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# Bäcklund transformation and its superposition principle of a Blaszak–Marciniak four-field lattice

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A four-field lattice furnished by Blaszak and Marciniak is transformed into bilinear form upon introducing two auxiliary independent variables. A Bäcklund transformation in bilinear form is found for the lattice and the corresponding nonlinear superposition formula is rigorously established. As a consequence, soliton solutions to the lattice are derived. © *1999 American Institute of Physics.* [S0022-2488(99)02811-X]

## I. INTRODUCTION

Recently, Blaszak and Marciniak have derived several three-field and four-field lattices[1] as an application of $r$-matrix formalism to the algebra of shift operators. Two examples are

$$a_t(n)=c(n+1)-c(n-1), \tag{1}$$

$$b_t(n)=a(n-1)c(n-1)-a(n)c(n), \tag{2}$$

$$c_t(n)=c(n)(b(n)-b(n+1)), \tag{3}$$

and

$$u_t(n)=u(n)(v(n)-v(n-1)), \tag{4}$$

$$v_t(n)=w(n)u(n+1)-u(n)w(n-1), \tag{5}$$

$$w_t(n)=q(n)u(n+2)-u(n)q(n-1), \tag{6}$$

$$q_t(n)=u(n+3)-u(n), \tag{7}$$

both of which have Abelian symmetry algebras of infinite dimensions. Recently, an integrable symplectic map connected with (1)–(3) and its hierarchy was obtained by Wu and Geng[2] and

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master symmetries were presented by the discrete zero curvature equation by one of the authors (Ma) and Fuchssteiner.[3] Moreover, (1)–(3) were transformed into bilinear equations by introducing an auxiliary independent variable, and thus a Bäcklund transformation and its nonlinear superposition formula were established by one of the authors (X.-B.H.) and Zhu.[4] As a result, soliton solutions to Eqs. (1)–(3) were found.

However, to the best of our knowledge, so far there have not been any results on solutions of the lattice (4)–(7) in the literature. In this paper, we would like to present a way to construct solutions to the Eqs. (4)–(7) by establishing a Bäcklund transformation and its nonlinear superposition formula for the lattice (4)–(7). As an application of the obtained results, soliton solutions of (4)–(7) are derived. The basic tool used in this paper is Hirota's bilinear formalism. As usual, the crucial step of using Hirota's method is to transform the system of equations under consideration into bilinear form, which is far from algorithmic and often highly technical. In Sec. II, through a long computation, we will show a way of deriving a bilinear form for the lattice (4)–(7). Then a Bäcklund transformation in bilinear form will be presented in Sec. III and the corresponding nonlinear superposition formula will be established in Sec. IV. Finally in Sec. V, a conclusion will be given. Some bilinear operator identities, which are fundamental and necessary for our discussion, are listed in Appendix A.

## II. BILINEAR FORM

In this section, we want to derive a bilinear form for the lattice (4)–(7). To that end, let us make

$$u(n)=\frac{f(n+1)f(n-1)}{f^2(n)},\qquad v(n)=\left(\ln\frac{f(n+1)}{f(n)}\right)_t. \tag{8}$$

Our choice of the above-mentioned transformation comes from an observation that the first Eq. (4) of the lattice can be transformed into the following form:

$$(\ln u(n))_t=v(n)-v(n-1).$$

Furthermore let us introduce an auxiliary independent variable $z$ such that

$$(D_tD_z-2e^{D_n}+2)f(n)\cdot f(n)=0, \tag{9}$$

where Hirota's bilinear differential operator $D_x^mD_t^k$, bilinear difference operator $\exp(\delta D_n)$ and bilinear differential-difference operator $D_x^mD_t^k\exp(\delta D_n)$ are defined as follows:[5–9]

$$D_x^mD_t^ka\cdot b\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^k a(x,t)b(x',t')|_{x'=x,t'=t},$$

$$\exp(\delta D_n)a(n)\cdot b(n)\equiv\exp\left[\delta\left(\frac{\partial}{\partial n}-\frac{\partial}{\partial n'}\right)\right]a(n)b(n')|_{n'=n}=a(n+\delta)b(n-\delta),$$

$$D_x^mD_t^k\exp(\delta D_n)a(n)\cdot b(n)\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^k a(n+\delta,x,t)b(n-\delta,x',t')|_{x'=x,t'=t}.$$

Then from (7) to (9), we know that

$$q_t(n)=\frac{f(n+4)f(n+2)}{f^2(n+3)}-\frac{f(n+1)f(n-1)}{f^2(n)}=\frac{\partial^2}{\partial t\partial z}\ln\frac{f(n+3)}{f(n)},$$

which implies that we can choose

$$q(n)=\frac{\partial}{\partial z}\ln\frac{f(n+3)}{f(n)}. \tag{10}$$

From here it is clear that the introduction of auxiliary variable $z$ makes it easy to solve $q(n)$ in terms of $f$ without containing any explicit integral. That is also our motivation for introducing the auxiliary variable $z$. Substituting (8) and (10) into Eq. (5) allows us to take

$$(\ln f(n+1))_{tt}=w(n)u(n+1),$$

from which it follows that

$$w(n)=\frac{1}{2}\frac{D_t^2 f(n+1)\cdot f(n+1)}{f(n+2)f(n)}. \tag{11}$$

Furthermore from (6) we have

$$\frac{1}{2}\frac{D_t(D_t^2 f(n+1)\cdot f(n+1))\cdot f(n+2)f(n)}{f^2(n+2)f^2(n)}=\frac{f(n+1)D_z f(n+3)\cdot f(n)}{f^2(n+2)f(n)}-\frac{f(n+1)D_z f(n+2)\cdot f(n-1)}{f^2(n)f(n+2)}$$

or

$$\begin{aligned}&\tfrac{1}{2}D_t(D_t^2 f(n)\cdot f(n))\cdot(e^{D_n}f(n)\cdot f(n))\\&=2\sinh(\tfrac{1}{2}D_n)(D_z e^{(3/2)D_n}f(n)\cdot f(n))\cdot(e^{(1/2)D_n}f(n)\cdot f(n)).\end{aligned} \tag{12}$$

By use of (9) and (A1)–(A3), we can compute that

$$\begin{aligned}D_t(D_t^2 f(n)\cdot f(n))\cdot(e^{D_n}f(n)\cdot f(n))=&D_t(D_t^2 f(n)\cdot f(n))\cdot f^2(n)+\tfrac{1}{6}D_t(D_t^3 D_z f(n)\cdot f(n))\cdot f^2(n)\\&-\tfrac{2}{3}\sinh(\tfrac{1}{2}D_n)[(D_t^3 e^{(1/2)D_n}f(n)\cdot f(n))\cdot(e^{(1/2)D_n}f(n)\cdot f(n))\\&+3(D_t e^{(1/2)D_n}f(n)\cdot f(n))\cdot(D_t^2 e^{(1/2)D_n}f(n)\cdot f(n))].\end{aligned} \tag{13}$$

Now from (12) and by use of (13), (A4), and (A5) we can have the following relation:

$$\begin{aligned}&2\sinh(\tfrac{1}{2}D_n)(D_z e^{(3/2)D_n}f(n)\cdot f(n))\cdot(e^{(1/2)D_n}f(n)\cdot f(n))\\&=\tfrac{3}{8}D_t(D_t^2 f(n)\cdot f(n))\cdot f^2(n)+\tfrac{1}{16}D_t(D_t^3 D_z f(n)\cdot f(n))\cdot f^2(n)\\&\quad+\tfrac{1}{8}D_t(D_t^2 e^{D_n}f(n)\cdot f(n))\cdot f^2(n)-\tfrac{1}{2}\sinh(\tfrac{1}{2}D_n)(D_t^3 e^{(1/2)D_n}f(n)\cdot f(n))\\&\quad\cdot(e^{(1/2)D_n}f(n)\cdot f(n))+\tfrac{1}{2}\{\sinh(\tfrac{1}{2}D_n)(D_t D_y e^{(1/2)D_n}f(n)\cdot f(n))\\&\quad\cdot(e^{(1/2)D_n}f(n)\cdot f(n))-\tfrac{1}{2}D_t(D_y e^{D_n}f(n)\cdot f(n))\cdot f^2(n)\},\end{aligned} \tag{14}$$

where we have introduced another auxiliary independent variable $y$ such that

$$D_t^2 e^{(1/2)D_n}f(n)\cdot f(n)=D_y e^{(1/2)D_n}f(n)\cdot f(n).$$

Finally, (14) can be decoupled into the following two bilinear equations:

$$(D_t^2 e^{D_n}+3D_t^2+\tfrac{1}{2}D_t^3 D_z-2D_y e^{D_n})f(n)\cdot f(n)=0,$$

$$(D_t^3 e^{(1/2)D_n}+4D_z e^{(3/2)D_n}-D_yD_t e^{(1/2)D_n})f(n)\cdot f(n)=0.$$

To sum up, we obtain the following bilinear form for the lattice (4)–(7):

$$(D_tD_z-2e^{D_n}+2)f(n)\cdot f(n)=0, \tag{15}$$

$$D_t^2 e^{(1/2)D_n}f(n)\cdot f(n)=D_y e^{(1/2)D_n}f(n)\cdot f(n), \tag{16}$$

$$(D_t^2e^{D_n}+3D_t^2+\tfrac{1}{2}D_t^3D_z-2D_ye^{D_n})f(n)\cdot f(n)=0, \tag{17}$$

$$(D_t^3 e^{(1/2)D_n}+4D_z e^{(3/2)D_n}-D_yD_t e^{(1/2)D_n})f(n)\cdot f(n)=0. \tag{18}$$

