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165

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Optimal Mixed-level Supersaturated Designs and a New Class of Combinatorial Designs

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Abstract. This paper considers the construction of optimal mixed-level supersaturated designs under the $E(\chi^2)$-optimality criterion. According to a lower bound of the $E(\chi^2)$ value, a combinatorial construction method is proposed and illustrated. Meanwhile, a new class of combinatorial designs is put forward. Furthermore, some methods for constructing these combinatorial designs which correspond to $E(\chi^2)$-optimal mixed-level supersaturated designs are provided. All of these results extend the catalogue of $E(\chi^2)$-optimal mixed-level supersaturated designs.

Key words and phrases: Balanced; $E(\chi^2)$-optimality; Mixed-level; Supersaturated design; Uniformly resolvable; Weighted design.

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1 Introduction

When there are many factors and the goal of an experiment is to detect the active factors which actually affect the response or output, a supersaturated design can help the ex-

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perimenter to screen the factors and choose a small number of active factors from a huge number of potentially active factors. Because of their run size economy, these designs have widely been adopted in practical complex system investigations. For detailed examples refer to Lin (1993, 1995), Wu (1993) and Nguyen (1996).

The research on supersaturated designs dates back to Satterthwaite (1959). Subsequently, Booth and Cox (1962) further examined this problem and proposed a criterion, called $E(s^2)$, for choosing good supersaturated designs. Based on this optimality criterion, many combinatorial and algorithmic construction methods have been put forward, see Lin (1993, 1995), Wu (1993), Nguyen (1996), Cheng (1997), Li and Wu (1997), Tang and Wu (1997), Liu and Zhang (2000), Butler et al. (2001) and Bulutoglu and Cheng (2004) for example. However, all of the work had been restricted to the two-level designs until Yamada and Lin (1999) and Yamada et al. (1999) extended the $E(s^2)$ criterion to the multi-level case and proposed the average χ^2 criterion. Fang et al. (2000), Lu et al. (2003) and Xu and Wu (2005) considered the construction of multi-level supersaturated designs.

Furthermore, the χ^2-efficiency criterion was introduced by Yamada and Lin (2002) and Yamada and Matsui (2002) for the case of mixed-level supersaturated designs. Fang et al. (2003, 2004) also defined the $E(f_{NOD})$-optimality and constructed several classes of $E(f_{NOD})$-optimal mixed-level supersaturated designs. Recently, Ai et al. (2006) introduced an optimality criterion, called $E(\chi^2)$, for mixed-level supersaturated designs, which is inspired by the χ^2 statistic in a contingency table. This criterion covers $E(s^2)$ for two-level designs and $ave(\chi^2)$ for symmetrical designs as two special cases. Based on the improved lower bound of the $E(\chi^2)$ value, they further presented several classes of $E(\chi^2)$-optimal mixed-level supersaturated designs derived from saturated orthogonal arrays.

Combinatorial designs have played an essential role in the construction of supersaturated designs [see Liu and Zhang (1998), Lu and Meng (2000), Fang, Ge, Liu and Qin (2004), and Fang, Lu, Tang and Yin (2004), for example]. This paper continues to study the issue of combinatorial construction of optimal mixed-level supersaturated designs. Section 2 first reviews the $E(\chi^2)$-optimality criterion for mixed-level supersaturated designs and presents a lower bound of the $E(\chi^2)$ value. Section 3 puts forward a new class of combinatorial designs, called uniformly resolvable balanced weighted designs (URBWD), and then proposes an approach to constructing $E(\chi^2)$-optimal mixed-level supersaturated designs via these URBWDs. Section 4 further provides some methods for constructing URBWDs corresponding to $E(\chi^2)$-optimal mixed-level supersaturated designs. All of these results extend the catalogue of $E(\chi^2)$-optimal mixed-level supersaturated designs. Section 5 concludes this article with some remarks.

