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Convergence of a difference scheme for two dimensional heat equation in unbounded domains by artificial boundary conditions*

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Abstract

The numerical solution of the two-dimensional heat-conduction equation in an unbounded domain is considered. An artificial boundary is introduced to make the computational domain finite. On the artificial boundary, an exact boundary condition is proposed to reduce the original problem to an initial-boundary value problem in a finite computational domain. A difference scheme is constructed by the method of reduction of order to solve the problem in the finite computational domain. It is proved that the difference scheme is uniquely solvable, unconditionally stable and convergent with the convergence order 2 in space and order 3/2 in time in an energy norm. A numerical example demonstrates the theoretical results.

Keywords: heat equation, artificial boundary condition, finite difference, convergence, solvability, stability

2000 Mathematics Subject Classification: Primary 65M12, 65M06, 65M15, 65M20

1 Introduction

The heat transfer problem in unbounded domains models some application problems. The numerical solution to this kind of problems usually faces the problem of introducing artificial boundaries and setting the boundary conditions on the artificial boundaries. Early work on using artificial boundary technique can be found in [1] by Engquist and Majda, they used absorbing boundary conditions for the numerical simulation of waves. In [6, 7], Han and Wu proposed artificial boundary

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conditions in series form for some elliptic problems. In [2, 9], Keller and Givoli used DtN boundary conditions for some problems. Some other approaches using artificial boundary conditions can be found from [3, 4, 5, 8, 10, 11].

In [14], we proposed a difference method for the one-dimensional heat transfer problem in unbounded domains. An artificial boundary condition is introduced to reduce the problem to an initial boundary value problem in a finite computational domain. Stability and convergence are obtained for the difference scheme. In this paper, we consider a two-dimensional heat transfer problem in an unbounded domain. By introducing an artificial boundary condition we can also reduce the original unbounded problem to an initial boundary value problem in a finite computational domain.

Consider the following initial-boundary value problem of the heat equation on a two dimensional unbounded domain:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad \text{in } \Omega \times (0, \mathrm{T}], \tag{1.1}$$

$$u = 0, \text{ on } \partial\Omega \times (0, \mathrm{T}], \tag{1.2}$$

$$u|_{t=0} = g(x,y), \quad (x,y) \in \overline{\Omega}, \tag{1.3}$$

$$u \to 0, \quad \text{when } x \to +\infty, \tag{1.4}$$

where $\Omega \subset R^2$ is an unbounded domain as shown in Fig.1, $f(x,y,t)$, $g(x,y)$ are two given functions, the Supp(f) and Supp(g) are compact. Suppose $f(x,y,t) \equiv 0$, $g(x,y) \equiv 0$ for $x \geq b_0$. Introduce the artificial boundary

$$\Gamma_b = \{(x,y) \mid x = b, 0 \leq y \leq l\}, \text{ with } \mathrm{b} \geq \mathrm{b}_0.$$

Γ_b divides the domain Ω into two parts, the unbounded part Ω_e and the bounded part Ω_i, where $\Omega_e = \{(x,y) \mid b < x < +\infty, 0 < y < l\}$, $\Omega_i = \Omega \backslash \overline{\Omega_e}$.

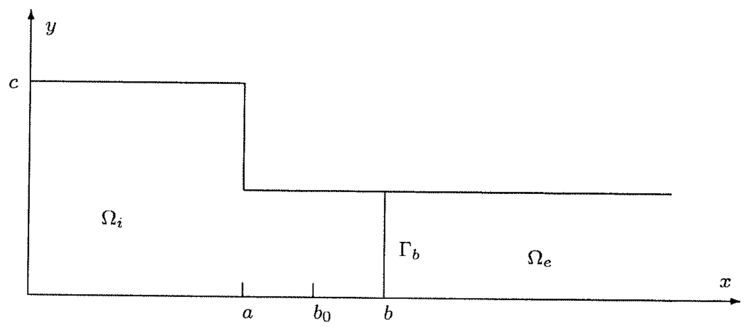

Fig. 1 The domain on which the solution to be solved

Now consider the restriction of $u(x,y,t)$, the solution of the problem (1.1)-(1.4), on the unbounded domain $\Omega_e \times (0,T]$. The function $u(x,y,t)$ satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad \text{in } \Omega_e \times (0, \mathrm{T}], \tag{1.5}$$

$$u|_{y=0,l} = 0, \quad b \leq x < +\infty, \ 0 < t \leq T, \tag{1.6}$$

$$u|_{x=b} = u(b,y,t), \quad 0 \leq y \leq l, \ 0 < t \leq T, \tag{1.7}$$

$$u|_{t=0} = 0, \quad b \le x < +\infty, \; 0 \le y \le l, \tag{1.8}$$

$$u \to 0, \quad \text{when} \quad x \to +\infty. \tag{1.9}$$

Since $u(b,y,t)$ is unknown, the problem (1.5)-(1.9) is an uncompleted posed problem, which can not be solved independently. Suppose the $u(b,y,t)$ is given, the problem (1.5)-(1.9) is a properly posed problem and the solution can be obtained in the following. Let

$$u(x,y,t) = \sum_{n=1}^{\infty} v_n(x,t)\sin(\mu_n y) \tag{1.10}$$

with $\mu_n = n\pi/l, \; n = 1,2,3,\cdots$. Substituting (1.10) into (1.5)-(1.9), we obtain

$$\begin{aligned}
&\frac{\partial v_n(x,t)}{\partial t} = \frac{\partial^2 v_n(x,t)}{\partial x^2} - \mu_n^2 v_n(x,t), \quad b < x < +\infty, \; 0 < t \le T,\\
&v_n(b,t) = \frac{2}{l}\int_0^l u(b,\xi,t)\sin(\mu_n \xi)d\xi, \quad 0 < t \le T,\\
&v_n(x,0) = 0, \quad b \le x < +\infty,\\
&v_n(x,t) \to 0, \quad \text{when} \quad x \to +\infty.
\end{aligned}$$

Let $w_n(x,t) = e^{\mu_n^2 t} v_n(x,t)$. Then $w_n(x,t)$ satisfies:

$$\begin{aligned}
&\frac{\partial w_n(x,t)}{\partial t} = \frac{\partial^2 w_n(x,t)}{\partial x^2}, \quad b < x < +\infty, 0 < t \le T,\\
&w_n(b,t) = \frac{2}{l}\, e^{\mu_n^2 t}\int_0^l u(b,\xi,t)\sin(\mu_n \xi)d\xi, \quad 0 < t \le T,\\
&w_n(x,0) = 0, \quad b \le x < +\infty,\\
&w_n(x,t) \to 0, \quad \text{when} \quad x \to +\infty.
\end{aligned}$$

This problem has been studied in the work by Han and Huang [1] and the exact boundary condition at $x = b$ for $w_n(x,t)$ is

$$\frac{\partial w_n(b,t)}{\partial x} = -\frac{1}{\sqrt{\pi}}\int_0^t \frac{\partial w_n(b,\lambda)}{\partial \lambda}\cdot\frac{d\lambda}{\sqrt{t-\lambda}}, \quad 0 \le t \le T.$$

For the function $v_n(x,t)$, we have:

$$\begin{aligned}
\frac{\partial v_n(b,t)}{\partial x} &= -\frac{2}{\sqrt{\pi} l} e^{-\mu_n^2 t}\int_0^t \left\{\int_0^l \frac{\partial}{\partial \lambda}\left[e^{\mu_n^2 \lambda} u(b,\xi,\lambda)\right]\sin\mu_n\xi \; d\xi\right\}\frac{d\lambda}{\sqrt{t-\lambda}}\\
&= -\frac{2}{\sqrt{\pi} l}\int_0^t \left\{\int_0^l \frac{\partial}{\partial \lambda}\left[e^{-\mu_n^2 (t-\lambda)} u(b,\xi,\lambda)\right]\sin\mu_n\xi \; d\xi\right\}\frac{d\lambda}{\sqrt{t-\lambda}}.
\end{aligned}$$

Finally we obtain the exact boundary condition of the given problem (1.1)-(1.4) on $(0,T] \times \Gamma_b$:

$$\frac{\partial u(b,y,t)}{\partial x} = -\frac{2}{\sqrt{\pi} l}\sum_{n=1}^{\infty}\int_0^t \left\{\int_0^l \frac{\partial}{\partial \lambda}\left[e^{-\mu_n^2 (t-\lambda)} u(b,\xi,\lambda)\right]\sin\mu_n\xi \; d\xi\right\}\frac{d\lambda}{\sqrt{t-\lambda}}\sin\mu_n y. \tag{1.11}$$

Using the exact boundary condition (1.11), we reduce the original problem (1.1)-(1.4) to a problem on the bounded domain Ω_i :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad \text{in } \Omega_i \times (0,T], \tag{1.12}$$

$$u(x,y,0) = g(x,y), \quad (x,y) \in \overline{\Omega_i}, \tag{1.13}$$

$$u = 0, \qquad \text{on } \{\Omega_i \setminus \Gamma_b\} \times (0,T], \tag{1.14}$$

$$\frac{\partial u(b,y,t)}{\partial x} = -\frac{2}{\sqrt{\pi} l} \sum_{n=1}^{\infty} \int_0^t \left\{ \int_0^l \frac{\partial}{\partial \lambda} \left[e^{-\mu_n^2 (t-\lambda)} u(b,\xi,\lambda) \right] \sin \mu_n \xi \, d\xi \right\} \frac{d\lambda}{\sqrt{t-\lambda}} \cdot \sin \mu_n y,$$
$$\text{on } \Gamma_b \times (0,T]. \tag{1.15}$$

The brief outline of this paper is as follows: In section 2, we give some notations and construct a difference scheme for (1.12)-(1.15) by the method of reduction of order [12, 13]. We prove the solvability, stability and convergence in section 3. The numerical example is given in section 4 to show that the approach given in this paper is feasible and effective.