## III. BÄCKLUND TRANSFORMATION

In this section, we derive a bilinear Bäcklund transformation (BT) for the system of the bilinear Eqs. (15)–(18). The concrete form of BT is presented as follows:

**Theorem 1:** The bilinear system of Eqs. (15)–(18) has the following Bäcklund transformation:

$$(D_z+\lambda^{-1}e^{-D_n}+\mu)f(n)\cdot g(n)=0, \tag{19}$$

$$(D_te^{-(1/2)D_n}-\lambda e^{(1/2)D_n}+\gamma e^{-(1/2)D_n})f(n)\cdot g(n)=0, \tag{20}$$

$$(D_ye^{-(1/2)D_n}-\lambda D_te^{(1/2)D_n}-\lambda\gamma e^{(1/2)D_n}+\omega e^{-(1/2)D_n})f(n)\cdot g(n)=0, \tag{21}$$

$$\begin{aligned}(\lambda^{-1}D_t^3e^{-(1/2)D_n}&+D_t^2e^{(1/2)D_n}+2\gamma D_te^{(1/2)D_n}+4\gamma^2e^{(1/2)D_n}+3\lambda^{-1}\gamma D_t^2e^{-(1/2)D_n}-2D_ye^{(1/2)D_n}\\&-2\omega e^{(1/2)D_n}+\nu e^{-(1/2)D_n})f(n)\cdot g(n)=0,\end{aligned} \tag{22}$$

$$\begin{aligned}(2D_t^3e^{-D_n}&+2D_tD_ye^{-D_n}+6\gamma D_t^2e^{-D_n}+\tfrac{1}{2}\lambda\nu e^{-D_n}+8\lambda^{-1}e^{-2D_n}+2\omega D_te^{-D_n}+2\gamma D_ye^{-D_n}\\&+4\lambda^2D_te^{D_n}+6\gamma^2D_te^{-D_n}+2\omega\gamma e^{-D_n}+3\gamma^3e^{-D_n}+\theta e^{D_n})f(n)\cdot g(n)=0,\end{aligned} \tag{23}$$

where $\lambda$, $\mu$, $\gamma$, $\omega$, $\nu$, and $\theta$ are arbitrary constants.

*Proof:* Let $f(n)$ be a solution of Eqs. (15)–(18). What we need to prove is that the function $g(n)$ satisfying (19)–(23) is another solution of Eqs. (15)–(18), i.e.,

$$P_1\equiv(D_zD_t-2e^{D_n}+2)g(n)\cdot g(n)=0,$$

$$P_2\equiv(D_t^2e^{(1/2)D_n}-D_ye^{(1/2)D_n})g(n)\cdot g(n)=0,$$

$$P_3\equiv(D_t^2e^{D_n}+3D_t^2+\tfrac{1}{2}D_t^3D_z-2D_ye^{D_n})g(n)\cdot g(n)=0,$$

$$P_4\equiv(D_t^3e^{(1/2)D_n}+4D_ze^{(3/2)D_n}-D_yD_te^{(1/2)D_n})g(n)\cdot g(n)=0.$$

In fact, a similar deduction to that in Refs. 10 and 11 can give rise to $P_1=0$, $P_2=0$. Thus we focus on showing that $P_3=0$ and $P_4=0$. Let us first prove that $P_3=0$. Making use of (A6) and (A7) can give

$$\begin{aligned}-2P_3f^2(n)=&2[(D_t^2e^{D_n}+3D_t^2+\tfrac{1}{2}D_t^3D_z-2D_ye^{D_n})f\cdot f]g^2\\&-2[(D_t^2e^{D_n}+3D_t^2+\tfrac{1}{2}D_t^3D_z-2D_ye^{D_n})g\cdot g]f^2+3[(D_zD_t-2e^{D_n}+2)f\cdot f]\\&\times(D_t^2g\cdot g)-3(D_t^2f\cdot f)[(D_zD_t-2e^{D_n}+2)g\cdot g]\end{aligned}$$

$$=I_1-8\sinh(\tfrac{1}{2}D_n)[(D_y e^{(1/2)D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g)-(e^{(1/2)D_n}f\cdot g)$$
$$\cdot(D_y e^{-(1/2)D_n}f\cdot g)]-4D_t\cosh(\tfrac{1}{2}D_n)[(D_t e^{(1/2)D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g)$$
$$-(e^{(1/2)D_n}f\cdot g)\cdot(D_t e^{-(1/2)D_n}f\cdot g)]-4\sinh(\tfrac{1}{2}D_n)[(D_t^2 e^{(1/2)D_n}f\cdot g)$$
$$\cdot(e^{-(1/2)D_n}f\cdot g)-2(D_t e^{(1/2)D_n}f\cdot g)\cdot(D_t e^{-(1/2)D_n}f\cdot g)$$
$$+(e^{(1/2)D_n}f\cdot g)\cdot(D_t^2 e^{-(1/2)D_n}f\cdot g)], \tag{24}$$

where the function $I_1$ is defined by

$$I_1\equiv[(D_t^3D_z+6D_t^2+6D_t^2e^{D_n})f\cdot f]g^2-f^2(D_t^3D_z+6D_t^2+6D_t^2e^{D_n})g\cdot g$$
$$+3[(D_zD_t-2e^{D_n}+2)f\cdot f](D_t^2g\cdot g)-3(D_t^2f\cdot f)[(D_zD_t-2e^{D_n}+2)g\cdot g].$$

Using a similar deduction to that in Ref. 12, we have

$$I_1=4\lambda^{-1}\sinh(\tfrac{1}{2}D_n)(D_t^3e^{-(1/2)D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g)+12\sinh(\tfrac{1}{2}D_n)(D_t^2e^{(1/2)D_n}f\cdot g)$$
$$\cdot(e^{-(1/2)D_n}f\cdot g)+24\gamma\sinh(\tfrac{1}{2}D_n)(D_te^{(1/2)D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g)+24\gamma^2\sinh(\tfrac{1}{2}D_n)$$
$$\times(e^{(1/2)D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g)+12\lambda^{-1}\gamma\sinh(\tfrac{1}{2}D_n)(D_t^2e^{-(1/2)D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g).$$

Thus by using (A8)–(A11), (20), and (21) equality (24) can be further reduced to the following:

$$-2P_3f^2(n)=4\lambda^{-1}\sinh(\tfrac{1}{2}D_n)(D_t^3e^{-(1/2)D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g)+4\sinh(\tfrac{1}{2}D_n)(D_t^2e^{(1/2)D_n}f\cdot g)$$
$$\cdot(e^{-(1/2)D_n}f\cdot g)+8\gamma\sinh(\tfrac{1}{2}D_n)(D_te^{(1/2)D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g)+16\gamma^2\sinh(\tfrac{1}{2}D_n)$$
$$\times(e^{(1/2D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g)+12\lambda^{-1}\gamma\sinh(\tfrac{1}{2}D_n)(D_t^2e^{-(1/2)D_n}f\cdot g)$$
$$\cdot(e^{-(1/2)D_n}f\cdot g)-8\sinh(\tfrac{1}{2}D_n)(D_ye^{(1/2)D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g)$$
$$-8\omega\sinh(\tfrac{1}{2}D_n)(e^{(1/2)D_n}f\cdot g)\cdot(e^{-(1/2)D_n}f\cdot g)=0,$$

which implies that $P_3=0$.

Second, let us prove that $P_4=0$. By using (A12)–(A25) and (19)–(22), we can deduce the following relation

$$-P_4e^{(3/2)D_n}f\cdot f=[(D_t^3e^{(1/2)D_n}+4D_ze^{(3/2)D_n}-D_yD_te^{(1/2)D_n})f\cdot f][e^{(3/2)D_n}g\cdot g]-[e^{(3/2)D_n}f\cdot f]$$
$$\times[(D_t^3e^{(1/2)D_n}+4D_ze^{(3/2)D_n}-D_yD_te^{(1/2)D_n})g\cdot g]$$
$$=\sinh(\tfrac{1}{2}D_n)(e^{D_n}f\cdot g)\cdot[(2D_t^3e^{-D_n}+2D_tD_ye^{-D_n}+6\gamma D_t^2e^{-D_n}+\tfrac{1}{2}\lambda\nu e^{-D_n}$$
$$+8\lambda^{-1}e^{-2D_n}+2\omega D_te^{-D_n}+2\gamma D_ye^{-D_n}+4\lambda^2D_te^{D_n}+6\gamma^2D_te^{-D_n}$$
$$+2\omega\gamma e^{-D_n})f\cdot g]-3\gamma^2\sinh(D_n)(e^{(1/2)D_n}f\cdot g)\cdot(-\gamma e^{-(1/2)D_n}f\cdot g)$$
$$=\sinh(\tfrac{1}{2}D_n)(e^{D_n}f\cdot g)\cdot[(2D_t^3e^{-D_n}+2D_tD_ye^{-D_n}+6\gamma D_t^2e^{-D_n}+\tfrac{1}{2}\lambda\nu e^{-D_n}$$
$$+8\lambda^{-1}e^{-2D_n}+2\omega D_te^{-D_n}+2\gamma D_ye^{-D_n}+4\lambda^2D_te^{D_n}+6\gamma^2D_te^{-D_n}+2\omega\gamma e^{-D_n}$$
$$+3\gamma^3e^{-D_n})f\cdot g]=0.$$

Thus the proof of Theorem 1 is completed.

