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The "source generation" procedure (SGP) proposed by Hu and Wang [Inverse Probl. **22**, 1903 (2006)] provides a new way to systematically generate so-called soliton equations with self-consistent sources. In this paper, we apply this SGP to a Davey-Stewartson (DS) equation based on the Hirota bilinear form, producing a system of equations which is called the DS equation with self-consistent sources (DSESCS). Meanwhile, we obtain the Gramm-type determinant solutions to the DSESCS. Since the DS equation is a (2+1)-dimensional integrable generalization of the nonlinear Schrödinger (NLS) equation, the DSESCS may be viewed as a (2+1)-dimensional integrable generalization of the nonlinear Schrödinger equation with self-consistent sources. These results indicate the commutativity of source generation procedure and (2+1)-dimensional integrable generalizations for the NLS equation. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2830432]

I. INTRODUCTION

Many integrable nonlinear evolution partial differential equations[1–4] that describe physically important cases of nonlinear dispersive wave propagation have received considerable attention in recent years. The best known example is the Davey-Stewartson (DS) equation,[5,6] which consists of a pair of coupled nonlinear equations expressed in terms of two variables u and v in the form

$$iu_t + (\partial_x^2 + \partial_y^2)u + vu = 0,$$

$$\partial_{xy}v = 2(\partial_x^2 + \partial_y^2)|u|^2,$$

where u denotes the complex amplitude and v denotes the velocity of an underlying mean flow, in the original physical context describing short surface waves.

By introducing new variables f and g determined by

$$u = \frac{g}{f}, \quad v = 2(\partial_x^2 + \partial_y^2)\ln f,$$

we can get the DS equation in bilinear form,[7,8]

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$$(iD_t + D_x^2 + D_y^2)g\cdot f = 0, \tag{1}$$

$$D_xD_yf\cdot f = 2|g|^2, \tag{2}$$

where g and f are a complex and a real function, respectively, and D denotes the usual Hirota operator[9]

$$D_x^nD_y^m a\cdot b = \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m a(x,y)b(x',y')\Bigg|_{x'=x,y'=y}.$$

Equations (1) and (2) are an important system of evolution equations, both from the perspectives of theory and physical applications. It can arise in fluid dynamics,[5,6] plasma physics,[10] self-dual Yang-Mills fields,[11] and quantum field.[12–14] Theoretically, many techniques of nonlinear waves are relevant, e.g., Hirota bilinear form,[15] Darboux transformations,[16] symmetries,[17] rich soliton, and related structures.[18,19] As an active area of soliton theory and integrable systems, soliton equations with self-consistent sources (SESCS) have attracted much attention lately (see Refs. 20–30), mainly due to their important role in many physical areas such as hydrodynamics, solid state physics, plasma physics,[31] and so on. Their soliton solutions can describe the interactions between different solitary waves. For instance, the nonlinear Schrödinger equation with self-consistent sources (NLSESCS) has been widely studied, which cannot only describe the soliton propagation in a medium with both resonant and nonresonant nonlinearities,[32–35] but also describe the nonlinear interaction of an electrostatic high-frequency wave with ion acoustic waves in a two component homogeneous plasma.[36]

Till now, many $(2+1)$-dimension integrable system with self-consistent sources have been proposed. However, the DS equation with self-consistent sources (ESCS) has not been found yet. Recently, A new "source generation" procedure (SGP) (Ref. 37) has been proposed to study soliton ESCS (Refs. 37–39) based on bilinear forms. In this paper, we will present and solve the DSESCS through the source generation procedure.