Before winding up this section, some notations and definitions are introduced for later use. Let $Z_s = \{1, 2, \ldots, s\}$. A mixed-level design of n runs and m factors with levels $s_1, \ldots, s_m$, denoted by $(n, s_1 \cdots s_m)$, is an $n \times m$ matrix $D = [x_{ij}]_{n\times m}$ in which each row represents a run, each column represents a factor and the jth column takes values from Z_{s_j}. In particular, an (n, s^m)-design is symmetrical. A mixed-level orthogonal array (OA) of n runs, m factors with levels $s_1, \ldots, s_m$ and strength t, denoted by $OA(n, s_1 \cdots s_m, t)$, is an $(n, s_1 \cdots s_m)$-design in which all possible level combinations for any t factors appear the same number of times. An OA of strength one is also called U-type or balanced.

2 E(χ^2)-optimality for mixed-level supersaturated designs

For an $(n, s_1 \cdots s_m)$-design D, let $D = [d_1, \ldots, d_m]$ be its column partition. For every pair of columns d_k and d_l, define the following χ^2 statistic

$$\chi^2(d_k, d_l) = \frac{s_k s_l}{n} \sum_{u=1}^{s_k} \sum_{v=1}^{s_l} [n_{uv}^{(kl)} - n/(s_k s_l)]^2, \tag{1}$$

where $n_{uv}^{(kl)}$ is the number of times that the level combination (u, v) appears as a row in columns d_k and d_l. It can be seen that the value $\chi^2(d_k, d_l)$ just measures the non-orthogonality between two columns d_k and d_l. Then the following $E(\chi^2)$ value

$$E(\chi^2) = \frac{2}{m(m-1)} \sum_{1 \le k < l \le m} \chi^2(d_k, d_l) \tag{2}$$

can be used to evaluate the overall non-orthogonality among the columns of D. The value $E(\chi^2) = 0$ if and only if design D is an OA of strength two. A supersaturated design is called $E(\chi^2)$-optimal if it minimizes the value $E(\chi^2)$ among all designs with the same parameters. Note that the value $E(\chi^2)$ is the same as $ave(\chi^2)$ for symmetrical designs in Yamada and Lin (1999) and a multiple of $E(s^2)$ for two-level designs in Booth and

Cox (1962). Thus, the $E(\chi^2)$ criterion covers both $E(s^2)$ and $ave(\chi^2)$ as two special cases. Moreover, Ai et al. (2006) showed that the $E(\chi^2)$ value is a linear function of the generalized wordlength pattern $A_2(D)$ introduced by Xu and Wu (2001). Actually, the $E(\chi^2)$ criterion can also be regarded as a special case of both minimum moment aberration and generalized minimum aberration; see Xu (2003) and Xu and Wu (2005). The justifications of the generalized minimum aberration criterion have been established from the points of view of the lower-order projection uniformity [Ai and Zhang (2004)], the design efficiency [Cheng et al. (2002), Ai et al. (2005)] and the interaction columns balance [Ai and He (2006)].

For an $(n, s_1 \cdots s_m)$-design $D = [x_{ij}]_{n\times m}$, let

$$\delta_{ij}(D) = \sum_{k=1}^{m} s_k \delta(x_{ik}, x_{jk}) \tag{3}$$

be the weighted coincidence number between the ith and jth rows, where $\delta(x_{ik}, x_{jk})$ is the Kronecker delta of two values x_{ik} and x_{jk}, which equals 1 if $x_{ik} = x_{jk}$ and 0 otherwise. Theorem 6 of Xu (2003) established a lower bound of the value $E(\chi^2)$, which is restated in the following theorem.

Theorem 1. *For any balanced $(n, s_1 \cdots s_m)$-design D,*

$$\begin{aligned} E(\chi^2) &= \frac{1}{nm(m-1)} \sum_{1\le i\ne j\le n} [\delta_{ij}(D)]^2 + C_1(n, s_1, \ldots, s_m) && (4) \\ &\ge \frac{1}{(n-1)m(m-1)} \left(nm - \sum_{k=1}^{m} s_k\right)^2 + C_1(n, s_1, \ldots, s_m), && (5) \end{aligned}$$

where $C_1(n, s_1, \ldots, s_m)$ is a constant depending only on the parameters $n, s_1, \ldots, s_m$. Equality in (5) holds if and only if all $\delta_{ij}(D)$'s $(1 \le i \ne j \le n)$ are the same.

