω of these m state variables. Let state variables 1 to ω denote variables with an integral constraint. In particular,

$$\int_{t_0}^{T} f_j\left[s, x(s), u_1(s), u_2(s), \ldots, u_N(s)\right] ds \leq x_j(t_0) - x_j^T, \tag{2.3}$$

where $j = 1, 2, \ldots, \omega$; x_j^T is the minimum level of state variable j that must be maintained. Specifically, the game would stop if any x_j^T is reached.

Condition (2.3) represents a terminal constraint, under which the integral (sum) of the changes in each of these ω state variables over time is constrained by the difference between its initial state and its minimum level. The game terminates when any of these constraints is met. It follows that the game horizon T is determined endogenously by the actions of the players. Examples of games with constraints of this nature are prevalent. As mentioned before, the extraction of exhaustible resource from a common-pool, competition for market shares under given demand, sharing of CPU time by competing computer users, allocations under fixed funding, claims on frontier land by homesteaders, and economic development under a global environmental constraint are examples.

The presence of a constraint coupling the controls of players significantly complicates the game situation. In a recent paper, Carlson and Haurie (2000) considered a class of infinite horizon games with coupled state constraints, which place some limits on the state variables at each instant of time. It is shown that the solution could be obtained by finding an overtaking Nash equilibrium for an associated infinite horizon decoupled game, with the asymptotic constraint being delineated by a regulator's tax scheme. In the present analysis, we derive theorems to characterize feedback Nash equilibrium solutions to games subject to the integral constraint (2.3). Since such constraint ties the game horizon T to the initial state x^0, the game equilibrium is expected to be, in general, dependent on the initial state. Given that FNE strategies depend on the current state x and the time t, serious technical difficulties would be expected to arise. Because of this, feedback solutions for differential games with an endogenous horizon in this nature have not been studied. However, it will be shown that the terminal constraint (2.3) has a special property, which allows us to establish a relationship between the game horizon and the current state at current time. It then becomes possible to characterize a feedback solution for differential games with such terminal constraints.

3 Identifying a feedback Nash equilibrium

To solve for a feedback solution for the endogenous game (2.1)–(2.3), we first consider a class of N-person differential games in which player i maximizes:

$$J^i = \int_0^T g^i \left[s, x(s), u_1(s), u_2(s), \ldots, u_N(s)\right] ds + \psi^{i(T)} \left[T, x(T)\right], \quad i = 1, 2, \ldots, N. \tag{3.1}$$

where $x(s) = [x_1(s), x_2(s), \ldots, x_m(s)] \in R^m$ is a vector of state variable at time s, $u_i(s) \in U_i \subset R^{m_i}$ is player i's control, and T is time when the game terminates.

The state of the game evolves according to the dynamical equation:

$$\dot{x}(s) = -f\left[s, x(s), u_1(s); u_2(s), \ldots, u_N(s)\right], \qquad \text{and} \qquad x(0) = x^0, \tag{3.2}$$

where $f = [f_1, f_2, \ldots, f_m]$ is a vector valued function. There are integral constraints on ω of these m state variables. Let state variables 1 to ω denote variables with an integral constraint. In particular,

$$\int_0^T f_j \left[s, x(s), u_1(s), u_2(s), \ldots, u_N(s)\right] ds \leq x_j(0) - x_j^T, \tag{3.3}$$

where $j = 1, 2, \ldots, \omega$; x_j^T is the minimum level of state variable j that must be maintained. Specifically, the game would stop if any x_j^T is reached.

Definition 3.1 *An N-tuple of strategies $\{u_i^*(t) = \phi_i^*(t, x); i = 1, 2, \ldots, N\}$ constitutes a Feedback Nash equilibrium (FNE) solution for the game (2.1)–(2.3), if there exist functionals $V^i(t, x)$ for $i \in [1, 2, \ldots, N]$, which satisfy the following conditions:*

$$\begin{aligned} &V^i(t, x) = \\ &\int_0^T g^i \left[s, x(s), \phi_1^*(s, x), \phi_2^*(s, x), \ldots, \phi_N^*(s, x)\right] ds + \psi^{i(T)} \left[T, x(T)\right] \\ &\geq \int_t^{\hat{T}} g^i \left[s, \hat{x}(s), \phi_1^*(s, \hat{x}), \phi_2^*(s, \hat{x}), \ldots, \phi_{i-1}^*(s, \hat{x}), \phi_i(s, \hat{x}), \phi_{i+1}^*(s, \hat{x}), \ldots, \phi_N^*(s, \hat{x})\right] ds \\ &+\psi^{i(\hat{T})} \left[\hat{T}, \hat{x}\left(\hat{T}\right)\right], \end{aligned}$$

$\forall \phi_i(t,x) \in \Omega_i$ *where on the interval* $[t,T]$,

$$\dot{x}(s) = -f[s, x(s), \phi_1^*(s,x), \phi_2^*(s,x), \ldots, \phi_N^*(s,x)], \qquad x(t) = x;$$

and on the interval $\left[t, \hat{T}\right]$,

$$\frac{d\hat{x}(s)}{ds} = -f\left[s, \hat{x}(s), \phi_1^*(s,\hat{x}), \phi_2^*(s,\hat{x}), \ldots, \phi_{i-1}^*(s,\hat{x}), \phi_i(s,\hat{x}), \phi_{i+1}^*(s,\hat{x}), \ldots, \phi_N^*(s,\hat{x})\right],$$

$$\hat{x}(t) = \hat{x}; \text{ and}$$

$$\int_t^T f_j[s, x(s), \phi_1^*(s,x), \phi_2^*(s,x), \ldots, \phi_N^*(s,x)]\, ds \le x_j(t) - x_j^T, \quad \text{for } j = 1, 2, \ldots, \omega;$$

$$\int_t^T f_j[s, \hat{x}(s), \phi_1^*(s,\hat{x}), \phi_2^*(s,\hat{x}), \ldots, \phi_{i-1}^*(s,\hat{x}), \phi_i(s,\hat{x}), \phi_{i+1}^*(s,\hat{x}), \ldots, \phi_N^*(s,\hat{x})]\, ds$$

$$\le \hat{x}_j(t) - x_j^T, \qquad \text{for } j = 1, 2, \ldots, \omega.$$

Let $\phi^*(t,x)$ stands for $[\phi_1^*(t,x), \phi_2^*(t,x), \ldots, \phi_N^*(t,x)]$, and $\phi^{\neq i*}(t,x)$ stands for $[\phi_1^*(t,x), \phi_2^*(t,x), \ldots, \phi_{i-1}^*(t,x), \phi_i(t,x), \phi_{i+1}^*(t,x), \ldots, \phi_N^*(t,x)]$. Invoking the results in Yeung (2000), we obtain the conditions characterizing a FNE solution of the game as follows:

Theorem 3.1 *An N-tuple of strategies* $\{u_i^*(t) = \phi_i^*(t,x); i = 1, 2, \ldots, N\}$ *constitutes a FNE solution for the game (3.1)–(3.3), if there exist functionals* $V^i(t,x)$ *for* $i \in [1, 2, \ldots, N]$, *which satisfy the following set of partial differential equations:*

$$-V_t^i = \max_{\phi_i \in \Omega_i} \left\{ g^i\left[t, x(t), \phi^{\neq i*}(t,x)\right] - V_x^t f\left[t, x(t), \phi^{\neq i*}(t,x)\right] \right\}, \tag{3.4a}$$

$$V^i(T,x) = \psi^i(T,x), \tag{3.4b}$$

$$g^i[T, x(t), \phi^*(T,x)] - V_x^i\{f[T, x(T), \phi^*(T,x)]\} + \psi_t^{i(T)}[T, x(T)] = 0, \tag{3.4c}$$

$$\int_t^T f_j[s, x(s), \phi^*(s,x)]\, ds \le x_j(t) - x_j^T, \text{ for } j = 1, 2, \ldots, \omega; \tag{3.4d}$$

and the terminal time T *is determined by a mapping,* $\tau(t,x) : t \times x \to T$, *which satisfies the subset of (3.4d) with at least one holding as equations.*

Proof. *Condition (3.4a) arises from the standard results of dynamic programming. (3.4b) follows from of the fact that there is no salvage value for the state variables at terminal time T. Condition (3.4c) is a boundary condition for a free terminal horizon problem.*

Condition (3.4d) results from the state dynamics (3.2) and constraints (3.3). To see this, first note that the equilibrium state path can be expressed as:

$$x_j(t) = \int_0^t -f_j[s, x(s), \phi^*(s, x)]\,ds,\ x_j(0) = x_j^0,\ \textit{for}\ j = 1, 2, \ldots, \omega.$$

Then, define

$$y_j(t) = \int_t^T f_j[s, x(s), \phi^*(s, x)]\,ds,$$
$$y_j(0) = x_j^0 - x_j^T = x_j(0) - x_j^T.$$

We then have $y_j(t) = x_j(t) - x_j^T$. Condition (3.4d) follows. Moreover, condition (3.4d) presents a relationship between t, x, and T. A given set of (t, x) determines the value of T via (3.4d). Hence, T is determined by a mapping, $\tau(t, x) : t \times x \rightarrow T$, satisfying (3.4d).

As noted above, a feedback equilibrium consists of strategies which are dependent upon the current state x and time t. The link between the game horizon T and the initial state x^0, given in (3.3), is expected to lead to difficulties in deriving the FNE strategies. However, as demonstrated in Theorem 3.1, an implicit functional relationship between T, t, and x can be found. It is precisely this property that allows us to characterize and derive, for the first time, a FNE solution for differential games with endogenously determined horizons.

Then we examine the subgames in the last time interval of the game (2.1)–(2.3). Consider the case where $\theta_{a_1}^1$ is realized in time interval $[t_1, t_2)$, $\theta_{a_2}^{2[(1,a_1)]}$ is realized in time interval $[t_2, t_3), \ldots,$ and $\theta_{a_\tau}^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}$ is realized in time interval $[t_\tau, t_{\tau+1}]$. In particular, constraint (2.3) will be met at a time $T_{\tau_{[a_1 a_2 \ldots a_\tau]}} \in [t_\tau, t_{\tau+1}]$ and the game will terminate. Hence we can write τ as $\tau_{[a_1 a_2 \ldots a_\tau]}$. In cases where there is no ambiguity, we just use τ.

Denoting $[u_1(s), u_2(s), \ldots, u_N(s)]$ by $u(s)$, the subgame in time interval $[t_\tau, t_{\tau+1}]$ becomes an N-person game in which player i maximizes:

$$E_{t_0}\left\{\int_{t_\tau}^{T_{\tau_{[a_1 a_2 \ldots a_\tau]}}} g^{i(\tau)}\left[\theta_{a_\tau}^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}, s, x(s), u(s)\right] ds \right.$$
$$\left. + q^{i(T)}\left[\theta^T, T_{\tau_{[a_1 a_2 \ldots a_\tau]}}, x\left(T_{\tau_{[a_1 a_2 \ldots a_\tau]}}\right)\right]\right\}$$

$$= \int_{t_\tau}^{T_{\tau[a_1a_2\ldots a_\tau]}} g^{i(\tau)} \left[\theta_{a_\tau}^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}, s, x(s), u(s)\right] ds$$
$$+ \sum_{a=1}^{\eta_T} \lambda_a^T q^{i(T)} \left[\theta_a^T, T_{\tau[a_1a_2\ldots a_\tau]}, x\left(T_{\tau[a_1a_2\ldots a_\tau]}\right)\right], \tag{3.5}$$

subject to the state dynamical equation:

$$\dot{x}(s) = -f[s, x(s), u(s)], \quad x(t_\tau) = x^\tau \equiv \{x_1^\tau, x_2^\tau, \ldots, x_m^\tau\} \in X, \tag{3.6}$$

where $\{x_1^\tau, x_2^\tau, \ldots, x_m^\tau\} > \left\{x_1^T, x_2^T, \ldots, x_m^T\right\}$, and

subject to the condition that the constrained state variables 1 to ω satisfy the following integral constraints:

$$\int_{t_\tau}^{T_{\tau[a_1a_2\ldots a_\tau]}} f_j[s, x(s), u(s)]\, ds \le x_j^\tau - x_j^T, \text{ for } j = 1, 2, \ldots, \omega, \tag{3.7}$$

and at least one $x_j\left(T_{\tau[a_1a_2\ldots a_\tau]}\right) = x_j^T$ at time $T_{\tau[a_1a_2\ldots a_\tau]}$, for $j = 1, 2, \ldots, \omega$, and the game will stop at time $T_{\tau[a_1a_2\ldots a_\tau]}$.

Following Theorem 3.1, we characterize the conditions of a feedback solution for the game (3.5)–(3.7) as:

Lemma 3.1 *An N-tuple of feedback strategies* $\left\{u_{a_\tau}^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t) = \phi_{a_\tau}^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x);\ i \in N\right\}$ *constitutes a Nash equilibrium solution for the game (3.5)–(3.7), if there exist functionals* $V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}(t,x)$: $\left[t_\tau, T_{\tau[a_1a_2\ldots a_\tau]}\right] \times R^n \to R$, *for* $i \in N$, *which satisfy the following set of partial differential equations*[1]*:*

$$-V_t^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} =$$
$$\max_{\phi_{a_\tau}^i \in U^i} \left\{ g^{i(\tau)} \left[\theta_{a_\tau}^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}, t, x(t), \phi_{a_\tau}^{\neq i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x)\right]\right.$$
$$\left. + V_x^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} f\left[t, x(t), \phi_{a_\tau}^{\neq i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x)\right]\right\}, \tag{3.8a}$$

$$V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}\left(T_{\tau[a_1a_2\ldots a_\tau]}, x\right) = \sum_{a=1}^{\eta_T} \lambda_a^T q^{i(T)} \left[\theta_a^T, T_{\tau[a_1a_2\ldots a_\tau]}, x\left(T_{\tau[a_1a_2\ldots a_\tau]}\right)\right], \tag{3.8b}$$

$$g^{i(\tau)} \left[\theta_{a_\tau}^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}, t, x(t), \phi_{a_\tau}^{\neq i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x)\right]$$
$$+ V_x^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} f\left[t, x(t), \phi_{a_\tau}^{\neq i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x)\right]$$

[1] τ is used to denote $\tau_{[a_1a_2\ldots a_\tau]}$ in short.