II. GRAMMIAN SOLUTIONS OF THE DS EQUATION

The Grammian solutions of the DS equations (1) and (2) can be expressed in the form[7,8]

$$f = |m| = |c + \Phi|,$$

$$g = \begin{vmatrix} 0 & \bar{s}^t \\ r & m \end{vmatrix}, \quad \bar{g} = \begin{vmatrix} 0 & \bar{r}^t \\ s & m \end{vmatrix},$$

where c is a Hermitian matrix, the bar denotes the complex conjugate, $r=(\varphi_1,\ldots,\varphi_M;0,\ldots,0)^t, s=(0,\ldots,0;\psi_1,\ldots,\psi_M)^t$, and Φ is an $(M+N)\times(M+N)$ matrix with block structure

$$\Phi = \begin{pmatrix} \int_{-\infty}^{x}\varphi_i\bar{\varphi}_j dx & 0 \\ 0 & \int_{y}^{+\infty}\psi_k\bar{\psi}_l dy \end{pmatrix}.$$

Here each φ_i is a function of x and t, and ψ_k is a function of y and t, where $i,j\in\{1,\ldots,M\}$ and $k,l\in\{1,\ldots,N\}$. Besides, φ_i and ψ_k satisfy the linear equations

$$i\varphi_t + \varphi_{xx} = 0, \tag{3}$$

$$i\psi_t - \psi_{yy} = 0. \tag{4}$$

III. CONSTRUCTION OF THE DS EQUATION WITH SELF-CONSISTENT SOURCES

Now, we construct the DSESCS following the source generation procedure. Firstly, we change solutions of Eqs. (1) and (2) to the following form:

$$F = |A| = |C + \Phi|, \tag{5}$$

$$G = \begin{vmatrix} 0 & \vec{s}^t \\ r & A \end{vmatrix}, \quad \bar{G} = \begin{vmatrix} 0 & \vec{r}^t \\ s & A \end{vmatrix}, \tag{6}$$

where the $(M+N)\times(M+N)$ Hermitian matrix C satisfies

$$C_{kj} = \begin{cases} C_k(t), & k=j \text{ and } 1 \leqslant k \leqslant K \leqslant M+N \\ c_{kj}, & \text{otherwise}, \end{cases}$$

and Φ, r, and s are defined as before. In this regard, we introduce other new functions defined by (for $k=1,2,\ldots,K$)

$$P_k = \sqrt{\dot{C}_k(t)} \begin{vmatrix} \vec{s}^t \\ A_{k,0} \end{vmatrix}, \quad Q_k = \sqrt{\dot{C}_k(t)} |rA_{0,k}|,$$
$$\bar{P}_k = \sqrt{\dot{C}_k(t)} |sA_{0,k}|, \quad \bar{Q}_k = \sqrt{\dot{C}_k(t)} \begin{vmatrix} \vec{r}^t \\ A_{k,0} \end{vmatrix}, \tag{7}$$

where A_{kj} denotes a matrix resulting from eliminating the kth row and jth column from the matrix A, and $A_{k,0}$, $A_{0,k}$ denote matrices resulting from eliminating the kth row and jth column, respectively, from the matrix A. We can show that functions in (5)–(7) satisfy the bilinear equations

$$(iD_t + D_x^2 + D_y^2) G \cdot F = i \sum_{k=1}^{K} P_k Q_k, \tag{8}$$

$$D_y D_x F \cdot F = 2|G|^2, \tag{9}$$

$$D_x F \cdot P_k + \bar{Q}_k G = 0, \tag{10}$$

$$D_y F \cdot Q_k - \bar{P}_k G = 0. \tag{11}$$

In the following, we prove that the functions in (5)–(7) are actually solutions of the bilinear equations (8)–(11). From (5)–(7), we know that F, G, and $\bar{G}$ obviously satisfy Eq. (9). Hence, we only need to prove that Eqs. (8), (10), and (11) hold for the functions in (5)–(7). In fact, according to (5)–(7), we have the differential formulas

$$\frac{\partial}{\partial t} F = \sum_{k=1}^{K} \dot{C}_k(t) |A_{k,k}| + i \left(- \begin{vmatrix} 0 & \vec{r}^t \\ r_x & A \end{vmatrix} + \begin{vmatrix} 0 & \vec{r}_x^t \\ r & A \end{vmatrix} - \begin{vmatrix} 0 & \vec{s}^t \\ s_y & A \end{vmatrix} + \begin{vmatrix} 0 & \vec{s}_y^t \\ s & A \end{vmatrix} \right), \tag{12}$$