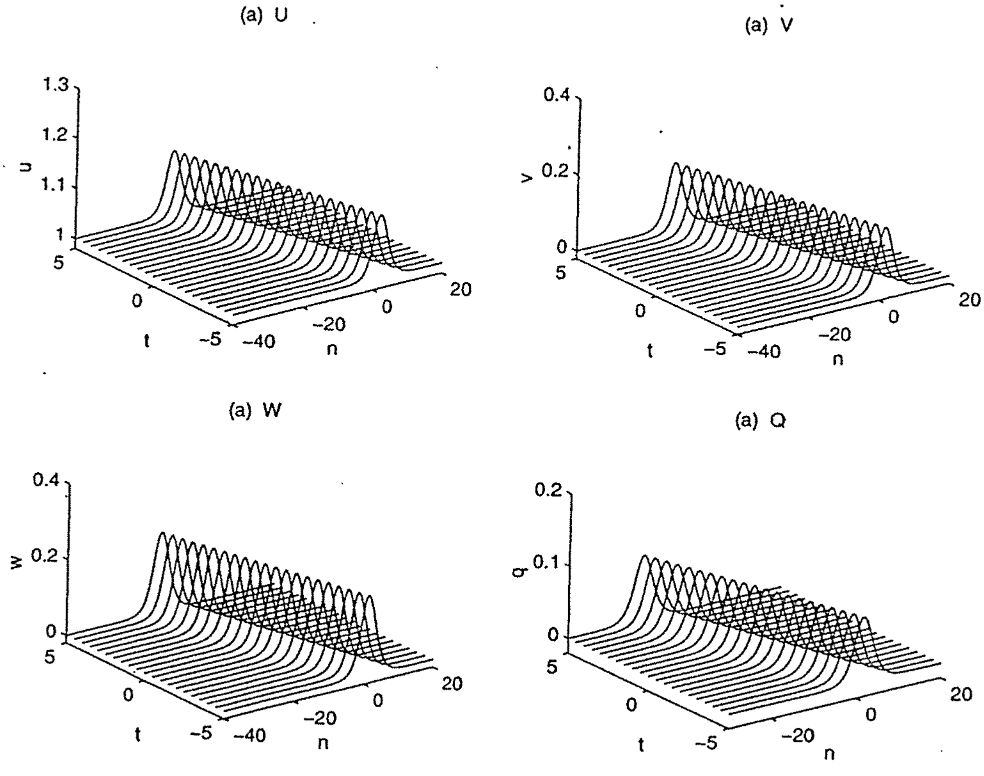



FIG. 1. 1-soliton solution of the lattice (4)–(7).

By using the BT given by (19)–(23), we can easily obtain the following solution from the trivial solution $f(n)=1$:

$$g(n)=1+\exp(\eta), \quad \eta=pn+qt+rz+sy+\eta^0 \tag{25}$$

with

$$\lambda^4=e^p(1+e^p+e^{2p}), \quad \mu=-\lambda^{-1}, \quad \gamma=\lambda, \quad \omega=\lambda^2, \quad \nu=-2\lambda^2, \quad \theta=-4\lambda^3-8\lambda^{-1},$$

where $p$ is an arbitrary constant, $q=\lambda(1-e^{-p})$, $r=\lambda^{-1}(e^p-1)$, $s=\lambda^2(1-e^{-2p})$ and $\eta^0$ is an arbitrary constant. Therefore the corresponding one-soliton solution of the lattice (4)–(7) is

$$\begin{aligned} u(n)&=\frac{g(n+1)g(n-1)}{g^2(n)}, \quad v(n)=\left(\ln\frac{g(n+1)}{g(n)}\right)_t, \\ q(n)&=\left(\ln\frac{g(n+3)}{g(n)}\right)_z, \quad w(n)=\frac{1}{2}\frac{D_t^2 g(n+1)\cdot g(n+1)}{g(n+2)g(n)}, \end{aligned} \tag{26}$$

with $g(n)$ being given by (25). The plot of the solution (26) is shown in Fig. 1, where we choose $p=0.7$, $\lambda\approx 1.94$, $\eta^0=0$, $z=1$, $y=1$.

## IV. NONLINEAR SUPERPOSITION FORMULA

In the following, we shall simply denote, without confusion, $f(n,t)$ by $f(n)$ or $f$. The corresponding nonlinear superposition formula is shown in the following theorem.

**Theorem 2:** Let $f_0$ be a solution of Eqs. (15)–(18). Suppose that $f_i$ $(i=1,2)$ are two other solutions of (15)–(18) which are related to $f_0$ under the BT(19)–(23) with parameters $(\lambda_i,\mu_i,\gamma_i,\omega_i,\nu_i,\theta_i)$, i.e., $f_0 \rightarrow^{(\lambda_i,\mu_i,\gamma_i,\omega_i,\nu_i,\theta_i)} f_i$ $(i=1,2)$, where $\lambda_1\lambda_2\neq 0$, $f_j\neq 0$ $(j=0,1,2)$. Then $f_{12}$ defined by

$$\exp(-\tfrac{1}{2}D_n)f_0\cdot f_{12}=k[\lambda_1\exp(-\tfrac{1}{2}D_n)-\lambda_2\exp(\tfrac{1}{2}D_n)]f_1\cdot f_2, \tag{27}$$

where $k$ is a nonzero constant, is a new solution which is related to $f_1$ and $f_2$ under the BT(19)-(23) with parameters $(\lambda_2,\mu_2,\gamma_2,\omega_2,\nu_2,\theta_2)$ and $(\lambda_1,\mu_1,\gamma_1,\omega_1,\nu_1,\theta_1)$, respectively.

In order to prove Theorem 2, we first establish some basic lemmas. In what follows, we always assume that the hypotheses of Theorem 2 are satisfied and $f_{12}$ is determined by (27). Besides we set

$$J_i(n)\equiv(D_te^{-(1/2)D_n}-\lambda_ie^{(1/2)D_n}+\gamma_ie^{-(1/2)D_n})f_0(n)\cdot f_i(n),\quad i=1,2$$

$$K_i(n)\equiv(D_ye^{-(1/2)D_n}-\lambda_iD_te^{(1/2)D_n}-\lambda_i\gamma_ie^{(1/2)D_n}+\omega_ie^{-(1/2)D_n})f_0(n)\cdot f_i(n),\quad i=1,2$$

$$\begin{aligned}L_i(n)\equiv{}&(\lambda_i^{-1}D_t^3e^{-(1/2)D_n}+D_t^2e^{(1/2)D_n}+2\gamma_iD_te^{(1/2)D_n}+4\gamma_i^2e^{(1/2)D_n}+3\lambda_i^{-1}\gamma_iD_t^2e^{-(1/2)D_n}\\&-2D_ye^{(1/2)D_n}-2\omega_ie^{(1/2)D_n}+\nu_ie^{-(1/2)D_n})f_0(n)\cdot f_i(n),\quad i=1,2.\end{aligned}$$

*Lemma 1:* The bilinear relations hold:

$$(D_z+\lambda_2^{-1}e^{-D_n}+\mu_2)f_1\cdot f_{12}=0, \tag{28}$$

$$(D_z+\lambda_1^{-1}e^{-D_n}+\mu_1)f_2\cdot f_{12}=0, \tag{29}$$

$$(D_te^{-(1/2)D_n}-\lambda_2e^{(1/2)D_n}+\gamma_2e^{-(1/2)D_n})f_1\cdot f_{12}=0, \tag{30}$$

$$(D_te^{-(1/2)D_n}-\lambda_1e^{(1/2)D_n}+\gamma_1e^{-(1/2)D_n})f_2\cdot f_{12}=0, \tag{31}$$

$$-D_zf_1\cdot f_2+(\mu_1-\mu_2)f_1f_2-\frac{1}{k\lambda_1\lambda_2}e^{-D_n}f_0\cdot f_{12}=0, \tag{32}$$

$$(\lambda_2D_te^{(1/2)D_n}+\lambda_1D_te^{-(1/2)D_n}-2\lambda_2\gamma_1e^{(1/2)D_n}+2\lambda_1\gamma_2e^{-(1/2)D_n})f_1\cdot f_2+\frac{1}{k}D_te^{-(1/2)D_n}f_0\cdot f_{12}=0. \tag{33}$$

*Proof:* (28)–(33) can be proved similarly as in Refs. 10 and 11.

*Lemma 2:* The bilinear relations hold:

$$-D_tf_1\cdot f_2+(\gamma_1-\gamma_2)f_1f_2-\frac{1}{k}f_0f_{12}=0, \tag{34}$$

$$D_ye^{-(1/2)D_n}f_1\cdot f_{12}=(\lambda_2D_te^{(1/2)D_n}+\lambda_2\gamma_2e^{(1/2)D_n}-\omega_2e^{-(1/2)D_n})f_1\cdot f_{12}, \tag{35}$$

$$D_ye^{-(1/2)D_n}f_2\cdot f_{12}=(\lambda_1D_te^{(1/2)D_n}+\lambda_1\gamma_1e^{(1/2)D_n}-\omega_1e^{-(1/2)D_n})f_2\cdot f_{12}. \tag{36}$$

*Proof:* First, according to the hypotheses of Theorem 2, we have

$$J_1(n)f_2(n+\tfrac{1}{2})-J_2(n)f_1(n+\tfrac{1}{2})=0,$$

from which, by use of (A26) and (27), it follows that (34) holds. Next, since $f_1$ and $f_2$ are two solutions of (15)–(18), we have

$$[(D_t^2 e^{(1/2)D_n} - D_y e^{(1/2)D_n}) f_1 \cdot f_1](e^{(1/2)D_n} f_2 \cdot f_2) - (e^{(1/2)D_n} f_1 \cdot f_1)[(D_t^2 e^{(1/2)D_n} - D_y e^{(1/2)D_n}) f_2 \cdot f_2] = 0,$$

which can be rewritten as

$$-\frac{1}{k\lambda_2}(e^{-(1/2)D_n} f_0 \cdot f_2)[(D_y e^{-(1/2)D_n} - \lambda_2 D_t e^{(1/2)D_n} - \lambda_2 \gamma_2 e^{(1/2)D_n} + \omega_2 e^{-(1/2)D_n}) f_1 \cdot f_{12}] = 0$$

by use of (A2), (A27)–(A30), (27), (30), and (34). Therefore (35) holds. Similarly we can prove that (36) also holds.

