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Published in:
Journal of Mathematical Physics

DOI:
10.1063/1.532896

Published: 01/07/1999

Document Version:
Publisher's PDF, also known as Version of record

[1]

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 40, NUMBER 7 JULY 1999

A finite-dimensional integrable system associated with the three-wave interaction equations

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(Received 2 November 1998; accepted for publication 17 March 1999)

Under a constraint between the potentials and the eigenfunctions, the 3×3 AKNS matrix spectral problem and its adjoint spectral problem associated with the three-wave interaction equations are nonlinearized so as to be a new finite-dimensional Hamiltonian system. A general scheme for generating involutive systems of conserved integrals and their two new generators are proposed, by which the finite-dimensional Hamiltonian system is further proved to be completely integrable in the Liouville sense. Moreover, the involutive solutions of the three-wave interaction equations are given. © *1999 American Institute of Physics.* [S0022-2488(99)00407-7]

I. INTRODUCTION

It has been known that there are several systematic approaches to obtain explicit solutions of soliton equations, such as the inverse scattering transformation, the Hirota technique, the algebra–geometric method, the polar expansion solution method, etc.[1–6] Some interesting explicit solutions have been found, the most important among which are pure-soliton solutions, finite-band solutions and polar expansion solutions.

The observation that all the explicit solutions mentioned above have a finite number of parameters, which means that they satisfy some kind of ordinary differential equations, suggests an important approach to get new finite-dimensional integrable systems from soliton equations. Recently an effective method, the so-called nonlinearization of eigenvalue problems or Lax pairs,[7–9] has been developed and applied to various soliton hierarchies associated with 2×2 zero-trace matrix spectral problems, from which a considerable number of new finite-dimensional systems are obtained that are completely integrable in the Liouville sense. Another important application of the nonlinearization method is that it provides a way of solving soliton equations, integrable nonlinear partial differential equations, by separation of spatial and temporal variables. At the same time the inter-relation between soliton equations and finite-dimensional integrable systems is revealed. The method is sometimes called the method of separation of variables for nonlinear partial differential equations, which generalizes the corresponding method for linear ones.

A similar method, the restricted flow technique, for bi-Hamiltonian soliton hierarchies is proposed in Refs. 10, 11 and bi-Hamiltonian structures for the resulting finite-dimensional integrable systems can also be worked out through a Miura map.[11,12] There are attempts to apply the nonlinearization method or the restricted flow technique to discrete systems in order to get integrable symplectic maps.[13–17]

Very recently, the nonlinearization method has been generalized to discuss Lax pairs and adjoint Lax pairs of soliton equations[18–21] so that it may also be suitable for the cases of 2×2 nonzero-trace matrix spectral problems. It has been applied successfully to the hierarchy of Harry

Dym type equations and the Blaszak discrete soliton hierarchy,[20,21] which correspond to 3×3 matrix spectral problems.

The key to the complete integrability of a finite-dimensional Hamiltonian system is the existence of an involutive system of conserved integrals. However, it is difficult for us to search for an involutive system of conserved integrals of a given finite-dimensional Hamiltonian system. In this paper, based on the above works we are going to discuss the nonlinearization for a 3×3 AKNS matrix spectral problem and its adjoint spectral problem associated with the three-wave interaction equations,[1,22] from which a new finite-dimensional Hamiltonian system is obtained. Resorting to the characteristic polynomial of solution matrix of the stationary zero-curvature equation, we propose a general scheme for generating involutive systems of enough conserved integrals of the resulting finite-dimensional Hamiltonian system. To prove the functional independence of conserved integrals, two new generators of involutive systems of conserved integrals are introduced, which are two natural generalizations of the 2×2 case.[23] This shows that the finite-dimensional Hamiltonian system is completely integrable in the Liouville sense.

Consider the $n\times n$ matrix spectral problem

$$\psi_x=U(u,\lambda)\psi,\ \ \psi=(\psi^1,\dots,\psi^n)^T. \tag{1.1}$$

In order to derive the isospectral hierarchy associated with Eq. (1.1), we proceed first to solve the stationary zero-curvature equation,

$$V_x-[U,V]=0,\ \ V=\sum_{j\geq0} V^{(j)}\lambda^{-j}, \tag{1.2}$$

which usually is equivalent to Lenard recursive equation

$$KG_{j-1}=JG_j,\ \ JG_{-1}=0,\ j\geq0. \tag{1.3}$$

Here K and J are two skew-symmetric operators. The soliton hierarchy $u_t=JG_m$ has a Lax pair, the spectral problem (1.1) and the auxiliary problem

