

Group magicness of complete n-partite graphs

Shiu, W.C.; Low, Richard M.

Published in:

Journal of Combinatorial Mathematics and Combinatorial Computing

Published: 01/07/2006

Document Version:

Peer reviewed version

[Link to publication](#)

Citation for published version (APA):

Shiu, W. C., & Low, R. M. (2006). Group magicness of complete n-partite graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 58, 129-134.

General rights

Copyright and intellectual property rights for the publications made accessible in HKBU Scholars are retained by the authors and/or other copyright owners. In addition to the restrictions prescribed by the Copyright Ordinance of Hong Kong, all users and readers must also observe the following terms of use:

- Users may download and print one copy of any publication from HKBU Scholars for the purpose of private study or research
- Users cannot further distribute the material or use it for any profit-making activity or commercial gain
- To share publications in HKBU Scholars with others, users are welcome to freely distribute the permanent publication URLs

Group magicness of complete n -partite graphs*

W.C. Shiu

Department of Mathematics,
Hong Kong Baptist University,
224 Waterloo Road, Kowloon Tong,
Hong Kong, China.
wshiu@hkbu.edu.hk

Richard M. Low

Department of Mathematics
San José State University
San José, CA 95192, USA
low@math.sjsu.edu

Abstract

Let A be a non-trivial Abelian group. We call a graph $G = (V, E)$ A -magic if there exists a labeling $f : E \rightarrow A^*$ such that the induced vertex set labeling $f^+ : V \rightarrow A$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$ is a constant map. In this paper, we show that K_{k_1, k_2, \dots, k_n} ($k_i \geq 2$) is A -magic, for all A where $|A| \geq 3$.

Keywords : integer-magic spectrum, group-magic, A -magic, n -partite graph
2000 MSC : 05C15

1 Introduction

Let $G = (V, E)$ be a connected, simple graph. For any nontrivial Abelian group A (written additively), let $A^* = A \setminus \{0\}$. A function $f : E \rightarrow A^*$ is called a *labeling* of G . Any such labeling induces a map $f^+ : V \rightarrow A$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$. If there exists a labeling f whose induced map on V is a constant map, we say that f is an A -magic labeling and that G is an A -magic graph. The *integer-magic spectrum* of a graph G is the set $\text{IM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-magic and } k \geq 1\}$. By convention, \mathbb{Z} -magic graphs are considered to be \mathbb{Z}_1 -magic.

\mathbb{Z} -magic graphs were considered by Stanley [19, 20], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1, 2, 3] and others [7, 8, 14] have studied A -magic graphs and \mathbb{Z}_k -magic graphs were investigated in [4, 6, 9, 10, 11, 12, 13].

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A -magic graph is due to J. Sedláček [15, 16], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [5] had introduced yet another definition of a magic graph. Over the years, there has been great research interest in graph labeling problems. The interested reader is directed to Wallis' [21] recent monograph on magic graphs.

*Supported by Faculty Research Grant, Hong Kong Baptist University.

2 Basic definitions and notation

In the study of edge-magic labelings, Shiu, Lam and Lee [17, 18] introduced the following notation. Suppose $f : E \rightarrow X$ is a mapping (i.e., an edge labeling of G), where X is a set. The *labeling matrix* for f , denoted by $\mathcal{L}_f(G)$, is the matrix whose rows and columns are named by the vertices of G and defined in the following way: the (u, v) -entry is $f(uv)$ if $uv \in E$, and is $*$ otherwise. If f is an A -magic labeling of G , then $\mathcal{L}_f(G)$ is an A -magic labeling matrix of G . Note that the row sum of an A -magic labeling matrix is called the A -magic value corresponding to the labeling f .

Thus, finding an A -magic labeling of G is equivalent to finding an A -magic labeling matrix $\mathcal{L}_f(G)$, where each row sum (as well as column sum) is the same constant value. In the context of row and column sums, entries with an $*$ are treated as 0.

A graph is called *fully magic* if it is A -magic, for every Abelian group A . A graph is called *non-magic* if for every abelian group A , it is not A -magic.

In this paper, we analyze the group-magicness property for the class of complete n -partite graphs.

3 Main results

First, let us make a few observations. They are straight-forward to verify and can be found in [14].

Observations:

1. A graph G is \mathbb{Z}_2 -magic if and only if every vertex of G is of the same parity.
2. An Eulerian graph G having an even number of edges is A -magic.
3. If A_1 is a subgroup of A and graph G is A_1 -magic, then G is A -magic.

We now characterize the abelian groups A , for which $K_{m,n}$ is A -magic. Let f be a labeling of the complete bipartite graph $K_{m,n}$. Then $\mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \star_m & B \\ B^T & \star_n \end{pmatrix}$, where B is an $m \times n$ matrix, \star_m and \star_n are square matrices of order m and n respectively with all entries are $*$.

