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A neural network for a class of convex quadratic minimax problems with constraints*

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Abstract—This paper presents a neural network model for solving a class of convex quadratic minimax problems with constraints. The structure of the proposing network is simpler than the corresponding ones in the literature. Four sufficient conditions are provided to ensure the asymptotic stability of the proposing network. The exponential stability of the new neural network is also shown under certain conditions. These new results are further extended to the globally projected dynamical system with asymmetric connection weights. Some new sufficient conditions ensuring the asymptotic stability of this dynamical system are also obtained. The validity and transient behavior of the proposing neural network are demonstrated by some simulation results.

Index Terms—Minimax problem, saddle point, neural network, convergence and stability, exponential stability.

1. Introduction

Consider the following quadratic function

$$f(x,y) = h^T x - s^T y + \frac{1}{2}x^T Hx - x^T Qy - \frac{1}{2}y^T Sy \tag{1}$$

where $H \in R^{m\times m}$ and $S \in R^{n\times n}$, $Q \in R^{m\times n}$, $h \in R^m$, $s \in R^n$ are given with H and S being symmetric and positive semi-definite, $U = \{x \in R^m \mid a_i \le x_i \le b_i \text{ for } i = 1,2,\cdots,m\}$, $V = \{y \in R^n \mid c_j \le y_j \le d_j \text{ for } j = 1,2,\cdots,n\}$, and some $-a_i, -c_j$ (or b_i, d_j) could be $+\infty$. Then we have the following minimax problem:

$$\min_{x\in U}\{\max_{y\in V}\{f(x,y)\}\}. \tag{2}$$

Minimax problems provide a useful reformulation of optimality conditions and also arise in a variety of engineering and economic contexts including game theory, military scheduling, automatic control, and so on. In particular, problem (2) includes

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- linear programming ($H = 0, S = 0$ and $Q = 0$);
- quadratic programming ($S = 0$, $H \neq 0$);
- linear and quadratic programming with bound constraints.

In many engineering and scientific applications, real-time on-line solutions of minimax problems are desired. However, traditional algorithms [6], [10] may not be applicable for a real-time on-line solution scheme. In recent years, the neural network approach for solving optimization problems has been studied by many researchers and many good results have been obtained [5, 8, 9, 12, 13, 14, 15, 16, 17]. In particular, Tao and Fang [12] proposed the following neural network model for (2):

$$\begin{cases} \frac{dx}{dt} = (I_m + H)(P_U(x - Hx - h + Qy) - x) - Q(P_V(y - Sy - s - Q^T x) - y), \\ \frac{dy}{dt} = (I_n + S)(P_V(y - Sy - s - Q^T x) - y) + Q^T(P_U(x - Hx - h + Qy) - x), \end{cases}$$

where I_n is the identity matrix of order n, and $P_\Omega : R^n \longrightarrow \Omega$ is a projection operator defined by $P_\Omega(u) = \arg\min_{v \in \Omega} \|u - v\|$, where Ω is a closed convex subset of R^n. This model can reach the exact saddle point of (1), and is globally and exponentially stable when H and S are positive definite. But its structure is rather complicated, and further simplification can be achieved.

In order to simplify the above network structure, a neural network model for solving problem (2) is proposed by means of the conditions of the saddle point of (1) in this paper. Four sufficient conditions which guarantee the asymptotic stability of the proposing network are given, and these results are further extended to the globally projected dynamical system with asymmetric connection weights. Furthermore, the exponential stability of the proposing network is also studied.

The solution of problem (2) is closely related to the saddle points of $f(x, y)$. A point $(x^*, y^*) \in U \times V$ is said to be a saddle point of $f(x, y)$ if

$$f(x^*, y) \leq f(x^*, y^*) \leq f(x, y^*), \qquad \forall (x, y) \in U \times V. \tag{3}$$

Throughout this paper, we assume that the set $\Omega^* = \{(x, y) \in U \times V \mid (x, y) \text{ is a saddle point of } f(x, y)\} \neq \emptyset$ and there exists a finite $z^* \in \Omega^*$. Obviously, if $(x^*, y^*) \in \Omega^*$, then (x^*, y^*) must be a solution of problem (2). Therefore, it would be sufficient to find a point in Ω^* for problem (2).

The following notation will be used in the later discussion. We let $\|\cdot\|$ denote the Euclidean norm, and $(a)_b^c = \min\{c, \max\{a, b\}\}$, where $c \geq b$. For any $n \times n$ symmetric matrix M, $\lambda_{\min}(M)$ denotes its minimum eigenvalue. A basic property of the projection mapping on a closed convex set is [7]

$$(u - P_\Omega(u))^T(P_\Omega(u) - v) \geq 0, \qquad \forall u \in R^n, v \in \Omega. \tag{4}$$

The rest of the paper is organized as follows. In Section 2, a neural network model for solving problem (2) is constructed. Section 3 analyzes the stability and convergence of the proposing network. Illustrative examples and numerical simulations are provided in Section 4. Finally some concluding remarks are drawn in Section 5.