*Lemma 3:* The bilinear relations hold:

$$(\lambda_2^{-1} D_t^3 e^{-(1/2)D_n} + D_t^2 e^{(1/2)D_n} + 2\gamma_2 D_t e^{(1/2)D_n} + 4\gamma_2^2 e^{(1/2)D_n} + 3\lambda_2^{-1}\gamma_2 D_t^2 e^{-(1/2)D_n} - 2D_y e^{(1/2)D_n} - 2\omega_2 e^{(1/2)D_n} + \nu_2 e^{-(1/2)D_n}) f_1 \cdot f_{12} = 0, \tag{37}$$

$$(\lambda_1^{-1} D_t^3 e^{-(1/2)D_n} + D_t^2 e^{(1/2)D_n} + 2\gamma_1 D_t e^{(1/2)D_n} + 4\gamma_1^2 e^{(1/2)D_n} + 3\lambda_1^{-1}\gamma_1 D_t^2 e^{-(1/2)D_n} - 2D_y e^{(1/2)D_n} - 2\omega_1 e^{(1/2)D_n} + \nu_1 e^{-(1/2)D_n}) f_2 \cdot f_{12} = 0. \tag{38}$$

*Proof:* Since $f_1$ and $f_2$ are two solutions of Eqs. (15)–(18), we have

$$[(D_t^3 D_z + 6D_t^2 + 2D_t^2 e^{D_n} - 4D_y e^{D_n}) f_1 \cdot f_1] f_2^2 - [(D_t^3 D_z + 6D_t^2 + 2D_t^2 e^{D_n} - 4D_y e^{D_n}) f_2 \cdot f_2] f_1^2 + 3[(D_z D_t - 2e^{D_n} + 2) f_1 \cdot f_1](D_t^2 f_2 \cdot f_2) - 3(D_t^2 f_1 \cdot f_1)[(D_z D_t - 2e^{D_n} + 2) f_2 \cdot f_2] = 0.$$

On the other hand, using a similar deduction as in Ref. 12, we obtain that

$$\begin{aligned}
&[(D_t^3 D_z + 6D_t^2 + 6D_t^2 e^{D_n}) f_1 \cdot f_1] f_2^2 - [(D_t^3 D_z + 6D_t^2 + 6D_t^2 e^{D_n}) f_2 \cdot f_2] f_1^2 \\
&\quad + 3[(D_z D_t - 2e^{D_n} + 2) f_1 \cdot f_1](D_t^2 f_2 \cdot f_2) - 3(D_t^2 f_1 \cdot f_1)[(D_z D_t - 2e^{D_n} + 2) f_2 \cdot f_2] \\
&= -\frac{2}{k\lambda_1\lambda_2} e^{-(1/2)D_n}\{[(D_t^3 e^{-(1/2)D_n} + 3\lambda_2 D_t^2 e^{(1/2)D_n} + 3\gamma_2 D_t^2 e^{-(1/2)D_n} + 6\lambda_2\gamma_2^2 e^{(1/2)D_n} \\
&\quad + 6\lambda_2\gamma_2 D_t e^{-(1/2)D_n}) f_0 \cdot f_2] \cdot (e^{-(1/2)D_n} f_1 \cdot f_{12}) - (e^{-(1/2)D_n} f_0 \cdot f_2) \cdot [(D_t^3 e^{-(1/2)D_n} \\
&\quad + 3\lambda_2 D_t^2 e^{(1/2)D_n} + 3\gamma_2 D_t^2 e^{(1/2)D_n} + 6\lambda_2\gamma_2^2 e^{(1/2)D_n} + 6\lambda_2\gamma_2 D_t e^{-(1/2)D_n}) f_1 \cdot f_{12}]\}.
\end{aligned}$$

Thus by using (A6), (A7), (A31)–(A33), (27), and (34), we have

$$\frac{2}{k\lambda_1} e^{-(1/2)D_n}(e^{-(1/2)D_n} f_0 \cdot f_2) \cdot [(\lambda_2^{-1} D_t^3 e^{-(1/2)D_n} + D_t^2 e^{(1/2)D_n} + 2\gamma_2 D_t e^{(1/2)D_n} + 4\gamma_2^2 e^{(1/2)D_n} + 3\lambda_2^{-1}\gamma_2 D_t^2 e^{-(1/2)D_n} - 2D_y e^{(1/2)D_n} - 2\omega_2 e^{(1/2)D_n} + \nu_2 e^{-(1/2)D_n}) f_1 \cdot f_{12}] = 0,$$

which implies that (37) holds. Similarly we can show that (38) holds.

*Lemma 4:* The bilinear relation holds:

$$(\lambda_1 D_t^2 e^{-(1/2)D_n} - \lambda_2 D_t^2 e^{(1/2)D_n} + 4\lambda_2\gamma_1 D_t e^{(1/2)D_n} + 4\lambda_1\gamma_2 D_t e^{-(1/2)D_n} - 4\lambda_2\gamma_1^2 e^{(1/2)D_n} + 4\lambda_1\gamma_2^2 e^{-(1/2)D_n}) f_1 \cdot f_2 - \frac{1}{k} D_t^2 e^{-(1/2)D_n} f_0 \cdot f_{12} + \frac{4}{k}\lambda_1\lambda_2 e^{(1/2)D_n} f_0 \cdot f_{12} = 0. \tag{39}$$

*Proof:* According to the hypotheses of Theorem 2, we have

$$\frac{1}{\lambda_1}[J_1(n)]_t f_2\left(n-\frac{1}{2}\right)-\frac{1}{\lambda_2}[J_2(n)]_t f_1\left(n-\frac{1}{2}\right)-\frac{\gamma_1}{\lambda_1}J_1(n)f_2\left(n-\frac{1}{2}\right)+\frac{\gamma_2}{\lambda_2}J_2(n)f_1\left(n-\frac{1}{2}\right)=0,$$

which implies that (39) holds by use of (27), (33), and (34).

*Lemma 5:* The bilinear relation holds:

$$(\lambda_1 D_y e^{-(1/2)D_n}+\lambda_2 D_y e^{(1/2)D_n}+2\lambda_1\omega_2 e^{-(1/2)D_n}-2\lambda_2\omega_1 e^{(1/2)D_n})f_1\cdot f_2 + \left(\frac{1}{k}D_y e^{-(1/2)D_n}+\frac{2}{k}\lambda_1\lambda_2 e^{(1/2)D_n}\right)f_0\cdot f_{12}=0. \quad (40)$$

*Proof:* Based on the hypotheses of Theorem 2, we have

$$\lambda_2 K_1(n)f_2(n-\tfrac{1}{2})-\lambda_1 K_2(n)f_1(n-\tfrac{1}{2})=0,$$

which implies that (40) holds upon using (27) and (33).

*Lemma 6:* The bilinear relation holds:

$$D_y f_1\cdot f_2+(\omega_2-\omega_1)f_1 f_2+\frac{1}{k}D_t f_0\cdot f_{12}+\frac{1}{k}(\gamma_1+\gamma_2)f_0 f_{12}=0. \quad (41)$$

*Proof:* According to the hypotheses of Theorem 2, we obtain

$$K_1(n)f_2(n+\tfrac{1}{2})-K_2(n)f_1(n+\tfrac{1}{2})=0,$$

which lead to (41) upon taking into account (27) and (33).