2 The difference scheme

Take three positive integers I, J and K. Divide the intervals $[0,b]$ into I-subintervals, $[0,c]$ into J-subintervals and $[0,T]$ into $K-$subintervals. Denote

$$x_i = ih_1, \ 0 \le i \le I, \ h_1 = b/I; \quad x_{i-\frac{1}{2}} = (x_i + x_{i-1})/2, \ 1 \le i \le I;$$
$$y_j = jh_2, \ 0 \le j \le J, \ h_2 = c/J; \quad y_{j-\frac{1}{2}} = (y_j + y_{j-1})/2, \ 1 \le j \le J;$$
$$t_k = k\tau, \ 0 \le k \le K, \ \tau = T/K; \quad t_{k-\frac{1}{2}} = (t_k + t_{k-1})/2, \ 1 \le k \le K.$$

For simplicity, assume that there exist two integers I_0 and J_0 such that $x_{I_0} = a$ and $y_{J_0} = l$. Let

$$\bar{\omega} = \{0 \le i \le I_0, 0 \le j \le J\} \cup \{I_0 + 1 \le i \le I, 0 \le j \le J_0\},$$
$$\omega_0 = \{1 \le i \le I_0, 1 \le j \le J\} \cup \{I_0 + 1 \le i \le I, 1 \le j \le J_0\},$$
$$\omega_0' = \{1 \le i \le I_0, 0 \le j \le J-1\} \cup \{I_0 + 1 \le i \le I-1, 0 \le j \le J_0 - 1\},$$
$$\omega_0'' = \{1 \le i \le I_0, 1 \le j \le J-1\} \cup \{I_0 + 1 \le i \le I, 1 \le j \le J_0 - 1\},$$
$$\omega_1 = \{1 \le i \le I_0 - 1, 1 \le j \le J-1\} \cup \{I_0 \le i \le I-1, 1 \le j \le J_0 - 1\},$$
$$\omega_1' = \{0 \le i \le I_0 - 1, 1 \le j \le J-1\} \cup \{I_0 \le i \le I-1, 1 \le j \le J_0 - 1\},$$
$$\omega_2 = \{0 \le i \le I, j = 0\} \cup \{i = 0, 1 \le j \le J\} \cup \{1 \le i \le I_0, j = J\}$$
$$\cup \{i = I_0, J_0 \le j \le J-1\} \cup \{I_0 + 1 \le i \le I, j = J_0\},$$
$$\Omega_h = \{(x_i, y_j) \mid (i,j) \in \bar{\omega}\}, \qquad \Omega_\tau = \{t_k \mid 0 \le k \le K\}.$$

If $w = \{w_{ij}^k\}$ is a net function on $\Omega_h \times \Omega_\tau$, introduce the following notations:

$$w_{i-\frac{1}{2},j}^k = \frac{1}{2}(w_{ij}^k + w_{i-1,j}^k), \quad \delta_x w_{i-\frac{1}{2},j}^k = \frac{1}{h_1}(w_{ij}^k - w_{i-1,j}^k),$$
$$w_{i,j-\frac{1}{2}}^k = \frac{1}{2}(w_{ij}^k + w_{i,j-1}^k), \quad \delta_y w_{i,j-\frac{1}{2}}^k = \frac{1}{h_2}(w_{ij}^k - w_{i,j-1}^k),$$
$$w_{ij}^{k-\frac{1}{2}} = \frac{1}{2}(w_{ij}^k + w_{ij}^{k-1}), \quad \delta_t w_{ij}^{k-\frac{1}{2}} = \frac{1}{\tau}(w_{ij}^k - w_{ij}^{k-1}),$$

$$\delta_x^2 w_{ij}^k = \frac{1}{h_1}\left(\delta_x w_{i+\frac{1}{2},j}^k - \delta_x w_{i-\frac{1}{2},j}^k\right), \quad \delta_y^2 w_{ij}^k = \frac{1}{h_2}\left(\delta_y w_{i,j+\frac{1}{2}}^k - \delta_y w_{i,j-\frac{1}{2}}^k\right),$$

$$\|w^k\| = \sqrt{h_1 h_2 \sum_{(i,j)\in\omega_0}\left(w_{i-\frac{1}{2},j-\frac{1}{2}}^k\right)^2},$$

$$\|\delta_x w^k\| = \sqrt{h_1 h_2 \sum_{(i,j)\in\omega_0}\left(\delta_x w_{i-\frac{1}{2},j-\frac{1}{2}}^k\right)^2}, \quad \|\delta_y w^k\| = \sqrt{h_1 h_2 \sum_{(i,j)\in\omega_0}\left(\delta_y w_{i-\frac{1}{2},j-\frac{1}{2}}^k\right)^2}.$$

If the dependence on the mesh sizes needs to be indicated, we denote $\|w^k\|$ by $\|w^k(h,\tau)\|$.

Our difference scheme for (1.12)-(1.15) is given in the following:

$$\begin{aligned}
&\frac{1}{4}\delta_t\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}} + u_{i+\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right) = \frac{1}{2}\delta_x^2\left(u_{i,j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{i,j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\\
&\quad + \frac{1}{2}\delta_y^2\left(u_{i-\frac{1}{2},j}^{k-\frac{1}{2}} + u_{i+\frac{1}{2},j}^{k-\frac{1}{2}}\right) + \frac{1}{4}\left(f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + f_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}} + f_{i+\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + f_{i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right),\\
&\qquad (i,j)\in\omega_1,\ k\geq 1,
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&\frac{1}{2}\delta_t\left(u_{I-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{I-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right) = \frac{2}{h_1}\left\{-\frac{4}{\sqrt{\pi\tau}l}\sum_{n=1}^{\infty}\left[a_0 H_n^k\right.\right.\\
&\quad \left.\left. - \sum_{m=1}^{k-1}(a_{k-m-1} - a_{k-m})e^{-\mu_n^2(t_k - t_m)}H_n^m\right] s(n,j) - \frac{1}{2}\delta_x\left(u_{I-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{I-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right\}\\
&\quad + \delta_y^2 u_{I-\frac{1}{2},j}^{k-\frac{1}{2}} + \frac{1}{2}\left(f_{I-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + f_{I-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right), \quad 1\leq j\leq J_0 - 1,\ k\geq 1,
\end{aligned} \tag{2.2}$$

$$u_{ij}^k = 0, \qquad (i,j)\in\omega_2,\ k\geq 0, \tag{2.3}$$

$$u_{ij}^0 = g(x_i, y_j), \quad (i,j)\in\omega_1, \tag{2.4}$$

where

$$\begin{aligned}
&a_m = \frac{1}{\sqrt{m+1}+\sqrt{m}},\\
&s(n,\nu) = \frac{1}{2}\left[\alpha_{n,\nu-\frac{1}{2}} + \alpha_{n,\nu+\frac{1}{2}}\right], \ \alpha_{n,\nu-\frac{1}{2}} = \frac{1}{h_2}\int_{y_{\nu-1}}^{y_\nu}\sin(\mu_n y)dy,\\
&H_n^k = h_2\sum_{\nu=1}^{J_0-1} u_{I,\nu}^{k-\frac{1}{2}} s(n,\nu).
\end{aligned} \tag{2.5}$$

The difference scheme (2.1)-(2.4) can be derived by the method of reduction of order. Suppose that $u(x,y,t)\in C_{x,y,t}^{4,4,3}(\overline{\Omega}_i\times[0,T])$ and let $v = \frac{\partial u}{\partial x}, w = \frac{\partial u}{\partial y}$. Then (1.12)-(1.15) is equivalent to

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} + f(x,y,t), \quad \text{in } \Omega_i\times(0,T], \tag{2.6}$$

$$v = \frac{\partial u}{\partial x}, \quad w = \frac{\partial u}{\partial y}, \quad \text{in } \Omega_i\times(0,T], \tag{2.7}$$

$$u(x,y,0) = g(x,y), \quad (x,y)\in\overline{\Omega}_i, \tag{2.8}$$

$$u = 0, \qquad \text{on } \{\partial\Omega_i\setminus\Gamma_b\}\times(0,T], \tag{2.9}$$

$$\begin{aligned}
&v(b,y,t) = -\frac{2}{\sqrt{\pi}l}\sum_{n=1}^{\infty}\int_0^t\left\{\int_0^l\frac{\partial}{\partial\lambda}\left[e^{-\mu_n^2(t-\lambda)}u(b,\xi,\lambda)\right]\sin\mu_n\xi\, d\xi\right\}\frac{d\lambda}{\sqrt{t-\lambda}}\cdot\sin\mu_n y,\\
&\qquad \text{on } \Gamma_b\times(0,T].
\end{aligned} \tag{2.10}$$

Define the net functions on $\Omega_h \times \Omega_\tau$,

$$U_{ij}^k = u(x_i, y_j, t_k), \qquad V_{ij}^k = v(x_i, y_j, t_k), \qquad W_{ij}^k = w(x_i, y_j, t_k).$$

For the discretization of the boundary condition (2.10), we need the following two lemmas.