$$+\sum_{a=1}^{\eta_T} \lambda_a^T q_t^T \left[\theta_a^T, T_{\tau_{[a_1 a_2 \ldots a_\tau]}}, x\left(T_{\tau_{[a_1 a_2 \ldots a_\tau]}}\right)\right] = 0,$$

$$\text{when } t = T_{\tau_{[a_1 a_2 \ldots a_\tau]}}, \tag{3.8c}$$

$$\int_t^{T_{\tau_{[a_1 a_2 \ldots a_\tau]}}} f_j \left[s, x(s), \phi_{a_\tau}^{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(s,x)\right] ds \leq x_j(t) - x_j^T,$$

$$\text{and } x_j(t_\tau) = x_j^\tau \in X, \qquad \text{for } j = 1, 2, \ldots, \omega, \tag{3.8d}$$

and the time interval $T_{\tau_{[a_1 a_2 \ldots a_\tau]}} - t$ *is determined by a mapping* $\zeta(t,x) : t \times x \to T_{\tau_{[a_1 a_2 \ldots a_\tau]}} - t$*, which satisfies the subset of (3.8d) holding as equations.*

We then consider the subgames in the time intervals preceding the interval in which the terminal time $T_{\tau_{[a_1 a_2 \ldots a_\tau]}}$ occurs. In order to formulate the subgame in the second last time interval $[t_{\tau-1}, t_\tau)$, it is necessary to identify the terminal payoffs at time t_τ. For the case where $\theta_{a_1}^1$ has occurred in time interval $[t_1, t_2)$, $\theta_{a_2}^{2[(1,a_1)]}$ has occurred in time interval $[t_2, t_3), \ldots$, and $\theta_{a_{\tau-2}}^{\tau-2[(1,a_1)(2,a_2)\ldots(\tau-3,a_{\tau-3})]}$ has occurred in time interval $[t_{\tau-2}, t_{\tau-1})$, and $\theta^{\tau-1}$ assumes the value $\theta_{a_{\tau-1}}^{\tau-1[(1,a_1)(2,a_2)\ldots(\tau-2,a_{\tau-2})]}$ in time interval $[t_{\tau-1}, t_\tau)$, the random variable θ^τ has a range

$$\left\{\theta_1^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}, \theta_2^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}, \ldots, \theta_{\eta_{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}}^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}\right\}$$

and corresponding probabilities

$$\left\{\lambda_1^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}, \lambda_2^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}, \ldots, \lambda_{\eta_{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}}^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}\right\}.$$

Given state x, in Lemma 2.1 player i's value functions at time t_τ equals $V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}(t_\tau, x)$, for $i \in N$ and $a_\tau \in \left[1, 2, \ldots, \eta_{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}\right]$. Taking mathematical expectation, the terminal payoffs of player i, for $i \in N$, in the subgame over the time interval $[t_{\tau-1}, t_\tau]$ are obtained as:

$$\sum_{a_\tau=1}^{\eta_{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}} \left[\lambda_{a_\tau}^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]} V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}(t_\tau, x)\right]. \tag{3.9}$$

In the time interval $[t_k, t_{k+1}]$, for $k = \tau - 1$, where $\theta_{a_k}^{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}$ is realized, for $a_k = 1, 2, \ldots, \eta_k$, player i, for $i \in N$, maximizes:

$$\int_{t_k}^{t_{k+1}} g^{i(k)} \left[\theta_{a_k}^{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}, s, x(s), u(s)\right] ds$$

$$+\sum_{a_{k+1}=1}^{\eta_{k+1[(1,a_1)(2,a_2)\dots(k,a_k)]}} \lambda_{a_{k+1}}^{k+1[(1,a_1)(2,a_2)\dots(k,a_k)]} V^{i[(1,a_1)(2,a_2)\dots(k,a_k)]a_{k+1}}\left(t_{k+1},x\left(t_{k+1}\right)\right), \tag{3.10}$$

subject to the dynamical equation:

$$\dot{x}(s) = -f\left[s, x(s), u(s)\right], \quad x(t_k) = x^k \in X. \tag{3.11}$$

We can characterize the conditions of a feedback solution for the game (3.10)–(3.11) as:

Lemma 3.2 *An N-tuple of feedback strategies* $\left\{u_{a_\tau}^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(t) = \phi_{a_\tau}^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(t,x);\ i \in N\right\}$ *constitutes a Nash equilibrium solution for the game (3.10)-(3.11), if there exist functionals* $V^{i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]a_k}(t,x) : [t_k, t_{k+1}] \times R^n \to R$*, for* $i \in N$*, which satisfy the following set of partial differential equations:*

$$\begin{aligned}
&-V_t^{i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]a_k} = \\
&\max_{\phi_{a_k}^i \in U^i} \left\{ g^{i(k)}\left[\theta_{a_k}^{k[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]}, t, x(t), \phi_{a_k}^{\neq i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]*}(t,x)\right] \right. \\
&\quad \left. + V_x^{i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]a_k} f\left[t, x(t), \phi_{a_k}^{\neq i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]*}(t,x)\right]\right\},
\end{aligned} \tag{3.12a}$$

$$\begin{aligned}
&V^{i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]a_k}(t_{k+1}, x) = \\
&\sum_{a_{k+1}=1}^{\eta_{k+1[(1,a_1)(2,a_2)\dots(k,a_k)]}} \left[\lambda_{a_{k+1}}^{k+1[(1,a_1)(2,a_2)\dots(k,a_k)]} V^{i[(1,a_1)(2,a_2)\dots(k,a_k)]a_{k+1}}(t_{k+1}, x)\right].
\end{aligned} \tag{3.12b}$$

Proof. This result follows readily from the optimality conditions for feedback solution of differential games due to Isaacs (1965) and Bellman (1957), and the definition of Nash equilibrium (1951).

Repeating the analysis in Lemma 4.2, we can characterize the solution for the subgames in the intervals $[t_k, t_{k+1})$, for $k = \tau-2, \tau-3, \dots, 2, 1$.

Finally, in the time interval $[t_0, t_1]$, player i, for $i \in N$, maximizes:

$$\int_{t_0}^{t_1} g^{i(0)}\left[s, x(s), u(s)\right] ds + \sum_{a_1=1}^{\eta_1} \lambda_{a_1}^1 V^{i(a_1)}\left(t_1, x(t_1)\right), \tag{3.13}$$

subject to the state dynamical equation:

$$\dot{x}(s) = -f[s, x(s), u(s)], \quad \text{with given initial states } x(t_0) = x^0 \in X. \tag{3.14}$$

The conditions of a feedback solution of the game (3.13)–(3.14) can be characterized as:

Lemma 3.3 *An N-tuple of feedback strategies $\{u_0^{i*}(t) = \phi_0^{i*}(t,x)\,;\ i \in N\}$ constitutes a Nash equilibrium solution for the game (3.13)–(3.14), if there exist functionals $V^{i(0)}(t,x) : [t_0, t_1] \times R^n \to R$, for $i \in N$, which satisfy the following set of partial differential equations:*

$$-V_t^{i(0)} = \max_{\phi_0^i \in U^i} \left\{ g^{i(0)}\left[t, x(t), \phi_0^{\neq i*}(t,x)\right] + V_x^{i(0)} f\left[t, x(t), \phi_0^{\neq i*}(t,x)\right] \right\}, \tag{3.15a}$$

$$V^{i(0)}(t_1, x) = \sum_{a_1=1}^{\eta_1} \left[\lambda_{a_{a_1}}^1 V^{1(a_1)}(t_1, x)\right]. \tag{3.15b}$$