2. A neural network model

First, we prove the following results, which builds the theoretical foundation in constructing a network for problem (2).

Theorem 1. $(x^*, y^*) \in \Omega^*$ if and only if

$$\begin{cases} (x - x^*)^T(Hx^* + h - Qy^*) \geq 0, & \forall x \in U, \\ (y - y^*)^T(Sy^* + s + Q^T x^*) \geq 0, & \forall y \in V. \end{cases} \tag{5}$$

Proof. According to [1], we know that $(x^*, y^*) \in \Omega^* \Longleftrightarrow$ (3) holds $\Longleftrightarrow$ x^* is a global minimizer of function $f(x, y^*)$ with respective to U and y^* is the global maximizer of function $f(x^*, y)$ with respective to $V \Longleftrightarrow$ (5) holds. This completes the proof. ∎

From (4) and Theorem 1, we have the following result.

Lemma 1. $(x^*, y^*) \in \Omega^*$ if and only if

$$\begin{cases} x^* = P_U[x^* - \alpha(Hx^* + h - Qy^*)], \\ y^* = P_V[y^* - \alpha(Sy^* + s + Q^T x^*)], \end{cases} \tag{6}$$

where $\alpha > 0$ is a constant, $P_U(x) = [(x_1)_{a_1}^{b_1}, (x_2)_{a_2}^{b_2}, \cdots, (x_m)_{a_m}^{b_m}]^T \in R^m$ and $P_V(y) = [(y_1)_{c_1}^{d_1}, (y_2)_{c_2}^{d_2}, \cdots, (y_n)_{c_n}^{d_n}]^T \in R^n$.

The result of Lemma 1 indicates that x^* is the projection of some vector on U and y^* is the projection of some vector on V. Let $z = (x^T, y^T)^T \in R^{m+n}$, then Lemma 1 suggests the following dynamical system for a neural network model:

$$\frac{dz}{dt} = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = -F(z) = -\begin{pmatrix} x - P_U[x - \alpha(Hx + h - Qy)] \\ y - P_V[y - \alpha(Sy + s + Q^T x)] \end{pmatrix}. \tag{7}$$

It should be noted that model (7) is a special case of the model in [17], thus it can be simply implemented by using simple hardware units such as adders, amplifies, integrators and some activation functions in [2, 13]. Obviously, the model in (7) consists of $m + n$ neurons and one hidden layer, but the model in [12] consists of $m + n$ neurons and two hidden layers.

In terms of model complexity, it is easy to see that the total numbers of multiplications/divisions and additions/subtractions per iteration for (7) are $(m+n)^2 + m + n$ and $(m+n)^2 + 2(m+n)$, respectively. While for the model of [12], the total numbers of multiplications/divisions and additions/subtractions per iteration are $2(m+n)^2$ and $2(m+n)^2 + 2(m+n)$, respectively. Thus the asymptotic complexity of our model (7) is about half of the model in [12].

From Lemma 1, we have the following result.

Corollary 1. Assume that $E = \{z \in R^{n+m} \mid F(z) = 0\}$ is the set of equilibrium points of network (7). Then $E = \Omega^*$.

3. Stability analysis

For simplicity, in the rest of this paper, we denote $\tilde{x} = P_U[x - \alpha(Hx + h - Qy)]$ and $\tilde{y} = P_V[y - \alpha(Sy + s + Q^T x)]$. First, we prove the following theorem which guarantees the existence and uniqueness for the solution $z(t)$ of network (7).

Theorem 2. $\forall z^0 \in R^{m+n}$, there exists a unique solution $z(t)$ of (7) for all $t \geq 0$ with $z(0) = z^0$.

Proof. See the appendix. ▮

The results in Theorem 2 and Corollary 1 show that network (7) is well defined. Furthermore, the following Lemma 2 ensures that the solution $z(t)$ of (7) will eventually stay in $U \times V$ as t becomes large enough.

Lemma 2. The solution $z(t)$ of (7) will approach to the feasible set $U \times V$ globally and exponentially when the initial point $z^0 \notin U \times V$. Moreover $z(t) \subset U \times V$ when $z^0 \in U \times V$.

Proof. From the proof of Theorem 2, we know that $F(z)$ in (7) is Lipschitz continuous on R^{m+n}. From Theorem 3.2 in [15], our results can be established. ▮

In order to analyze the convergence of the solution $z(t)$ of (7), we define the following function for any $z \in U \times V$,

$$\begin{aligned} V(z) &= \alpha(x - \tilde{x})^T (Hx + h - Qy) - \frac{1}{2}\|x - \tilde{x}\|^2 + \alpha(y - \tilde{y})^T (Sy + s + Q^T x) \\ &\quad - \frac{1}{2}\|y - \tilde{y}\|^2 + \frac{\beta}{2}\|z - z^*\|^2, \end{aligned} \tag{8}$$

where $z^* \in \Omega^*$ is finite, α is given in (6), and β is a nonnegative constant. Obviously, if $\beta = 1$, $V(z)$ is just the Lyapunov function introduced by Xia *et al.* [17]. But their $V(z)$ ($\beta = 1$) can not be used to prove the exponential convergence for (7). As we will show later that our $V(z)$ in (8) is actually a Lyapunov function for certain β, but for now we first explore some properties for $V(z)$.