$$\psi_{t_m}=V_m\psi,\ \ V_m=(\lambda^mV)_+, \tag{1.4}$$

where the symbol + stands for the choice of non-negative power of λ. The introduction of the adjoint problem of Eq. (1.1),

$$\phi_x=-U(u,\lambda)^T\phi,\ \ \phi=(\phi^1,\dots,\phi^n)^T \tag{1.5}$$

allows the calculation of the functional gradient of the eigenvalues with regard to the potential u (see, e.g., Sec. III). Usually such a functional gradient $\nabla\lambda_j$ satisfies the following equation:

$$K\nabla\lambda_j=\rho(\lambda_j)J\nabla\lambda_j,\ \ \rho(\lambda_j)=c_1\lambda_j+c_2\lambda_j^2,\ \ 1\leq j\leq N, \tag{1.6}$$

where $\lambda_1,\dots,\lambda_N$ are the eigenvalues of Eqs. (1.1) and (1.5), c_1 and c_2 are constants. The following two kinds of constraints:

$$G_0=\sum_{j=1}^{N}\nabla\lambda_j,\ \ G_{-1}=\sum_{j=1}^{N}\nabla\lambda_j, \tag{1.7}$$

which are called the Bargmann and Neumann constraints, respectively, play a central role in the process of nonlinearization of the eigenvalue problems (1.1) and (1.5). From (1.7) we can obtain the relations

$$u=f(q,p)\ \ \text{and}\ \ g(q,p)=0,\ u=f(q,p), \tag{1.8}$$

where $q=(q_1^1,\ldots,q_N^1,\ldots,q_1^n,\ldots,q_N^n)^T$, $p=(p_1^1,\ldots,p_N^1,\ldots,p_1^n,\ldots,p_N^n)^T$, $q_j^i=\psi^i(\lambda_j)$, $p_j^i=\phi^i(\lambda_j)$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant N$. Under the two constraints, N replicas of the spectral problems (1.1) and (1.5) associated with $\lambda_1,\ldots,\lambda_N$ are nonlinearized into two finite-dimensional Hamiltonian systems

$$q_x=U(f(q,p),\Lambda)q,\quad p_x=-U(f(q,p),\Lambda)^Tp,\quad \Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_N), \tag{1.9}$$

$$q_x=U(f(q,p),\Lambda)q,\quad p_x=-U(f(q,p),\Lambda)^Tp,\quad g(q,p)=0, \tag{1.10}$$

which are called the Bargmann and Neumann systems, respectively. In the following, we propose a general scheme for generating conserved integrals of Eq. (1.9) or (1.10). Noticing the matrix $\mu I-V$ is also a solution of the stationary zero-curvature equation (1.2), which implies that $\det(\mu I-V)$ is a constant with respect to the variable x, for each value of the spectral parameter λ. Here μ is a parameter, I is an $n\times n$ matrix. Let us consider the characteristic polynomial of solution matrix V of Eq. (1.2),

$$\det(\mu I-V)=\mu^n-\mathcal{F}_\lambda^{(0)}\mu^{n-1}+\mathcal{F}_\lambda^{(1)}\mu^{n-2}+\cdots+(-1)^n\mathcal{F}_\lambda^{(n-1)}, \tag{1.11}$$

where

$$\mathcal{F}_\lambda^{(0)}=\mathrm{tr}\,V,\quad \mathcal{F}_\lambda^{(1)}=\sum_{1\leqslant i<j\leqslant n}\begin{vmatrix}V_{ii} & V_{ij}\\ V_{ji} & V_{jj}\end{vmatrix},$$

$$\mathcal{F}_\lambda^{(2)}=\sum_{1\leqslant i<j<k\leqslant n}\begin{vmatrix}V_{ii} & V_{ij} & V_{ik}\\ V_{ji} & V_{jj} & V_{jk}\\ V_{ki} & V_{kj} & V_{kk}\end{vmatrix},\ldots,\quad \mathcal{F}_\lambda^{(n-1)}=\det V. \tag{1.12}$$

By using Eqs. (1.3), (1.6) and the corresponding constraint, Eqs. (1.12) are reduced to generating functions of the conserved integrals of the Bargmann system (1.9), or the Neumann system (1.10). Sometimes some modifications are made, especially for the Neumann system. Thus we easily get the conserved integrals of the system (1.9) or (1.10) from their generating functions.