Proof. *Again, this result follows readily from the optimality conditions for feedback solution of differential games due to Isaacs (1965) and Bellman (1953), and the definition of Nash equilibrium (1953).*

A feedback solution of the game (2.1)–(2.3) can be obtained as:

Theorem 3.2 *A set of feedback strategies $\left\{\left[u_{a_k}^{i[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]*}(t) = \phi_{a_k}^{i[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]*}(t,x)\right], [u_0^{i*}(t) = \phi_0^{i*}(t,x)],\ i \in N \text{ and } k = 1, 2, \ldots, \tau_{[a_1 a_2 \ldots a_\tau]}\right\}$ contingent upon the event that $\theta^1 = \theta_{a_1}^1, \theta^2 = \theta_{a_2}^{2[(1,a_1)]}, \ldots, \theta^k = \theta_{a_2}^{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}$, for $a_1 \in [1, 2, \ldots, \eta_1]$, and $a_2 \in \left[1, 2, \ldots, \eta_{2[(1,a_1)]}\right]$, and $a_3 \in \left[1, 2, \ldots, \eta_{3[(1,a_1)(2,a_2)]}\right]$, and $a_k \in [1, 2, \ldots \ldots, \eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}\Big]$, constitutes a Nash equilibrium solution for the game (2.1)–(2.3), if there exist functionals $V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}(t,x) : \left[t_{\tau_{[a_1 a_2 \ldots a_\tau]}}, T_{\tau_{[a_1 a_2 \ldots a_\tau]}}\right] \times R^n \to R$, for $i \in N$, which satisfy the following set of partial differential equations:*

$$-V_t^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} =$$

$$\max_{\phi_{a_\tau}^i \in U^i} \left\{ g^{i(\tau)}\left[\theta_{a_\tau}^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}, t, x(t), \phi_{a_\tau}^{\neq i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x)\right]\right.$$
$$\left. + V_x^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} f\left[t, x(t), \phi_{a_\tau}^{\neq i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x)\right]\right\}, \tag{3.16a}$$

$$V^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}\left(T_{\tau_{[a_1a_2\dots a_\tau]}},x\right)=\sum_{a=1}^{\eta_T}\lambda_a^T q^{i(T)}\left[\theta_a^T,T_{\tau_{[a_1a_2\dots a_\tau]}},x\left(T_{\tau_{[a_1a_2\dots a_\tau]}}\right)\right], \tag{3.16b}$$

$$\begin{aligned}&g^{i(\tau)}\left[\theta_{a_\tau}^{\tau[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]},t,x(t),\phi_{a_\tau}^{\neq i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(t,x)\right]\\&\quad+V_x^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}f\left[t,x(t),\phi_{a_\tau}^{\neq i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(t,x)\right]\\&\quad+\sum_{a=1}^{\eta_T}\lambda_a^T q_t^T\left[\theta_a^T,T_{\tau_{[a_1a_2\dots a_\tau]}},x\left(T_{\tau_{[a_1a_2\dots a_\tau]}}\right)\right]=0,\\&\qquad\qquad \textit{when } t=T_{\tau_{[a_1a_2\dots a_\tau]}},\end{aligned} \tag{3.16c}$$

$$\begin{aligned}&\int_t^{T_{\tau_{[a_1a_2\dots a_\tau]}}} f_j\left[s,x(s),\phi_{a_\tau}^{[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(s,x)\right]ds\le x_j(t)-x_j^T,\\&\qquad \textit{and } x_j(t_\tau)=x_j^T\in X,\qquad \textit{for } j=1,2,\dots,\omega,\end{aligned} \tag{3.16d}$$

and the time interval $T_{\tau_{[a_1a_2\dots a_\tau]}}-t$ *is determined by a mapping* $\zeta(t,x):t\times x\to T_{\tau_{[a_1a_2\dots a_\tau]}}-t$*, which satisfies the subset of (3.8d) holding as equations; and functionals* $V^{i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]a_k}(t,x):[t_k,t_{k+1}]\times R^n\to R$*, for* $i\in N$ *and* $k\in\left[1,2,\dots,\tau_{[a_1a_2\dots a_\tau]}-1\right]$*, which satisfy the following set of partial differential equations:*

$$\begin{aligned}&-V_t^{i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]a_k}=\\&\max_{\phi_{a_k}^i\in U^i}\left\{g^{i(k)}\left[\theta_{a_k}^{k[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]},t,x(t),\phi_{a_k}^{\neq i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]*}(t,x)\right]\right.\\&\qquad\left.+V_x^{i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]a_k}f\left[t,x(t),\phi_{a_k}^{\neq i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]*}(t,x)\right]\right\},\end{aligned} \tag{3.17a}$$

$$\begin{aligned}&V^{i[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]a_k}(t_{k+1},x)=\\&\sum_{a_{k+1}=1}^{\eta_{k+1[(1,a_1)(2,a_2)\dots(k,a_k)]}}\left[\lambda_{a_{k+1}}^{k+1[(1,a_1)(2,a_2)\dots(k,a_k)]}V^{i[(1,a_1)(2,a_2)\dots(k,a_k)]a_{k+1}}(t_{k+1},x)\right];\end{aligned} \tag{3.17b}$$

and functionals $V^{i(0)}(t,x):[t_0,t_1]\times R^n\to R$*, for* $i\in N$*, which satisfy the following set of partial differential equations:*

$$-V_t^{i(0)}=\max_{\phi_0^i\in U^i}\left\{g^{i(0)}\left[t,x(t),\phi_0^{\neq i*}(t,x)\right]+V_x^{i(0)}f\left[t,x(t),\phi_0^{\neq i*}(t,x)\right]\right\}, \tag{3.18a}$$

$$V^{i(0)}(t_1,x)=\sum_{a_1=1}^{\eta_1}\left[\lambda_{a_1}^1 V^{1(a_1)}(t_1,x)\right]; \tag{3.18b}$$

and the conditions

$$\int_{t_0}^{t_1} f_j\left[s, x(s), \phi_0^*(s,x)\right] ds + \sum_{k=1}^{\tau[a_1 a_2 \ldots a_\tau]} \int_{t_k}^{t_{k+1}} f_j\left[s, x(s), \phi_{a_k}^{[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]*}(s,x)\right] ds$$
$$+ \int_{t_{\tau[a_1 a_2 \ldots a_\tau]}}^{T_{\tau[a_1 a_2 \ldots a_\tau]}} f_j\left[s, x(s), \phi_{a_\tau}^{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(s,x)\right] ds \leq x_j^0 - x_j^T,$$
$$\text{for } j = 1, 2, \ldots, \omega. \tag{3.19}$$

Proof. The requirements imposed by (3.16a) to (3.16d) follow from Theorem 3.1 and Lemma 3.1. The results in (3.17) and (3.18) are obtained from Bellamn's technique of optimality as indicated in Lemmas 3.2 and 3.3. Finally, (3.19) yields the requirement of the integral constraints.

