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Drift of rigidly rotating spirals under periodic and noisy illuminations

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Under the weak deformation approximation, the motion of rigidly rotating spirals induced by periodic and noisy illuminations are investigated analytically. We derive an approximate but explicit formula of the spiral drift velocity directly from the original reaction-diffusion equation. With this formula we are able to explain the main features in the periodic and noisy illuminations induced spiral drift problems. Numerical computations of the Oregonator model are carried out as well, and they agree with the main qualitative conclusions of our analytical results. © 2004 American Institute of Physics. [DOI: 10.1063/1.1795711]

I. INTRODUCTION

Excitable media derive much of their interest from varied and sometimes unexpected spatiotemporal wave patterns owing to their nonlinearity. Spiral excitation waves in two spatial dimensions are among the most paradigmatic examples of spatiotemporal self-organizing structures in excitable media, and have attracted much attention.^{1,2} They have been observed in very different systems: the heart muscle,³ where spiral waves associate with cardiac arrhythmia; the catalytic surface processes;⁴ the prototypical Belousov-Zhabotinsky (BZ) reaction.⁵ Spiral waves of electrical activity occurring in cardiac tissue are life threatening because they act as high frequency sources of waves which take control over the heart's natural pacemaker and induce tachycardia.³ So, the control of spiral is a question of interest for all excitable media.⁶ One can induce rotating waves within a fibrillating heart to drift to a boundary (heart's surface) where they can no longer be sustained.

All realistic media are embedded in some environment and thus undergo external forces and field. Under this aspect it becomes important to investigate the evolution of spiral wave under external forcing.⁷⁻¹⁹ Agladze, Davydov, and Mikhailov⁷ reported the first experimental observation of a drift (or resonance) of spiral waves during periodic modulation of the excitability of an excitable medium with a frequency close to the nature rotation frequency of the spiral waves. This resonant drift phenomenon for rigidly rotating spirals has been confirmed in numerous experiments and numerical simulations.¹⁰⁻¹⁴

Based on the case of a weakly excitable media Davydov, Zykov, and Mikhailov¹⁵ provided an initial mathematical treatment of resonant drift for rigidly rotating spiral waves under homogeneous periodic forcing. Their approach is based on a kinematic model of spiral dynamics in which one disregards the thickness of the excited area and models the spiral as a one-dimensional curve. As pointed in their paper, if one tries to develop a perturbation theory proceeding directly from the reaction-diffusion equations, this meets serious difficulties. They arise because translation and rotation perturbation modes for the spiral waves are not spatially localized, and the authors bypass such difficulties by using the quasisteady approximation.¹⁵ Mantel and Barkley¹⁶ considered the periodic forcing of spiral waves from a dynamical-systems view point and showed that much of the spiral behavior can be deduced simply from the interaction of dynamics with system symmetries. They based their analysis on a system of ordinary differential equations. This system has been shown to model well the dynamics of spiral wave, though it has not been obtained by rigorous reduction of a partial differential equation model of excitable media.

On the other hand, the noise-induced drift of spiral waves has recently attracted much attention.¹⁷⁻¹⁹ Sendina-Nadal *et al.*¹⁹ experimentally and numerically investigated spiral chemical waves subjected to a spatiotemporal random excitability in relation to the light-sensitive Belousov-Zhabotinsky reaction. Brownian motion is identified and characterized by an effective diffusion coefficient which shows a rather complex dependence on the time and length scales of the noise relative to those of the spiral. Using the kinematical model based on a kinematic approach,¹⁵ the authors studied the noise-induced drift of spirals. In their drift

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velocity formula, the drift velocity has no direct relation with external forcing. And so far no analytical results have been obtained for the drift of spirals under noisy illuminations. Therefore, analytical results are of crucial importance for a deeper and more comprehensive understanding of the general mechanism of the drift and for a wider application of spiral wave control.

Recently we derived, directly from the original reaction-diffusion equation and its rigidly rotating spiral solution, an approximate formula of the perturbation-induced spiral wave drift velocity that includes the drifts of a mechanical deformation and an electric field as special cases.²⁰ With this approximate formula we are able to explain the main features appearing in the directional spiral-tip-drift problems, such as the constant drift of periodic mechanical deformation at $\omega = \omega_0$, the dc electric field net drift and the ac electric field drift at $\omega = 2\omega_0$, and the phase and chirality relations in these drift processes. In particular, we predict a triple frequency resonant drift for the mechanical deformation, which is confirmed by direct simulations of a reaction-diffusion model.