Lemma 3. Let $V(z)$ be the function in (8). Then the followings are true.

(i) $V(z) \geq \frac{\beta}{2}\|z - z^*\|^2$ for all $z \in U \times V$.

(ii) $\nabla V(z)^T F(z) \geq 0$ for $0 \leq \beta \leq 1$ and all $z \in U \times V$.

(iii) Let $\lambda = \min\{\lambda_{\min}(H), \lambda_{\min}(S)\}$ and $0 \leq \beta \leq 1$, then

$$\nabla V(z)^T F(z) \geq \gamma V(z), \qquad \forall z \in U \times V, \tag{9}$$

where $\gamma = \min\{1 - \beta, 2\alpha\lambda\} \geq 0$.

Proof. See the appendix. ▮

The results in Lemma 3 pave a way for the following stability results of neural network (7).

Theorem 3. If one of the following conditions is satisfied:

(i) at least one of the matrices H and S is positive definite;

(ii) $\forall z' = (x'^T, y'^T)^T, z \in U \times V$, if $Hx = Hx'$ and $Sy = Sy'$, then $D^T x = D^T x'$ and $Dy = Dy'$;

(iii) let $z^* = ((x^*)^T, (y^*)^T)^T \in \Omega^*$ and is finite, $\forall z \in U \times V$, if $Hx = H\tilde{x}$, $Sy = S\tilde{y}$ and

$$(z - z^*)^T \begin{pmatrix} Hx + h - Qy \\ Sy + s + Q^T x \end{pmatrix} = 0,$$

then $z \in \Omega^*$;

(iv) let $z^* = ((x^*)^T, (y^*)^T)^T \in \Omega^*$ and is finite, $\forall z \in U \times V$, if

$$(z - z^*)^T \begin{pmatrix} Hx + h - Qy \\ Sy + s + Q^T x \end{pmatrix} = 0,$$

then $z \in \Omega^*$;

then neural network (7) is Lyapunov stable, and the trajectory $z(t)$ of (7) with $z(0) = z^0 \in U \times V$ will converge to a saddle point of $f(x, y)$. In particular, if problem (2) has a unique solution, then neural network (7) is globally and asymptotically stable.

Proof. See the appendix. ∎

Theorem 4. Assume that H and S are positive definite. Then neural network (7) with initial point $z^0 \in U \times V$ is globally and exponentially stable.

Proof. See the appendix. ∎

Since neural network (7) is a special case of the following globally projected dynamical system proposed by Friesz *et al.* [3]

$$\frac{du}{dt} = \lambda\{P_\Omega[u - \alpha(Nu + q)] - u\} \tag{10}$$

(λ and α are positive constants, Ω is a closed convex subset of R^n, N is an $n \times n$ matrix and $q \in R^n$), and it has been conjectured that (10) is a very useful computational paradigm for solving finite-dimensional variational inequalities [15], then it would be of both theoretical and application interests to extend the results of (7) to (10). Similar to the analysis for Theorem 3 and Theorem 4, we have the following two corollaries.

Corollary 2. If one of the following conditions is satisfied:

(i) let

$$N = \begin{pmatrix} N_1 & -N_3 \\ N_3^T & N_2 \end{pmatrix},$$

where $N_1 \in R^{n_1 \times n_1}$, $N_2 \in R^{n_2 \times n_2}$, $N_3 \in R^{n_1 \times n_2}$, $n_1 + n_2 = n$, N_2 is symmetric and positive semi-definite, and N_1 is positive definite but not necessarily symmetric (or N_1 is symmetric and positive semi-definite, and N_2 is positive definite but not necessarily symmetric);

(ii) $\forall u, v \in \Omega$, if $(u-v)^T N(u-v) = 0$, then $Nu = Nv$;

(iii) $\forall u \in \Omega$, if $(N^T+N)(u-\tilde{u}) = 0$ and $(u-u^*)(Nu+q) = 0$, then u is an equilibrium point of (10), where $u^* \in \Omega$ is a finite equilibrium point of (10), and $\tilde{u} = P_\Omega[u-\alpha(Nu+q)]$;

(iv) $\forall u \in \Omega$, if $(u-u^*)(Nu+q) = 0$, then u is an equilibrium point of (10), where $u^* \in \Omega$ is a finite equilibrium point of (10);

then system (10) is Lyapunov stable, and the trajectory $u(t)$ of (10) with $u(0) = u^0 \in \Omega$ will converge to one of its equilibrium points. In particular, if system (10) has a unique equilibrium point, then system (10) is globally and asymptotically stable.

Corollary 3. If N is positive definite, then system (10) with initial point $u^0 \in \Omega$ is globally and exponentially stable.