4 A two-state problem with endogenous horizon and uncertain payoffs

Consider a 2-person game in which player $i \in [1,2]$ minimizes the cost function:

$$\begin{aligned} J^i &= E_0\left\{\int_0^{t_1} \left[b u_{i1}(s) u_{i2}(s) + B_0 x_1(s) x_2(s)\right] e^{-rs} ds \right. \\ &+ \sum_{h=1}^{\tau-1} \int_{t_h}^{t_{h+1}} \left[b u_{i1}(s) u_{i2}(s) + B^h x_1(s) x_2(s)\right] e^{-rs} ds \\ &+ \left. \int_{t_\tau}^{T} \left[b u_{i1}(s) u_{i2}(s) + B^\tau x_1(s) x_2(s)\right] e^{-rs} ds - \frac{1}{6r}\left(kb\sqrt{r^2 + \frac{12B^\tau}{b}} - krb\right) e^{-rt}\right\}, \end{aligned} \tag{4.1}$$

subject to

$$\dot{x}_1(s) = -\sum_{j=1}^{2} u_{j1}(s) - \frac{k}{x_2(s)}, \qquad x_1(0) = x_1^0 > 0, \tag{4.2}$$

$$\dot{x}_2(s) = -\sum_{j=1}^{2} u_{j2}(s), \qquad x_2(0) = x_2^0 > 0, \tag{4.3}$$

where $u_{i1}(s) \in U_{i1} \in R^+$ and $u_{i2}(s) \in U_{i2} \in R^+$ are the controls of player i, and r, k are positive parameters. While the value of B_0 is known with certainty in the time interval $[t_0, t_1)$, B^h is a random variable realizable in the time interval $[t_h, t_{h+1})$, for $h = 1, 2, \ldots$, with range $[B_1, B_2, \ldots, B_\eta]$ and corresponding probabilities $[\lambda_1, \lambda_2, \ldots, \lambda_\eta]$.

The game would end when any of the state variables reaches zero. Consider the case when $B^1_{a_1}$ is realized in time interval $[t_1, t_2)$, $B^{2[(1,a_1)]}_{a_2}$ is realized in time interval $[t_2, t_3), \ldots$, and $B^{\tau[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]}_{a_\tau}$ is realized in time interval $[t_\tau, t_{\tau+1})$. In particular, either $x_1(s)$ or $x_2(s)$ or both will reach zero a time $s = T_{\tau_{[a_1a_2\ldots a_\tau]}} \in [t_\tau, t_{\tau+1})$, when the game will terminate. Hence we can write τ as $\tau_{[a_1a_2\ldots a_\tau]}$. In cases when there is no ambiguity, we just use τ.

Invoking Theorem 3.2, the conditions characterizing a feedback solution of the game (4.1)–(4.3) can be obtained as follows.

Corollary 4.1 *A set of feedback strategies* $\Big\{\Big[u^{i[(1,a_1)(2,a_2)\ldots(h-1,a_{h-1})]*}_{a_h[1]}(t) = \phi^{i[(1,a_1)(2,a_2)\ldots(h-1,a_{h-1})]*}_{a_h[1]}(t,x)\Big], \Big[u^{i[(1,a_1)(2,a_2)\ldots(h-1,a_{h-1})]*}_{a_h[2]}(t) = \phi^{i[(1,a_1)(2,a_2)\ldots(h-1,a_{h-1})]*}_{a_h[2]}(t,x)\Big],$ $\Big[u^{i*}_{0[1]}(t) = \phi^{i*}_{0[1]}(t,x)\Big], \Big[u^{i*}_{0[2]}(t) = \phi^{i*}_{0[1]}(t,x)\Big]; i \in [1,2]$ *and* $h = 1, 2, \ldots, \tau_{[a_1a_2\ldots a_\tau]}\Big\}$, *contingent upon the events that* $B^1 = B_{a_1}, B^2 = B_{a_2}, \ldots, B^{\tau_{[a_1a_2\ldots a_\tau]}} = B_{a_{\tau_{[a_1a_2\ldots a_\tau]}}}$ *for* $a_h \in [1, 2, \ldots, \eta]$ *and* $h \in \Big[1, 2, \ldots, \tau_{[a_1a_2\ldots a_\tau]}\Big]$, *constitutes a Nash equilibrium solution for the game (4.1)–(4.3), if there exist functionals* $V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}(t,x) : \Big[t_{\tau_{[a_1a_2\ldots a_\tau]}}, T_{\tau_{[a_1a_2\ldots a_\tau]}}\Big] \times R^n \to R$, *for* $i \in [1,2]$, *which satisfy the following set of partial differential equations*[2]:

$$
\begin{aligned}
&-V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}_t \\
&= \max_{u_{i1},u_{i2}} \Bigg\{ \Big[b u_{i1}(t) u_{i2}(t) + B_{a_{\tau_{[a_1a_2\ldots a_\tau]}}} x_1 x_2 \Big] e^{-rt} \\
&\quad - V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}_{x_1} \left[\sum_{\substack{j=1\\ j\neq i}}^{2} u^{j[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}_{a_\tau[1]}(t) + u_{i1}(t) + \frac{k}{x_2} \right] \\
&\quad - V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}_{x_2} \left[\sum_{\substack{j=1\\ j\neq i}}^{2} u^{j[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}_{a_\tau[2]}(t) + u_{i2}(t) \right] \Bigg\},
\end{aligned}
\tag{4.4a}
$$

$$
\begin{aligned}
&V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}\left(T_{\tau_{[a_1a_2\ldots a_\tau]}}, x_1, x_2\right) = \\
&\quad -\frac{1}{6r}\left(kb\sqrt{r^2 + 12B_{a_{\tau_{[a_1a_2\ldots a_\tau]}}}/b} - krb\right) e^{-rT_{\tau_{[a_1a_2\ldots a_\tau]}}},
\end{aligned}
\tag{4.4b}
$$

$$
\Big[b u^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}_{a_\tau[1]}(t)\, u^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}_{a_\tau[1]}(t) + B_{a_{\tau_{[a_1a_2\ldots a_\tau]}}} x_1 x_2\Big] e^{-rt}
$$

[2] τ is used to denote $\tau_{[a_1a_2\ldots a_\tau]}$ in short.