The outline of the paper is as follows: In Sec. II, we shall reconstruct the soliton hierarchy associated with the 3×3 AKNS spectral problem and establish their Hamiltonian structures. In Sec. III, we shall introduce the Bargmann constraint between the potentials and eigenfunctions. Under the constraint, a new finite-dimensional Hamiltonian system is obtained by nonlinearization of the 3×3 AKNS spectral problem and its adjoint one. In Sec. IV, we shall show how the scheme is applied to generate involutive systems of conserved integrals of the finite-dimensional Hamiltonian system. Further we prove that the finite-dimensional Hamiltonian system is completely integrable in the Liouville sense. In Sec. V, the representation of involutive solutions of the three-wave interaction equations and the soliton hierarchy is given. Finally in the Appendix, by means of the generating functions of conserved integrals, we give the involutivity of conserved integrals. Then we introduce two new generators of involutive systems of conserved integrals and prove the functional independence of conserved integrals.

II. THE SOLITION HIERARCHY AND HAMILTONIAN STRUCTURES

Let us consider the 3×3 AKNS matrix spectral problem

$$\psi_x=U(u,\lambda)\psi,\quad \psi=\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix},\quad U=\begin{pmatrix}\alpha_1\lambda & u_{12} & u_{13}\\ u_{21} & \alpha_2\lambda & u_{23}\\ u_{31} & u_{32} & \alpha_3\lambda\end{pmatrix}, \tag{2.1}$$

where the potential $u=(u_{12},u_{21},u_{13},u_{31},u_{23},u_{32})^T$, λ is a constant spectral parameter, $\alpha_i's(1\leqslant i\leqslant 3)$ are three distinct constants. Our aim is to engender the soliton hierarchy from the spectral problem (2.1). To this end, we first solve the stationary zero-curvature equation,

$$V_x-[U,V]=0,\quad V=(V_{ij})_{3\times 3}, \tag{2.2}$$

which is equivalent to

$$V_{ijx}+u_{ij}(V_{ii}-V_{jj})+\sum_{\substack{k=1\\k\neq i,j}}^{3}(u_{kj}V_{ik}-u_{ik}V_{kj})-\lambda(\alpha_i-\alpha_j)V_{ij}=0,\quad i\neq j, \tag{2.3a}$$

$$V_{iix}=\sum_{\substack{k=1\\k\neq i}}^{3}(u_{ik}V_{ki}-u_{ki}V_{ik}),\quad 1\leqslant i,j\leqslant 3. \tag{2.3b}$$

Substitution the expansion

$$V_{ij}=\sum_{n\geqslant 0}V_{ij}^{(n)}\lambda^{-n} \tag{2.4}$$

into Eq. (2.3), we obtain the recurrence relations

$$V_{iix}^{(0)}=0,\quad V_{ij}^{(0)}=0,\quad (i\neq j),$$

$$V_{ijx}^{(n)}+u_{ij}(V_{ii}^{(n)}-V_{jj}^{(n)})+\sum_{\substack{k=1\\k\neq i,j}}^{3}(u_{kj}V_{ik}^{(n)}-u_{ik}V_{kj}^{(n)})-(\alpha_i-\alpha_j)V_{ij}^{(n+1)}=0,\quad i\neq j,$$

$$V_{iix}^{(n)}=\sum_{\substack{k=1\\k\neq i}}^{3}(u_{ik}V_{ki}^{(n)}-u_{ki}V_{ik}^{(n)}),\quad 1\leqslant i,j\leqslant 3, n\geqslant 0. \tag{2.5}$$

By Eq. (2.5) we have

$$V_{ii}^{(0)}=\beta_i(\text{constant}),\qquad V_{ij}^{(0)}=0,\quad i\neq j,$$

$$V_{ii}^{(1)}=0,\quad V_{ij}^{(1)}=\frac{\beta_i-\beta_j}{\alpha_i-\alpha_j}u_{ij},\quad i\neq j, \tag{2.6}$$

and require that

$$\beta_i\neq\beta_j(i\neq j),\qquad V_{ij}^{(n)}|_{u=0}=0,\quad n\geqslant 1, \tag{2.7}$$

where the condition (2.7) means to identify constants of the integration to be zero. Hence $V_{ij}^{(n)}$ is uniquely determined. It is easy to calculate that

$$V_{ij}^{(2)}=\frac{\beta_i-\beta_j}{(\alpha_i-\alpha_j)^2}u_{ijx}+\frac{1}{\alpha_i-\alpha_j}\sum_{\substack{k=1\\k\neq i,j}}^{3}\left(\frac{\beta_k-\beta_i}{\alpha_k-\alpha_i}-\frac{\beta_k-\beta_j}{\alpha_k-\alpha_j}\right)u_{ik}u_{kj},\quad i\neq j, \tag{2.8a}$$

$$V_{ii}^{(2)}=\sum_{\substack{k=1\\k\neq i}}^{3}\frac{\beta_k-\beta_i}{(\alpha_k-\alpha_i)^2}u_{ik}u_{ki}. \tag{2.8b}$$