$$\frac{\partial}{\partial t} G = \sum_{k=1}^{K} \dot{C}_k(t) \begin{vmatrix} 0 & \hat{\vec{s}}_k^t \\ \hat{r}_k & A_{k,k} \end{vmatrix} + i \left(\begin{vmatrix} 0 & \vec{s}_{yy}^t \\ r & A \end{vmatrix} + \begin{vmatrix} 0 & \vec{s}^t \\ r_{xx} & A \end{vmatrix} - \begin{vmatrix} 0 & 0 & \vec{s}^t \\ 0 & 0 & \vec{r}^t \\ r & r_x & A \end{vmatrix} + \begin{vmatrix} 0 & 0 & \vec{s}_y^t \\ 0 & 0 & \vec{s}^t \\ s & r & A \end{vmatrix} \right), \tag{13}$$

where $\hat{r}_k$ and $\hat{s}_k$ denote vectors resulting from eliminating the kth element from r and s, respectively.

Substituting (5)–(7), (12), and (13) into (8), we get a determinant identity that is the sum of Jacobi's identity of determinants,[9]

$$0 = i\sum_{k=1}^{K} \dot{C}_k(t)\left(\begin{vmatrix} 0 & \hat{\vec{s}}_k^t \\ \hat{r}_k & A_{k,k}\end{vmatrix}|A| - \begin{vmatrix} \vec{s}^t \\ A_{k,0}\end{vmatrix}|rA_{0,k}| - \begin{vmatrix} 0 & \vec{s}^t \\ r & A\end{vmatrix}|A_{k,k}|\right) + 2\left(\begin{vmatrix} 0 & 0 & \vec{s}^t \\ 0 & 0 & \vec{r}^t \\ r & r_x & A\end{vmatrix}|A| - \begin{vmatrix} 0 & \vec{s}^t \\ r & A\end{vmatrix}\right.$$
$$\left.\times\begin{vmatrix} 0 & \vec{r}^t \\ r_x & A\end{vmatrix} + \begin{vmatrix} 0 & \vec{s}^t \\ r_x & A\end{vmatrix}\begin{vmatrix} 0 & \vec{r}^t \\ r & A\end{vmatrix}\right) + 2\left(\begin{vmatrix} 0 & 0 & \vec{s}^t \\ 0 & 0 & \vec{s}_y^t \\ s & r & A\end{vmatrix}|A| - \begin{vmatrix} 0 & \vec{s}_y^t \\ r & A\end{vmatrix}\begin{vmatrix} 0 & \vec{s}^t \\ s & A\end{vmatrix} + \begin{vmatrix} 0 & \vec{s}^t \\ r & A\end{vmatrix}\begin{vmatrix} 0 & \vec{s}_y^t \\ s & A\end{vmatrix}\right),$$

which indicates that Eq. (8) holds. Similarly, from expressions (7), we have the differential formulas

$$(\partial/\partial x)P_k = \sqrt{\dot{C}_k(t)}\begin{vmatrix} 0 & \vec{r}^t \\ 0 & \vec{s}^t \\ -\hat{r}_k & A_{k,0}\end{vmatrix},$$
$$(\partial/\partial y)Q_k = \sqrt{\dot{C}_k(t)}\begin{vmatrix} 0 & 0 & \hat{\vec{s}}_k^t \\ s & r & A_{0,k}\end{vmatrix}. \tag{14}$$