It is known that if a design D is a saturated $OA(n, s_1 \cdots s_m, 2)$, then $\delta_{ij}(D) = m - 1$ for all $1 \le i \ne j \le n$ [Mukerjee and Wu (1995)]. It follows from this that a supersaturated design constructed by juxtaposition of the columns of two or more saturated OA's of strength two with the same number of runs is $E(\chi^2)$-optimal. Ai et al. (2006) presented some improved lower bounds of the $E(\chi^2)$ value and gave some series of $E(\chi^2)$-optimal supersaturated designs derived from saturated orthogonal arrays.

3 A new class of combinatorial designs

Combinatorial approaches have been useful in the construction of fractional factorial designs. In this section, a new class of combinatorial designs is put forward and an approach to constructing $E(\chi^2)$-optimal mixed-level supersaturated designs via these combinatorial designs is also proposed.

Definition 1. *Let n be a positive integer, K be a set of positive integers and W be a set of nonnegative real numbers. A balanced weighted design of index λ with a weight function $w : K \mapsto W$, denoted by $(n, K, \lambda; w)$-BWD, is a triple $(V, \mathcal{B}, w)$ satisfying:*

(1) $V = \{v_1, \ldots, v_n\}$ is a set of n distinct points;

(2) $\mathcal{B}$ is a family of subsets (called blocks) of V such that $|B| \in K$ for each $B \in \mathcal{B}$;

(3) For any two distinct points $v_i, v_j \in V$, $\sum_{B\in\mathcal{B},\{v_i,v_j\}\subseteq B} w(|B|) = \lambda$.

Here, we call $W = \{w(k) : k \in K\}$ the weight set of K and λ the weight index of $(n, K, \lambda; w)$-BWD. It is easy to see that if the weight set W only contains a single value 1, then the balanced weighted designs are indeed the well-known pairwise balanced designs in combinatorial design theory. When the assigned weights are inversely proportional to the block size, i.e., $w(k) = c/k$, where c is a constant, we call them natural weights. As the constant c does not affect the properties of designs, throughout this paper, we only consider the balanced weighted designs with natural weights $w(k) = 1/k$, which are simply denoted by (n, K, λ)-BWDs. A balanced weighted design is called *resolvable*, if its blocks can be partitioned into classes (called *parallel classes*), so that each element of V occurs exactly once among the blocks of each class. Furthermore, if the blocks in the same parallel class are of the same size, then the design is called a uniformly resolvable balanced weighted design, abbreviated by URBWD.

Example 1. Let $V = \{Z_6 \times Z_3, \infty_1, \infty_2\}$ be the point set. Denote

$$\mathcal{A} = \left\{ \begin{array}{l} \{(0,0),(3,0)\},\{(0,1),(3,1)\},\{(0,2),(3,2)\}, \\ \{(1,0),(4,0)\},\{(1,1),(4,1)\},\{(1,2),(4,2)\}, \\ \{(2,0),(5,0)\},\{(2,1),(5,1)\},\{(2,2),(5,2)\},\{\infty_1,\infty_2\} \end{array} \right\},$$

$$\mathcal{B}_{i+} = \left\{ \begin{array}{l} \{(4+i,2),(1+i,1),(i,1),(2+i,1)\} \\ \{(2+i,2),(i,2),(5+i,0),(1+i,2)\} \\ \{(i,0),(4+i,0),(3+i,2),(4+i,1)\} \\ \{(5+i,2),(5+i,1),(1+i,0),\infty_1\} \\ \{(2+i,0),(3+i,0),(3+i,1),\infty_2\} \end{array} \right\}$$

and

$$\mathcal{B}_{i-} = \left\{ \begin{array}{l} \{(i,0),(1+i,0),(3+i,1),(1+i,2)\} \\ \{(2+i,0),(1+i,1),(i,2),(5+i,1)\} \\ \{(3+i,0),(5+i,2),(4+i,1),(5+i,0)\} \\ \{(4+i,0),(3+i,2),(i,1),\infty_1\} \\ \{(2+i,1),(2+i,2),(4+i,2),\infty_2\} \end{array} \right\},$$

where $i \in Z_6$. Then $\mathcal{A} \cup_{i=0}^{5} \mathcal{B}_{i+} \cup_{i=0}^{5} \mathcal{B}_{i-}$ forms the block set of a $(20, \{2,4\}, 1/2)$-URBWD with natural weights.