Theorem 3.1 *Let m and n be even. Then, $K_{m,n}$ has an A -magic labeling with magic value 0, for all A .*

Proof: Suppose $a \in A^*$. Let S be an $m \times n$ matrix defined by $S_{i,j} = (-1)^{i+j}a$, where $S_{i,j}$ denotes the (i, j) -entry of S . Then, the row sums and the column sums of S are zero. Clearly, $\begin{pmatrix} \star_m & S \\ S^T & \star_n \end{pmatrix}$ is an A -magic labeling matrix of $K_{m,n}$, with A -magic value 0. \square

The matrix S defined in the proof above is called an $m \times n$ *zero-sum* $(a, -a)$ -matrix.

The integer-magic spectrum of $K_{1,n}$ has been found [11]. For convenience, we state the result here.

Theorem A $K_{1,1}$ is fully magic and $K_{1,2}$ is non-magic. For $n \geq 3$, $\text{IM}(K_{1,n}) = \bigcup_{p|(n-1)} p\mathbb{N}$.

It is straight-forward to verify the following lemma.

Lemma 3.2 For $n \geq 3$, $K_{1,n}$ is V_4 -magic if and only if n is odd.

So we may assume $m \geq 3$.

Lemma 3.3 Let A be an abelian group of order at least 3. Then, there exist $a, b, c \in A \setminus \{0\}$ (not necessary distinct) such that $a + b + c = 0$.

Proof: It suffices to consider three cases, namely: $A = \mathbb{Z}$, \mathbb{Z}_k for $k \geq 3$, or V_4 .

If $A = \mathbb{Z}$, then it is obvious. If $A = \mathbb{Z}_k$, then choose $a = b = 1$ and $c = -2$. If $A = V_4$, then choose $a = (1, 0)$, $b = (0, 1)$ and $c = (1, 1)$. \square

Theorem 3.4 Suppose m is odd, with $m \geq 3$ and $n \geq 2$. For any abelian group A where $|A| \geq 3$, $K_{m,n}$ has an A -magic labeling with magic value 0.

Proof: Let $a, b, c \in A \setminus \{0\}$ be chosen in the same manner as discussed in the proof of Lemma 3.3.

Case 1. n is even.

Let $B = \begin{pmatrix} C \\ D \end{pmatrix}$, where C is an $(m-3) \times n$ zero-sum $(a, -a)$ -matrix and D is a $3 \times n$ matrix defined by

$$D_{i,j} = \begin{cases} (-1)^j a & \text{if } i = 1; \\ (-1)^j b & \text{if } i = 2; \\ (-1)^j c & \text{if } i = 3. \end{cases}$$

Note that if $m = 3$, then C does not appear. Then, $\mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \star_m & B \\ B^T & \star_n \end{pmatrix}$ is an A -magic labeling matrix of $K_{m,n}$, for $A = \mathbb{Z}, \mathbb{Z}_k$ ($k \geq 3$), and V_4 . By Observation 3, $K_{m,n}$ is A -magic, for all A where $|A| \geq 3$

Case 2. n is odd.

Then, $n \geq 3$. Let B be a matrix of the following form:

$$B = \begin{pmatrix} C_1 & D_1^T \\ D_1 & E \end{pmatrix},$$

where C_1 is an $(m-3) \times (n-3)$ zero-sum $(a, -a)$ -matrix, D_1 is a $3 \times (n-3)$ matrix defined similarly as in the proof of the previous case, and E is a Latin square of order 3 defined as follows:

$$E = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

Then, $\mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \star_m & B \\ B^T & \star_n \end{pmatrix}$ is an A -magic labeling matrix of $K_{m,n}$, for $A = \mathbb{Z}, \mathbb{Z}_k$ ($k \geq 3$), and V_4 . By Observation 3, $K_{m,n}$ is A -magic, for all A where $|A| \geq 3$.

It is clear that the row sums and the column sums of these A -magic labeling matrices are zero. \square

Here a few examples which illustrate Theorem 3.4.