$$-V_{x_1}^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}\left[\sum_{j=1}^{2}u_{a_\tau[1]}^{j[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(t)+\frac{k}{x_2}\right]$$
$$-V_{x_2}^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}\left[\sum_{j=1}^{2}u_{a_\tau[1]}^{j[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(t)\right]$$
$$+\frac{1}{6}\left(kb\sqrt{r^2+12B_{a_{\tau[a_1a_2\dots a_\tau]}}/b}-krb\right)e^{-rT_{\tau[a_1a_2\dots a_\tau]}}=0,\ \textit{at time}\ t=T_{\tau[a_1a_2\dots a_\tau]},\tag{4.4c}$$

and for $t\in\left[t_{\tau[a_1a_2\dots a_\tau]},T_{\tau[a_1a_2\dots a_\tau]}\right]$,

$$\int_t^{T_{\tau[a_1a_2\dots a_\tau]}}\left[\sum_{j=1}^{2}u_{a_\tau[1]}^{j[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(s)+\frac{k}{x_2(s)}\right]ds\leq x_1(t),\tag{4.4d}$$
$$\int_t^{T_{\tau[a_1a_2\dots a_\tau]}}\left[\sum_{j=1}^{2}u_{a_\tau[2]}^{j[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(s)\right]ds\leq x_2(t),\tag{4.4e}$$

and the time interval $T_{\tau[a_1a_2\dots a_\tau]}-t$ *is determined by a mapping* $\zeta(t,x):t\times x\rightarrow T_{\tau[a_1a_2\dots a_\tau]}-t$, *which satisfies the subset of (4.4d) and (4.4e) holding as equations; and functionals* $V^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t,x):[t_h,t_{h+1}]\times R^n\rightarrow R$, *for* $i\in[1,2]$ *and* $h\in\left[1,2,\dots,\tau_{[a_1a_2\dots a_\tau]}-1\right]$, *which satisfy the following set of partial differential equations:*

$$-V_t^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}$$
$$=\max_{u_{i1},u_{i2}}\Bigg\{[bu_{i1}(t)u_{i2}(t)+B_{a_h}x_1x_2]e^{-rt}$$
$$-V_{x_1}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}\left[\sum_{\substack{j=1\\j\neq i}}^{2}u_{a_h[1]}^{j[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]*}(t)+u_{i1}(t)+\frac{k}{x_2}\right]$$
$$-V_{x_2}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}\left[\sum_{\substack{j=1\\j\neq i}}^{2}u_{a_h[2]}^{j[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]*}(t)+u_{i2}(t)\right]\Bigg\},\tag{4.5a}$$

$$V^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t_{h+1},x)=\sum_{a_{h+1}=1}^{\eta}\left[\lambda_{a_{h+1}}V^{i[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{h+1},x)\right];\tag{4.5b}$$

and functionals $V^{i(0)}(t,x):[0,t_1]\times R^n\rightarrow R$, *for* $i\in[1,2]$, *which satisfy the following set of partial differential equations:*

$$-V_t^{i(0)}=\max_{u_{i1},u_{i2}}\left\{[bu_{i1}(t)u_{i2}(t)+B_0x_1x_2]e^{-rt}-V_{x_1}^{i(0)}\left[\sum_{\substack{j=1\\j\neq i}}^{2}u_{0[1]}^{j*}(t)+u_{i1}(t)+\frac{k}{x_2}\right]\right.$$

$$-V_{x_2}^{i(0)}\left[\sum_{\substack{j=1\\j\neq i}}^{2} u_{0[2]}^{j*}(t)+u_{i2}(t)\right]\Bigg\},$$

(4.6a)

$$V^{i(0)}(t_1,x),\ \sum_{a_1=1}^{\eta}\left[\lambda_{a_1}V^{i(a_1)}(t_1,x)\right], \tag{4.6b}$$

and the conditions

$$\int_0^{t_1}\left[\sum_{j=1}^{2}u_{0[1]}^{j*}(s)+\frac{k}{x_2(s)}\right]ds+\sum_{k=1}^{\tau_{[a_1a_2\ldots a_\tau]}}\int_{t_h}^{t_{h+1}}\left[\sum_{j=1}^{2}u_{a_h[1]}^{j[(1,a_1)(2,a_2)\ldots(h-1,a_{h-1})]*}(s)+\frac{k}{x_2(s)}\right]ds$$
$$+\int_{t_{\tau_{[a_1a_2\ldots a_\tau]}}}^{T_{\tau_{[a_1a_2\ldots a_\tau]}}}\left[\sum_{j=1}^{2}u_{a_\tau[1]}^{j[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(s)+\frac{k}{x_2(s)}\right]ds\leq x_1^0, \tag{4.7a}$$

$$\int_0^{t_1}\left[\sum_{j=1}^{2}u_{0[2]}^{j*}(s)\right]+\sum_{k=1}^{\tau_{[a_1a_2\ldots a_\tau]}}\int_{t_h}^{t_{h+1}}\left[\sum_{j=1}^{2}u_{a_h[2]}^{j[(1,a_1)(2,a_2)\ldots(h-1,a_{h-1})]*}(s)\right]ds$$
$$+\int_{t_{\tau_{[a_1a_2\ldots a_\tau]}}}^{T_{\tau_{[a_1a_2\ldots a_\tau]}}}\left[\sum_{j=1}^{2}u_{a_\tau[2]}^{j[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(s)\right]ds\leq x_2^0. \tag{4.7b}$$

Proposition 4.1 *The functionals* $V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}(t,x)$, *for* $i\in[1,2]$ *and* $t\in\left[t_{\tau_{[a_1a_2\ldots a_\tau]}},T_{\tau_{[a_1a_2\ldots a_\tau]}}\right]$, *can be obtained as:*

$$V^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}(t,x)=A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}\left[x_1x_2-\frac{k}{r}\right]e^{-rt}, \tag{4.8}$$

and $T_{\tau_{[a_1a_2\ldots a_\tau]}}$ *is determined by the mapping*

$$T_{\tau_{[a_1a_2\ldots a_\tau]}}=t+\frac{b}{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}}\ln\left[\frac{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}x_1x_2}{kb}+1\right], \tag{4.9}$$

where $A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}=\frac{1}{6}\left(b\sqrt{r^2+12B_{a_{\tau_{[a_1a_2\ldots a_\tau]}}}/b}-rb\right)$.

Proof. See Appendix.

For each realization $B_{a_{\tau_{[a_1a_2\ldots a_\tau]}}}$, for $a_{\tau_{[a_1a_2\ldots a_\tau]}}\in[1,2,\ldots\eta]$, a unique value of $\left(T_{\tau_{[a_1a_2\ldots a_\tau]}}-t_{\tau_{[a_1a_2\ldots a_\tau]}}\right)$ and a corresponding positive value of $A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}$ are obtained.

The game equilibrium strategies in time interval $\left[t_{\tau_{[a_1a_2\ldots a_\tau]}},T_{\tau_{[a_1a_2\ldots a_\tau]}}\right]$ are obtained as:

$$u_{a_\tau[1]}^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x)=\frac{A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}x_1}{b},\text{ and}$$
$$u_{a_\tau[2]}^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x)=\frac{A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}x_2}{b}. \tag{4.10}$$

The value functions $V^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t,x) : [t_h, t_{h+1}] \times R^n \to R$, for $i \in [1,2]$ and $h \in \left[1, 2, \dots, \tau_{[a_1 a_2 \dots a_\tau]} - 1\right]$, are obtained as follows.