Let $(V, \mathcal{B})$ be a uniformly resolvable balanced weighted design based on an n-point set $V = \{v_1, \dots, v_n\}$. Suppose $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_m$, where each $\mathcal{B}_i = \cup_j \{B_{ij}\}$, $j = 1, 2, \dots, s_i$; $i = 1, 2, \dots, m$, represents the ith parallel class with block size k_i. Obviously, for each $i = 1, 2, \dots, m$, $k_i s_i = n$. Thus, a U-type design can be constructed from a URBWD according to the following steps:

(1) Assign a natural order $1, 2, \dots, s_i$ to s_i blocks B_{ij}'s in $\mathcal{B}_i$, $i = 1, 2, \dots, m$;

(2) For each $\mathcal{B}_i$, $i = 1, 2, \dots, m$, construct the corresponding n-vector d_i, whose tth component takes on the value j if the tth point $v_t \in V$ occurs in the jth block B_{ij} in $\mathcal{B}_i$;

(3) Juxtapose these d_i's to form an $n \times m$ matrix D.

Example 2. Following the above construction method, we can obtain the corresponding U-type design $\mathrm{D}(20, 10^1 5^{12})$ (Table 1) from the $(20, \{2,4\}, 1/2)$-URBWD with natural weights in Example 1.

Moreover, the following theorem shows that the resulting U-type design from the URBWD is $E(\chi^2)$-optimal.

Theorem 2. *The U-type design $D(n, s_1 \dots s_m)$ obtained via the above construction method from an (n, K, λ)-URBWD with natural weights is $E(\chi^2)$-optimal.*

Proof. Let $(V, \mathcal{B})$ be the URBWD based on $V = \{v_1, \ldots, v_n\}$ and $D = [x_{it}]_{n\times m}$ be the U-type design matrix. By Theorem 1, we need only to prove that for any two distinct rows i and j, the weighted coincidence number is a constant.

According to the definition of URBWD, for any two distinct points v_i and v_j in V,

$$\sum_{B\in\mathcal{B},\{v_i,v_j\}\subseteq B} w(|B|) = \sum_{B\in\mathcal{B},\{v_i,v_j\}\subseteq B} 1/|B| = \lambda.$$

Additionally, based on the construction method of the U-type design D, $\delta(x_{it}, x_{jt}) = 1$, i.e., the rows i and j take on the same value in the tth column if and only if there exists $B \in \mathcal{B}_t$ such that $\{v_i, v_j\} \subseteq B$. So the weighted coincidence number between the ith and jth rows is

$$\sum_{t=1}^{m} s_t\delta(x_{it}, x_{jt}) = \sum_{t=1}^{m}\left(\sum_{B\in\mathcal{B}_t,\{v_i,v_j\}\subseteq B} s_t\right) = \sum_{B\in\mathcal{B},\{v_i,v_j\}\subseteq B} \frac{n}{|B|} = n\lambda,$$

which is a constant. This completes the proof of Theorem 2.

4 Construction of URBWDs

This section is devoted to providing some methods for constructing URBWDs. To begin with, we introduce some terminology and related results from combinatorial design theory, which will be used later.

Definition 2. *Let n be a positive integer, K be a set of positive integers. A group divisible design of index λ, denoted by (K, λ)-GDD is a triple $(V, \mathcal{G}, \mathcal{B})$ satisfying:*

(1) V is a set of n points;

(2) $\mathcal{G}$ is a partition of set V into subsets (called groups);

(3) $\mathcal{B}$ is a collection of subsets of V with sizes from K (called blocks), such that a group and a block contain at most one common point;

(4) Every pair of points from distinct groups occurs in exactly λ blocks.