In this paper, we will develop this study for the drift of rigidly rotating spirals under periodic and noisy illuminations, which have been studied extensively.⁷⁻¹⁹ Under the approximation that the deformation of the spiral in the tip region is weak, we will give explicit formulas for the drift velocities of the rigidly rotating spiral induced by periodic and noisy illuminations. These formulas can explain the drift phenomena of spirals observed in experiments and numerical simulations in excitable media. Numerical computations are also carried out, and the agreement of the numerical realizations with our theoretical predictions is good, qualitatively.

II. AN APPROXIMATE DRIFT VELOCITY OF SPIRALS UNDER WEAK ILLUMINATIONS

In view of the experimental motivation, we examine a light-sensitive excitable media such as the ruthenium-catalyzed BZ reaction. Wave propagation in this medium can be simulated by the Oregonator model^{21,22} with an additional term $\Phi(t, x, y)$ describing the effects of an external illumination.^{13,12}

$$\begin{aligned} u_t &= \frac{1}{\epsilon} \left[u - u^2 - (fv + \Phi) \frac{u-q}{u+q} \right] + D_u \nabla^2 u, \\ v_t &= u - v + D_v \nabla^2 v. \end{aligned} \quad (1)$$

Here variables u and v represent the concentrations of the autocatalytic species HBrO_2 and the catalyst. For certain parameter values and $\Phi(t, x, y) = \Phi_0$, the system (1) has a spiral wave solution with rigid rotation, and in the neighborhood of the tip the solution can be generally expressed as^{23,24}

$$\begin{aligned} \hat{u} &= u_0 + a_1 r \cos(-\sigma\theta + \omega_0 t - \beta) + \dots, \\ \hat{v} &= v_0 + a_2 r \cos(-\sigma\theta + \omega_0 t - \alpha) + \dots, \end{aligned} \quad (2)$$

where $u_0, v_0, a_1 > 0$, and $a_2 > 0$ are four coefficients; α and β are two phase shifts and $0 < \alpha - \beta < \pi/2$,²⁴ $\sigma = \pm 1$ is the

chirality of the spiral ($\sigma = +1$ for anticlockwise rotating spiral, while $\sigma = -1$ for clockwise one); ω_0 is the rotating frequency of the spiral.

Now we consider an external time dependent and space inhomogeneous illumination $\Phi(t, x, y) = \Phi_0 + I(t, x, y)$ and the two-variable Oregonator model is changed into

$$\begin{aligned} u_t &= \frac{1}{\epsilon} \left[u - u^2 - [fv + \Phi_0 + I(t, x, y)] \frac{u-q}{u+q} \right] + D_u \nabla^2 u, \\ v_t &= u - v + D_v \nabla^2 v. \end{aligned} \quad (3)$$

Supposing $I(t, x, y)$ induces a drift of the spiral wave tip with velocity $\vec{V}(t) = [V_1(t), V_2(t)]$, we can then rewrite the perturbed Eq. (3) in the moving frame of \vec{V} as

$$\begin{aligned} u_t &= \frac{1}{\epsilon} \left[u - u^2 - (fv + \Phi_0) \frac{u-q}{u+q} \right] + D_u \nabla'^2 u + \vec{V} \cdot \nabla' u \\ &\quad - \frac{1}{\epsilon} \frac{u-q}{u+q} I, \\ v_t &= u - v + D_v \nabla'^2 v + \vec{V} \cdot \nabla' v, \end{aligned} \quad (4)$$

where $\nabla' = (\partial/\partial x', \partial/\partial y')$, $x' = x - \int_0^t V_1(t_1) dt_1$, and $y' = y - \int_0^t V_2(t_2) dt_2$. We assume that for small I , the deformation of the spiral solution around the tip is weak, and all the values of u, v , and their derivatives at the tip in the moving frame are kept unchanged compared to those values of the unperturbed spiral (2) at the tip. Under this approximation and from Eq. (4), we can get that $[\vec{V} \cdot \nabla' \hat{u} - 1/\epsilon(\hat{u} - q/\hat{u} + q)I]_{\text{tip}} = 0$ and $[\vec{V} \cdot \nabla' \hat{v}]_{\text{tip}} = 0$ (for the details, see Ref. 20). Thus one can give an approximate drift velocity of spirals under weak illumination,