Equations (2.5)–(2.7) can be equivalently written as the Lenard form

$$KG_{n-1}=JG_n,\quad G_{n-1}=(V_{21}^{(n)},V_{12}^{(n)},V_{31}^{(n)},V_{13}^{(n)},V_{32}^{(n)},V_{23}^{(n)})^T,\quad n\geq 1, \tag{2.9a}$$

$$G_0=\left(\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}u_{21},\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}u_{12},\frac{\beta_1-\beta_3}{\alpha_1-\alpha_3}u_{31},\frac{\beta_1-\beta_3}{\alpha_1-\alpha_3}u_{13},\frac{\beta_2-\beta_3}{\alpha_2-\alpha_3}u_{32},\frac{\beta_2-\beta_3}{\alpha_2-\alpha_3}u_{23}\right)^T \tag{2.9b}$$

with $G_n|_{u=0}=0$. Here J and K are two skew-symmetric operators,

$$J=\begin{pmatrix} 0 & \alpha_1-\alpha_2 & 0 & 0 & 0 & 0\\ \alpha_2-\alpha_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \alpha_1-\alpha_3 & 0 & 0\\ 0 & 0 & \alpha_3-\alpha_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \alpha_2-\alpha_3\\ 0 & 0 & 0 & 0 & \alpha_3-\alpha_2 & 0 \end{pmatrix},$$

$$K=(K_{lm})_{6\times 6},\quad K_{ml}^*=-K_{lm}$$

with

$$K_{11}=2u_{12}\partial^{-1}u_{12},\quad K_{12}=\partial-2u_{12}\partial^{-1}u_{21},\quad K_{13}=u_{12}\partial^{-1}u_{13},\quad K_{14}=u_{32}-u_{12}\partial^{-1}u_{31},$$

$$K_{15}=-u_{13}-u_{12}\partial^{-1}u_{23},\quad K_{16}=u_{12}\partial^{-1}u_{32},\quad K_{22}=2u_{21}\partial^{-1}u_{21},\quad K_{23}=-u_{23}-u_{21}\partial^{-1}u_{13},$$

$$K_{24}=u_{21}\partial^{-1}u_{31},\quad K_{25}=u_{21}\partial^{-1}u_{23},\quad K_{26}=u_{31}-u_{21}\partial^{-1}u_{32},\quad K_{33}=2u_{13}\partial^{-1}u_{13},$$

$$K_{34}=\partial-2u_{13}\partial^{-1}u_{31},\quad K_{35}=u_{13}\partial^{-1}u_{23},\quad K_{36}=-u_{12}-u_{13}\partial^{-1}u_{32},\quad K_{44}=2u_{31}\partial^{-1}u_{31},$$

$$K_{45}=u_{21}-u_{31}\partial^{-1}u_{23},\quad K_{46}=u_{31}\partial^{-1}u_{32},\quad K_{55}=2u_{23}\partial^{-1}u_{23},\quad K_{56}=\partial-2u_{23}\partial^{-1}u_{32},$$

$$K_{66}=2u_{32}\partial^{-1}u_{32},\quad \partial=\partial/\partial x,\quad \partial\partial^{-1}=\partial^{-1}\partial=1.$$

Using the trace identity technique,[24] we have

$$\frac{\delta H_n}{\delta u_{ij}}=V_{ji}^{(n)},\quad H_n=-\frac{1}{n}(\alpha_1V_{11}^{(n+1)}+\alpha_2V_{22}^{(n+1)}+\alpha_3V_{33}^{(n+1)}), \tag{2.10}$$

where the potentials $u_{ij},i\neq j$, are assumed to belong to the Schwartz space $\mathcal{S}(\Omega)$, $\Omega=(-\infty,\infty)$.

Now we introduce the auxiliary problem of the spectral problem (2.1),

$$\psi_{t_m}=V^{(m)}\psi,\quad V^{(m)}=V^{(m)}(u,\lambda)=(\lambda^mV)_+,\quad m\geq 1. \tag{2.11}$$

The compatibility condition between Eqs. (2.1) and (2.11) leads to the zero-curvature equation, $U_t-V_x+[U,V]=0$, that is the hierarchy of soliton equations with bi-Hamiltonian forms

$$u_{t_m}=X_m=K\frac{\delta H_m}{\delta u}=J\frac{\delta H_{m+1}}{\delta u},\quad m\geq 1, \tag{2.12}$$

where the vector field $X_m=KG_{m-1}=JG_m$ and Eq. (2.10) is used. The typical nonlinear system in the hierarchy is the famous three-wave interaction equations,

$$u_{ijt_1}=\frac{\beta_i-\beta_j}{\alpha_i-\alpha_j}u_{ijx}+\sum_{\substack{k=1\\k\neq i,j}}^{3}\left(\frac{\beta_k-\beta_i}{\alpha_k-\alpha_i}-\frac{\beta_k-\beta_j}{\alpha_k-\alpha_j}\right)u_{ik}u_{kj},\quad i\neq j,\ 1\leqslant i,\ j\leqslant 3, \tag{2.13}$$

which has important applications in physics.[1,22,25]