Usually, we use an "exponential notation" to describe the *group type* of a GDD: A GDD of type ${t_1}^{u_1}{t_2}^{u_2}\cdots{t_l}^{u_l}$ is a GDD which has u_i groups of size t_i for $i = 1, \ldots, l$.

When $K = \{k\}$, in particular, the notation (k, λ)-GDD is used. If for all $i = 1, 2, \cdots, l$, $t_i = 1$, then a (K, λ)-GDD of type 1^n is called a *pairwise balanced design (PBD)*, or a $\mathrm{B}(n, K, \lambda)$. Furthermore, when $K = \{k\}$, then a pairwise balanced design becomes the well-known balanced incomplete block design. As with the case of balanced weighted designs, the notions of resolvable and uniformly resolvable can be naturally extended to group divisible designs, pairwise balanced designs and balanced incomplete block designs. Moreover, as with the case of URBWDs, the notations of URGDDs and URPBDs stand for uniformly resolvable GDDs and PBDs, respectively. For resolvable BIBDs, since $K = \{k\}$ is a one-element set, we simply denote them by RBIBDs.

A simple construction method of URBWDs is to juxtapose some resolvable balanced incomplete block design, which is presented in the following theorem.

Theorem 3. *Let v be a positive integer. Suppose there exist m resolvable balanced incomplete block designs based on v points, each denoted by RBIBD(v, k_i, λ_i), $i = 1, 2, \ldots, m$. Then for any weight set $W = \{w_1, w_2, \ldots, w_m\}$, there exists a uniformly resolvable balanced weighted design $(v, K, \lambda; w)$-URBWD, where $K = \{k_1, k_2, \ldots, k_m\}$, $w(k_i) = w_i$ and $\lambda = \sum_{i=1}^{m} w_i \lambda_i$.*

Remark: For statistical purpose, here we are mainly interested in designs which have mutually distinct parallel classes, each of them being uniform. To ensure that the derived factorial designs are not fully aliased, it is natural to require that the original URBWDs contain no identical parallel classes. This requirement is always imposed in this section.

For URBWDs constructed via the method in Theorem 3, it is obvious that any pair of points occurs evenly in the design, as it does in PBDs. However, from the definition it can be seen that a balanced weighted design only needs the weighted index to be a constant, while it does not put any restriction on the occurrence of the point pair.

For example, in a balanced weighted design with natural weights, it may happen that some pairs of points occur in two blocks of size two, three blocks of size three, and one block of size four, while other pairs of points occur in three blocks of size two and three blocks of size four. Since $2 \times 1/2 + 3 \times 1/3 + 1 \times 1/4 = 3 \times 1/2 + 3 \times 1/4$, it makes the situation complicated. Here we only present a construction method for URBWDs with natural weights satisfying the following two conditions:

(1) The sizes of blocks only take two values, k_1 or k_2;

(2) Any pair of two points either occurs in μ_1 blocks of size k_1 or in μ_2 blocks of size k_2.

Theorem 4. *Let s be a positive integer. Suppose that*

(1) There exists a resolvable group divisible design (K, λ_1)-URGDD with group type $m_1 m_2 \cdots m_g$;

(2) For any $k \in K$, there exists a resolvable group divisible design (k_1, λ_2)-RGDD of group type s^k;

(3) For any m_j, $j = 1, 2, \ldots, g$, there exists a resolvable balanced incomplete block design $(m_j s, k_2, \mu_2)$-RBIBD;

(4) Both μ_1/k_1 and μ_2/k_2 are equal to the same weighted index λ, where $\mu_1 = \lambda_1\lambda_2$.

Then there exists a $(n, \{k_1, k_2\}, \lambda)$-URBWD with natural weights, where $n = s\sum_{i=1}^{g} m_i$.