$$\begin{aligned} V_1(t) &= \left[\frac{1}{\epsilon} \frac{u-q}{u+q} I \hat{v}_y / (\hat{u}_x \hat{v}_y - \hat{u}_y \hat{v}_x) \right]_{\text{tip}}, \\ V_2(t) &= \left[-\frac{1}{\epsilon} \frac{u-q}{u+q} I \hat{v}_x / (\hat{u}_x \hat{v}_y - \hat{u}_y \hat{v}_x) \right]_{\text{tip}}, \end{aligned} \quad (5)$$

where $[\hat{v}_x]_{\text{tip}} = a_2 \cos(\omega_0 t - \alpha)$, $[\hat{v}_y]_{\text{tip}} = \sigma a_2 \sin(\omega_0 t - \alpha)$, and $[\hat{u}_x \hat{v}_y - \hat{u}_y \hat{v}_x]_{\text{tip}} = -\sigma a_1 a_2 \sin(\alpha - \beta)$ according to (2). Simplifying above equations, one can get

$$\begin{aligned} V_1(t) &= -D \sin(\omega_0 t - \alpha) I(t, x, y)_{\text{tip}}, \\ V_2(t) &= \sigma D \cos(\omega_0 t - \alpha) I(t, x, y)_{\text{tip}}, \end{aligned} \quad (6)$$

where $D = (u_0 - q)/[\epsilon a_1 \sin(\alpha - \beta)(u_0 + q)]$. Then, from above equations, one can study the drift phenomena of spiral under all kinds of illuminations.

III. RESONANT DRIFT OF SPIRALS INDUCED BY PERIODIC ILLUMINATION

From the general result Eq. (6), we will first discuss the case of periodic forcing $I(t, x, y) = A \cos(\omega t - \phi)$ and the drift velocity of spiral tip induced by the periodic illumination then reads

$$\begin{aligned} V_1(t) &= -DA \sin(\omega_0 t - \alpha) \cos(\omega t - \phi), \\ V_2(t) &= \sigma DA \cos(\omega_0 t - \alpha) \cos(\omega t - \phi). \end{aligned} \quad (7)$$

To get a net drift velocity, ω should take the resonant frequency ω_0 . For $\omega = \omega_0$, the constant (or average) drift velocity reads

$$\begin{aligned}\bar{V}_1 &= \frac{1}{2}DA \sin(\alpha - \phi), \\ \bar{V}_2 &= \sigma \frac{1}{2}DA \cos(\alpha - \phi),\end{aligned}\quad (8)$$

where the oscillatory parts in (7) are averaged out. It is clear that the direction of the drift can be adjusted by changing the phase shift ϕ of the periodic illumination and the drift velocity is in proportion to the illumination intensity. In Figs. 1(a) and 1(b), we present the tip motion by numerical simulation of Eq. (3) for anticlockwise ($\sigma = +1$) and clockwise ($\sigma = -1$) spirals at $\omega = \omega_0$ for different phase shifts. One can see that the analytical formula (8) agrees with the numerical results and satisfactorily explains the experimental observations.⁷

From Eqs. (7) and (8) and without knowing the particular constants of Eq. (2), one can give some quantitative relations that are independent of the special models. Defining the constant drift velocity amplitude $|\bar{V}|$ and the drift angle as

$$|\bar{V}| = \sqrt{V_1^2 + V_2^2}, \quad \tan \Theta = \bar{V}_2 / \bar{V}_1. \quad (9)$$

From Eq. (8), one gets that

$$|\bar{V}| = \left| \frac{1}{2}DA \right| = \left| \frac{A(u_0 - q)}{2\epsilon a_1 \sin(\alpha - \beta)(u_0 + q)} \right|, \quad (10)$$

which is independent of ϕ , and

$$\tan \Theta = \sigma \cot(\alpha - \phi) = \sigma \tan(\phi - \alpha + \pi/2),$$

that is, Θ is linear in ϕ ,

$$\Theta = \sigma(\phi - \alpha + \pi/2) + k\pi, \quad (11)$$

where k is an integer. These results are confirmed by numerical simulations of reaction-diffusion equations in Fig. 2.