*Lemma 7:* The bilinear relation holds:

$$\begin{aligned}&-\frac{1}{k}D_t^3 e^{-(1/2)D_n}f_0\cdot f_{12}+\frac{4\lambda_1\lambda_2}{k}D_t e^{(1/2)D_n}f_0\cdot f_{12}-\frac{8\lambda_1\lambda_2}{k}(\gamma_1+\gamma_2)e^{(1/2)D_n}f_0\cdot f_{12}\\&+6\lambda_2\gamma_1 D_t^2 e^{(1/2)D_n}f_1\cdot f_2-6\lambda_1\gamma_2 D_t^2 e^{-(1/2)D_n}f_1\cdot f_2+2\lambda_1\lambda_2\nu_1 e^{(1/2)D_n}f_1\cdot f_2\\&-2\lambda_1\lambda_2\nu_2 e^{-(1/2)D_n}f_1\cdot f_2-12\lambda_1\gamma_2^2 D_t e^{-(1/2)D_n}f_1\cdot f_2-12\lambda_2\gamma_1^2 D_t e^{(1/2)D_n}f_1\cdot f_2\\&+12\lambda_2\gamma_1^3 e^{(1/2)D_n}f_1\cdot f_2-12\lambda_1\gamma_2^3 e^{-(1/2)D_n}f_1\cdot f_2\\&-\lambda_1 D_t^3 e^{-(1/2)D_n}f_1\cdot f_2-\lambda_2 D_t^3 e^{(1/2)D_n}f_1\cdot f_2=0.\end{aligned} \quad (42)$$

*Proof:* It follows from the hypotheses of Theorem 2 that

$$\begin{aligned}&\frac{1}{4}L_1(n)f_2\left(n-\frac{1}{2}\right)-\frac{1}{4}L_2(n)f_1\left(n-\frac{1}{2}\right)+\frac{3}{2}\frac{\gamma_1^2}{\lambda_1}J_1(n)f_2\left(n-\frac{1}{2}\right)-\frac{3}{2}\frac{\gamma_2^2}{\lambda_2}J_2(n)f_1\left(n-\frac{1}{2}\right)\\&-\frac{3}{2}\frac{\gamma_1}{\lambda_1}[J_1(n)]_t f_2\left(n-\frac{1}{2}\right)+\frac{3}{2}\frac{\gamma_2}{\lambda_2}[J_2(n)]_t f_1\left(n-\frac{1}{2}\right)+\frac{3}{4}\lambda_1^{-1}[J_1(n)]_{tt}f_2\left(n-\frac{1}{2}\right)\\&-\tfrac{3}{4}\lambda_2^{-1}[J_2(n)]_{tt}f_1(n-\tfrac{1}{2})=0.\end{aligned}$$

This tells that (42) holds by using (27), (33), (A34), and (A35).

*Lemma 8:* The bilinear relation holds:

$$\begin{aligned}&[-D_y D_t e^{-(1/2)D_n}+2\lambda_1\lambda_2 D_t e^{(1/2)D_n}+4\lambda_1\lambda_2(\gamma_1+\gamma_2)e^{(1/2)D_n}]f_0\cdot f_{12}+k(2\lambda_2\omega_1 D_t e^{(1/2)D_n}\\&+2\lambda_1\omega_2 D_t e^{-(1/2)D_n}+2\lambda_2\gamma_1 D_y e^{(1/2)D_n}+2\lambda_1\gamma_2 D_y e^{-(1/2)D_n}-\lambda_2 D_y D_t e^{(1/2)D_n}\\&+\lambda_1 D_y D_t e^{-(1/2)D_n}-4\lambda_2\gamma_1\omega_1 e^{(1/2)D_n}+4\lambda_1\gamma_2\omega_2 e^{-(1/2)D_n})f_1\cdot f_2=0.\end{aligned} \quad (43)$$

*Proof:* According to the hypotheses of Theorem 2, we have

$$\lambda_1^{-1}[K_1(n)]_t f_2\left(n-\frac{1}{2}\right)-\lambda_2^{-1}[K_2(n)]_t f_1\left(n-\frac{1}{2}\right)-\frac{\gamma_1}{\lambda_1}K_1(n)f_2\left(n-\frac{1}{2}\right)$$
$$+\frac{\gamma_2}{\lambda_2}K_2(n)f_1\left(n-\frac{1}{2}\right)=0,$$

which shows that (43) holds by using (27), (33), (A36), and (A37).

*Lemma 9:* The bilinear relation holds:

$$2\gamma_2\sinh(D_n)(e^{(1/2)D_n}f_0\cdot f_{12})\cdot(e^{(1/2)D_n}f_1\cdot f_2)-D_t\cosh(D_n)(e^{(1/2)D_n}f_0\cdot f_{12})\cdot(e^{(1/2)D_n}f_1\cdot f_2)$$
$$+\sinh(D_n)[(D_te^{(1/2)D_n}f_0\cdot f_{12})\cdot(e^{(1/2)D_n}f_1\cdot f_2)+(e^{(1/2)D_n}f_0\cdot f_{12})\cdot(D_te^{(1/2)D_n}f_1\cdot f_2)]$$
$$=-\frac{\lambda_2}{\lambda_1}e^{-(1/2)D_n}[(e^{D_n}f_0\cdot f_2)\cdot(D_te^{D_n}f_1\cdot f_{12})-(D_te^{D_n}f_0\cdot f_2)\cdot(e^{D_n}f_1\cdot f_{12})]. \tag{44}$$

*Proof.* On the one hand, by use of (A38), (A39), and (30), we can have

$$2\gamma_2\sinh(D_n)(e^{(1/2)D_n}f_0\cdot f_{12})\cdot(e^{(1/2)D_n}f_1\cdot f_2)-D_t\cosh(D_n)(e^{(1/2)D_n}f_0\cdot f_{12})\cdot(e^{(1/2)D_n}f_1\cdot f_2)$$
$$+\sinh(D_n)[(D_te^{(1/2)D_n}f_0\cdot f_{12})\cdot(e^{(1/2)D_n}f_1\cdot f_2)+(e^{(1/2)D_n}f_0\cdot f_{12})\cdot(D_te^{(1/2)D_n}f_1\cdot f_2)]$$
$$=\lambda_2[(e^{(3/2)D_n}f_0\cdot f_2)(e^{(1/2)D_n}f_1\cdot f_{12})-(e^{(1/2)D_n}f_0\cdot f_2)(e^{(3/2)D_n}f_1\cdot f_{12}).$$

On the other hand, by use of (A40) and (30), we can have

$$e^{-(1/2)D_n}[(e^{D_n}f_0\cdot f_2)\cdot(D_te^{D_n}f_1\cdot f_{12})-(D_te^{D_n}f_0\cdot f_2)\cdot(e^{D_n}f_1\cdot f_{12}]$$
$$=-\lambda_1[(e^{(3/2)D_n}f_0\cdot f_2)(e^{(1/2)D_n}f_1\cdot f_{12})-(e^{(1/2)D_n}f_0\cdot f_2)(e^{(3/2)D_n}f_1\cdot f_{12})].$$

Therefore combining these two equalities leads to the required equality (44).

We now turn to the proof of Theorem 2. Based on Lemma 1, Lemma 2, and Lemma 3, it suffices to show that

$$(2D_t^3e^{-D_n}+2D_tD_ye^{-D_n}+6\gamma_2D_t^2e^{-D_n}+\tfrac{1}{2}\lambda_2\nu_2e^{-D_n}+8\lambda_2^{-1}e^{-2D_n}+2\omega_2D_te^{-D_n}+2\gamma_2D_ye^{-D_n}$$
$$+4\lambda_2^2D_te^{D_n}+6\gamma_2^2D_te^{-D_n}+2\omega_2\gamma_2e^{-D_n}+3\gamma_2^3e^{-D_n}+\theta_2e^{D_n})f_1\cdot f_{12}=0, \tag{45}$$

$$(2D_t^3e^{-D_n}+2D_tD_ye^{-D_n}+6\gamma_1D_t^2e^{-D_n}+\tfrac{1}{2}\lambda_1\nu_1e^{-D_n}+8\lambda_1^{-1}e^{-2D_n}+2\omega_1D_te^{-D_n}+2\gamma_1D_ye^{-D_n}$$
$$+4\lambda_1^2D_te^{D_n}+6\gamma_1^2D_te^{-D_n}+2\omega_1\gamma_1e^{-D_n}+3\gamma_1^3e^{-D_n}+\theta_1e^{D_n})f_2\cdot f_{12}=0. \tag{46}$$

Since $f_1$ and $f_2$ are two solutions of Eqs. (15)–(18), we have

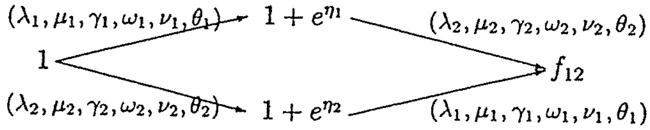
$$[(D_t^3e^{(1/2)D_n}+4D_ze^{(3/2)D_n}-D_yD_te^{(1/2)D_n})f_1\cdot f_1][e^{(3/2)D_n}f_2\cdot f_2]$$
$$-[(D_t^3e^{(1/2)D_n}+4D_ze^{(3/2)D_n}-D_yD_te^{(1/2)D_n})f_2\cdot f_2][e^{(3/2)D_n}f_1\cdot f_1]=0. \tag{47}$$

Taking advantage of (A12)–(A14), (A41)–(A43), (27), (32)–(34), and (39)–(44), we can rewrite (47) as

$$-\frac{1}{k\lambda_1}e^{-(1/2)D_n}(e^{D_n}f_0\cdot f_2)\cdot\Bigg[\Bigg(D_t^3e^{-D_n}+D_yD_te^{-D_n}+3\gamma_2D_t^2e^{-D_n}+\frac{1}{4}\lambda_2\nu_2e^{-D_n}$$
$$+4\lambda_2^{-1}e^{-2D_n}+\omega_2D_te^{-D_n}+\gamma_2D_ye^{-D_n}+2\lambda_2^2D_te^{D_n}+3\gamma_2^2D_te^{-D_n}+\omega_2\gamma_2e^{-D_n}$$
$$+\frac{3}{2}\gamma_2^3e^{-D_n}+\frac{1}{2}\theta_2e^{D_n}\Bigg)f_1\cdot f_{12}\Bigg]=0, \tag{48}$$

which implies that (45) holds. Similarly we can prove that (46) also holds. Therefore we have completed the proof of theorem 2.