III. A FINITE-DIMENSIONAL HAMILTONIAN SYSTEM

In order to give the constraint between the potentials and the eigenfunctions, it is necessary to calculate the functional gradient of the eigenvalue with respect to the potential. We introduce the adjoint spectral problem of (2.1),

$$\phi_x=-U(u,\lambda)^T\phi,\quad \phi=(\phi^1,\phi^2,\phi^3)^T. \tag{3.1}$$

Suppose that $u_{ij}\rightarrow u_{ij}+\epsilon\delta u_{ij}$, $1\leqslant i\neq j\leqslant 3$, denote $\partial/\partial\epsilon|_{\epsilon=0}$ by a dot. The underlying interval Ω is $(-\infty,\infty)$ under the decaying condition at infinity. A direct calculation shows by Eqs. (2.1) and (3.1) that

$$(\phi^T\dot{\psi})_x=\phi^T\dot{U}\psi. \tag{3.2}$$

If λ is an eigenvalue of the spectral problems (2.1) and (3.1), the integration of the left-hand side of equality (3.2) vanishes because of the boundary conditions. Then we have

$$\int_\Omega \phi^T\dot{U}\psi dx=0. \tag{3.3}$$

Let $\lambda_1,\ldots,\lambda_N$ be N distinct eigenvalues. Then the systems associated with spectral problems (2.1) and (3.1) can be written in the form

$$(q_l^1,q_l^2,q_l^3)_x=(q_l^1,q_l^2,q_l^3)U(u,\lambda_l)^T,\quad (p_l^1,p_l^2,p_l^3)_x=-(p_l^1,p_l^2,p_l^3)U(u,\lambda_l), \tag{3.4}$$

where $q_l^i=\psi^i(\lambda_l)$, $p_l^i=\phi^i(\lambda_l)$, $1\leqslant i\leqslant 3$, $1\leqslant l\leqslant N$, are eigenfunctions. Noticing Eq. (3.4) and $\dot{U}(u,\lambda_l)=U(\delta u,\delta\lambda_l)$, we obtain by Eq. (3.3) that

$$\int_\Omega (p_l^1,p_l^2,p_l^3)U(\delta u,\delta\lambda_l)(q_l^1,q_l^2,q_l^3)^T dx=0,$$

which implies that the functional gradient of the eigenvalue λ_l with regard to the potential u is

$$\nabla\lambda_l=\frac{\delta\lambda_l}{\delta u}=\left(\frac{\delta\lambda_l}{\delta u_{12}},\frac{\delta\lambda_l}{\delta u_{21}},\frac{\delta\lambda_l}{\delta u_{13}},\frac{\delta\lambda_l}{\delta u_{31}},\frac{\delta\lambda_l}{\delta u_{23}},\frac{\delta\lambda_l}{\delta u_{32}}\right)^T=(q_l^2p_l^1,q_l^1p_l^2,q_l^3p_l^1,q_l^1p_l^3,q_l^3p_l^2,q_l^2p_l^3)^T, \tag{3.5}$$

where we assume that

$$\int_\Omega(\alpha_1q_l^1p_l^1+\alpha_2q_l^2p_l^2+\alpha_3q_l^3p_l^3)dx=-1.$$

A direct calculation shows that the gradient $\nabla\lambda_l$ satisfies the following equation:

$$K\nabla\lambda_l=\lambda_lJ\nabla\lambda_l. \tag{3.6}$$

As a matter of fact, we obtain from Eq. (3.4) that

$$(q_l^1p_l^1)_x-(q_l^2p_l^2)_x=2u_{12}q_l^2p_l^1-2u_{21}q_l^1p_l^2+u_{13}q_l^3p_l^1-u_{31}q_l^1p_l^3-u_{23}q_l^3p_l^2+u_{32}q_l^2p_l^3, \tag{3.7}$$

$$(q_l^1 p_l^2)_x + u_{32} q_l^1 p_l^3 - u_{13} q_l^3 p_l^2 = u_{12}(q_l^2 p_l^2 - q_l^1 p_l^1) + \lambda_l(\alpha_1 - \alpha_2) q_l^1 p_l^2. \tag{3.8}$$