Proposition 4.2

$$V^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t,x) =$$
$$A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)\left[x_1 x_2 - \frac{k}{r}\right] e^{-rt} + C_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t) \quad (4.11)$$

where $A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)$ *and* $C_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)$ *satisfy:*

$$\dot{A}_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t) =$$
$$\frac{1}{b}\left(3A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)^2\right) + rA_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t) - B_{a_h},$$
$$\dot{C}_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t) = \frac{k}{r}\dot{A}_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)\exp(-rt),$$
$$A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t_{h+1}) = \sum_{a_{h+1}=1}^{\eta} \lambda_{a_{h+1}} A_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{k+1}),$$
$$C_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t_{h+1}) = \sum_{a_{h+1}}^{\eta} \lambda_{h+1} C_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{h+1}). \quad (4.12)$$

Proof. Performing the indicated maximization in (4.5a), we obtain:

$$u_{a_h[1]}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]*}(t,x) = V_{x_2}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}\frac{e^{rt}}{b}$$
$$u_{a_h[2]}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]*}(t,x) = V_{x_1}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}\frac{e^{rt}}{b}, \text{ for } i = 1,2.$$

Substitute $u_{a_h[1]}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]*}(t,x)$ *and* $u_{a_h[2]}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]*}(t,x)$ *for* $i \in [1,2]$ *above into (4.5a)–(4.5b). Obtain the partial derivatives* $V_{x_1}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}V(t,x)$, $V_{x_2}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}V(t,x)$, *and* $V_t^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}V(t,x)$ *using (4.12), and then substitute them into (4.5a)–(4.5b). Proposition 4.2 follows.*

From (4.12), we readily observe that $\dot{A}_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)$ is positive/negative if $A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)$ is greater/less than $\frac{1}{6}\left(b\sqrt{r^2 + 12B_{a_h}/b} - rb\right)$. For given $\sum_{a_{h+1}=1}^{\eta} \lambda_{a_{h+1}} A_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{k+1}) < \frac{1}{6}\left(b\sqrt{r^2 + 12B_{a_h}/b} - rb\right)$, a time path $\left\{A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)\right\}_{t=t_h}^{t_{h+1}}$, starting from a level $A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t_h)$ which is greater than $\sum_{a_{h+1}=1}^{\eta} \lambda_{a_{h+1}} A_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{k+1})$ and less than $\frac{1}{6}\left(b\sqrt{r^2 + 12B_{a_h}/b} - rb\right)$, can be found, to yield $A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t_{h+1}) = \sum_{a_{h+1}=1}^{\eta} \lambda_{a_{h+1}} A_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{k+1})$. In particular, $A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)$ will decrease over time, and reach the level $\sum_{a_{h+1}=1}^{\eta} \lambda_{a_{h+1}} A_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{k+1})$ at time t_{h+1}. On the other hand, for given $\sum_{a_{h+1}=1}^{\eta} \lambda_{a_{h+1}} A_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{k+1}) > \frac{1}{6}\left(b\sqrt{r^2+12B_{a_h}/b} - rb\right)$, a time path $\left\{A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)\right\}_{t=t_h}^{t_{h+1}}$, starting from a level $A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t_h)$ which is less than $\sum_{a_{h+1}=1}^{\eta} \lambda_{a_{h+1}} A_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{k+1})$ and greater than $\frac{1}{6}\left(b\sqrt{r^2+12B_{a_h}/b} - rb\right)$, can be found, to yield $A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t_{h+1}) = \sum_{a_{h+1}=1}^{\eta} \lambda_{a_{h+1}} A_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{k+1})$. In particular, $A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)$ will increase over time and reach the level $\sum_{a_{h+1}=1}^{\eta} \lambda_{a_{h+1}} A_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{k+1})$ at time t_{h+1}.

Since $\dot{C}_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t) = \frac{k}{r}\dot{A}_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)\exp(-rt)$, its sign is the same as that of $\dot{A}_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)$. When $\dot{A}_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)$ is negative/positive, the time path $\left\{C_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)\right\}_{t=t_h}^{t_{h+1}}$ therefore represents a monotonically decreasing/increasing trajectory, which would reach the level $\sum_{a_{h+1}}^{\eta} \lambda_{h+1} C_{[(1,a_1)(2,a_2)\dots(h,a_h)]a_{h+1}}(t_{k+1})$ at time t_{h+1}.

The game equilibrium strategies are obtained as:

$$
\begin{aligned}
u_{a_h[1]}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]*}(t,x) &= \frac{A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)\,x_1}{b}, \text{ and} \\
u_{a_h[2]}^{i[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]*}(t,x) &= \frac{A_{[(1,a_1)(2,a_2)\dots(h-1,a_{h-1})]a_h}(t)\,x_2}{b}, \\
&\text{for } i=1,2, \text{ and } h=1,2,\dots,\tau_{[a_1a_2\dots a_\tau]}-1.
\end{aligned} \tag{4.13}
$$

The value functions $V^{i(0)}(t,x) : [0,t_1] \times R^n \to R$, for $i \in [1,2]$, can be obtained as:

Proposition 4.3

$$
V^{i[0]}(t,x_1,x_2) = A_{[0]}(t)\left[x_1x_2 - \frac{k}{r}\right]e^{-rt} + C_{[0]}(t), \quad \textit{for } i \in [1,2]; \tag{4.14}
$$

where $A_{[0]}(t)$ *and* $C_{[0]}(t)$ *satisfy:*

$$
\dot{A}_{[0]}(t) = \frac{3A_{[0]}(t)^2}{b} + rA_{[0]}(t) - B_0,
$$

$$\dot{C}_{[0]}(t) = \frac{k}{r}\dot{A}_{[0]}(t)\exp(-rt),$$
$$A_{[0]}(t_1) = \sum_{a_1=1}^{\eta} \lambda_{a_1} A_{[1,a_1]}(t_1),$$
$$C_{[0]}(t_1) = \sum_{a_1=1}^{\eta} \lambda_{a_1} C_{[1,a_1]}(t_1). \tag{4.15}$$

Proof. Follow the proof of Proposition 4.2.

The game equilibrium strategy for player i in the time interval $[0, t_1)$ is obtained as:

$$u^{i*}_{0[1]}(t) = \frac{A_{[0]}(t)\, x_2(t)}{b}, \text{ and } u^{i*}_{0[2]}(t) = \frac{A_{[0]}(t)\, x_1(t)}{b}, \text{ for } i = 1, 2, \tag{4.16}$$

if neither $x_1(t)$ nor $x_2(t)$ reaches 0 at a time $t \in [0, t_1]$.

If either $x_1(t)$ or $x_2(t)$ reaches 0 at a time $t \in [0, t_1]$, the analysis reverts back to the endogenous horizon game (3.1)–(3.2). At time t_1, if $B^1 = B_{a_1} \in [B_1, B_2, \ldots, B_\eta]$ and either $x_1(t)$ or $x_2(t)$ reaches 0 at a time $t \in [t_1, t_2]$, player i, for $i \in [1, 2]$, would adopt game equilibrium strategies specified in (4.10). If neither $x_1(t)$ nor $x_2(t)$ reaches 0 at a time before t_2 and $B^1 = B_{a_1} \in [B_1, B_2, \ldots, B_\eta]$, the game will continue with player i, for $i \in [1, 2]$, adopting game equilibrium strategies specified in (4.13). The game will continue with players using the strategies (4.13) over time intervals when neither $x_1(t)$ nor $x_2(t)$ reaches 0. Finally in the time interval when either $x_1(t)$ or $x_2(t)$ reaches 0, both players will employ strategies (4.10) and the game will end at time $T_{\tau[a_1 a_2 \ldots a_\tau]}$.

5 Conclusions

In this paper, we propose a new class of games - designated as *endogenous-horizon randomly furcating stochastic differential games.* The scope of endogenous-horizon differential games is widened by the introduction of uncertainty in the sense of random shocks in the players' future payoffs. In particular, the term "randomly furcating" is introduced to emphasize that a particularly useful way to analyze such situations is to assume that payoffs change at any instant of time according to known probability distributions defined in terms of multiple-branching stochastic processes. Exploiting this new approach, complicated game problems with endogenous horizons can be readily analyzed and theorems characterizing feedback Nash equilibrium solutions derived for the first time.