As an application of the nonlinear superposition formula (27), we can construct soliton solutions of the Blaszak–Marciniak lattice of the Eqs. (15)–(18). Choose for example $f_0=1,k = 1/\lambda_1-\lambda_2$. It is easily verified that




where $f_{12}$ is given by

$$f_{12}=1+\frac{\lambda_1e^{-p_1}-\lambda_2}{\lambda_1-\lambda_2}e^{\eta_1}+\frac{\lambda_1-\lambda_2e^{-p_2}}{\lambda_1-\lambda_2}e^{\eta_2}+\frac{\lambda_1e^{-p_1}-\lambda_2e^{-p_2}}{\lambda_1-\lambda_2}e^{\eta_1+\eta_2} \tag{49}$$

with

$$\eta_i=p_in+q_it+r_iz+s_iy+\eta_i^0,\quad q_i=\lambda_i(1-e^{-p_i}),\quad r_i=\lambda_i^{-1}(e^{p_i}-1),\quad s_i=\lambda_i^2(1-e^{-2p_i})$$

and

$$\lambda_i^4=e^{p_i}(1+e^{p_i}+e^{2p_i}),\quad \mu_i=-\lambda_i^{-1},\quad \gamma_i=\lambda_i,\quad \omega_i=\lambda_i^2,\quad \nu_i=-2\lambda_i^2,\quad \theta_i=-4\lambda_i^3-8\lambda_i^{-1}$$

in which the $p_i$ $(i=1,2)$ are arbitrary constants. Thus the corresponding 2-soliton solution of (4)–(7) is

$$u(n)=\frac{f_{12}(n+1)f_{12}(n-1)}{f_{12}^2(n)},\quad v(n)=\left(\ln\frac{f_{12}(n+1)}{f_{12}(n)}\right)_t,$$
$$q(n)=\left(\ln\frac{f_{12}(n+3)}{f_{12}(n)}\right)_z,\quad w(n)=\frac{1}{2}\frac{D_t^2f_{12}(n+1)\cdot f_{12}(n+1)}{f_{12}(n+2)f_{12}(n)} \tag{50}$$

with $f_{12}(n)$ being given by (49). The plot of (50) is shown in Fig. 2 where we chose $p_1=1.2$, $p_2=1.42$, $\lambda_1\approx-2.67$, $\lambda_2\approx3.10$, $\eta_1^0=\eta_2^0=0$, $z=1$, $y=1$.

In general, along this line, we can generate multisoliton solutions for the Blaszak–Marciniak lattice (4)–(7) successively.

## V. CONCLUSION

By introducing two auxiliary variables, a four-field lattice introduced by Blaszak and Marciniak[1] is transformed into Hirota's bilinear form. The transformation of the dependent variables are given by (8), (10), and (11). A bilinear Bäcklund transformation (Theorem 1) is found and its corresponding nonlinear superposition formula (Theorem 2) is rigorously proved. As a

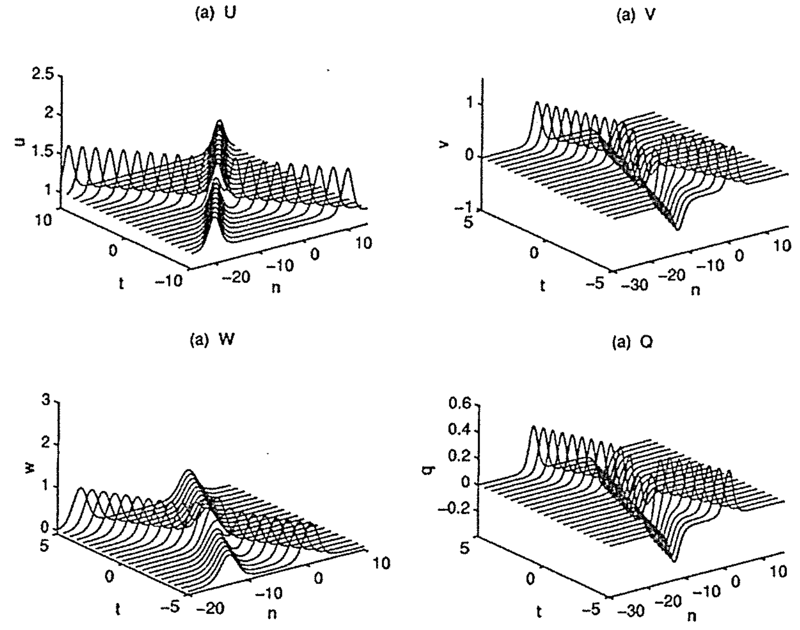



FIG. 2. 2-soliton solution of the lattice (4)–(7).

consequence, one-soliton and two-soliton solutions to the lattice are constructed. In principle, the resulted nonlinear superposition formula guarantees the existence of multisoliton solutions and tells us how to construct them explicitly.

## ACKNOWLEDGMENTS

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## APPENDIX: HIROTA BILINEAR OPERATOR IDENTITIES

The following bilinear operator identities hold for arbitrary functions $a,b,c$ and $d$:

$$D_t^3(D_zD_ta\cdot a)\cdot a^2=D_t[(D_t^3D_za\cdot a)\cdot a^2+3(D_tD_za\cdot a)\cdot(D_t^2a\cdot a)], \tag{A1}$$

$$D_t^{2n+1}a\cdot a=0, \quad n=0,1,2,\ldots, \tag{A2}$$

$$\begin{aligned}D_t^3(e^{D_n}a\cdot a)\cdot a^2=&2\sinh(\tfrac{1}{2}D_n)[(D_t^3e^{(1/2)D_n}a\cdot a)\cdot(e^{(1/2)D_n}a\cdot a)\\&+3(D_te^{(1/2)D_n}a\cdot a)\cdot(D_t^2e^{(1/2)D_n}a\cdot a)],\end{aligned} \tag{A3}$$

$$\begin{aligned}&D_t[(D_t^2e^{D_n}a\cdot a)\cdot a^2+(e^{D_n}a\cdot a)\cdot(D_t^2a\cdot a)]\\&=2\sinh(\tfrac{1}{2}D_n)[(D_t^3e^{(1/2)D_n}a\cdot a)\cdot(e^{(1/2)D_n}a\cdot a)+(D_t^2e^{(1/2)D_n}a\cdot a)\cdot(D_te^{(1/2)D_n}a\cdot a)],\end{aligned} \tag{A4}$$

$$\begin{aligned}&2\sinh(\tfrac{1}{2}D_n)[(D_yD_te^{(1/2)D_n}a\cdot a)\cdot(e^{(1/2)D_n}a\cdot a)+(D_te^{(1/2)D_n}a\cdot a)\cdot(D_ye^{(1/2)D_n}a\cdot a)]\\&=D_t(D_ye^{D_n}a\cdot a)\cdot a^2,\end{aligned} \tag{A5}$$

$$(D_y e^{D_n} a\cdot a)b^2 - a^2 D_y e^{D_n} b\cdot b = 2\sinh(\tfrac{1}{2}D_n)[(D_y e^{(1/2)D_n} a\cdot b)\cdot(e^{-(1/2)D_n} a\cdot b) - (e^{(1/2)D_n} a\cdot b)\cdot(D_y e^{-(1/2)D_n} a\cdot b)]$$
$$= 2D_y\cosh(\tfrac{1}{2}D_n)(e^{(1/2)D_n} a\cdot b)\cdot(e^{-(1/2)D_n} a\cdot b), \tag{A6}$$

$$(D_t^2 e^{D_n} a\cdot a)b^2 - a^2 D_t^2 e^{D_n} b\cdot b = D_t\cosh(\tfrac{1}{2}D_n)[(D_t e^{(1/2)D_n} a\cdot b)\cdot(e^{-(1/2)D_n} a\cdot b) - (e^{(1/2)D_n} a\cdot b)\cdot(D_t e^{-(1/2)D_n} a\cdot b)] + \sinh(\tfrac{1}{2}D_n)[(D_t^2 e^{(1/2)D_n} a\cdot b)\cdot(e^{-(1/2)D_n} a\cdot b) - 2(D_t e^{(1/2)D_n} a\cdot b)\cdot(D_t e^{-(1/2)D_n} a\cdot b) + (e^{(1/2)D_n} a\cdot b)\cdot(D_t^2 e^{-(1/2)D_n} a\cdot b)]$$
$$= D_t\cosh(\tfrac{1}{2}D_n)[(D_t e^{(1/2)D_n} a\cdot b)\cdot(e^{-(1/2)D_n} a\cdot b) - (e^{(1/2)D_n} a\cdot b)\cdot(D_t e^{-(1/2)D_n} a\cdot b)] + D_t^2\sinh((1/2)D_n)(e^{(1/2)D_n} a\cdot b)\cdot(e^{-(1/2)D_n} a\cdot b), \tag{A7}$$

$$D_t\cosh(\tfrac{1}{2}D_n)[(D_t e^{(1/2)D_n} a\cdot b)\cdot(e^{-(1/2)D_n} a\cdot b) + (e^{(1/2)D_n} a\cdot b)\cdot(D_t e^{-(1/2)D_n} a\cdot b)]$$
$$= \sinh(\tfrac{1}{2}D_n)[(D_t^2 e^{(1/2)D_n} a\cdot b)\cdot(e^{-(1/2)D_n} a\cdot b) - (e^{(1/2)D_n} a\cdot b)\cdot(D_t^2 e^{-(1/2)D_n} a\cdot b)], \tag{A8}$$