Equations (8) and (10) show that the drift velocity $|\bar{V}|$ is directly proportional to the amplitude A of the periodic forcing. In Fig. 3, we give the dependences of drift velocity $|\bar{V}|$ on A that are obtained from a direct simulation of Eq. (3). The numerical results support our analytic results for small amplitude A . For $A \geq 0.016$, spiral begins to breakup. When we increase A further (for example, $A = 0.04$) spiral waves will be suppressed, which is consistent with the experimental results that at large A all waves in the photosensitive BZ reaction are suppressed (for example, see Ref. 25).

Furthermore, according to Eq. (11), one can draw following conclusions: for $\sigma = +1$, the drift direction of spiral waves will change *clockwise* when we increase the phase shift ϕ of the periodic illumination, while for $\sigma = -1$, the drift direction will change *anticlockwise*. And these results are confirmed by the numerical simulations in Figs. 1 and 2. In Ref. 20, we give analytically and numerically that these simple relations also hold for the $\omega = 2\omega_0$ resonance for ac electric field and the $\omega = 3\omega_0$ resonance for periodic mechanical deformation, while they fail for the $\omega = \omega_0$ resonance for periodic mechanical deformation.

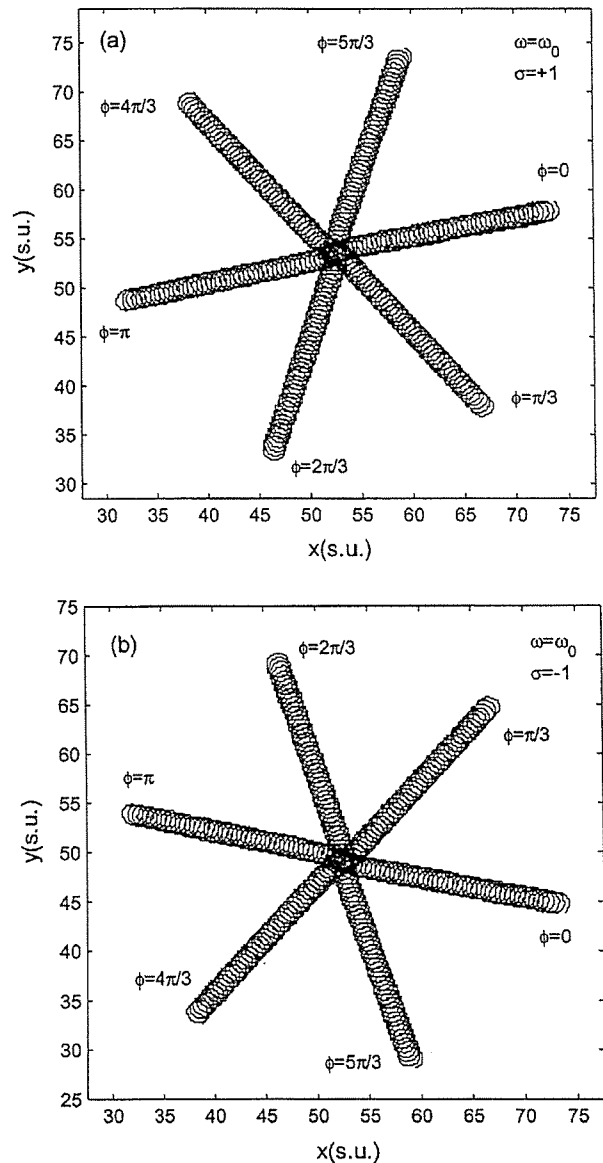


FIG. 1. Dependence of the drift by periodic illuminations on the phase shift at resonant frequency $\omega = \omega_0$. (a) anticlockwise ($\sigma = +1$) spiral; (b) clockwise ($\sigma = -1$) spiral. Parameters in the Oregonator model are $f = 1.4$, $q = 0.002$, $\epsilon = 0.05$, $D_u = 1$, $D_v = 0.6$, $\Phi_0 = 0.01$, and $A = 0.001$. The system size is 102.4×102.4 , grid 512×512 points, and $\Delta t = 0.001$ are used in our simulations. The spiral tip position is identified by maximizing the vector product $|\nabla u \times \nabla v|$ (Ref. 26). The no-flux boundary conditions are imposed. We refer to the space and time units of the Oregonator model as s.u. and t.u. respectively.