Lemma 1 Suppose $g(y) \in C^2[0, l]$. Then

$$\max_{1 \le j \le J_0} \left| \sum_{n=1}^{\infty} \frac{2}{l} \left(\int_0^l g(\xi) \sin \omega\xi d\xi \right) \alpha_{n,j-\frac{1}{2}} - \sum_{n=1}^{\infty} \frac{2}{l} \left(h_2 \sum_{\nu=1}^{J_0} \frac{g(y_{\nu-1}) + g(y_\nu)}{2} \alpha_{n,\nu-\frac{1}{2}} \right) \alpha_{n,j-\frac{1}{2}} \right| \le \frac{1}{12} \max_{0 \le y \le l} |g''(y)| \, h_2^2. \tag{2.11}$$

Proof According to the sine expansion, we have

$$g(y) = \sum_{n=1}^{\infty} \left[\frac{2}{l} \int_0^l g(\xi) \sin \mu_n \xi d\xi \right] \sin \mu_n y, \quad 0 \le y \le l,$$

which follows that

$$\begin{aligned}
&\frac{1}{2} [g(y_{j-1}) + g(y_j)] \\
&= \frac{1}{h_2} \int_{y_{j-1}}^{y_j} g(y) dy + \frac{1}{12} g''(\eta_j) h_2^2 \\
&= \sum_{n=1}^{\infty} \left[\frac{2}{l} \int_0^l g(\xi) \sin \mu_n \xi d\xi \right] \frac{1}{h_2} \int_{y_{j-1}}^{y_j} \sin \mu_n y dy + \frac{1}{12} g''(\eta_j) h_2^2 \\
&= \sum_{n=1}^{\infty} \left[\frac{2}{l} \int_0^l g(\xi) \sin \mu_n \xi d\xi \right] \alpha_{n,j-\frac{1}{2}} + \frac{1}{12} g''(\eta_j) h_2^2, \quad 1 \le j \le J_0,
\end{aligned} \tag{2.12}$$

where $\eta_j \in (y_{j-1}, y_j)$. On the other hand, define the function $g_I(y)$:

$$g_I(y) = \frac{1}{2} [g(y_{j-1}) + g(y_j)], \quad y_{j-1} < y < y_j, \quad 1 \le j \le J_0.$$

According to sine expansion again, we have

$$\begin{aligned}
g_I(y) &= \sum_{n=1}^{\infty} \left[\frac{2}{l} \int_0^l g_I(\xi) \sin \mu_n \xi d\xi \right] \sin \mu_n y \\
&= \sum_{n=1}^{\infty} \left[\frac{2}{l} \sum_{\nu=1}^{J_0} \int_{y_{\nu-1}}^{y_\nu} g_I(\xi) \sin \mu_n \xi d\xi \right] \sin \mu_n y \\
&= \sum_{n=1}^{\infty} \left[\frac{2}{l} \sum_{\nu=1}^{J_0} \frac{g(y_{\nu-1}) + g(y_\nu)}{2} \int_{y_{\nu-1}}^{y_\nu} \sin \mu_n \xi d\xi \right] \sin \mu_n y \\
&= \sum_{n=1}^{\infty} \left[\frac{2}{l} h_2 \sum_{\nu=1}^{J_0} \frac{g(y_{\nu-1}) + g(y_\nu)}{2} \alpha_{n,\nu-\frac{1}{2}} \right] \sin \mu_n y, \; y_{j-1} < y < y_j, \; 1 \le j \le J_0.
\end{aligned} \tag{2.13}$$

and the Parseval's equality

$$\sum_{n=1}^{\infty}\left[\frac{2}{l}h_2\sum_{\nu=1}^{J_0}\frac{g(y_{\nu-1})+g(y_\nu)}{2}\alpha_{n,\nu-\frac{1}{2}}\right]^2=\frac{2}{l}\int_0^l [y_I(y)]^2\,dy$$

$$=\frac{2}{l}\cdot h_2\sum_{j=1}^{J_0}\left[\frac{g(y_{j-1})+g(y_j)}{2}\right]^2. \tag{2.14}$$

Integrating the both sides of (2.13) on the interval (y_{j-1}, y_j) and then dividing the result by h_2, we obtain

$$\frac{g(y_{j-1})+g(y_j)}{2}=\sum_{n=1}^{\infty}\left[\frac{2}{l}h_2\sum_{\nu=1}^{J_0}\frac{g(y_{\nu-1})+g(y_\nu)}{2}\;\alpha_{n,\nu-\frac{1}{2}}\right]\frac{1}{h_2}\int_{y_{j-1}}^{y_j}\sin\mu_n y\,dy$$

$$=\sum_{n=1}^{\infty}\left[\frac{2}{l}h_2\sum_{\nu=1}^{J_0}\frac{g(y_{\nu-1})+g(y_\nu)}{2}\;\alpha_{n,\nu-\frac{1}{2}}\right]\alpha_{n,j-\frac{1}{2}},\quad 1\le j\le J_0. \tag{2.15}$$

Taking the difference of (2.12) and (2.15) we obtain the result.

Lemma 2 Suppose $f(t)\in C^2[0,t_n]$. Then

$$\left|\int_0^{t_n} f'(t)\frac{dt}{\sqrt{t_n-t}}-\sum_{k=1}^{n}\frac{f(t_k)-f(t_{k-1})}{\tau}\int_{t_{k-1}}^{t_k}\frac{dt}{\sqrt{t_n-t}}\right|\le\frac{1}{6}(10\sqrt{2}-11)\max_{0\le t\le t_n}|f''(t)|\,\tau^{3/2}.$$

The proof of this lemma can be found from [14].

Using Lemma 1 and Lemma 2 we obtain

$$V_{I,j-\frac{1}{2}}^k=\frac{1}{h_2}\int_{y_{j-1}}^{y_j} v(b,y,t_k)dy+O(h_2^2)$$

$$=-\frac{2}{\sqrt{\pi}l}\sum_{n=1}^{\infty}\int_0^l\left\{\int_0^{t_k}\frac{\partial}{\partial\lambda}\left[e^{-\mu_n^2(t_k-\lambda)}u(b,\xi,\lambda)\right]\frac{d\lambda}{\sqrt{t_k-\lambda}}\right\}\sin(\mu_n\xi)d\xi\;\alpha_{n,j-\frac{1}{2}}+O(h_2^2)$$

$$=-\frac{2}{\sqrt{\pi}l}\sum_{n=1}^{\infty}\left\{h_2\sum_{\nu=1}^{J_0}\int_0^{t_k}\frac{\partial}{\partial\lambda}\left[e^{-\mu_n^2(t_k-\lambda)}\;\frac{u(b,y_\nu,\lambda)+u(b,y_{\nu-1},\lambda)}{2}\right]\frac{d\lambda}{\sqrt{t_k-\lambda}}\;\alpha_{n,\nu-\frac{1}{2}}\right\}\alpha_{n,j-\frac{1}{2}}$$

$$+O(h_2^2)$$

$$=-\frac{2}{\sqrt{\pi}l}\int_0^{t_k}\frac{\partial}{\partial\lambda}\left\{\sum_{n=1}^{\infty}\left[h_2\sum_{\nu=1}^{J_0}\left(e^{-\mu_n^2(t_k-\lambda)}\;\frac{u(b,y_\nu,\lambda)+u(b,y_{\nu-1},\lambda)}{2}\right)\;\alpha_{n,\nu-\frac{1}{2}}\alpha_{n,j-\frac{1}{2}}\right]\right\}\cdot$$

$$\frac{d\lambda}{\sqrt{t_k-\lambda}}+O(h_2^2)$$

$$=-\frac{2}{\sqrt{\pi}l}\sum_{m=1}^{k}\frac{1}{\tau}\left[\sum_{n=1}^{\infty}h_2\sum_{\nu=1}^{J_0}e^{-\mu_n^2(t_k-t_m)}U_{I,\nu-\frac{1}{2}}^m\alpha_{n,\nu-\frac{1}{2}}\;\alpha_{n,j-\frac{1}{2}}\right.$$

$$\left.-\sum_{n=1}^{\infty}h_2\sum_{\nu=1}^{J_0}e^{-\mu_n^2(t_k-t_{m-1})}U_{I,\nu-\frac{1}{2}}^{m-1}\alpha_{n,\nu-\frac{1}{2}}\;\alpha_{n,j-\frac{1}{2}}\right](2\sqrt{\tau}a_{k-m})+O(\tau^{3/2})+O(h_2^2)$$

$$=-\frac{4}{\sqrt{\pi\tau}l}\sum_{n=1}^{\infty}h_2\sum_{\nu=1}^{J_0}\left[\sum_{m=1}^{k}a_{k-m}e^{-\mu_n^2(t_k-t_m)}U_{I,\nu-\frac{1}{2}}^m\right.$$

$$- \sum_{m=1}^{k} a_{k-m} e^{-\mu_n^2 (t_k - t_{m-1})} U_{I,\nu-\frac{1}{2}}^{m-1} \Bigg] \quad \alpha_{n,\nu-\frac{1}{2}} \ \alpha_{n,j-\frac{1}{2}} + O(\tau^{3/2}) + O(h_2^2)$$

$$= -\frac{4}{\sqrt{\pi\tau} l} \sum_{n=1}^{\infty} h_2 \sum_{\nu=1}^{J_0} \Bigg[a_0 U_{I,\nu-\frac{1}{2}}^{k}$$

$$- \sum_{m=1}^{k-1} (a_{k-m-1} - a_{k-m}) e^{-\mu_n^2 (t_k - t_m)} U_{I,\nu-\frac{1}{2}}^{m} \Bigg] \quad \alpha_{n,\nu-\frac{1}{2}} \ \alpha_{n,j-\frac{1}{2}} + O(\tau^{3/2}) + O(h_2^2)$$

$$= -\frac{4}{\sqrt{\pi\tau} l} \sum_{n=1}^{\infty} \Bigg[a_0 \hat{H}_n^k - \sum_{m=1}^{k-1} (a_{k-m-1} - a_{k-m}) e^{-\mu_n^2 (t_k - t_m)} \hat{H}_n^m \Bigg] \alpha_{n,j-\frac{1}{2}} + O(\tau^{3/2}) + O(h_2^2),$$

where,

$$\hat{H}_n^k = h_2 \sum_{\nu=1}^{J_0} U_{I,\nu-\frac{1}{2}}^{k} \alpha_{n,\nu-\frac{1}{2}}.$$

According to the above equality and Taylor expansion, from (2.6)-(2.10), we have

$$\delta_t U_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x V_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + \delta_y W_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + p_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}, (i,j) \in \omega_0, \ k \geq 1, \tag{2.16}$$

$$V_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x U_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + q_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}, \quad (i,j) \in \omega_0, \ k \geq 1, \tag{2.17}$$

$$W_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_y U_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + r_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}, \quad (i,j) \in \omega_0, \ k \geq 1, \tag{2.18}$$

$$V_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} = -\frac{4}{\sqrt{\pi \ \tau} \ l} \sum_{n=1}^{\infty} \Bigg[a_0 \hat{H}_n^k - \sum_{m=1}^{k-1} (a_{k-m-1} - a_{k-m}) e^{-\mu_n^2 (t_k - t_m)} \hat{H}_n^m \Bigg] \alpha_{n,j-\frac{1}{2}} + s_{j-\frac{1}{2}}^{k-\frac{1}{2}},$$
$$1 \leq j \leq J_0, \ k \geq 1, \tag{2.19}$$

$$U_{ij}^k = 0, \ (i,j) \in \omega_2, \ k \geq 0, \quad U_{ij}^0 = g(x_i, y_j), \ (i,j) \in \omega_1. \tag{2.20}$$