$$D_t\cosh(\tfrac{1}{2}D_n)a\cdot a = 0, \tag{A9}$$

$$\sinh(\tfrac{1}{2}D_n)a\cdot a = 0, \tag{A10}$$

$$D_t\cosh(\tfrac{1}{2}D_n)(e^{(1/2)D_n} a\cdot b)\cdot(e^{-(1/2)D_n} a\cdot b) = \sinh(\tfrac{1}{2}D_n)[(D_t e^{(1/2)D_n} a\cdot b)\cdot(e^{-(1/2)D_n} a\cdot b) - (e^{(1/2)D_n} a\cdot b)\cdot(D_t e^{-(1/2)D_n} a\cdot b)], \tag{A11}$$

$$(D_t^3 e^{(1/2)D_n} a\cdot a)(e^{(3/2)D_n} b\cdot b) - (e^{(3/2)D_n} a\cdot a)(D_t^3 e^{(1/2)D_n} b\cdot b)$$
$$= 2\sinh(\tfrac{1}{2}D_n)(e^{D_n} a\cdot b)\cdot(D_t^3 e^{-D_n} a\cdot b) + \tfrac{1}{2}\sinh(D_n)[(D_t^3 e^{-(1/2)D_n} a\cdot b)\cdot(e^{(1/2)D_n} a\cdot b) + 3(D_t e^{-(1/2)D_n} a\cdot b)\cdot(D_t^2 e^{(1/2)D_n} a\cdot b)] - 3D_t\cosh(D_n)\times(D_t e^{-(1/2)D_n} a\cdot b)\cdot(D_t e^{(1/2)D_n} a\cdot b) + \tfrac{3}{2}D_t^2\sinh(D_n)(D_t e^{-(1/2)D_n} a\cdot b)\cdot(e^{(1/2)D_n} a\cdot b)$$
$$= \tfrac{1}{4}D_t^3\cosh(D_n)(e^{-(1/2)D_n} a\cdot b)\cdot(e^{(1/2)D_n} a\cdot b) + \tfrac{3}{4}D_t^2\sinh(D_n)\times[(D_t e^{-(1/2)D_n} a\cdot b)\cdot(e^{(1/2)D_n} a\cdot b) - (e^{-(1/2)D_n} a\cdot b)\cdot(D_t e^{(1/2)D_n} a\cdot b)]$$
$$+ \tfrac{3}{4}D_t\cosh(D_n)[(D_t^2 e^{-(1/2)D_n} a\cdot b)\cdot(e^{(1/2)D_n} a\cdot b) + (e^{-(1/2)D_n} a\cdot b)\cdot(D_t^2 e^{(1/2)D_n} a\cdot b) - 2(D_t e^{-(1/2)D_n} a\cdot b)\cdot(D_t e^{(1/2)D_n} a\cdot b)] + \tfrac{1}{4}\sinh(D_n)[(D_t^3 e^{-(1/2)D_n} a\cdot b)\cdot(e^{(1/2)D_n} a\cdot b) + 3(D_t e^{-(1/2)D_n} a\cdot b)\cdot(D_t^2 e^{(1/2)D_n} a\cdot b) - (e^{-(1/2)D_n} a\cdot b)\cdot(D_t^3 e^{(1/2)D_n} a\cdot b) - 3(D_t^2 e^{-(1/2)D_n} a\cdot b)\cdot(D_t e^{(1/2)D_n} a\cdot b)], \tag{A12}$$

$$(D_z e^{(3/2)D_n} a\cdot a)(e^{(3/2)D_n} b\cdot b) - (e^{(3/2)D_n} a\cdot a)(D_z e^{(3/2)D_n} b\cdot b) = 2\sinh(\tfrac{3}{2}D_n)(D_z a\cdot b)\cdot ab, \tag{A13}$$

$$(D_yD_te^{(1/2)D_n}a\cdot a)(e^{(3/2)D_n}b\cdot b)-(e^{(3/2)D_n}a\cdot a)(D_yD_te^{(1/2)D_n}b\cdot b)$$
$$=-2\sinh(\tfrac{1}{2}D_n)(e^{D_n}a\cdot b)\cdot(D_tD_ye^{-D_n}a\cdot b)-\sinh(D_n)[(D_ye^{-(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}a\cdot b)$$
$$+(D_te^{-(1/2)D_n}a\cdot b)\cdot(D_ye^{(1/2)D_n}a\cdot b)]+D_y\cosh(D_n)(D_te^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)$$
$$+D_t\cosh(D_n)(D_ye^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)=\tfrac{1}{2}D_yD_t\sinh(D_n)(e^{-(1/2)D_n}a\cdot b)$$
$$\cdot(e^{(1/2)D_n}a\cdot b)+\tfrac{1}{2}D_t\cosh(D_n)[(D_ye^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)-(e^{-(1/2)D_n}a\cdot b)$$
$$\cdot(D_ye^{(1/2)D_n}a\cdot b)]+\tfrac{1}{2}D_y\cosh(D_n)[(D_te^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)-(e^{-(1/2)D_n}a\cdot b)$$
$$\cdot(D_te^{(1/2)D_n}a\cdot b)]+\tfrac{1}{2}\sinh(D_n)[(D_yD_te^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)+(e^{-(1/2)D_n}a\cdot b)$$
$$\cdot(D_yD_te^{(1/2)D_n}a\cdot b)-(D_ye^{-(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}a\cdot b)-(D_te^{-(1/2)D_n}a\cdot b)$$
$$\cdot(D_ye^{(1/2)D_n}a\cdot b)], \qquad \text{(A14)}$$

$$2\sinh(D_n)(D_t^2e^{(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)$$
$$=D_t[(D_te^{(3/2)D_n}a\cdot b)\cdot(e^{-(1/2)D_n}a\cdot b)+(e^{(3/2)D_n}a\cdot b)\cdot(D_te^{-(1/2)D_n}a\cdot b)], \qquad \text{(A15)}$$

$$2D_t\cosh(D_n)(D_te^{(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)$$
$$=D_t[(D_te^{(3/2)D_n}a\cdot b)\cdot(e^{-(1/2)D_n}a\cdot b)-(e^{(3/2)D_n}a\cdot b)\cdot(D_te^{-(1/2)D_n}a\cdot b)], \qquad \text{(A16)}$$

$$\sinh(D_n)[(D_t^2e^{(1/2)D_n}a\cdot b)\cdot(e^{-(1/2)D_n}a\cdot b)+(e^{(1/2)D_n}a\cdot b)\cdot(D_t^2e^{-(1/2)D_n}a\cdot b)$$
$$+2(D_te^{(1/2)D_n}a\cdot b)\cdot(D_te^{-(1/2)D_n}a\cdot b)]$$
$$=\sinh(\tfrac{1}{2}D_n)[(D_t^2e^{D_n}a\cdot b)\cdot(e^{-D_n}a\cdot b)+(e^{D_n}a\cdot b)\cdot(D_t^2e^{-D_n}a\cdot b)+2(D_te^{D_n}a\cdot b)$$
$$\cdot(D_te^{-D_n}a\cdot b)], \qquad \text{(A17)}$$

$$D_t\cosh(D_n)[(D_te^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)1+(e^{-(1/2)D_n}a\cdot b)\cdot(D_te^{-(1/2)D_n}a\cdot b)]$$
$$=-\sinh(\tfrac{1}{2}D_n)[(D_t^2e^{D_n}a\cdot b)\cdot(e^{-D_n}a\cdot b)-(e^{D_n}a\cdot b)\cdot(D_t^2e^{-D_n}a\cdot b)], \qquad \text{(A18)}$$

$$D_t^2\sinh(D_n)(e^{-q(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)$$
$$=-\sinh(\tfrac{1}{2}D_n)[(D_t^2e^{D_n}a\cdot b)\cdot(e^{-D_n}a\cdot b)-2(D_te^{D_n}a\cdot b)\cdot(D_te^{-D_n}a\cdot b)$$
$$+(e^{D_n}a\cdot b)\cdot(D_t^2e^{-D_n}a\cdot b)], \qquad \text{(A19)}$$

$$\sinh(D_n)(e^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)=-\sinh(\tfrac{1}{2}D_n)(e^{D_n}a\cdot b)\cdot(e^{-D_n}a\cdot b), \qquad \text{(A20)}$$

$$\sinh(\tfrac{3}{2}D_n)(e^{-D_n}a\cdot b)\cdot ab=-\sinh(\tfrac{1}{2}D_n)(e^{D_n}a\cdot b)\cdot(e^{-2D_n}a\cdot b), \qquad \text{(A21)}$$

$$\sinh(D_n)[(D_te^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)+(e^{-(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}a\cdot b)]$$
$$=-\sinh(\tfrac{1}{2}D_n)[(D_te^{D_n}a\cdot b)\cdot(e^{-D_n}a\cdot b)+(e^{D_n}a\cdot b)\cdot(D_te^{-D_n}a\cdot b)], \qquad \text{(A22)}$$

$$D_t\cosh(D_n)(e^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)$$
$$=-\sinh(\tfrac{1}{2}D_n)[(D_te^{D_n}a\cdot b)\cdot(e^{-D_n}a\cdot b)-(e^{D_n}a\cdot b)\cdot(D_te^{-D_n}a\cdot b)], \qquad \text{(A23)}$$

$$D_t(e^{(3/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b)=-2\sinh(\tfrac{1}{2}D_n)(e^{D_n}a\cdot b)\cdot(D_te^{D_n}a\cdot b), \qquad \text{(A24)}$$

$$D_t(e^{(3/2)D_n}a\cdot b)\cdot(e^{-(1/2)D_n}a\cdot b)=2\sinh(\tfrac{1}{2}D_n)(D_te^{(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}a\cdot b), \qquad \text{(A25)}$$