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be a (K, λ_1)-URGDD with group type $m_1 m_2 \cdots m_g$, where $X = \cup_{i=1}^{g} X_i$, $X_i = \{(i, j) : j \in Z_{m_i}\}$ and $\mathcal{G} = \{X_1, X_2, \ldots, X_g\}$. We will construct the required URBWD on the point set $V = X \times Z_s$. For any block $A \in \mathcal{A}$, since $|A| \in K$, there exists a (k_1, λ_2)-RGDD, $(V_A, \mathcal{G}_A, \mathcal{B}_A)$, which is based on the point set $V_A = A \times Z_s$ with $\mathcal{G}_A = \cup_{a \in A}\{a \times Z_s\}$. For any group $X_i \in \mathcal{G}$, since $|X_i| = m_i$, there exists an $(m_j s, k_2, \mu_2)$-RBIBD, $(V_{X_i}, \mathcal{B}_{X_i})$, which is based on the point set $V_{X_i} = X_i \times Z_s$. Now combine all the blocks in $\mathcal{B}_A$ and $\mathcal{B}_{X_i}$, i.e., let

$$\mathcal{B} = (\cup_{A \in \mathcal{A}} \mathcal{B}_A) \bigcup (\cup_{X_i \in \mathcal{G}} \mathcal{B}_{X_i}),$$

then it is readily checked that $(V, \mathcal{B})$ is the required design.

Theorem 5. *For any $v \equiv 16 \pmod{48}$, there exists a $(v, \{2, 4\}, 1/2)$-URBWD with natural weights.*

Proof. Denote $v = 48n + 16$. It is known from Hanani et al. (1972) that for any $h = 12n+4$, there exists an $(h, 4, 1)$-RBIBD, i.e., $(4, 1)$-RGDD with type 1^h . Additionally, it is obvious that there exists a $(4, 2, 1)$-RBIBD. Thus according to Theorem 4, we need only to prove that there exists a $(4, 2)$-RGDD of type 4^4.

Let the point set be $I_{16} = \{1, 2, \ldots, 16\}$ and the groups be $(1, 2, 3, 4)$, $(5, 6, 7, 8)$, $(9, 10, 11, 12)$ and $(13, 14, 15, 16)$. Then the following eight parallel classes form the required $(4, 2)$-RGDD.

$$\left\{\begin{matrix}\{1\ 5\ 9\ 13\}\\ \{2\ 6\ 10\ 14\}\\ \{3\ 7\ 11\ 15\}\\ \{4\ 8\ 12\ 16\}\end{matrix}\right\}\quad\left\{\begin{matrix}\{1\ 6\ 11\ 16\}\\ \{2\ 5\ 12\ 15\}\\ \{3\ 8\ 9\ 14\}\\ \{4\ 7\ 10\ 13\}\end{matrix}\right\}\quad\left\{\begin{matrix}\{1\ 7\ 12\ 14\}\\ \{2\ 8\ 11\ 13\}\\ \{3\ 5\ 10\ 16\}\\ \{4\ 6\ 9\ 15\}\end{matrix}\right\}\quad\left\{\begin{matrix}\{1\ 8\ 10\ 15\}\\ \{2\ 7\ 9\ 16\}\\ \{3\ 6\ 12\ 13\}\\ \{4\ 5\ 11\ 14\}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\{1\ 5\ 12\ 14\}\\ \{2\ 6\ 9\ 15\}\\ \{3\ 7\ 10\ 16\}\\ \{4\ 8\ 11\ 13\}\end{matrix}\right\}\quad\left\{\begin{matrix}\{1\ 6\ 10\ 13\}\\ \{2\ 5\ 11\ 16\}\\ \{3\ 8\ 12\ 15\}\\ \{4\ 7\ 9\ 14\}\end{matrix}\right\}\quad\left\{\begin{matrix}\{1\ 7\ 11\ 15\}\\ \{2\ 8\ 10\ 14\}\\ \{3\ 5\ 9\ 13\}\\ \{4\ 6\ 12\ 16\}\end{matrix}\right\}\quad\left\{\begin{matrix}\{1\ 8\ 9\ 16\}\\ \{2\ 7\ 12\ 13\}\\ \{3\ 6\ 11\ 14\}\\ \{4\ 5\ 10\ 15\}\end{matrix}\right\}$$

The following corollary follows immediately from Theorem 5.