And there exists a constant c_1 such that

$$|p_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}| \leq c_1(\tau^2 + h_1^2 + h_2^2), \quad (i,j) \in \omega_0, \ k \geq 1, \tag{2.21}$$

$$|q_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}| \leq c_1(\tau^2 + h_1^2 + h_2^2), \quad (i,j) \in \omega_0, \ k \geq 1, \tag{2.22}$$

$$|r_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}| \leq c_1(\tau^2 + h_1^2 + h_2^2), \quad (i,j) \in \omega_0, \ k \geq 1, \tag{2.23}$$

$$|s_{j-\frac{1}{2}}^{k-\frac{1}{2}}| \leq c_1(\tau^{3/2} + h_2^2), \quad 1 \leq j \leq J_0, \ k \geq 1. \tag{2.24}$$

We construct the difference scheme for (2.6)-(2.10) as follows:

$$\delta_t u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x v_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + \delta_y w_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}, \quad (i,j) \in \omega_0, k \geq 1, \tag{2.25}$$

$$v_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}, \quad (i,j) \in \omega_0, k \geq 1, \tag{2.26}$$

$$w_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_y u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}, \quad (i,j) \in \omega_0, k \geq 1, \tag{2.27}$$

$$v_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} = -\frac{4}{\sqrt{\pi \ \tau} \ l} \sum_{n=1}^{\infty} \Bigg[a_0 H_n^k - \sum_{m=1}^{k-1} (a_{k-m-1} - a_{k-m}) e^{-\mu_n^2 (t_k - t_m)} H_n^m \Bigg] \alpha_{n,j-\frac{1}{2}},$$

$$1 \le j \le J_0, \ k \ge 1, \tag{2.28}$$

$$u_{ij}^k = 0, \ (i,j) \in \omega_2, \ k \ge 0, \tag{2.29}$$

$$u_{ij}^0 = g(x_i, y_j), \qquad (i,j) \in \omega_1. \tag{2.30}$$

Theorem 1 Difference scheme (2.25)-(2.30) is equivalent to (2.1)-(2.4) and

$$w_{i-\frac{1}{2},j}^{k-\frac{1}{2}} = \delta_y u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + \frac{h_2}{2}\left(\delta_t u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} - \delta_x v_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} - f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}\right), (i,j) \in \omega_0, \ k \ge 1, \tag{2.31}$$

$$w_{i,0}^{k-\frac{1}{2}} = \delta_y u_{i-\frac{1}{2},\frac{1}{2}}^{k-\frac{1}{2}} - \frac{1}{2}h_2\left(\delta_t u_{i-\frac{1}{2},\frac{1}{2}}^{k-\frac{1}{2}} - \delta_x v_{i-\frac{1}{2},\frac{1}{2}}^{k-\frac{1}{2}} - f_{i-\frac{1}{2},\frac{1}{2}}^{k-\frac{1}{2}}\right), \quad 1 \le i \le I, \ k \ge 1, \tag{2.32}$$

$$v_{i-\frac{1}{2},\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x u_{i-\frac{1}{2},\frac{1}{2}}^{k-\frac{1}{2}}, \quad 1 \le i \le I, \ k \ge 1, \tag{2.33}$$

$$\begin{aligned}\frac{1}{2}\left(v_{i,j-\frac{1}{2}}^{k-\frac{1}{2}} + v_{i,j+\frac{1}{2}}^{k-\frac{1}{2}}\right) &= \frac{1}{2}\delta_x\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right) + \frac{1}{2}h_1\left[\frac{1}{2}\delta_t\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right.\\ &\left.-\delta_y^2 u_{i-\frac{1}{2},j}^{k-\frac{1}{2}} - \frac{1}{2}\left(f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + f_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right], (i,j) \in \omega_0'', k \ge 1,\end{aligned} \tag{2.34}$$

$$\begin{aligned}\frac{1}{2}\left(v_{0,j-\frac{1}{2}}^{k-\frac{1}{2}} + v_{0,j+\frac{1}{2}}^{k-\frac{1}{2}}\right) &= \frac{1}{2}\delta_x\left(u_{\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right) - \frac{1}{2}h_1\left[\frac{1}{2}\delta_t\left(u_{\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right.\\ &\left.-\delta_y^2 u_{\frac{1}{2},j}^{k-\frac{1}{2}} - \frac{1}{2}\left(f_{\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + f_{\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right], \quad 1 \le j \le J, \ k \ge 1,\end{aligned} \tag{2.35}$$

$$v_{I,\frac{1}{2}}^{k-\frac{1}{2}} = -\frac{4}{\sqrt{\pi\,\tau}\,l}\sum_{n=1}^{\infty}\left[a_0 H_n^k - \sum_{m=1}^{k-1}(a_{k-m-1} - a_{k-m})e^{-\mu_n^2(t_k - t_m)}H_n^m\right]\alpha_{n,\frac{1}{2}}, \quad k \ge 1. \tag{2.36}$$

Proof Multiplying (2.25) by $\frac{1}{2}h_2$ and adding the result with (2.27), we have

$$w_{i-\frac{1}{2},j}^{k-\frac{1}{2}} = \delta_y u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + \frac{1}{2}h_2\left(\delta_t u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} - \delta_x v_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} - f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}\right), (i,j) \in \omega_0. \tag{2.37}$$

Similarly, we arrive at

$$w_{i-\frac{1}{2},j-1}^{k-\frac{1}{2}} = \delta_y u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} - \frac{1}{2}h_2\left(\delta_t u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} - \delta_x v_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} - f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}\right), (i,j) \in \omega_0,$$

namely,

$$w_{i-\frac{1}{2},j}^{k-\frac{1}{2}} = \delta_y u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}} - \frac{1}{2}h_2\left(\delta_t u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}} - \delta_x v_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}} - f_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right), (i,j) \in \omega_0'. \tag{2.38}$$

¿From equations (2.37) and (2.38) it follows that

$$\begin{aligned}&\frac{1}{2}\delta_x\left(v_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + v_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\\ &= \frac{1}{2}\delta_t\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right) - \delta_y^2 u_{i-\frac{1}{2},j}^{k-\frac{1}{2}} - \frac{1}{2}\left(f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + f_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right), (i,j) \in \omega_0''.\end{aligned} \tag{2.39}$$

The equation (2.26) is equivalent to

$$v_{i-\frac{1}{2},\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x u_{i-\frac{1}{2},\frac{1}{2}}^{k-\frac{1}{2}}, \ 1 \le i \le I, \tag{2.40}$$

$$\frac{1}{2}\left(v_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + v_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right) = \frac{1}{2}\delta_x\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right), (i,j) \in \omega_0''. \tag{2.41}$$

Multiplying (2.39) by $\frac{1}{2}h_1$ and adding the result with (2.41), we obtain

$$\frac{1}{2}\left(v_{i,j-\frac{1}{2}}^{k-\frac{1}{2}}+v_{i,j+\frac{1}{2}}^{k-\frac{1}{2}}\right)=\frac{1}{2}\delta_x\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)$$
$$+\frac{1}{2}h_1\left[\frac{1}{2}\delta_t\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)-\delta_y^2 u_{i-\frac{1}{2},j}^{k-\frac{1}{2}}-\frac{1}{2}\left(f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+f_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right],(i,j)\in\omega_0''. \quad (2.42)$$

Similarly, we can arrive at

$$\frac{1}{2}\left(v_{i-1,j-\frac{1}{2}}^{k-\frac{1}{2}}+v_{i-1,j+\frac{1}{2}}^{k-\frac{1}{2}}\right)=\frac{1}{2}\delta_x\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)$$
$$-\frac{1}{2}h_1\left[\frac{1}{2}\delta_t\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)-\delta_y^2 u_{i-\frac{1}{2},j}^{k-\frac{1}{2}}-\frac{1}{2}\left(f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+f_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right],(i,j)\in\omega_0'',$$

namely,

$$\frac{1}{2}\left(v_{i,j-\frac{1}{2}}^{k-\frac{1}{2}}+v_{i,j+\frac{1}{2}}^{k-\frac{1}{2}}\right)=\frac{1}{2}\delta_x\left(u_{i+\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)$$
$$-\frac{1}{2}h_1\left[\frac{1}{2}\delta_t\left(u_{i+\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)-\delta_y^2 u_{i+\frac{1}{2},j}^{k-\frac{1}{2}}-\frac{1}{2}\left(f_{i+\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+f_{i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right],(i,j)\in\omega_1'. \quad (2.43)$$

Combining (2.42) and (2.43) we obtain

$$\frac{1}{2}\delta_x\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)+\frac{1}{2}h_1\left[\frac{1}{2}\delta_t\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right.$$
$$\left.-\delta_y^2 u_{i-\frac{1}{2},j}^{k-\frac{1}{2}}-\frac{1}{2}\left(f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+f_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right]$$
$$=\frac{1}{2}\delta_x\left(u_{i+\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)-\frac{1}{2}h_1\left[\frac{1}{2}\delta_t\left(u_{i+\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right.$$
$$\left.-\delta_y^2 u_{i+\frac{1}{2},j}^{k-\frac{1}{2}}-\frac{1}{2}\left(f_{i+\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+f_{i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\right],(i,j)\in\omega_1,$$

namely,

$$\frac{1}{4}\delta_t\left(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}+u_{i+\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right)=\frac{1}{2}\delta_x^2\left(u_{i,j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i,j+\frac{1}{2}}^{k-\frac{1}{2}}\right)$$
$$+\frac{1}{2}\delta_y^2\left(u_{i-\frac{1}{2},j}^{k-\frac{1}{2}}+u_{i+\frac{1}{2},j}^{k-\frac{1}{2}}\right)+\frac{1}{4}\left(f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+f_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}+f_{i+\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+f_{i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}}\right),(i,j)\in\omega_1,$$

which is (2.1). The boundary condition (2.28) is equivalent to

$$v_{I,\frac{1}{2}}^{k-\frac{1}{2}}=-\frac{4}{\sqrt{\pi\tau}\,l}\sum_{n=1}^{\infty}\left[a_0H_n^k-\sum_{m=1}^{k-1}(a_{k-m-1}-a_{k-m})e^{-\mu_n^2(t_k-t_m)}H_n^m\right]\alpha_{n,\frac{1}{2}}, \quad (2.44)$$

$$\frac{1}{2}\left(v_{I,j-\frac{1}{2}}^{k-\frac{1}{2}}+v_{I,j+\frac{1}{2}}^{k-\frac{1}{2}}\right)=-\frac{4}{\sqrt{\pi\tau}\,l}\sum_{n=1}^{\infty}\left[a_0H_n^k-\sum_{m=1}^{k-1}(a_{k-m-1}-a_{k-m})e^{-\mu_n^2(t_k-t_m)}H_n^m\right]s(n,j),$$
$$1\le j\le J_0-1. \quad (2.45)$$