$$(D_ta\cdot b)c-(D_ta\cdot c)b=-aD_tb\cdot c, \qquad \text{(A26)}$$

$$(D_t^2e^{(1/2)D_n}a\cdot a)(e^{(1/2)D_n}b\cdot b)-(e^{(1/2)D_n}a\cdot a)(D_t^2e^{(1/2)D_n}b\cdot b)=2D_t\cosh(\tfrac{1}{2}D_n)(D_ta\cdot b)\cdot ab, \qquad \text{(A27)}$$

$$(D_ye^{(1/2)D_n}a\cdot a)(e^{(1/2)D_n}b\cdot b)-(e^{(1/2)D_n}a\cdot a)(D_ye^{(1/2)D_n}b\cdot b) =D_y(e^{(1/2)D_n}a\cdot b)\cdot(e^{-(1/2)D_n}a\cdot b), \qquad \text{(A28)}$$

$$2D_t\cosh(\tfrac{1}{2}D_n)ab\cdot cd=(D_te^{(1/2)D_n}a\cdot d)(e^{-(1/2)D_n}c\cdot b)-(e^{(1/2)D_n}a\cdot d)(D_te^{-(1/2)D_n}c\cdot b) +(D_te^{-(1/2)D_n}a\cdot d)(e^{(1/2)D_n}c\cdot b)-(e^{-(1/2)D_n}a\cdot d)(D_te^{(1/2)D_n}c\cdot b), \qquad \text{(A29)}$$

$$D_y(e^{-(1/2)D_n}a\cdot b)\cdot(e^{-(1/2)D_n}c\cdot d)=(D_ye^{-(1/2)D_n}a\cdot d)(e^{-(1/2)D_n}c\cdot b)-(e^{-(1/2)D_n}a\cdot d) \times(D_ye^{-(1/2)D_n}c\cdot b), \qquad \text{(A30)}$$

$$2D_y\cosh((\tfrac{1}{2}D_n)(e^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d) =e^{-(1/2)D_n}[(D_ye^{(1/2)D_n}a\cdot d)\cdot(e^{-(1/2)D_n}c\cdot b)-(e^{(1/2)D_n}a\cdot d)\cdot(D_ye^{-(1/2)D_n}c\cdot b) +(D_ye^{-(1/2)D_n}a\cdot d)\cdot(e^{(1/2)D_n}c\cdot b)-(e^{-(1/2)D_n}a\cdot d)\cdot(D_ye^{(1/2)D_n}c\cdot b)], \qquad \text{(A31)}$$

$$2D_t^2\sinh(\tfrac{1}{2}D_n)(e^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d) =e^{-(1/2)D_n}[(D_t^2e^{(1/2)D_n}a\cdot d)\cdot(e^{-(1/2)D_n}c\cdot b)+(e^{(1/2)D_n}a\cdot d)\cdot(D_t^2e^{-(1/2)D_n}c\cdot b) -2(D_te^{(1/2)D_n}a\cdot d)\cdot(D_te^{-(1/2)D_n}c\cdot b)-(D_t^2e^{-(1/2)D_n}a\cdot d)\cdot(e^{(1/2)D_n}c\cdot b) -(e^{-(1/2)D_n}a\cdot d)\cdot(D_t^2e^{(1/2)D_n}c\cdot b)+2(D_te^{-(1/2)D_n}a\cdot d)\cdot(D_te^{(1/2)D_n}c\cdot b)], \qquad \text{(A32)}$$

$$2D_t\cosh(\tfrac{1}{2}D_n)[(D_te^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d)+(e^{-(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}c\cdot d)] =e^{-(1/2)D_n}[(D_t^2e^{(1/2)D_n}a\cdot d)\cdot(e^{-(1/2)D_n}c\cdot b)-(e^{(1/2)D_n}a\cdot d)\cdot(D_t^2e^{-(1/2)D_n}c\cdot b) +(D_t^2e^{-(1/2)D_n}a\cdot d)\cdot(e^{(1/2)D_n}c\cdot b)-(e^{-(1/2)D_n}a\cdot d)\cdot(D_t^2e^{(1/2)D_n}c\cdot b)], \qquad \text{(A33)}$$

$$a_{ttt}b=\tfrac{1}{8}[D_t^3a\cdot b+(ab)_{ttt}+3(D_ta\cdot b)_{tt}+3(D_t^2a\cdot b)_t], \qquad \text{(A34)}$$

$$ab_{ttt}=\tfrac{1}{8}[-D_t^3a\cdot b+(ab)_{ttt}-3(D_ta\cdot b)_{tt}+3(D_t^2a\cdot b)_t], \qquad \text{(A35)}$$

$$a_{yt}b=\tfrac{1}{4}[D_yD_ta\cdot b+(ab)_{yt}+(D_ya\cdot b)_t+(D_ta\cdot b)_y], \qquad \text{(A36)}$$

$$ab_{yt}=\tfrac{1}{4}[D_yD_ta\cdot b+(ab)_{yt}-(D_ya\cdot b)_t-(D_ta\cdot b)_y], \qquad \text{(A37)}$$

$$2\sinh(D_n)(e^{(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d)=(e^{(3/2)D_n}a\cdot d)(e^{-(1/2)D_n}c\cdot b)-(e^{-(1/2)D_n}a\cdot d) \times(e^{(3/2)D_n}c\cdot b), \qquad \text{(A38)}$$

$$\sinh(D_n)[(D_te^{(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d)+(e^{(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}c\cdot d)] \\ -D_t\cosh(D_n)(e^{(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d) \\ =(e^{(3/2)D_n}a\cdot d)(D_te^{-(1/2)D_n}c\cdot b)-(D_te^{-(1/2)D_n}a\cdot d)(e^{(3/2)D_n}c\cdot b), \tag{A39}$$

$$e^{-(1/2)D_n}[(e^{D_n}a\cdot b)\cdot(D_te^{D_n}c\cdot d)-(D_te^{D_n}a\cdot b)\cdot(e^{D_n}c\cdot d)] \\ =-e^{D_n}[(D_te^{-(1/2)D_n}a\cdot c)\cdot(e^{(1/2)D_n}d\cdot b)-(e^{-(1/2)D_n}a\cdot c)\cdot(D_te^{(1/2)D_n}d\cdot b)], \tag{A40}$$

$$2D_t\cosh(D_n)(e^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d)-2\sinh(D_n)[(D_te^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d) \\ +(e^{-(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}c\cdot d)] \\ =2e^{-(1/2)D_n}[(D_te^{-D_n}a\cdot d)\cdot(e^{D_n}c\cdot b)-(e^{D_n}a\cdot d)\cdot(D_te^{-D_n}c\cdot b)], \tag{A41}$$

$$\tfrac{1}{4}D_t^3\cosh(D_n)(e^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d)-\tfrac{3}{4}D_t^2\sinh(D_n)[(D_te^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d) \\ +(e^{-(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}c\cdot d)]+\tfrac{3}{4}D_t\cosh(D_n)[(D_t^2e^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d) \\ +(e^{-(1/2)D_n}a\cdot b)\cdot(D_t^2e^{(1/2)D_n}c\cdot d)+2(D_te^{-(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}c\cdot d)] \\ -\tfrac{1}{4}\sinh(D_n)[(D_t^3e^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d)+(e^{-(1/2)D_n}a\cdot b)\cdot(D_t^3e^{(1/2)D_n}c\cdot d) \\ +3(D_te^{-(1/2)D_n}a\cdot b)\cdot(D_t^2e^{(1/2)D_n}c\cdot d)+3(D_t^2e^{-(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}c\cdot d)] \\ =-e^{-(1/2)D_n}[(e^{D_n}a\cdot d)\cdot(D_t^3e^{-D_n}c\cdot b)-(D_t^3e^{-D_n}a\cdot d)\cdot(e^{D_n}c\cdot b)], \tag{A42}$$

$$-\tfrac{1}{2}D_yD_t\sinh(D_n)(e^{-(1/2)D_n}a\cdot b)\cdot(e^{(1/2)D_n}c\cdot d)+\tfrac{1}{2}D_y\cosh(D_n)[(D_te^{-(1/2)D_n}a\cdot b) \\ \cdot(e^{(1/2)D_n}c\cdot d)+(e^{-(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}c\cdot d)]+\tfrac{1}{2}D_t\cosh(D_n)[(D_ye^{-(1/2)D_n}a\cdot b) \\ \cdot(e^{(1/2)D_n}c\cdot d)+(e^{-(1/2)D_n}a\cdot b)\cdot(D_ye^{(1/2)D_n}c\cdot d)]-\tfrac{1}{2}\sinh(D_n)[(D_yD_te^{-(1/2)D_n}a\cdot b) \\ \cdot(e^{(1/2)D_n}c\cdot d)+(e^{-(1/2)D_n}a\cdot b)\cdot(D_yD_te^{(1/2)D_n}c\cdot d)+(D_ye^{-(1/2)D_n}a\cdot b)\cdot(D_te^{(1/2)D_n}c\cdot d) \\ +(D_te^{-(1/2)D_n}a\cdot b)\cdot(D_ye^{(1/2)D_n}c\cdot d)] \\ =-e^{-(1/2)D_n}[(e^{D_n}a\cdot d)\cdot(D_yD_te^{-D_n}c\cdot b)-(D_yD_te^{-D_n}a\cdot d)\cdot(e^{D_n}c\cdot b)]. \tag{A43}$$


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