Corollary 1. *For any* $v = 48n+16$, *there exists an* $E(\chi^2)$*-optimal* $(v, (12n+4)^{32n+8}(24n+8)^3)$*-design.*

Example 3. In the proof of Theorem 5, if we combine all the blocks of the listed $(4, 2)$-RGDD and four copies of a $(4, 2, 1)$-RBIBD based on each group of the $(4, 2)$-RGDD, we can obtain a $(16, \{2, 4\}, 1/2)$-URBWD with natural weights. And its corresponding U-type design $D(16, 4^8 8^3)$ is listed in Table 2.

5 Concluding remarks

In this paper, we revisit the $E(\chi^2)$-optimality criterion for measuring the mixed-level supersaturated designs. Based on the equivalent conditions for designs achieving a lower bound of the $E(\chi^2)$ value, we define a new type of configuration in combinatorial design theory, named uniformly resolvable balanced weighted designs, to construct $E(\chi^2)$-optimal mixed-level supersaturated designs. To this end, certain infinite classes of $E(\chi^2)$-optimal designs are obtained.

In recent years, many authors succeeded in using combinatorial configurations to construct experimental designs with specific properties. This approach can directly give various optimal experimental designs without any computation, and is thus quite effective. The present paper also takes this vantage point. However, different from the previous approaches, the balanced weighted designs defined in this paper are a new type of configuration. Construction methods and existence results of balanced weighted designs are themselves interesting on top of the application to experimental design construction. Subsequent discussion about BWDs can also be expected to enrich the contemporary design theory.

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Table 1: A U-type design $D(20, 10^1 5^{12})$

Run	Columns												
(0,0)	1	3	2	3	5	5	4	1	3	4	3	2	1
(1,0)	4	4	3	2	3	5	5	1	1	3	4	3	2
(2,0)	7	5	4	3	2	3	5	2	1	1	3	4	3
(3,0)	1	5	5	4	3	2	3	3	2	1	1	3	4
(4,0)	4	3	5	5	4	3	2	4	3	2	1	1	3
(5,0)	7	2	3	5	5	4	3	3	4	3	2	1	1
(0,1)	2	1	4	3	5	1	1	4	2	3	1	5	2
(1,1)	5	1	1	4	3	5	1	2	4	2	3	1	5
(2,1)	8	1	1	1	4	3	5	5	2	4	2	3	1
(3,1)	2	5	1	1	1	4	3	1	5	2	4	2	3
(4,1)	5	3	5	1	1	1	4	3	1	5	2	4	2
(5,1)	8	4	3	5	1	1	1	2	3	1	5	2	4
(0,2)	3	2	4	1	3	2	2	2	3	5	4	5	1
(1,2)	6	2	2	4	1	3	2	1	2	3	5	4	5
(2,2)	9	2	2	2	4	1	3	5	1	2	3	5	4
(3,2)	3	3	2	2	2	4	1	4	5	1	2	3	5
(4,2)	6	1	3	2	2	2	4	5	4	5	1	2	3
(5,2)	9	4	1	3	2	2	2	3	5	4	5	1	2
∞_1	A	4	4	4	4	4	4	4	4	4	4	4	4
∞_2	A	5	5	5	5	5	5	5	5	5	5	5	5

Table 2: A U-type design $D(16, 4^8 8^3)$

Run	Columns										
1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	1	2	2
3	3	3	3	3	3	3	3	3	2	1	2
4	4	4	4	4	4	4	4	4	2	2	1
5	1	2	3	4	1	2	3	4	3	3	3
6	2	1	4	3	2	1	4	3	3	4	4
7	3	4	1	2	3	4	1	2	4	3	4
8	4	3	2	1	4	3	2	1	4	4	3
9	1	3	4	2	2	4	3	1	5	5	5
10	2	4	3	1	3	1	2	4	5	6	6
11	3	1	2	4	4	2	1	3	6	5	6
12	4	2	1	3	1	3	4	2	6	6	5
13	1	4	2	3	4	1	3	2	7	7	7
14	2	3	1	4	1	4	2	3	7	8	8
15	3	2	4	1	2	3	1	4	8	7	8
16	4	1	3	2	3	2	4	1	8	8	7