It follows from (2.42) for $i=I$ that (2.45) is equivalent to

$$-\frac{4}{\sqrt{\pi\tau}\,l}\sum_{n=1}^{\infty}\left[a_0H_n^k-\sum_{m=1}^{k-1}(a_{k-m-1}-a_{k-m})e^{-\mu_n^2(t_k-t_m)}H_n^m\right]s(n,j)$$

$$
= \frac{1}{2}\delta_x\left(u^{k-\frac{1}{2}}_{I-\frac{1}{2},j-\frac{1}{2}} + u^{k-\frac{1}{2}}_{I-\frac{1}{2},j+\frac{1}{2}}\right) + \frac{1}{2}h_1\left[\frac{1}{2}\delta_t\left(u^{k-\frac{1}{2}}_{I-\frac{1}{2},j-\frac{1}{2}} + u^{k-\frac{1}{2}}_{I-\frac{1}{2},j+\frac{1}{2}}\right)\right.
$$

$$
\left. -\delta_y^2 u^{k-\frac{1}{2}}_{I-\frac{1}{2},j} - \frac{1}{2}\left(f^{k-\frac{1}{2}}_{I-\frac{1}{2},j-\frac{1}{2}} + f^{k-\frac{1}{2}}_{I-\frac{1}{2},j+\frac{1}{2}}\right)\right], \quad 1 \le j \le J_0 - 1,
$$

or,

$$
\frac{1}{2}\delta_t\left(u^{k-\frac{1}{2}}_{I-\frac{1}{2},j-\frac{1}{2}} + u^{k-\frac{1}{2}}_{I-\frac{1}{2},j+\frac{1}{2}}\right) = \frac{2}{h_1}\left\{-\frac{4}{\sqrt{\pi\tau}\, l}\sum_{n=1}^{\infty}\left[a_0 H_n^k\right.\right.
$$

$$
\left. - \sum_{m=1}^{k-1}(a_{k-m-1} - a_{k-m})e^{-\mu_n^2(t_k - t_m)}H_n^m\right] s(n,j)
$$

$$
\left. -\frac{1}{2}\delta_x\left(u^{k-\frac{1}{2}}_{I-\frac{1}{2},j-\frac{1}{2}} + u^{k-\frac{1}{2}}_{I-\frac{1}{2},j+\frac{1}{2}}\right)\right\} + \delta_y^2 u^{k-\frac{1}{2}}_{I-\frac{1}{2},j} + \frac{1}{2}(f^{k-\frac{1}{2}}_{I-\frac{1}{2},j-\frac{1}{2}} + f^{k-\frac{1}{2}}_{I-\frac{1}{2},j+\frac{1}{2}}), 1 \le j \le J_0 - 1,
$$

which is (2.2).

It is straightforward to check the following equivalent relations:

$$
\left.\begin{matrix}(2.25)\\(2.27)\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}(2.31)\\(2.32)\\(2.39)\end{matrix}\right. \quad (2.26) \Longleftrightarrow \left\{\begin{matrix}(2.40) = (2.33)\\(2.41)\end{matrix}\right. \quad \left.\begin{matrix}(2.39)\\(2.41)\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}(2.34)\\(2.35)\\(2.1)\end{matrix}\right.
$$

$$
(2.28) \Longleftrightarrow \left\{\begin{matrix}(2.44) = (2.36)\\(2.45) \Longleftrightarrow (2.2)\end{matrix}\right. \qquad \begin{matrix}(2.29) = (2.3)\\(2.30) = (2.4)\end{matrix}
$$

This completes the proof.

Based on the above theorem, at the $k-$th time level, we regard (2.25)-(2.30) as a system of linear algebraic equations about the unknowns

$$
\left\{u_{ij}^k,\ (i,j) \in \bar{\omega}\right\} \bigcup \left\{w^{k-\frac{1}{2}}_{i-\frac{1}{2},j},\ (i,j) \in \omega_0\right\} \bigcup \left\{w^{k-\frac{1}{2}}_{i,0},\ 1 \le i \le I\right\} \bigcup \left\{v^{k-\frac{1}{2}}_{i-\frac{1}{2},\frac{1}{2}},\ 1 \le i \le I\right\}
$$

$$
\bigcup \left\{\frac{1}{2}\left(v^{k-\frac{1}{2}}_{i,j-\frac{1}{2}} + v^{k-\frac{1}{2}}_{i,j+\frac{1}{2}}\right),\ (i,j) \in \omega_0''\right\} \bigcup \left\{\frac{1}{2}\left(v^{k-\frac{1}{2}}_{0,j-\frac{1}{2}} + v^{k-\frac{1}{2}}_{0,j+\frac{1}{2}}\right),\ 1 \le j \le J\right\} \bigcup \left\{v^{k-\frac{1}{2}}_{I,\frac{1}{2}}\right\}.
$$

If we have determined $\left\{u_{ij}^k,\ (i,j) \in \bar{\omega}\right\}$, we then obtain all other unknowns directly from (2.31)-(2.36).

3 The analysis of the difference scheme

In this section, we discuss the stability and convergence of the difference scheme proposed in section 2. First, we need the following lemma:

Lemma 3 For any $F = \{F_1, F_2, F_3, \cdots\}$, we have

$$
\sum_{k=1}^{N}\left[a_0 F_k - \sum_{m=1}^{k-1}(a_{k-m-1} - a_{k-m})e^{-\mu_n^2(t_k - t_m)}F_m\right]F_k \ge \frac{1}{\sqrt{2N}}\sum_{k=1}^{N}F_k^2,\ N = 1, 2, 3, \cdots,
$$

where a_m is defined in (2.5).

Proof Using Cauchy-Schwartz inequality, we have

$$\sum_{k=1}^{N}\left[a_0F_k-\sum_{m=1}^{k-1}(a_{k-m-1}-a_{k-m})e^{-\mu_n^2(t_k-t_m)}F_m\right]F_k$$
$$\geq\sum_{k=1}^{N}a_0F_k^2-\frac{1}{2}\sum_{k=2}^{N}\sum_{m=1}^{k-1}(a_{k-m-1}-a_{k-m})(F_m^2+F_k^2)$$
$$=\sum_{k=1}^{N}\left[a_0-\frac{1}{2}(a_0-a_{k-1})\right]F_k^2-\frac{1}{2}\sum_{m=1}^{N-1}\sum_{k=m+1}^{N}(a_{k-m-1}-a_{k-m})\,F_m^2$$
$$=\frac{1}{2}\sum_{k=1}^{N}(a_0+a_{k-1})\,F_k^2-\frac{1}{2}\sum_{m=1}^{N-1}(a_0-a_{N-m})\,F_m^2=\frac{1}{2}\sum_{k=1}^{N}(a_{k-1}+a_{N-k})F_k^2.$$

Noticing that, for $1\leq k\leq N$,

$$a_{k-1}+a_{N-k}=\frac{1}{\sqrt{k}+\sqrt{k+1}}+\frac{1}{\sqrt{N-k+1}+\sqrt{N-k}}$$
$$\geq\frac{1}{2\sqrt{k+\frac{1}{2}}}+\frac{1}{2\sqrt{N-k+\frac{1}{2}}}\geq\frac{1}{\sqrt{N/2}},$$

we obtain the result needed.

Theorem 2 Let $\{u_{ij}^k,\ (i,j)\in\bar{\omega},0\leq k\leq K\}$ be the solution of (2.1)-(2.4). Then, when $\tau\leq 2/9$,

$$\|u^n\|^2+2\tau\sum_{k=1}^{n}\left(\|\delta_xu^{k-\frac{1}{2}}\|^2+\|\delta_yu^{k-\frac{1}{2}}\|^2\right)\leq\frac{9}{8}e^{\frac{9}{8}n\tau}\left(\|u^0\|^2+\tau\sum_{k=1}^{n}\|f^{k-\frac{1}{2}}\|^2\right),\quad n\geq 1.\qquad(3.1)$$

Proof According to Theorem 1, it suffices to analyze the difference scheme (2.25)-(2.30). Multiplying (2.25), (2.26) and (2.27) by $u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}$, $v_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}$ and $w_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}$ respectively, and then adding the results, we obtain

$$\frac{1}{2\tau}\left[(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k})^2-(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-1})^2\right]+(v_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2+(w_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2$$
$$=u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}\delta_xv_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+(\delta_xu_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})v_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}\delta_yw_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}$$
$$+(\delta_yu_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})w_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}+u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}$$
$$\leq\frac{1}{h_1}\left(u_{i,j-\frac{1}{2}}^{k-\frac{1}{2}}v_{i,j-\frac{1}{2}}^{k-\frac{1}{2}}-u_{i-1,j-\frac{1}{2}}^{k-\frac{1}{2}}v_{i-1,j-\frac{1}{2}}^{k-\frac{1}{2}}\right)+\frac{1}{h_2}\left(u_{i-\frac{1}{2},j}^{k-\frac{1}{2}}w_{i-\frac{1}{2},j}^{k-\frac{1}{2}}-u_{i-\frac{1}{2},j-1}^{k-\frac{1}{2}}w_{i-\frac{1}{2},j-1}^{k-\frac{1}{2}}\right)$$
$$+\frac{1}{2}(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2+\frac{1}{2}(f_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2,\quad(i,j)\in\omega_0.$$