Substituting (5)–(7) and (14) into (10) and (11) yields the following determinant identities, respectively:

$$-\begin{vmatrix} 0 & \vec{r}^t \\ r & A\end{vmatrix}\begin{vmatrix} \vec{s}^t \\ A_{k,0}\end{vmatrix} - \begin{vmatrix} 0 & \vec{r}^t \\ 0 & \vec{s}^t \\ -\hat{r}_k & A_{k,0}\end{vmatrix}|A| + \begin{vmatrix} \vec{r}^t \\ A_{k,0}\end{vmatrix}\begin{vmatrix} 0 & \vec{s}^t \\ r & A\end{vmatrix} = 0,$$

$$\begin{vmatrix} 0 & \vec{s}^t \\ s & A\end{vmatrix}|rA_{0,k}| - \begin{vmatrix} 0 & 0 & \hat{\vec{s}}_k^t \\ s & r & A_{0,k}\end{vmatrix}|A| - \begin{vmatrix} 0 & \vec{s}^t \\ r & A\end{vmatrix}|sA_{0,k}| = 0.$$

These determinant identities are special cases of the Pfaffian identity,[9]

$$\begin{aligned}&\mathrm{pf}(a_1,a_2,a_3,1,\cdots,2n-1)\mathrm{pf}(1,\cdots,2n)\\ &\quad = \mathrm{pf}(a_1,1,\cdots,2n-1)\mathrm{pf}(a_2,a_3,1,\cdots,2n) - \mathrm{pf}(a_2,1,\cdots,2n-1)\mathrm{pf}(a_1,a_3,1,\cdots,2n)\\ &\qquad + \mathrm{pf}(a_3,1,\cdots,2n-1)\mathrm{pf}(a_1,a_2,1,\cdots,2n).\end{aligned}$$

In an analogous way, the complex conjugate equations of Eqs. (8), (10), and (11) can also be proved. Therefore, functions (5)–(7) are N-soliton solutions of the bilinear equations (8)–(11), and Eqs. (8)–(11) construct the DS equation with K (where $M+N \geqslant K$) pairs of self-consistent sources.

If we apply the dependent variable transformations,

$$U = 2\ln F, \quad W = \frac{G}{F}, \quad \Phi_{1,k} = \frac{P_k}{F}, \quad \Phi_{2,k} = \frac{Q_k}{F},$$

the bilinear equations (8)–(11) are transformed to the nonlinear equations

$$iW_t + W_{xx} + W_{yy} + WU_{xx} + WU_{yy} = \sum_{k=1}^{K} i\Phi_{1,k}\Phi_{2,k}, \tag{15}$$

$$U_{xy} = 2|W|^2, \tag{16}$$

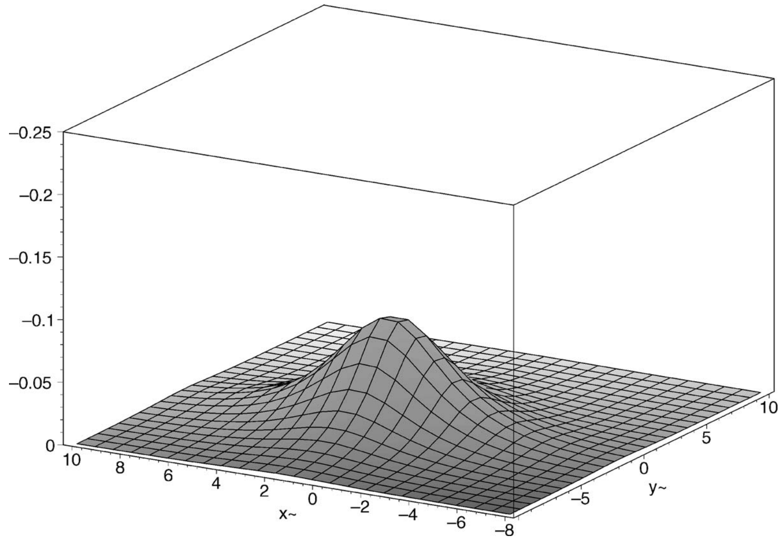

FIG. 1. The real part of W.

$$\Phi_{1,k_x} = \bar{\Phi}_{2,k} W, \quad \Phi_{2,k_y} = -\bar{\Phi}_{1,k} W. \tag{17}$$