It is worth of mentioning that Example 3.1 in [15] does not satisfy the condition of Corollary 2. The unique equilibrium point of Example 3.1 was shown to be unstable in [15].

4. Illustrative examples

In this section, 5 examples will be provided to illustrate both the theoretical results achieved in Section 3 and the simulation performance of these dynamical systems.

Example 1. Consider the following convex quadratic programming problems [14]:

$$\begin{cases} \min & \frac{1}{2}x^T Hx + h^T x \\ \text{s.t.} & Dx = b, \qquad x \in \Omega. \end{cases} \tag{11}$$

By the saddle theorem of convex program [1], Example 1 is equivalent to

$$\min_{x\in\Omega}\max_{y\in R^n} \left(\frac{1}{2}x^T Hx + h^T x - y^T(Dx-b)\right).$$

Then neural network (7) can be used to solve problem (11) when one of the conditions in Theorem 3 is satisfied.

Example 2. Consider the following linear complementarity problem:

$$x \geq 0, \qquad Nx + q \geq 0, \qquad x^T(Nx+q) = 0, \tag{12}$$

where $N \in R^{n\times n}$ and $q \in R^n$. Obviously, problem (12) can be solved by system (10). Taking an example with

$$N = \begin{pmatrix} 4 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 \\ 3 & 0 & 2 & 2 \\ -1 & -1 & -2 & 0 \end{pmatrix} \qquad \text{and} \qquad q = \begin{pmatrix} -8 \\ -6 \\ -4 \\ 3 \end{pmatrix},$$

its solution is $x^* = (14/15,\ 11/9,\ 19/45,\ 8/45)^T$. Let $n_1 = 3, n_2 = 1$, then this problem satisfies Corollary 2 (i).

For $\alpha = 1$, Figures 2(a) and 2(b) depict the trajectories of neural network (10) with different initial points $(2, 0.5, 1, 2.5)^T$ and $(0.5, 0.9, 0, -0.2)^T$ converging to x^*.

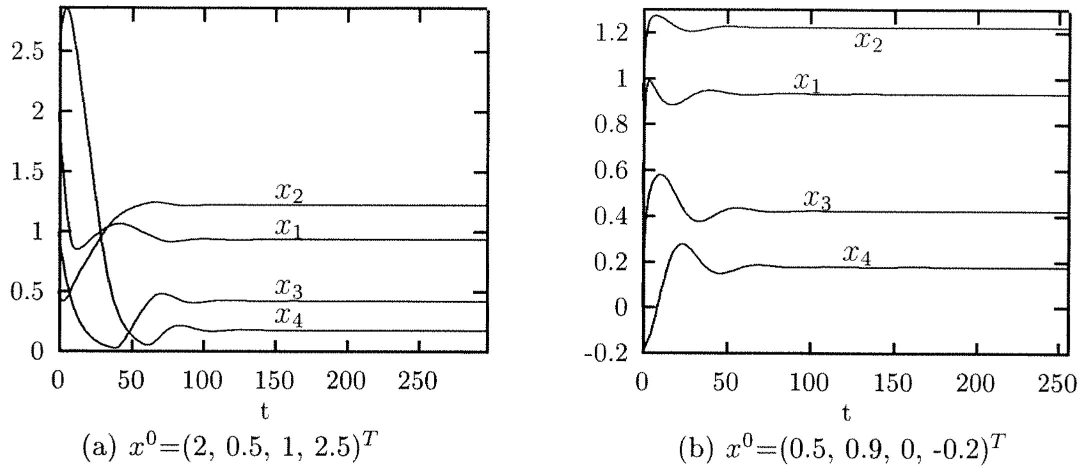


Figure 1. Transient behavior of network (10) for Example 2

Example 3. Consider problem (2) with $U = V = \{x \in R^2 \mid x \geq 0\}$ and $f(x,y)=(x_1 - x_2)^2 - (x_1 - x_2)(y_1 - y_2) - (y_1 - y_2)^2$. This problem has infinitely many saddle points $x^* = (r,r)^T$ $(r \geq 0)$ and $y^* = (w,w)^T$ $(w \geq 0)$. It is easy to verify that this problem satisfies Theorem 3 (ii).

For $\alpha = 1$, Figures 3(a) and 3(b) depict the trajectories of neural network (7) with different initial points $(3, 1.8, 1.6, 0)^T$ and $(0.2, 1.2, 0.6, 1.2)^T$ converging to solutions $((x^*)^T, (y^*)^T)$ = (2.4, 2.4, 1.0516, 1.0516) and $((x^*)^T, (y^*)^T)$=(0.7, 0.7, 1.192, 1.192), respectively.