Multiplying the above inequality by h_1h_2 and summing up for $(i,j)\in\omega_0$, then using (2.29), we get

$$\frac{1}{2\tau}\left(\|u^k\|^2-\|u^{k-1}\|^2\right)+\|\delta_xu^{k-\frac{1}{2}}\|^2+\|\delta_yu^{k-\frac{1}{2}}\|^2$$
$$\leq h_2\sum_{j=1}^{J_0}u_{I,j-\frac{1}{2}}^{k-\frac{1}{2}}v_{I,j-\frac{1}{2}}^{k-\frac{1}{2}}+\frac{1}{2}\|u^{k-\frac{1}{2}}\|^2+\frac{1}{2}\|f^{k-\frac{1}{2}}\|^2$$
$$\leq h_2\sum_{j=1}^{J_0}u_{I,j-\frac{1}{2}}^{k-\frac{1}{2}}v_{I,j-\frac{1}{2}}^{k-\frac{1}{2}}+\frac{1}{4}(\|u^k\|^2+\|u^{k-1}\|^2)+\frac{1}{2}\|f^{k-\frac{1}{2}}\|^2,\quad k\geq 1.\qquad(3.2)$$

Denote

$$G^k = \|u^k\|^2 + 2\tau \sum_{l=1}^{k} (\|\delta_x u^{l-\frac{1}{2}}\|^2 + \|\delta_y u^{l-\frac{1}{2}}\|^2), \quad k \ge 1.$$

It follows from (3.2) that

$$\frac{1}{2\tau}(G^k - G^{k-1}) \le h_2 \sum_{j=1}^{J_0} u_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} v_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} + \frac{1}{4}(G^k + G^{k-1}) + \frac{1}{2}\|f^{k-\frac{1}{2}}\|^2, \ k \ge 1.$$

Summing up for k from 1 to N, we have

$$\frac{1}{2\tau}(G^N - G^0) \le h_2 \sum_{k=1}^{N}\sum_{j=1}^{J_0} u_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} v_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} + \frac{1}{4}\sum_{k=1}^{N}(G^k + G^{k-1}) + \frac{1}{2}\sum_{k=1}^{N}\|f^{k-\frac{1}{2}}\|^2, \quad N \ge 1. \quad (3.3)$$

Using (2.28) and Lemma 3, we have

$$\begin{aligned} & h_2 \sum_{k=1}^{N}\sum_{j=1}^{J_0} u_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} v_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} && (3.3) \\ &= h_2 \sum_{k=1}^{N}\sum_{j=1}^{J_0} u_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \left\{ -\frac{4}{\sqrt{\pi\tau}\, l} \sum_{n=1}^{\infty} \left[a_0 H_n^k - \sum_{m=1}^{k-1} (a_{k-m-1} - a_{k-m}) e^{-\mu_n^2(t_k - t_m)} H_n^m \right] \alpha_{n,j-\frac{1}{2}} \right\} \\ &= -\frac{4}{\sqrt{\pi\tau}\, l} \sum_{n=1}^{\infty}\sum_{k=1}^{N} \left[a_0 H_n^k - \sum_{m=1}^{k-1} (a_{k-m-1} - a_{k-m}) e^{-\mu_n^2(t_k - t_m)} H_m \right] H_n^k \le 0, \quad N \ge 1. && (3.4) \end{aligned}$$

Inserting (3.4) into the right hand side of (3.3), we get

$$\frac{1}{2\tau}(G^N - G^0) \le \frac{1}{4}\sum_{k=1}^{N}(G^k + G^{k-1}) + \frac{1}{2}\sum_{k=1}^{N}\|f^{k-\frac{1}{2}}\|^2, \quad N \ge 1,$$

namely,

$$G^N \le G^0 + \frac{1}{2}\tau \sum_{k=1}^{N}(G^k + G^{k-1}) + \tau \sum_{k=1}^{N}\|f^{k-\frac{1}{2}}\|^2, \quad N \ge 1.$$

Gronwall inequality gives (when $\tau \le 2/9$)

$$G^k \le \frac{9}{8} e^{\frac{9}{8}k\tau} \left(G^0 + \tau \sum_{l=1}^{k} \|f^{l-\frac{1}{2}}\|^2 \right), \quad N \ge 1,$$

which is (3.1).

¿From Theorem 2, we obtain directly:

Theorem 3 The difference scheme (2.1)-(2.4) is uniquely solvable.

For the error estimate between the exact solution of the partial differential equation and the approximate solution of the difference scheme, we have:

Theorem 4 Assume that $u(x,y,t) \in C_{x,y,t}^{4,4,3}(\Omega_i \times [0,T])$ is the solution of problem (1.12)-(1.15) and $\{u_{ij}^k\}$ is the solution of (2.1)-(2.4). Let

$$\tilde{u}_{ij}^k = u(x_i, y_j, t_k) - u_{ij}^k, \quad (i,j) \in \omega_0, k \ge 0.$$

Then, when $\tau \leq 2/9$, we have the following error estimate:

$$\|\tilde{u}^k\|^2 + \frac{1}{2}\tau \sum_{l=1}^{n} \left(\|\delta_x \tilde{u}^{k-\frac{1}{2}}\|^2 + \|\delta_y \tilde{u}^{k-\frac{1}{2}}\|^2 \right) \leq \frac{9}{4}\bar{c}Te^{\frac{9}{8}T}(\tau^{3/2} + h_1^2 + h_2^2)^2, \quad 1 \leq k \leq K, \tag{3.5}$$

where

$$\bar{c} = \left(\frac{5bc}{2} + \frac{\sqrt{2\pi T}}{8} \right) c_1^2.$$

Proof According to Theorem 1, it suffices to analyze the error of the difference scheme (2.25)-(2.30). Let

$$\tilde{v}_{ij}^k = V_{ij}^k - v_{ij}^k, \quad \tilde{w}_{ij}^k = W_{ij}^k - w_{ij}^k, \quad (i,j) \in \bar{\omega}, \ k \geq 0.$$

Subtracting (2.25)-(2.30) from (2.16)-(2.20) respectively, we obtain the error equations:

$$\delta_t \tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x \tilde{v}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + \delta_y \tilde{w}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + p_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}, \quad (i,j) \in \omega_0, \ 1 \leq k \leq K, \tag{3.6}$$

$$\tilde{v}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x \tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + q_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}, \quad (i,j) \in \omega_0, \ 1 \leq k \leq K, \tag{3.7}$$

$$\tilde{w}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_y \tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + r_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}, \quad (i,j) \in \omega_0, \ 1 \leq k \leq K, \tag{3.8}$$

$$\tilde{v}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} = -\frac{4}{\sqrt{\pi\tau}\, l} \sum_{n=1}^{\infty} \left[a_0 \tilde{H}_n^k - \sum_{m=1}^{k-1} (a_{k-m-1} - a_{k-m}) e^{-\mu_n^2 (t_k - t_m)} \tilde{H}_n^m \right] \alpha_{n,j-\frac{1}{2}} + s_{j-\frac{1}{2}}^{k-\frac{1}{2}},$$
$$1 \leq j \leq J_0, \ 1 \leq k \leq K, \tag{3.9}$$

$$\tilde{u}_{ij}^k = 0, \quad (i,j) \in \omega_2, \quad 0 \leq k \leq K, \tag{3.10}$$

$$\tilde{u}_{ij}^0 = 0, \qquad (i,j) \in \omega_1. \tag{3.11}$$

where

$$\tilde{H}_n^k = h_2 \sum_{\nu=1}^{J_0} \tilde{u}_{I,\nu-\frac{1}{2}}^{k-\frac{1}{2}} \alpha_{n,\nu-\frac{1}{2}}.$$

Multiplying (3.6), (3.7) and (3.8) by $\tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}$, $\tilde{v}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}$ and $\tilde{w}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}$ respectively, and summing up the results, we obtain

$$\begin{aligned}
&\frac{1}{2\tau}\left[(\tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k})^2 - (\tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-1})^2 \right] + (\tilde{v}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2 + (\tilde{w}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2 \\
&= \tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} \delta_x \tilde{v}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + (\delta_x \tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}) \tilde{v}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + \tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} \delta_y \tilde{w}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} \\
&\quad + (\delta_y \tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}}) \tilde{w}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + \tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} \tilde{p}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + \tilde{v}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} \tilde{q}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} + \tilde{w}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} r_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}} \\
&\leq \frac{1}{h_1} \left(\tilde{u}_{i,j-\frac{1}{2}}^{k-\frac{1}{2}} \tilde{v}_{i,j-\frac{1}{2}}^{k-\frac{1}{2}} - \tilde{u}_{i-1,j-\frac{1}{2}}^{k-\frac{1}{2}} \tilde{v}_{i-1,j-\frac{1}{2}}^{k-\frac{1}{2}} \right) + \frac{1}{h_2} \left(\tilde{u}_{i-\frac{1}{2},j}^{k-\frac{1}{2}} \tilde{w}_{i-\frac{1}{2},j}^{k-\frac{1}{2}} - \tilde{u}_{i-\frac{1}{2},j-1}^{k-\frac{1}{2}} \tilde{w}_{i-\frac{1}{2},j-1}^{k-\frac{1}{2}} \right) \\
&\quad + \frac{1}{2}(\tilde{u}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2 + \frac{1}{2}(\tilde{p}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2 + \frac{1}{2}(\tilde{v}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2 + \frac{1}{2}(\tilde{q}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2 \\
&\quad + \frac{1}{2}(\tilde{w}_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2 + \frac{1}{2}(r_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})^2, \quad (i,j) \in \omega_0,
\end{aligned}$$

or,

$$\frac{1}{2\tau}\left[(\tilde{u}^{k}_{i-\frac{1}{2},j-\frac{1}{2}})^2-(\tilde{u}^{k-1}_{i-\frac{1}{2},j-\frac{1}{2}})^2\right]+\frac{1}{2}(\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+\frac{1}{2}(\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2$$
$$\leq\frac{1}{h_1}\left(\tilde{u}^{k-\frac{1}{2}}_{i,j-\frac{1}{2}}\tilde{v}^{k-\frac{1}{2}}_{i,j-\frac{1}{2}}-\tilde{u}^{k-\frac{1}{2}}_{i-1,j-\frac{1}{2}}\tilde{v}^{k-\frac{1}{2}}_{i-1,j-\frac{1}{2}}\right)+\frac{1}{h_2}\left(\tilde{u}^{k-\frac{1}{2}}_{i-\frac{1}{2},j}\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2},j}-\tilde{u}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-1}\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-1}\right)$$
$$+\frac{1}{2}(\tilde{u}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+\frac{1}{2}(\tilde{p}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+\frac{1}{2}(\tilde{q}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+\frac{1}{2}(r^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2,\quad (i,j)\in\omega_0. \tag{3.12}$$