To construct the soliton solution, we choose a simple solution of Eqs. (3) and (4),

$$\varphi_i(x,t) = e^{p_i x + i p_i^2 t}, \quad \psi_k(y,t) = e^{q_k y - i q_k^2 t} \quad (i = 1 \cdots M,\ k = 1 \cdots N). \tag{18}$$

Then the one-dromion solution of the DS equation with two self-consistent sources is obtained by setting $M=N=1$, $K=2$ in Eqs. (8)–(11) and choosing $\varphi_i(x,t), \psi_k(y,t)$ given by Eq. (18). In such a case, we have

$$F = \left(C_1(t) + \frac{e^{2(\mathrm{Re}\, p_1)[x - 2\, \mathrm{Im}\, p_1 t]}}{2\, \mathrm{Re}\, p_1} \right) \times \left(C_2(t) - \frac{e^{2(\mathrm{Re}\, q_1)[y + 2\, \mathrm{Im}\, q_1 t]}}{2\, \mathrm{Re}\, q_1} \right) - c_{12}\bar{c}_{12},$$

$$G = \bar{c}_{12} e^{(p_1 x + i\bar{p}_1^2 t)} e^{(\bar{q}_1 y + i\bar{q}_1^2 t)},$$

$$P_1 = -\bar{c}_{12}\sqrt{\dot{C}_1(t)}\, e^{(\bar{q}_1 y + i\bar{q}_1^2 t)}$$

$$Q_1 = \sqrt{\dot{C}_1(t)}\, e^{(p_1 x + i\bar{p}_1^2 t)} \times \left(C_2(t) - \frac{e^{2(\mathrm{Re}\, q_1)[y + 2\, \mathrm{Im}\, q_1 t]}}{2\, \mathrm{Re}\, q_1} \right),$$

$$P_2 = -\sqrt{\dot{C}_2(t)}\, e^{(\bar{q}_1 y + i\bar{q}_1^2 t)} \times \left(C_1(t) + \frac{e^{2(\mathrm{Re}\, p_1)[x - 2\, \mathrm{Im}\, p_1 t]}}{2\, \mathrm{Re}\, p_1} \right),$$

$$Q_2 = \bar{c}_{12}\sqrt{\dot{C}_2(t)}\, e^{(p_1 x + i\bar{p}_1^2 t)}.$$

To ensure that our solution has no singularities, we must impose additional conditions on the matrix C and on the value of p_1, q_1. In this particular case, we can take

$$\mathrm{Re}\, p_1 > 0, \quad \mathrm{Re}\, q_1 < 0, \quad C_1(t) > 0, \quad C_2(t) > 0, \quad C_1(t)C_2(t) > |c_{12}|^2.$$

The W field forms dromion solution. The U field is unbounded, but the derivatives of the U field form line solitons. In Figs. 1–6, we plot the one-dromion solution in the xy plane with $p_1 = \frac{1}{2} + \frac{1}{100}i$, $q_1 = -\frac{1}{2} + \frac{1}{100}i$, $C_1 = C_2 = \frac{4}{3} + \left(t + \frac{4}{3}\right)^2$, and $c_{12} = -\frac{3}{2}$, at time $t = \frac{2}{5}$.