Noticing Eqs. (3.7) and (3.8), we have

$$\begin{aligned} &K_{11} q_l^2 p_l^1 + K_{12} q_l^1 p_l^2 + K_{13} q_l^3 p_l^1 + K_{14} q_l^1 p_l^3 + K_{15} q_l^3 p_l^2 + K_{16} q_l^2 p_l^3 \\ &\quad = u_{12} \partial^{-1}(2u_{12} q_l^2 p_l^1 - 2u_{21} q_l^1 p_l^2 + u_{13} q_l^3 p_l^1 - u_{31} q_l^1 p_l^3 - u_{23} q_l^3 p_l^2 + u_{32} q_l^2 p_l^3) \\ &\quad + (q_l^1 p_l^2)_x + u_{32} q_l^1 p_l^3 - u_{13} q_l^3 p_l^2 = \lambda_l(\alpha_1 - \alpha_2) q_l^1 p_l^2, \end{aligned} \tag{3.9}$$

which implies that the sign of equality in the first row of Eq. (3.6) holds. In a similar way, we may prove that other rows of Eq. (3.6) are also identical equations.

Now we consider the Bargmann constraint

$$G_0 = \sum_{l=1}^{N} \nabla \lambda_l, \tag{3.10}$$

which implies

$$u_{ij} = \frac{\alpha_i - \alpha_j}{\beta_i - \beta_j} \langle q^i, p^j \rangle, \quad i \neq j, \ 1 \leq i,j \leq 3, \tag{3.11}$$

where $\langle \cdot,\cdot \rangle$ is the standard inner-product in $\mathcal{R}^N$, $q^i = (q_1^i, \ldots, q_N^i)^T$, $p^i = (p_1^i, \ldots, p_N^i)^T$. Substituting Eq. (3.11) into Eq. (3.4), we obtain a finite-dimensional Hamiltonian system

$$q_x^i = \frac{\partial H}{\partial p^i}, \quad p_x^i = -\frac{\partial H}{\partial q^i}, \quad 1 \leq i \leq 3, \tag{3.12}$$

with the Hamiltonian

$$\begin{aligned} H = {} & \alpha_1 \langle \Lambda q^1, p^1 \rangle + \alpha_2 \langle \Lambda q^2, p^2 \rangle + \alpha_3 \langle \Lambda q^3, p^3 \rangle + \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} \langle q^1, p^2 \rangle \langle q^2, p^1 \rangle \\ & + \frac{\alpha_1 - \alpha_3}{\beta_1 - \beta_3} \langle q^1, p^3 \rangle \langle q^3, p^1 \rangle + \frac{\alpha_2 - \alpha_3}{\beta_2 - \beta_3} \langle q^2, p^3 \rangle \langle q^3, p^2 \rangle, \end{aligned}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Moreover, we have from Eq. (3.4) that

$$[(q_l^1, q_l^2, q_l^3)(p_l^1, p_l^2, p_l^3)^T]_x = 0,$$

which implies

$$q_l^1 p_l^1 + q_l^2 p_l^2 + q_l^3 p_l^3 = \text{constant}. \tag{3.13}$$

It is easy to see that Eq. (3.13) are the conserved integrals of the system (3.12).

IV. THE INTEGRABILITY

In this section, we shall show how the characteristic polynomial of the solution matrix of Eq. (2.2) is used to generate the involutive systems of conserved integrals of the finite-dimensional Hamiltonian system (3.12). To this end, we first consider the characteristic polynomial

$$\det(\mu I - V) = \mu^3 - \mathcal{F}_\lambda^{(0)} \mu^2 + \mathcal{F}_\lambda^{(1)} \mu - \mathcal{F}_\lambda^{(2)}, \tag{4.1}$$

where

$$\mathcal{F}_\lambda^{(0)}=\mathrm{tr}\,V,\ \ \mathcal{F}_\lambda^{(1)}=\sum_{1\leq i<j\leq 3}\begin{vmatrix} V_{ii} & V_{ij}\\ V_{ji} & V_{jj}\end{vmatrix},\ \ \mathcal{F}_\lambda^{(2)}=\begin{vmatrix} V_{11} & V_{12} & V_{13}\\ V_{21} & V_{22} & V_{23}\\ V_{31} & V_{32} & V_{33}\end{vmatrix}. \tag{4.2}$$

It is easy to see that $\mathcal{F}_\lambda^{(0)},\mathcal{F}_\lambda^{(1)},\mathcal{F}_\lambda^{(2)}$ are constants with respect to the variable x. For the sake of convenience, we introduce a bilinear function $\overset{ij}{Q}_\lambda$ on $\mathcal{R}^N$ and its partial-fraction expansion and Laurent expansion,

$$\overset{ij}{Q}_\lambda=\langle(\lambda-\Lambda)^{-1}q^i,p^j\rangle=\sum_{l=1}^{N}\frac{q_l^i p_l^j}{\lambda-\lambda_l}=\sum_{n\geq 0}\lambda^{-n-1}\langle\Lambda^n q^i,p^j\rangle.$$