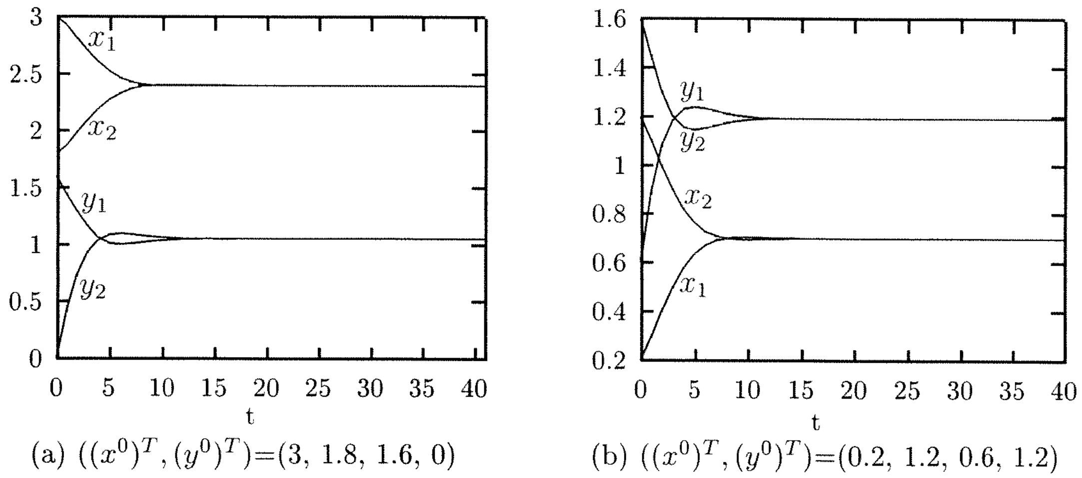
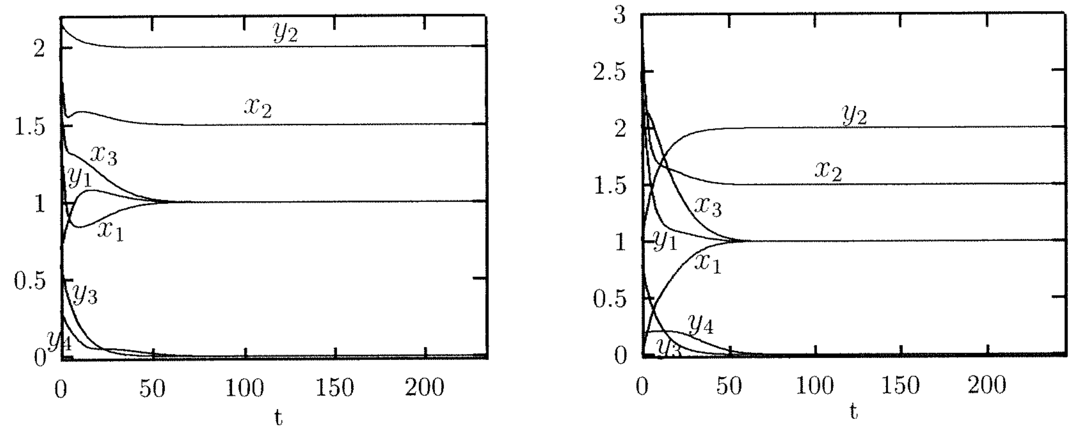


Figure 2. Transient behavior of network (7) for Example 3

Example 4. Consider problem (2) with $U = \{x \in R^3 \mid 0 \leq x \leq 3\}$, $V = \{y \in R^4 \mid 0 \leq y \leq 2\}$ and $f(x,y) = -8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 - y_1^2 - 0.5y_2^2 - y_3^2 - 0.5y_4^2 + y_1y_2 - y_3y_4 + y_1 + 3y_2 - y_3 + y_4 - x_1y_1 - x_2y_2 - x_3y_3 - x_1y_4$. Its solution is x^* $=(1, 1.5, 1)^T$, $y^*= (1, 2, 0, 0)^T$. It is easy to show that both H and S are positive definite. Therefore, the condition of Theorem 4 is satisfied.

For $\alpha = 1$, Figures 4(a) and 4(b) depict the trajectories of neural network (7) with different initial points $(1.4,\ 2,\ 1.7,\ 0.7,\ 2.2,\ 0.6,\ 0.3)^T$ and $(0,\ 3,\ 2,\ 2.5,\ 1,\ 0.8,\ 0.2)^T$ converging to the exact solution x^* and y^*.