IV. BROWNIAN MOTION OF SPIRALS DRIVEN BY NOISY ILLUMINATION

The general result of Eq. (6) can also be applied to the case of the noise-induced drift of spiral wave, which has been a topic of extensive investigation.¹⁹ In that case, the illumination field consisted of an array of square cells of size ℓ whose light intensity was varied. With their random forcing, the light-induced flow in the cell (i, j) is $\Phi(t, x, y)$

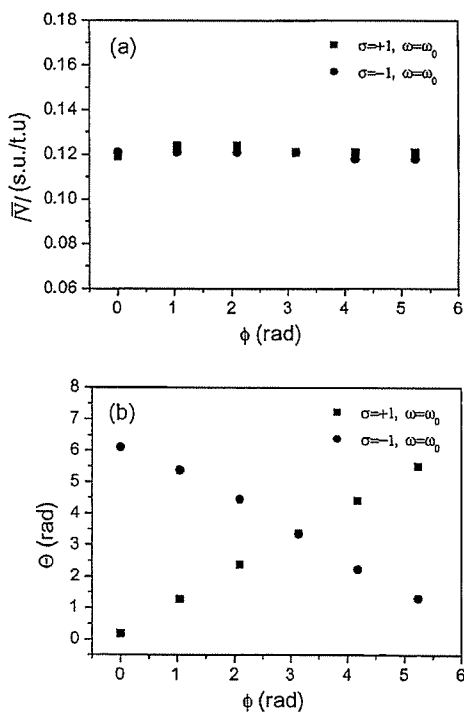


FIG. 2. The quantities $\overline{|V|}$ and Θ defined in Eq. (9) vs ϕ for the resonant drift of spiral induced by the periodic illuminations at $\omega=\omega_0$. These numerical results confirm the predictions of Eq. (8) that $\overline{|V|}$ is independent of ϕ and Θ is linear in ϕ .

$=\Phi_0 + \xi_{ij}(t)$, where the random term $\xi_{ij}(t)$ stands for a spatiotemporal Gaussian process, with a time independent zero mean average and a correlation function

$$\langle \xi_{ij}(t) \xi_{lm}(t') \rangle = 2B \delta_{il} \delta_{jm} \delta(t-t'), \quad (12)$$

where B characterizes the noise strength. The position of the center of the cell (i, j) is $x_i = (i-1/2)\ell$, $y_j = (j-1/2)\ell$.

Replacing $I(t, x, y)$ with $\xi_{ij}(t)$ in (6), one can obtain an approximate formula for the drift velocity of spiral under noisy illuminations:

$$\begin{aligned} V_1(t) &= -D \sin(\omega_0 t - \alpha) \xi_{ij}(t)_{\text{tip}}, \\ V_2(t) &= \sigma D \cos(\omega_0 t - \alpha) \xi_{ij}(t)_{\text{tip}}. \end{aligned} \quad (13)$$

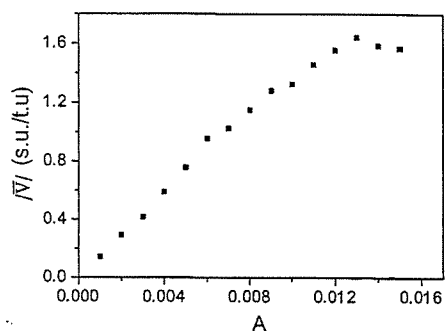


FIG. 3. The dependence of resonant drift velocity $\overline{|V|}$ on the amplitude A of the periodic illumination.

Then an analytical formula is obtained directly from the original partial differential equation for the noise-induced drift where the effects of the external forcing and the chirality of the spiral are given explicitly.

Based on a kinematic approach,¹⁵ Sendiña-Nadal *et al.* also gave a drift velocity formula for spiral driven by spatiotemporal structured noise,¹⁹

$$V_1(t) = -[V_0 + \xi_{V_0}(t)] \sin \omega_0 t,$$

$$V_2(t) = [V_0 + \xi_{V_0}(t)] \cos \omega_0 t,$$

where the authors assume that $\xi_{V_0}(t)$ represents the noise-induced fluctuation of the kinematic parameter V_0 . However, their formulas have no direct relation with external forcing since they are based on a kinematic approach.¹⁹