Example 3.1 $m = 3$ and $n = 4$. Then,

$$B = \begin{pmatrix} -a & a & -a & a \\ -b & b & -b & b \\ -c & c & -c & c \end{pmatrix}, \text{ and } \mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \star_3 & B \\ B^T & \star_4 \end{pmatrix}.$$

Example 3.2 $m = 3$ and $n = 3$. Then,

$$B = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}, \text{ and } \mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \star_3 & B \\ B^T & \star_3 \end{pmatrix}.$$

Example 3.3 $m = 5$ and $n = 4$. Then,

$$B = \begin{pmatrix} a & -a & a & -a \\ -a & a & -a & a \\ -b & b & -b & b \\ -c & c & -c & c \end{pmatrix}, \text{ and } \mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \star_5 & B \\ B^T & \star_4 \end{pmatrix}.$$

Example 3.4 $m = 5$ and $n = 5$. Then,

$$B = \begin{pmatrix} a & -a & -a & -b & -c \\ -a & a & a & b & c \\ -a & a & a & b & c \\ -b & b & c & a & b \\ -c & c & b & c & a \end{pmatrix}, \text{ and } \mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \star_5 & B \\ B^T & \star_5 \end{pmatrix}.$$

We conclude by showing that K_{k_1, k_2, \dots, k_n} ($k_i \geq 2$) is A -magic, for all A where $|A| \geq 3$. First, recall the following definitions and notation.

A graph G is n -partite, $n \geq 1$, if it is possible to partition $V(G)$ into n subsets V_1, V_2, \dots, V_n such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j , $i \neq j$.

A complete n -partite graph G is an n -partite graph with partite sets V_1, V_2, \dots, V_n having the added property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$. A complete n -partite graph G with partite sets V_1, V_2, \dots, V_n , where $|V_i| = k_i$, is denoted by K_{k_1, k_2, \dots, k_n} .

We now establish the following result.

Theorem 3.5 For $n \geq 2$, the complete n -partite graph K_{k_1, k_2, \dots, k_n} with $k_i \geq 2$, is A -magic, for all A where $|A| \geq 3$.

Proof: There are $\binom{n}{2}$ ways of choosing a pair from the partite sets V_1, V_2, \dots, V_n . For each pair, apply a labeling on the corresponding edge set, using either Theorem 3.1 or Theorem 3.4. \square

References

- [1] M. Doob, On the construction of magic graphs, *Proc. Fifth S.E. Conference on Combinatorics, Graph Theory and Computing*, (1974), 361-374.
- [2] M. Doob, Generalizations of magic graphs, *J. Combin. Theory, Ser. B*, **17** (1974), 205-217.
- [3] M. Doob, Characterizations of regular magic graphs, *J. Combin. Theory, Ser. B*, **25** (1978), 94-104.
- [4] M.C. Kong, S-M Lee, and H. Sun, On magic strength of graphs, *Ars Combin.*, **45** (1997), 193-200.
- [5] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), 451-461.
- [6] S-M Lee, Yong-Song Ho and R.M. Low, On the integer-magic spectra of maximal planar and maximal outerplanar graphs, preprint.
- [7] S-M Lee, A. Lee, Hugo Sun, and Ixin Wen, On group-magic graphs, *JCMCC*, **38** (2001), 197-207.
- [8] S-M Lee, F. Saba, E. Salehi, and H. Sun, On the V_4 -group magic graphs, *Cong. Numer.*, **156** (2002), 59-67.
- [9] S-M Lee, F. Saba, and G. C. Sun, Magic strength of the k -th power of paths, *Cong. Numer.*, **92** (1993), 177-184.
- [10] S-M Lee and E. Salehi, Integer-magic spectra of amalgamations of stars and cycles, *Ars Combin.*, **67** (2003), 199-212.
- [11] S-M Lee, E. Salehi and H. Sun, Integer-magic spectra of trees with diameters at most four, preprint.
- [12] S-M Lee, L. Valdes, and Yong-Song Ho, On group-magic spectra of trees, double trees and abbreviated double trees, *JCMCC*, **46** (2003), 85-96.
- [13] S-M Lee and J. Wang, On the integer-magic spectra of honeycomb graphs, preprint.
- [14] R.M. Low and S-M Lee, On group-magic eulerian graphs, *JCMCC*, **50** (2004), 141-148.
- [15] J. Sedlacék, On magic graphs, *Math. Slov.*, **26** (1976), 329-335.
- [16] J. Sedlacék, Some properties of magic graphs, in *Graphs, Hypergraph, and Bloc Syst. 1976, Proc. Symp. Comb. Anal., Zielona Gora (1976)*, 247-253.
- [17] W.C. Shiu, P.C.B. Lam and S-M Lee, Edge-magicness of the composition of a cycle with a null graph, *Cong. Numer.*, **132** (1998), 9-18.
- [18] W.C. Shiu, P.C.B. Lam and S-M Lee, On a Construction of Supermagic Graphs, *JCMCC*, **42** (2002), 147-160.
- [19] R.P. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.*, **40** (1973), 607-632.
- [20] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings, *Duke Math. J.*, **40** (1976), 511-531.
- [21] W.D. Wallis, *Magic Graphs*, Birkhauser Boston, 2001.