(a) The first initial point (b) The second initial point

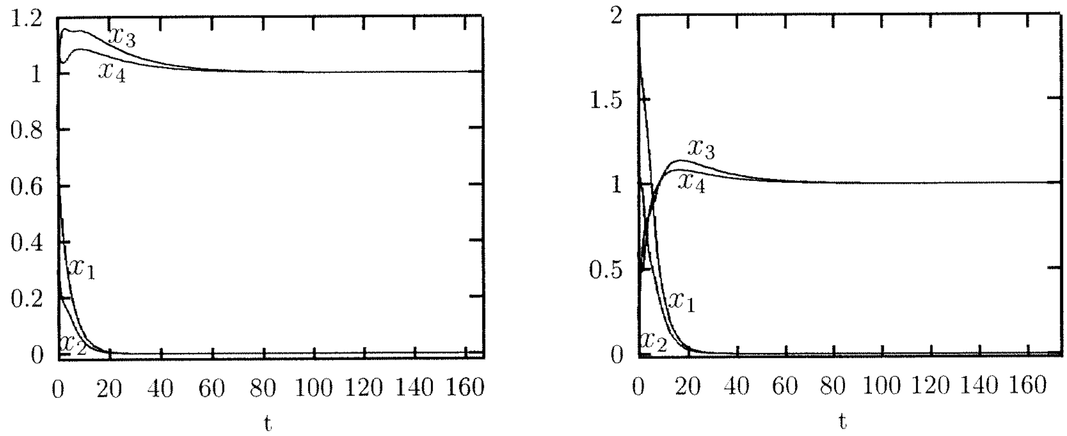
Figure 3. Transient behavior of network (7) for Example 4

Example 5. Consider the linear complementarity problem (12) with

$$N = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ -1 & 1 & 2 & -3 \\ 1 & -2 & -1 & 2 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Its solution is $x^*=(0,\ 0,\ 1,\ 1)^T$. It can be verified that this problem satisfies Corollary 2 (iii), but Corollary 2 (iv) is not satisfied.

For $\alpha = 1$, Figures 5(a) and 5(b) depict trajectories of neural network (10) with different initial points $(0.7,\ 0.3,\ 0.8,\ 1.2)^T$ and $(2,\ 1,\ 0.6,\ 0.2)^T$ converging to the exact solution x^*.



(a) The first initial point (b) The second initial point

Figure 4. Transient behavior of network (10) for Example 5

5. Concluding remarks

In this paper, we have proposed a neural network model for solving problem (2). Four sufficient conditions are provided to guarantee the asymptotical stability of the proposing neural network. Furthermore, the proposing network is shown to be exponentially stable under certain conditions. Moreover, these results are extended to the globally projected dynamical system proposed by Friesz et al. [3] with asymmetric connection weights. Four new sufficient conditions for this dynamical system to be asymptotically stable are given. Since the conditions in this paper can be checked easily in practice, these new results have both theoretical and application values.

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The appendix

Proof of Theorem 2. Since the projection mappings $P_U(\cdot)$ and $P_V(\cdot)$ are nonexpansive, for any $z, z' \in R^{m+n}$ we have

$$\begin{aligned} \|x - \tilde{x} - (x' - \tilde{x'})\| &\leq \|x - x' - \alpha[H(x - x') - Q(y - y')]\| + \|x - x'\| \\ &\leq (1 + \|I_m - \alpha H\|)\|x - x'\| + \alpha\|Q\| \cdot \|y - y'\| \end{aligned}$$

and

$$\|y - \tilde{y} - (y' - \tilde{y'})\| \leq (1 + \|I_n - \alpha S\|)\|y - y'\| + \alpha\|Q\| \cdot \|x - x'\|.$$

From the above two inequalities and

$$\|F(z) - F(z')\| \leq \|x - \tilde{x} - (x' - \tilde{x'})\| + \|y - \tilde{y} - (y' - \tilde{y'})\|, \quad \forall z, z' \in R^{m+n},$$

we can see that $F(z)$ is Lipschitz continuous on R^{m+n}. Then from Theorem 1 in [5], the result can be established. ∎

Proof of Lemma 3. Similar to the proof in [17], (i) can be established.

(ii) Similar to the proof in [17], we have

$$\begin{aligned}\nabla V(z)^T F(z) \geq\ & \alpha(x-\tilde{x})^T H(x-\tilde{x}) + \alpha(y-\tilde{y})^T S(y-\tilde{y}) \\ & +\alpha\beta[(x-x^*)^T H(x-x^*) + (x-x^*)^T(Hx^* + h - Qy^*) \\ & +(y-y^*)^T S(y-y^*) + (y-y^*)^T(Sy^* + s + Q^T x^*)]. \end{aligned} \tag{13}$$

Since $(x,y) \in U \times V$ and the matrices H and S are positive semi-definite, (5) and (13) imply (ii).

(iii) Obviously, if $\lambda = 0$ then $\gamma = 0$ and (9) holds from (ii).

Suppose $\lambda > 0$, i.e., H and S are positive definite, then similar to the proof in [17], we have