Appendix: Proof of Proposition 4.1.

Performing the indicated maximization in (4.4a), we obtain:

$$u_{a_\tau[1]}^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(t,x) = V_{x_2}^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}\frac{e^{rt}}{b}$$
$$u_{a_\tau[2]}^{j[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]*}(t,x) = V_{x_1}^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}\frac{e^{rt}}{b}, \text{ for } i = 1,2. \quad \text{(A.1)}$$

Using the proposed solution of the value functions in Proposition 4.1 we obtain

$$V_{x_1}^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}(t,x) = A_{[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}(t)\,x_2 e^{-rt},$$
$$V_{x_2}^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}(t,x) = A_{[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}(t)\,x_1 e^{-rt}, \text{ and}$$
$$V_{t}^{i[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}(t,x) = -rA_{[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}(t)\left[x_1x_2 - \frac{k}{r}\right].e^{-rt}. \quad \text{(A.2)}$$

Upon substituting results in (A.1) and (A.2) into (4.4a)–(4.4b), we have:

$$-rA_{[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}(t)\left[x_1x_2 - \frac{k}{r}\right]e^{-rt}$$
$$= B_{a_{\tau[a_1a_2\dots a_\tau]}}x_1x_2e^{-rT} - \frac{3A^2_{[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a}}{b}x_1x_2e^{-rT}A_{[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}ke^{-rt}, \quad \text{(A.3a)}$$

$$A_{[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}[x_1x_2 - k/r]\,e^{-rT_{\tau[a_1a_2\dots a_\tau]}} =$$
$$-\frac{1}{6r}\left(kb\sqrt{r^2 + 12B_{a_{\tau[a_1a_2\dots a_\tau]}}/b} - kbr\right)e^{-rT_{\tau[a_1a_2\dots a_\tau]}}, \qquad \text{at time } T_{\tau[a_1a_2\dots a_\tau]}, \quad \text{(A.3b)}$$

$$B_{a_{\tau[a_1a_2\dots a_\tau]}}x_1x_2e^{-rT_{\tau[a_1a_2\dots a_\tau]}} - \frac{3A^2_{[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a}}{b}x_1x_2e^{-rT_{\tau[a_1a_2\dots a_\tau]}}$$
$$-A_{[(1,a_1)(2,a_2)\dots(\tau-1,a_{\tau-1})]a_\tau}ke^{-rT_{\tau[a_1a_2\dots a_\tau]}}$$
$$+\frac{1}{6}\left(kb\sqrt{r^2 + 12B_{a_{\tau[a_1a_2\dots a_\tau]}}/b} - kbr\right)e^{-rT_{\tau[a_1a_2\dots a_\tau]}} = 0, \qquad \text{at time } T_{\tau[a_1a_2\dots a_\tau]}, \quad \text{(A.3c)}$$

for $i=1,2,$ and

$$\int_t^{T_{\tau[a_1a_2\ldots a_\tau]}} \frac{2A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}}{b} x_1(s) + \frac{k}{x_2(s)} ds \le x_1(t), \tag{A.3d}$$

$$\int_t^{T_{\tau[a_1a_2\ldots a_\tau]}} \frac{2A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}}{b} x_2(s)\, ds \le x_2(t). \tag{A.3e}$$

Substituting $u_{a_\tau[1]}^{i[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x) = u_{a_\tau[1]}^{j[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]*}(t,x) = 2A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} x_2 \frac{1}{b}$ into (4.3) and solving, we obtain the game equilibrium level of x_2 at time s as:

$$x_2(s) = x_2^0 \exp\left(\frac{-2A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} s}{b}\right). \tag{A.4}$$

Substituting $x_2(s)$ from (A.4) into (4.2), the dynamical equation for x_1 follows as:

$$\dot{x}_1(s) = \frac{-2A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}}{b} x_1(s) - \frac{k}{x_2^0} \exp\left(\frac{2A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} s}{b}\right). \tag{A.5}$$

Solving (A.5) then yields game equilibrium level of x_1 at time s as:

$$x_1(s) = \left[x_1^0 + \frac{bk}{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} x_2^0}\right] \exp\left(\frac{-2A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} s}{b}\right)$$
$$- \frac{bk}{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} x_2^0} \exp\left(\frac{2A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} s}{b}\right). \tag{A.6}$$

According to (A.4), $x_2(s)$ will be positive in finite time s. On the other hand, it is possible for $x_1(s)$ to decrease to zero at a finite time s. In particular, if $x_1\left(T_{\tau[a_1a_2\ldots a_\tau]}\right) = 0$ at time $T_{\tau[a_1a_2\ldots a_\tau]}$, we have:

$$x_1\left(T_{\tau[a_1a_2\ldots a_\tau]}\right) =$$
$$\left[x_1^0 + \frac{bk}{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} x_2^0}\right] \exp\left(\frac{-2A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} T_{\tau[a_1a_2\ldots a_\tau]}}{b}\right)$$
$$- \frac{bk}{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} x_2^0} \exp\left(\frac{2A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} T_{\tau[a_1a_2\ldots a_\tau]}}{b}\right) = 0. \tag{A.7}$$

The value of $T_{\tau[a_1a_2\ldots a_\tau]}$ that satisfies (A.7) can be obtained as:

$$T_{\tau[a_1a_2\ldots a_\tau]} = \frac{b}{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}} \ln\left[\frac{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} x_1^0 x_2^0}{b} + 1\right]. \tag{A.8}$$

Taking $x_1(t)$ and $x_2(t)$ as initial states and exploiting the above derivation, we can obtain the relationship between $x_1(t)$, $x_2(t)$ and the remaining time $T_{\tau[a_1a_2\ldots a_\tau]} - t$ for any time $t \in \left[0, T_{\tau[a_1a_2\ldots a_\tau]}\right]$ as:

$$T_{\tau[a_1a_2\ldots a_\tau]} - t = \frac{b}{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}} \ln\left[\frac{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} x_1(t)\, x_2(t)}{b} + 1\right]. \tag{A.9}$$

Therefore, at any time $t \in \left[0, T_{\tau_{[a_1 a_2 \ldots a_\tau]}}\right]$, there exists a mapping $\tau(t, x_1, x_2) : t \times x_1 \times x_2 \rightarrow T_{\tau_{[a_1 a_2 \ldots a_\tau]}}$:

$$T_{\tau_{[a_1 a_2 \ldots a_\tau]}} = t + \frac{b}{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}} \ln\left[\frac{4A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} x_1 x_2}{b} + 1\right], \quad (A.10)$$

which maps $[t, x_1, x_2]$ to a $T_{\tau_{[a_1 a_2 \ldots a_\tau]}}$.

Finally the candidate for $A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau}$ that satisfies system (A.3) is $A_{[(1,a_1)(2,a_2)\ldots(\tau-1,a_{\tau-1})]a_\tau} = \frac{1}{6}\left(b\sqrt{r^2 + 12B_{a_{\tau_{[a_1 a_2 \ldots a_\tau]}}}/b} - br\right)$.

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