By using Eqs. (2.9), (3.6) and the constraint (3.10), we take the following restriction:

$$G_n=\sum_{l=1}^{N}\lambda_l^n\nabla\lambda_l, \tag{4.3}$$

which is a special solution of Eq. (2.9) and can be written as follows:

$$V_{ij}^{(n)}=\langle\Lambda^{n-1}q^i,p^j\rangle,\ \ i\neq j,\quad 1\leq i,\ j\leq 3,\ \ n\geq 1. \tag{4.4a}$$

From Eq. (4.4a) and the third expression of Eq. (2.5), we have

$$V_{ii}^{(n)}=\langle\Lambda^{n-1}q^i,p^i\rangle,\quad 1\leq i\leq 3, n\geq 1. \tag{4.4b}$$

By utilizing Eqs. (2.4) and (4.4), we get

$$V_{ij}=\sum_{n\geq 1}\langle\Lambda^{n-1}q^i,p^j\rangle\lambda^{-n}=\overset{ij}{Q}_\lambda,\quad i\neq j,\ 1\leq i,j\leq 3, \tag{4.5a}$$

$$V_{ii}=\beta_i+\sum_{n\geq 1}\langle\Lambda^{n-1}q^i,p^i\rangle\lambda^{-n}=\beta_i+\overset{ii}{Q}_\lambda,\ \ 1\leq i\leq 3. \tag{4.5b}$$

Substituting Eq. (4.5) into Eq. (4.2) yields generating functions of integrals of motion for Eq. (3.12),

$$\hat{\mathcal{F}}_\lambda^{(0)}=\overset{11}{Q}_\lambda+\overset{22}{Q}_\lambda+\overset{33}{Q}_\lambda, \tag{4.6}$$

$$\hat{\mathcal{F}}_\lambda^{(1)}=(\beta_2+\beta_3)\overset{11}{Q}_\lambda+(\beta_1+\beta_3)\overset{22}{Q}_\lambda+(\beta_1+\beta_2)\overset{33}{Q}_\lambda+\sum_{1\leq i<j\leq 3}(\overset{ii}{Q}_\lambda\overset{jj}{Q}_\lambda-\overset{ij}{Q}_\lambda\overset{ji}{Q}_\lambda), \tag{4.7}$$

$$\hat{\mathcal{F}}_\lambda^{(2)}=\beta_2\beta_3\overset{11}{Q}_\lambda+\beta_1\beta_3\overset{22}{Q}_\lambda+\beta_1\beta_2\overset{33}{Q}_\lambda+\sum_{1\leq i<j\leq 3}\beta_{6-i-j}(\overset{ii}{Q}_\lambda\overset{jj}{Q}_\lambda-\overset{ij}{Q}_\lambda\overset{ji}{Q}_\lambda)+\begin{vmatrix}\overset{11}{Q}_\lambda & \overset{12}{Q}_\lambda & \overset{13}{Q}_\lambda\\ \overset{21}{Q}_\lambda & \overset{22}{Q}_\lambda & \overset{23}{Q}_\lambda\\ \overset{31}{Q}_\lambda & \overset{32}{Q}_\lambda & \overset{33}{Q}_\lambda\end{vmatrix}, \tag{4.8}$$

where $\hat{\mathcal{F}}_\lambda^{(0)}$, $\hat{\mathcal{F}}_\lambda^{(1)}$, and $\hat{\mathcal{F}}_\lambda^{(2)}$ are defined by

$$\hat{\mathcal{F}}_\lambda^{(0)}=\mathcal{F}_\lambda^{(0)}-\beta_1-\beta_2-\beta_3,\quad \hat{\mathcal{F}}_\lambda^{(1)}=\mathcal{F}_\lambda^{(1)}-\beta_1\beta_2-\beta_1\beta_3-\beta_2\beta_3,$$

$$\hat{\mathcal{F}}_\lambda^{(2)}=\mathcal{F}_\lambda^{(2)}-\beta_1\beta_2\beta_3.$$