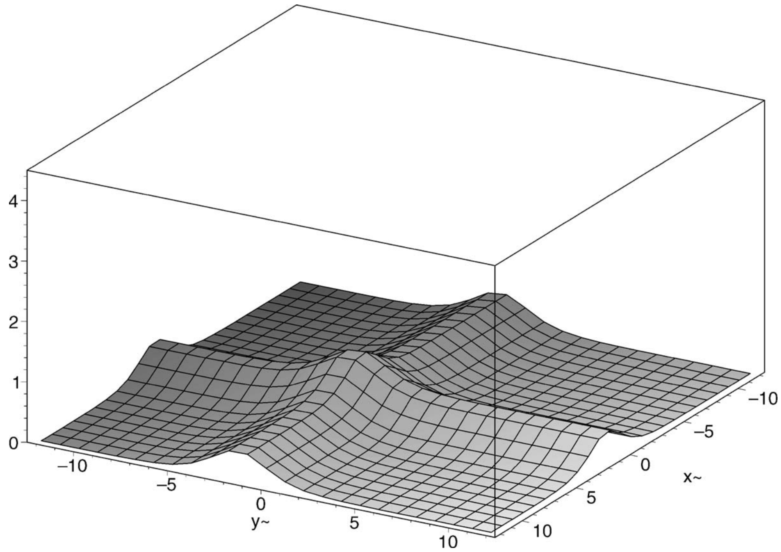

FIG. 2. $U_{xx}+U_{yy}$.

IV. CONCLUSION AND DISCUSSIONS

In this paper, we have proposed the DS equation with self-consistent sources by using the source generation procedure. Grammian type solutions of this coupled system have also been derived. If we set each $C_i(t)$ to be constant, the sources P_k, Q_k become zero. Then the bilinear DS equation with self-consistent sources is reduced to the bilinear DS equation without sources. In this case, the functions in (8)–(11) are reduced to the Grammian-type solution of the DS equation (1) and (2). Therefore, the DS equation with self-consistent sources is a generalization of the DS equation. On the other hand, the DS equation with self-consistent sources (DSESCS) can be thought of a (2+1)-dimension generalization of the nonlinear Schrödinger equation with self-consistent sources (NLSESCS) in some sense.

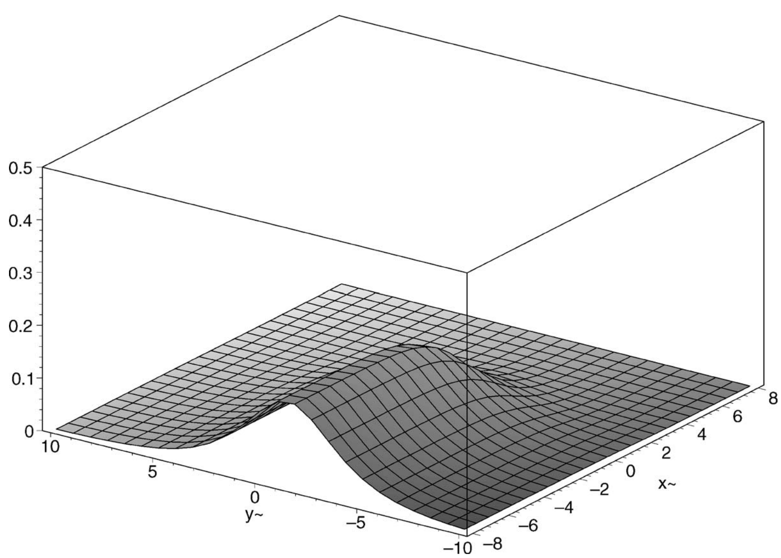

FIG. 3. The real part of $\Phi_{1,1}$.

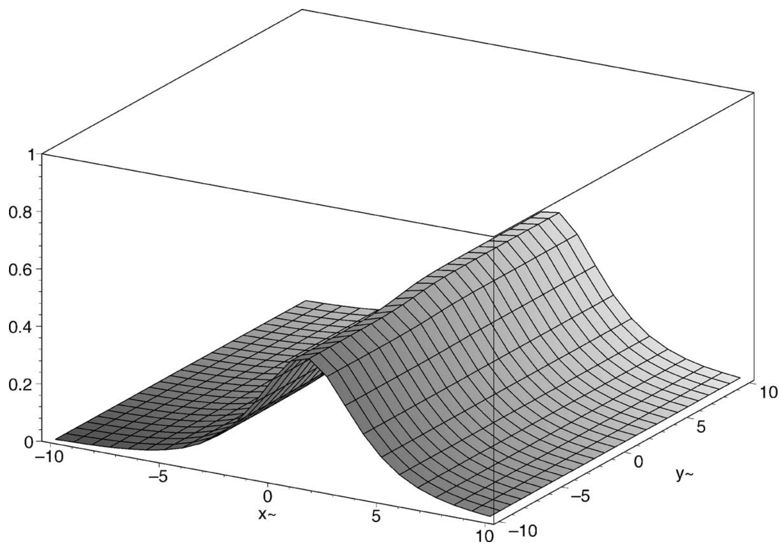

FIG. 4. The real part of $\Phi_{2,1}$.