$$\begin{aligned}\nabla V(z)^T F(z) \geq\ & \alpha(x-\tilde{x})^T H(x-\tilde{x}) + \alpha(y-\tilde{y})^T S(y-\tilde{y}) \\ & +(1-\beta)[\alpha(x-\tilde{x})^T(Hx+h-Qy) - \|x-\tilde{x}\|^2] \\ & +(1-\beta)[\alpha(y-\tilde{y})^T(Sy+s+Q^T x) - \|y-\tilde{y}\|^2] \\ & +\alpha\beta[(x-x^*)^T H(x-x^*) + (y-y^*)^T S(y-y^*)] \\ \geq\ & (1-\beta)[\alpha(x-\tilde{x})^T(Hx+h-Qy) - \frac{1}{2}\|x-\tilde{x}\|^2] \\ & +(1-\beta)[\alpha(y-\tilde{y})^T(Sy+s+Q^T x) - \frac{1}{2}\|y-\tilde{y}\|^2] \\ & +[\alpha\lambda_{\min}(H) - \frac{1-\beta}{2}]\|x-\tilde{x}\|^2 + [\alpha\lambda_{\min}(S) - \frac{1-\beta}{2}]\|y-\tilde{y}\|^2 \\ & +\alpha\beta[\lambda_{\min}(H)\|x-x^*\|^2 + \lambda_{\min}(S)\|y-y^*\|^2] \\ \geq\ & (1-\beta)[\alpha(x-\tilde{x})^T(Hx+h-Qy) - \frac{1}{2}\|x-\tilde{x}\|^2] \\ & +(1-\beta)[\alpha(y-\tilde{y})^T(Sy+s+Q^T x) - \frac{1}{2}\|y-\tilde{y}\|^2] \\ & +\alpha\beta\lambda\|z-z^*\|^2 + [\alpha\lambda - \frac{1-\beta}{2}][\|x-\tilde{x}\|^2 + \|y-\tilde{y}\|^2] \\ \geq\ & \gamma V(z). \end{aligned}$$

This completes the proof. ∎

Proof of Theorem 3. From Theorem 2, $\forall z^0 \in R^{m+n}$, let $z(t)$ be the unique and continuous solution of (7) with $z(0) = z^0$ over $[0, +\infty)$. Now, we fix β with $0 < \beta \leq 1$ in (8), then from Lemma 3 (i) and (ii), we know that the function $V(z)$ defined in (8) is nonnegative for all $z \in U \times V$, and that

$$\frac{d}{dt} V(z(t)) = -\nabla V(z(t))^T F(z(t)) \leq 0, \qquad \forall t \geq 0. \tag{14}$$

Inequality (14) indicates that along the trajectory $z(t)$ of (7), the function $V(z)$ is nonincreasing. Therefore, from Lemma 3, we can see that function $V(z)$ in (8) is a Lyapunov function. Thus neural network (7) is Lyapunov stable from Theorem 3.2 in [11].

From Lemma 3 (i), it is easy to see that set $\Omega_0 = \{z \in U \times V \mid V(z) \leq V(z^0)\}$ is bounded and $\{z(t) | t \geq 0\} \subseteq \Omega_0$. Thus there exist a limit point $\bar{z}$ and a sequence $\{t_n\}$ with

$0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots$ and $t_n \longrightarrow +\infty$ such that

$$\lim_{n \longrightarrow +\infty} z(t_n) = \bar{z}. \tag{15}$$

(15) indicates that $\bar{z} \in U \times V$ is an ω-limit point of $\{z(t) \mid t \geq 0\}$. From Theorem 3.4 (Local Invariant Set Theorem) in [11], the solution $z(t)$ of (7) converges to M as $t \to \infty$, where M is the largest invariant set in $R = \{z \in \Omega_0 \mid \frac{dV(z)}{dt} = 0\}$. Therefore $\bar{z} \in M$.

(i) Suppose that H is positive definite. If $dV(z)/dt = 0$, then $\nabla V(z)^T F(z) = 0$. From (13) and (5), it follows that

$$\begin{cases} (x - \tilde{x})^T H(x - \tilde{x}) = (x - x^*)^T H(x - x^*) = 0, \\ (y - \tilde{y})^T S(y - \tilde{y}) = (y - y^*)^T S(y - y^*) = 0, \\ (x - x^*)^T (Hx^* + h - Qy^*) = (y - y^*)^T (Sy^* + s + Q^T x^*) = 0. \end{cases} \tag{16}$$

Since H is positive definite, and S is positive semi-definite, the above equations imply

$$x = \tilde{x} = x^*, \qquad Sy = Sy^* = S\tilde{y}.$$

Thus $\forall v \in V$, from Theorem 1, we have

$$\begin{aligned} (v - y)^T (Sy + s + Q^T x) &= (v - y)^T (Sy^* + s + Q^T x^*) \\ &= [(v - y^*) - (y - y^*)]^T (Sy^* + s + Q^T x^*) \\ &= (v - y^*)^T (Sy^* + s + Q^T x^*) \geq 0. \end{aligned}$$

On the other hand, $\forall w \in U$, by setting $u = x - \alpha(Hx + h - Qy)$, $v = w \in U$ in (4), and $x = \tilde{x}$, we have

$$(w - x)^T (Hx + h - Qy) \geq 0.$$

The above two inequalities and Theorem 1 indicate that $(x, y) \in \Omega^*$. So $dz/dt = 0$ from Lemma 1 and (7).

Conversely, if $dz/dt = 0$, then $dV(z)/dt = 0$. Therefore, $dz/dt = 0 \iff dV(z)/dt = 0$. From this result, Lemma 1 and (7), we know that $R \subseteq \Omega^*$. So $\bar{z} \in \Omega^*$.