It follows from (3.7) and (3.8) that

$$\frac{1}{4}(\delta_x\tilde{u}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2=\frac{1}{4}(\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}}-\tilde{q}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2\leq\frac{1}{2}(\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+\frac{1}{2}(\tilde{q}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2$$

and

$$\frac{1}{4}(\delta_y\tilde{u}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2=\frac{1}{4}(\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}}-r^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2\leq\frac{1}{2}(\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+\frac{1}{2}(r^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2.$$

Adding $\frac{1}{2}(\tilde{q}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+\frac{1}{2}(r^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2$ on two sides of (3.12) and using the above two inequalities, we obtain

$$\frac{1}{2\tau}\left[(\tilde{u}^{k}_{i-\frac{1}{2},j-\frac{1}{2}})^2-(\tilde{u}^{k-1}_{i-\frac{1}{2},j-\frac{1}{2}})^2\right]+\frac{1}{4}(\delta_x\tilde{u}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+\frac{1}{4}(\delta_y\tilde{u}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2$$
$$\leq\frac{1}{h_1}\left(\tilde{u}^{k-\frac{1}{2}}_{i,j-\frac{1}{2}}\tilde{v}^{k-\frac{1}{2}}_{i,j-\frac{1}{2}}-\tilde{u}^{k-\frac{1}{2}}_{i-1,j-\frac{1}{2}}\tilde{v}^{k-\frac{1}{2}}_{i-1,j-\frac{1}{2}}\right)+\frac{1}{h_2}\left(\tilde{u}^{k-\frac{1}{2}}_{i-\frac{1}{2},j}\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2},j}-\tilde{u}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-1}\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-1}\right)$$
$$+\frac{1}{2}(\tilde{u}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+\frac{1}{2}(\tilde{p}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+(\tilde{q}^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2+(r^{k-\frac{1}{2}}_{i-\frac{1}{2},j-\frac{1}{2}})^2,\ (i,j)\in\omega_0.$$

Multiplying the inequality just obtained by h_1h_2 and summing up for $(i,j)\in\omega_0$, then using (3.10), we have

$$\frac{1}{2\tau}\left(\|\tilde{u}^k\|^2-\|\tilde{u}^{k-1}\|^2\right)+\frac{1}{4}(\|\delta_x\tilde{u}^{k-\frac{1}{2}}\|^2+\|\delta_y\tilde{u}^{k-\frac{1}{2}}\|^2)$$
$$\leq h_2\sum_{j=1}^{J_0}\tilde{u}^{k-\frac{1}{2}}_{I,j-\frac{1}{2}}\tilde{v}^{k-\frac{1}{2}}_{I,j-\frac{1}{2}}+\frac{1}{2}\|\tilde{u}^{k-\frac{1}{2}}\|^2+\frac{1}{2}\|\tilde{p}^{k-\frac{1}{2}}\|^2+\|\tilde{q}^{k-\frac{1}{2}}\|^2+\|r^{k-\frac{1}{2}}\|^2. \tag{3.13}$$

Using (2.21), (2.22) and (2.23), we have

$$\frac{1}{2}\|\tilde{p}^{k-\frac{1}{2}}\|^2+\|\tilde{q}^{k-\frac{1}{2}}\|^2+\|r^{k-\frac{1}{2}}\|^2\leq\frac{5}{2}bcc_1^2(\tau^{3/2}+h_1^2+h_2^2)^2.$$

Substituting the above inequality to (3.13), we obtain

$$\frac{1}{2\tau}\left(\|\tilde{u}^k\|^2-\|\tilde{u}^{k-1}\|^2\right)+\frac{1}{4}(\|\delta_x\tilde{u}^{k-\frac{1}{2}}\|^2+\|\delta_y\tilde{u}^{k-\frac{1}{2}}\|^2)$$
$$\leq h_2\sum_{j=1}^{J_0}\tilde{u}^{k-\frac{1}{2}}_{I,j-\frac{1}{2}}\tilde{v}^{k-\frac{1}{2}}_{I,j-\frac{1}{2}}+\frac{1}{4}(\|\tilde{u}^k\|^2+\|\tilde{u}^{k-1}\|^2)+\frac{5}{2}bcc_1^2(\tau^{3/2}+h_1^2+h_2^2)^2. \tag{3.14}$$

Denote

$$G^k=\|\tilde{u}^k\|^2+\frac{1}{2}\tau\sum_{l=1}^{k}(\|\delta_x\tilde{u}^{l-\frac{1}{2}}\|^2+\|\delta_y\tilde{u}^{l-\frac{1}{2}}\|^2),\quad 1\leq k\leq K.$$

Then, it follows from (3.14) that

$$\frac{1}{2\tau}\left(G^k - G^{k-1}\right) \le h_2 \sum_{j=1}^{J_0} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \tilde{v}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} + \frac{1}{4}(G^k + G^{k-1}) + \frac{5}{2} bcc_1^2 (\tau^2 + h_1^2 + h_2^2)^2, 1 \le k \le K.$$

Summing up the above inequality for k from 1 to N, we get

$$\frac{G^N - G^0}{2\tau} \le h_2 \sum_{j=1}^{J_0} \sum_{k=1}^{N} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \tilde{v}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} + \frac{1}{4} \sum_{k=1}^{N} (G^k + G^{k-1}) + \frac{5Nbcc_1^2}{2} (\tau^2 + h_1^2 + h_2^2)^2. \tag{3.15}$$

Using (3.9) and Lemma 3, we have

$$\begin{aligned}
& h_2 \sum_{j=1}^{J_0} \sum_{k=1}^{N} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \tilde{v}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \\
&= h_2 \sum_{j=1}^{J_0} \sum_{k=1}^{N} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \left\{ \frac{-4}{\sqrt{\pi\tau} l} \sum_{n=1}^{\infty} \left[a_0 \tilde{H}_n^k - \sum_{m=1}^{k-1} (a_{k-m-1} - a_{k-m}) e^{-\mu_n^2 (t_k - t_m)} \tilde{H}_n^m \right] \alpha_{n,j-\frac{1}{2}} + s_{j-\frac{1}{2}}^{k-\frac{1}{2}} \right\} \\
&\le \frac{-4}{\sqrt{\pi\tau}\, l} \sum_{n=1}^{\infty} \sum_{k=1}^{N} \left[a_0 \tilde{H}_n^k - \sum_{m=1}^{k-1} (a_{k-m-1} - a_{k-m}) e^{-\mu_n^2 (t_k - t_m)} \tilde{H}_n^m \right] \tilde{H}_n^k + h_2 \sum_{j=1}^{J_0} \sum_{k=1}^{N} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} s_{j-\frac{1}{2}}^{k-\frac{1}{2}} \\
&\le -\frac{4}{\sqrt{\pi\tau}\, l} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2N}} \sum_{k=1}^{N} \left(\tilde{H}_n^k\right)^2 + h_2 \sum_{j=1}^{J_0} \sum_{k=1}^{N} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} s_{j-\frac{1}{2}}^{k-\frac{1}{2}}, \\
&\le -\frac{2}{\sqrt{2\pi t_N}} \frac{2}{l} \sum_{k=1}^{N} \sum_{n=1}^{\infty} \left(h_2 \sum_{j=1}^{J_0} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \alpha_{n,j-\frac{1}{2}} \right)^2 + h_2 \sum_{j=1}^{J_0} \sum_{k=1}^{N} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} s_{j-\frac{1}{2}}^{k-\frac{1}{2}} \; 1 \le N \le K. \qquad (3.16)
\end{aligned}$$

Similarly to the derivation of (2.14), we have

$$\sum_{n=1}^{\infty} \left[\frac{2}{l} h_2 \sum_{j=1}^{J_0} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \alpha_{n,j-\frac{1}{2}} \right]^2 = \frac{2}{l} h_2 \sum_{j=1}^{J_0} \left(\tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \right)^2, \tag{3.17}$$

Substituting the above equality into (3.16) and using (2.24), we get

$$\begin{aligned}
& h_2 \sum_{j=1}^{J_0} \sum_{k=1}^{N} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \tilde{v}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \\
&\le -\frac{2}{\sqrt{2\pi t_N}} \sum_{k=1}^{N} h_2 \sum_{j=1}^{J_0} \left(\tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \right)^2 + h_2 \sum_{j=1}^{J_0} \sum_{k=1}^{N} \tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} s_{j-\frac{1}{2}}^{k-\frac{1}{2}} \\
&\le -\frac{2}{\sqrt{2\pi t_N}} \sum_{k=1}^{N} h_2 \sum_{j=1}^{J_0} \left(\tilde{u}_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \right)^2 + \frac{2}{\sqrt{2\pi t_N}} \sum_{k=1}^{N} h_2 \sum_{j=1}^{J_0} \left(u_{I,j-\frac{1}{2}}^{k-\frac{1}{2}} \right)^2 + \frac{\sqrt{2\pi t_N}}{8} \sum_{k=1}^{N} h_2 \sum_{j=1}^{J_0} \left(s_{j-\frac{1}{2}}^{k-\frac{1}{2}} \right)^2 \\
&= \frac{\sqrt{2\pi t_N}}{8} \sum_{k=1}^{N} h_2 \sum_{j=1}^{J_0} \left(s_{j-\frac{1}{2}}^{k-\frac{1}{2}} \right)^2 \qquad (3.17) \\
&\le \frac{\sqrt{2\pi t_N}}{8} \, l \, N c_1^2 \left(\tau^{3/2} + h_2^2 \right)^2. \qquad (3.18)
\end{aligned}$$