From the drift velocity formula (13), we can study the motion properties of spiral tip under noisy illumination: the coordinates of spiral tip then reads

$$\begin{aligned} x(t) - x(0) &= \int_0^t V_1(\tau) d\tau \\ &= \int_0^t -D \sin(\omega_0 \tau - \alpha) \xi_{i_0 j_0}(\tau) d\tau, \end{aligned}$$

$$\begin{aligned} y(t) - y(0) &= \int_0^t V_2(\tau) d\tau \\ &= \int_0^t \sigma D \cos(\omega_0 \tau - \alpha) \xi_{i_0 j_0}(\tau) d\tau, \end{aligned}$$

where the spiral tip is located in the cell (i_0, j_0) . From above equations and (12), we can derive a diffusion law for the spiral tip,

$$\langle r^2(t) \rangle = \langle [x(t) - x(0)]^2 + [y(t) - y(0)]^2 \rangle = 2D^2 B t. \quad (14)$$

So, under the weak deformation approximation, the spiral tip performs Brownian motion when driven by white noise. In Fig. 4, we give an example of the trajectory of spiral tip under noisy illuminations numerically: besides the inherent rigid rotation, the spiral tip performs random walk induced by noise. Recently, Aranson, Chaté, and Tang¹⁷ investigated spiral motion in the noisy complex Ginzburg-Landau equation using a linear response assumption, and they also gave a diffusion law for the spiral tip, while the diffusion constant in (14) is determined analytically, i.e., the diffusion constant is $2D^2 B$, where $D = [u_0 - q] / [\epsilon a_1 \sin(\alpha - \beta)(u_0 + q)]$ and B is the noise strength.

V. CONCLUSIONS

In conclusion, we have studied the motion of rigidly rotating spiral waves under periodic and noisy illuminations analytically, since analytical results are of crucial importance for a deeper and more comprehensive understanding of the mechanism of the drift of spiral waves. Directly from the original partial differential equation, approximate drift velocities of spiral waves driven by periodic and noisy illuminations are obtained analytically under the assumption of a weak deformation around the spiral wave tip, and the effects

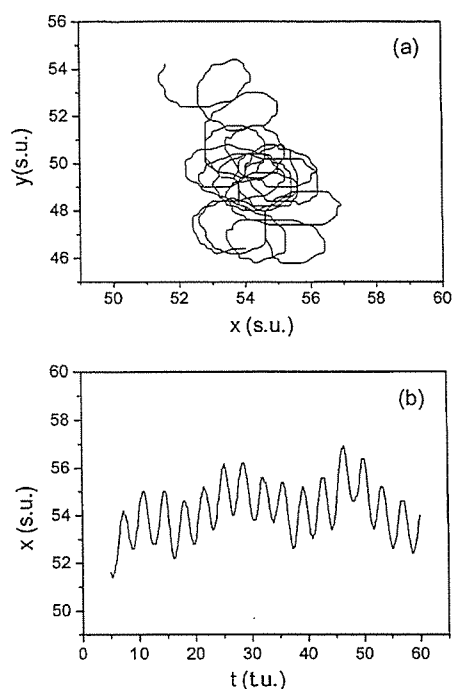


FIG. 4. (a) The trajectory of spiral tip in noisy illuminations. (b) X position of spiral tip. The spiral tip performs random walk except for the inherent rigid rotation. Parameters in the Oregonator model are $f=1.4$, $q=0.002$, $\epsilon=0.05$, $D_v=1$, $D_o=0.6$, $\Phi_0=0.01$, $2B=(0.003)^2$, and $\ell=6.4$. Grid 512×512 points, system size 102.4×102.4 , and $\Delta t=0.001$.

of the external illuminations and the chirality of the spiral are given explicitly. Using these approximate drift velocities, we can explain the main features appearing in the periodic and noisy illuminations induced spiral drift problems. It is shown that the direction of the drift under periodic illuminations ($\omega=\omega_0$) can be adjusted by changing the phase shift ϕ of the periodic illumination, and the drift velocity is in proportion to the illumination intensity. More interestingly, when the reaction-diffusion system is driven by white noisy illumination, a diffusion law for the spiral wave tip is derived. Numerical simulations are performed and yield a qualitative agreement with the analytical results. Furthermore, we hope some new results will be verified in the experiments (e.g., in the light-sensitive BZ reaction): for example, from Eq. (11), it is easy to see that the drift direction of spiral waves ($\sigma=+1$) will change clockwise when we increase the phase shift ϕ of the periodic illumination, while for $\sigma=-1$, the drift direction will change anticlockwise. Finally, it should be noted that in our analytical study, the deformation of the spiral wave around the tip is neglected and any effects caused by the deformation of the spiral wave are not taken into account. It will be interesting and important to consider the deformation of the spiral waves in studying the drift problems of the spirals further.

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