Solutions of DS equation have significant applications in hydrodynamics, plasma physics, nonlinear optics, etc. For example, in nonlinear optics, the solution of DS equation can describe the interaction between a spatiotemporal optical plus and adequately matched microwaves.[40] In addition, its solution can also well describe the propagation of two-dimensional water waves of finite depth[6] in hydrodynamics and an electrostatic ion wave propagating perpendicularly to applied magetic fields[41] in plasma physics. Since the DSESCS is an integrable generalization of the DS equation, we expect that solutions of DSESCS (similar to solutions of DS equation) can also describe some nonlinear phenomena in physics fields.

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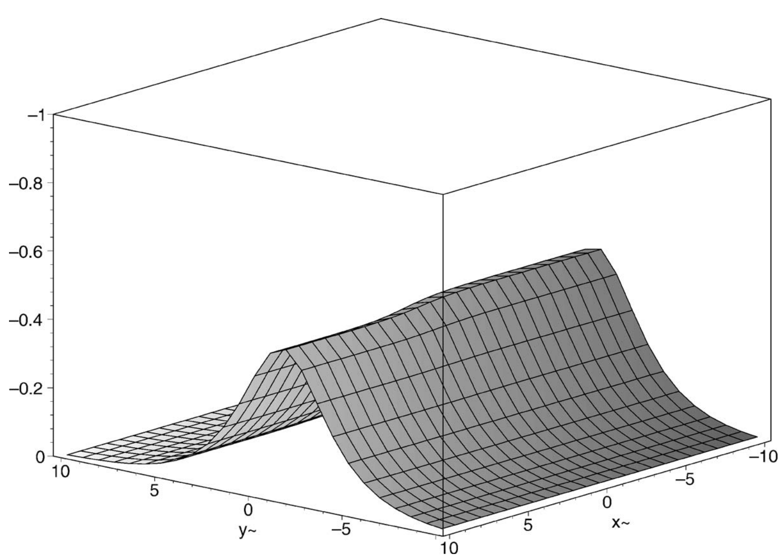

FIG. 5. The real part of $\Phi_{1,2}$.

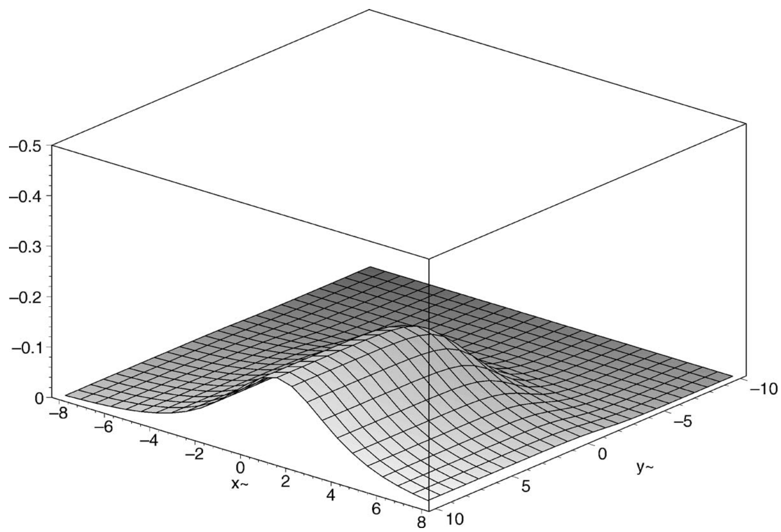

FIG. 6. The real part of $\Phi_{2,2}$.

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