Inserting (3.18) into (3.15) yields

$$\frac{1}{2\tau}(G^N - G^0) \le \frac{1}{4}\sum_{k=1}^{N}(G^k + G^{k-1}) + \left[\frac{5bc}{2} + \frac{\sqrt{2\pi t_N}}{8}\, l\right] N c_1^2(\tau^{3/2} + h_1^2 + h_2^2)^2$$

$$= \frac{1}{4}\sum_{k=1}^{N}(G^k + G^{k-1}) + N\bar{c}(\tau^{3/2} + h_1^2 + h_2^2)^2, \ 1 \le k \le K, \qquad (3.19)$$

or,

$$G^N \le G^0 + \frac{1}{2}\tau\sum_{k=1}^{N}(G^k + G^{k-1}) + 2T\bar{c}(\tau^{3/2} + h_1^2 + h_2^2)^2, \quad 1 \le N \le K.$$

When $\tau \le 2/9$, Gronwall inequality gives

$$G^k \le \frac{9}{8}e^{\frac{9}{8}k\tau}\left[G^0 + 2T\bar{c}(\tau^{3/2} + h_1^2 + h_2^2)^2\right], \quad 1 \le k \le K,$$

or,

$$\|\tilde{u}^k\|^2 + \frac{1}{2}\tau\sum_{l=1}^{k}\left(\|\delta_x\tilde{u}^{l-\frac{1}{2}}\|^2 + \|\delta_y\tilde{u}^{l-\frac{1}{2}}\|^2\right) \le \frac{9}{4}T\bar{c}e^{\frac{9}{8}T}(\tau^{3/2} + h_1^2 + h_2^2)^2, \quad 1 \le k \le K.$$

This completes the proof.

4 Numerical example

In order to demonstrate the effectiveness of the finite difference scheme (2.1)-(2.4) using artificial boundary conditions, we compute the following problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x < +\infty, 0 < y < 1, 0 < t \le 0.1, \qquad (4.1)$$

$$u(x, y, 0) = \phi(x, y), 0 \le x < \infty, 0 \le y \le 1, \qquad (4.2)$$

$$u|_{x=0} = 0, \quad u|_{y=0} = 0, \quad u|_{y=1} = 0, \quad 0 \le t \le 0.1, \qquad (4.3)$$

$$u \to 0, \text{ when } x \to +\infty. \qquad (4.4)$$

where

$$\phi(x, y) = \begin{cases} \sin(\pi x)\sin(\pi y), & 0 \le x \le 1 \\ 0, \quad x > 1 \end{cases} \qquad (4.5)$$

satisfies the requirement for the artificial boundary condition (compact support). Using the method of separation of variables, the solution of the above problem can be expressed as

$$u(x, y, t) = \sum_{n=1}^{\infty} v_n(x, t)\sin(n\pi y),$$

where $v_n(x, t)$ satisfies

$$\frac{\partial v_n}{\partial t} = \frac{\partial^2 v_n}{\partial x^2} - n^2\pi^2 v_n,$$

$$v_n(0, t) = 0,$$

$$v_n(x, 0) = 2\int_0^1 u(x, \eta, 0)\sin(n\pi\eta)d\eta = 2\int_0^1 \phi(x, \eta)\sin(n\pi\eta)d\eta,$$

$$v_n \to 0, \text{ as } x \to +\infty.$$

Let

$$w_n(x,t) = e^{n^2\pi^2 t} v_n(x,t)$$

then $w_n(x,t)$ satisfies

$$\frac{\partial w_n}{\partial t} = \frac{\partial^2 w_n}{\partial x^2},$$
$$w_n(0,t) = 0,$$
$$w_n(x,0) = 2\int_0^1 \phi(x,\eta)\sin(n\pi\eta)d\eta,$$
$$w_n \to 0, \text{ as } x \to \infty.$$

The solution of this problem is

$$\begin{aligned} w_n(x,t) &= \frac{1}{\sqrt{4\pi t}}\int_0^\infty w_n(\xi,0)\left[e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right]d\xi \\ &= \frac{2}{\sqrt{4\pi t}}\int_0^\infty \left[\int_0^1 \phi(\xi,\eta)\sin(n\pi\eta)d\eta\right]\cdot\left[e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right]d\xi \end{aligned}$$

Thus the solution of (4.1)-(4.4) becomes:

$$\begin{aligned} u(x,y,t) &= \frac{2}{\sqrt{4\pi t}}\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\sin(n\pi y)\int_0^\infty\left[\int_0^1 \varphi(\xi,\eta)\sin(n\pi\eta)d\eta\right]\left[e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right]d\xi \\ &= \frac{1}{\sqrt{4\pi t}}e^{-\pi^2 t}\sin(\pi y)\int_0^1 \sin(\pi\xi)\left[e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right]d\xi. \end{aligned}$$

Table 1 Some numerical results at points $(x_i, 0.5, 0.1)$ with $h = 1/50, \tau = 1/500, b = 1$

x_i	u	$\lvert u - u_h\rvert$	$\lvert u - u_{h/2}\rvert$	$\lvert u - u_{h/4}\rvert$
0.00	0.00000D+0	0.000D+0	0.000D+0	0.000D+0
0.08	0.36528D-1	0.565D-4	0.131D-4	0.157D-6
0.16	0.71204D-1	0.109D-3	0.254D-4	0.211D-6
0.24	0.10232D+0	0.154D-3	0.359D-4	0.191D-6
0.32	0.12844D+0	0.189D-3	0.441D-4	0.137D-6
0.40	0.14850D+0	0.211D-3	0.493D-4	0.875D-7
0.48	0.16189D+0	0.219D-3	0.515D-4	0.670D-7
0.56	0.16846D+0	0.215D-3	0.507D-4	0.817D-7
0.64	0.16854D+0	0.199D-3	0.473D-4	0.117D-6
0.72	0.16281D+0	0.174D-3	0.419D-4	0.139D-6
0.80	0.15232D+0	0.145D-3	0.352D-4	0.109D-6
0.88	0.13825D+0	0.154D-3	0.279D-4	0.473D-8
0.96	0.12189D+0	0.365D-2	0.526D-3	0.231D-5
1.00	0.11325D+0	0.563D-2	0.281D-2	0.140D-2

Table 2 The errors in the norm $\|\cdot\|$ with $b = 1$

h	τ	$\|\tilde{u}^K(h,\tau)\|$	$\|\tilde{u}^K(h,\tau)\|/\|\tilde{u}^K(\frac{h}{2},\frac{\tau}{2})\|$
1/25	1/250	0.240942D-2	2.79
1/50	1/500	0.864842D-3	2.81
1/100	1/1000	0.307893D-3	2.81
1/200	1/2000	0.109264D-3	

Take the artificial boundary $b = 1$. Table 1 gives some numerical results at $y = 0.5$ and $t = 0.1$. In this table, $u(x_i)$ stands for $u(x_i, 0.5, 0.1)$, $u_h(x_i)$ or $u_h(x_i, 0.5, 0.1)$ represents the numerical solution at the point $(x_i, 0.5, 0.1)$ by the difference scheme (2.1)-(2.4) and $err_h(x_i) = |u(x_i, 0.5, 0.1) - u_h(x_i, 0.5, 0.1)|$ with $h = 1/50, \tau = 1/500$. The global errors in norm are shown in Table 2, where $\|\tilde{u}^K\|$ is defined in Theorem 4.

Table 3 and Table 4 present the corresponding results for the artificial boundary $b = 2$. The accuracy of the difference solutions is almost same by taking the artificial boundary $b = 1$ and by taking the artificial boundary $b = 2$. This is because the artificial boundary conditions are exact.

Table 3 Some numerical results at points $(x_i, 0.5, 0.1)$ with $h = 1/50, \tau = 1/500, b = 2$

x_i	u	$\|u - u_h\|$	$\|u - u_{h/2}\|$	$\|u - u_{h/4}\|$
0.00	0.00000D+0	0.000D+0	0.000D+0	0.000D+0
0.16	0.71204D-1	0.113D-3	0.268D-4	0.319D-6
0.32	0.12844D+0	0.197D-3	0.467D-4	0.813D-6
0.48	0.16189D+0	0.230D-3	0.546D-4	0.104D-5
0.64	0.16854D+0	0.209D-3	0.498D-4	0.726D-6
0.80	0.15232D+0	0.153D-3	0.363D-4	0.439D-7
0.96	0.12189D+0	0.202D-2	0.305D-3	0.189D-5
1.00	0.11325D+0	0.288D-2	0.142D-2	0.700D-3
1.04	0.10448D+0	0.199D-2	0.299D-3	0.193D-5
1.20	0.70669D-1	0.267D-4	0.625D-5	0.107D-6
1.36	0.42859D-1	0.146D-4	0.364D-5	0.707D-6
1.52	0.23310D-1	0.156D-4	0.397D-5	0.138D-5
1.68	0.11360D-1	0.175D-4	0.441D-5	0.155D-5
1.84	0.49549D-2	0.151D-4	0.365D-5	0.119D-5
2.00	0.19314D-2	0.864D-5	0.185D-5	0.514D-6

Table 4 The errors in the norm $\|\cdot\|$ with $b = 2$

h	τ	$\|\tilde{u}^K(h,\tau)\|$	$\|\tilde{u}^K(h,\tau)\|/\|\tilde{u}^K(\frac{h}{2},\frac{\tau}{2})\|$
1/25	1/250	0.167154D-2	2.89
1/50	1/500	0.577801D-3	2.86
1/100	1/1000	0.201773D-3	2.86
1/200	1/2000	0.706137D-4	

5 Conclusions

An exact artificial boundary condition is applied to reduce the unbounded domain problem into a finite domain problem. The method of reduction of order is then used to derive a difference scheme for the reduced problem. It is proved that the difference scheme is unconditionally stable and convergent with the convergence order 2 in space and order 3/2 in time. The numerical example verifies the convergence rate.

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