By replacing z^* with $\bar{z}$ in $V(z)$, we let

$$\begin{aligned} \bar{V}(z) &= \alpha(x - \tilde{x})^T (Hx + h - Qy) - \frac{1}{2}\|x - \tilde{x}\|^2 + \alpha(y - \tilde{y})^T (Sy + s + Q^T x) \\ &\quad - \frac{1}{2}\|y - \tilde{y}\|^2 + \frac{\beta}{2}\|z - \bar{z}\|^2, \end{aligned}$$

then similar to the proofs of Lemma 3 (i) and (ii), we can conclude that $\bar{V}(z) \geq \frac{\beta}{2}\|z - \bar{z}\|^2$ for $z \in U \times V$ and $\bar{V}(z(t))$ is monotone decreasing on $[0, +\infty)$. From the continuity of the function $\bar{V}(z)$, it follows that $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$\bar{V}(z) < \frac{\beta\varepsilon^2}{2}, \qquad \text{if } \|z - \bar{z}\| \leq \delta. \tag{17}$$

From (15), (17) and the monotonicity of $\bar{V}(z(t))$, there exists a natural number N such that

$$\|z(t) - \bar{z}\|^2 \leq \frac{2}{\beta}\bar{V}(z(t)) \leq \frac{2}{\beta}\bar{V}(z(t_N)) < \varepsilon^2, \quad \text{when } t \geq t_N.$$

That is, $\lim_{t \longrightarrow +\infty} z(t) = \bar{z}$. This indicates that the solution $z(t)$ of (7) converges to a point in Ω^*, i.e. the solution $z(t)$ of (7) converges to a saddle point of $f(x, y)$.

Similarly, we can prove the solution $z(t)$ of (7) converges to a point in Ω^* when S is positive definite, i.e. the solution $z(t)$ of (7) converges to a saddle point of $f(x, y)$.

(ii) If $dV(z)/dt = 0$, then (16) holds. Since H and S are symmetric and positive semi-definite, the above equations imply

$$Hx = Hx^*, \qquad Sy = Sy^*.$$

From the assumptions, we have

$$D^T x = D^T x^*, \qquad Dy = Dy^*.$$

Thus similar to the above proof for (i), we can conclude that the solution $z(t)$ of (7) converges to a point in Ω^*.

(iii) If $dV(z)/dt = 0$, then (16) holds. Thus $\forall (x, y) \in U \times V$, we have

$$(x - x^*)^T(Hx + h - Qy) + (y - y^*)^T(Sy + s + Q^T x) = 0.$$

Hence $dz/dt = 0$ by the hypothesis and Corollary 1. Conversely, if $dz/dt = 0$, then $dV(z)/dt = 0$. Following the same arguments as in (i), we can say that for each $z^0 \in U \times V$, the trajectory $z(t)$ of (7) with $z(0) = z^0$ will converge to a saddle point of $f(x, y)$.

(iv) Similar to the proof for (iii), we can say that for each $z^0 \in U \times V$, the trajectory $z(t)$ of (7) with $z(0) = z^0$ will converge to a saddle point of $f(x, y)$.

In particular, if problem (2) has a unique solution z^*, then $\Omega^* = \{z^*\}$ since $\Omega^* \neq \emptyset$, and for each $z^0 \in U \times V$, the trajectory $z(t)$ with $z(0) = z^0$ will approach to z^*. So network (7) is globally and asymptotically stable. This completes the proof. ∎

Proof of Theorem 4. Since $\Omega^* \neq \emptyset$, from the hypothesis and the proof of Theorem 3, we have $dz/dt = 0 \iff dV(z)/dt = 0$. From (9), we know that $F(z) = 0 \iff z = z^*$. Therefore, z^* is the unique saddle point for $f(x, y)$. From Theorem 2 and Lemma 2, $z(t) \subset U \times V$ is the unique solution of system (7) for all $t \geq 0$ with $z(0) = z^0 \in U \times V$.

If we fix a β such that $0 < \beta < 1$ in (8), then, $\gamma = \min\{1 - \beta, 2\alpha\lambda\} > 0$, where λ is defined in Lemma 3. From Lemma 3 (iii), we have

$$\frac{dV(z(t))}{dt} = -\nabla V(z(t))^T F(z(t)) \leq -\gamma V(z(t)), \qquad \forall t \geq 0.$$

Thus

$$V(z(t)) \leq V(z^0)e^{-\gamma t}, \qquad \forall t \geq 0.$$

From Lemma 3 (i), we have

$$\|z(t) - z^*\| \leq \sqrt{\frac{2}{\beta} V(z^0)} e^{-\frac{\gamma}{2} t}, \qquad \forall t \geq 0.$$

This completes the proof. ∎

Figure Captions

Figure 1. Transient behavior of network (10) for Example 2.

Figure 2. Transient behavior of network (7) for Example 3.

Figure 3. Transient behavior of network (7) for Example 4.

Figure 4. Transient behavior of network (10) for Example 5.