Substituting the Laurent expansion of Q_λ^{ik} into Eqs. (4.6)–(4.8), respectively, we have

$$\hat{\mathcal{F}}_\lambda^{(0)}=\sum_{m\geqslant 0}\lambda^{-m-1}F_m^{(0)},\quad \hat{\mathcal{F}}_\lambda^{(1)}=\sum_{m\geqslant 0}\lambda^{-m-1}F_m^{(1)},\quad \hat{\mathcal{F}}_\lambda^{(2)}=\sum_{m\geqslant 0}\lambda^{-m-1}F_m^{(2)}, \tag{4.9}$$

where

$$F_m^{(0)}=\langle\Lambda^m q^1,p^1\rangle+\langle\Lambda^m q^2,p^2\rangle+\langle\Lambda^m q^3,p^3\rangle,\quad m\geqslant 0, \tag{4.10}$$

$$F_0^{(1)}=(\beta_2+\beta_3)\langle q^1,p^1\rangle+(\beta_1+\beta_3)\langle q^2,p^2\rangle+(\beta_1+\beta_2)\langle q^3,p^3\rangle, \tag{4.11a}$$

$$\begin{aligned}F_m^{(1)}=&(\beta_2+\beta_3)\langle\Lambda^m q^1,p^1\rangle+(\beta_1+\beta_3)\langle\Lambda^m q^2,p^2\rangle+(\beta_1+\beta_2)\langle\Lambda^m q^3,p^3\rangle\\&+\sum_{1\leqslant i<j\leqslant 3}\sum_{l=1}^{m}\begin{vmatrix}\langle\Lambda^{l-1}q^i,p^i\rangle & \langle\Lambda^{m-l}q^j,p^i\rangle\\ \langle\Lambda^{l-1}q^i,p^j\rangle & \langle\Lambda^{m-l}q^j,p^j\rangle\end{vmatrix},\quad m\geqslant 1,\end{aligned} \tag{4.11b}$$

$$F_0^{(2)}=\beta_2\beta_3\langle q^1,p^1\rangle+\beta_1\beta_3\langle q^2,p^2\rangle+\beta_1\beta_2\langle q^3,p^3\rangle, \tag{4.12a}$$

$$\begin{aligned}F_1^{(2)}=&\beta_2\beta_3\langle\Lambda q^1,p^1\rangle+\beta_1\beta_3\langle\Lambda q^2,p^2\rangle+\beta_1\beta_2\langle\Lambda q^3,p^3\rangle\\&+\sum_{1\leqslant i<j\leqslant 3}\beta_{6-i-j}\begin{vmatrix}\langle q^i,p^i\rangle & \langle q^j,p^i\rangle\\ \langle q^i,p^j\rangle & \langle q^j,p^j\rangle\end{vmatrix},\end{aligned} \tag{4.12b}$$

$$\begin{aligned}F_m^{(2)}=&\beta_2\beta_3\langle\Lambda^m q^1,p^1\rangle+\beta_1\beta_3\langle\Lambda^m q^2,p^2\rangle+\beta_1\beta_2\langle\Lambda^m q^3,p^3\rangle\\&+\sum_{1\leqslant i<j\leqslant 3}\sum_{\substack{l+n=m-1\\ l,n\geqslant 0}}\beta_{6-i-j}\begin{vmatrix}\langle\Lambda^l q^i,p^i\rangle & \langle\Lambda^n q^j,p^i\rangle\\ \langle\Lambda^l q^i,p^j\rangle & \langle\Lambda^n q^j,p^j\rangle\end{vmatrix}\\&+\sum_{\substack{l+n+s=m-2\\ l,n,s\geqslant 0}}\begin{vmatrix}\langle\Lambda^l q^1,p^1\rangle & \langle\Lambda^n q^1,p^2\rangle & \langle\Lambda^s q^1,p^3\rangle\\ \langle\Lambda^l q^2,p^1\rangle & \langle\Lambda^n q^2,p^2\rangle & \langle\Lambda^s q^2,p^3\rangle\\ \langle\Lambda^l q^3,p^1\rangle & \langle\Lambda^n q^3,p^2\rangle & \langle\Lambda^s q^3,p^3\rangle\end{vmatrix},\quad m\geqslant 2.\end{aligned} \tag{4.12c}$$

In this way, we obtain the conserved integrals $\{F_m^{(i)}\}$, $0\leqslant i\leqslant 2$, of the Hamiltonian system (3.12). The Poisson bracket of two functions in the symplectic space $(\mathcal{R}^{6N},\Sigma_{i=1}^3 dp^i\wedge dq^i)$ is defined as

$$\{f,g\}=\sum_{j=1}^{N}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial q_j^i}\frac{\partial g}{\partial p_j^i}-\frac{\partial f}{\partial p_j^i}\frac{\partial g}{\partial q_j^i}\right)=\sum_{i=1}^{3}\left(\left\langle\frac{\partial f}{\partial q^i},\frac{\partial g}{\partial p^i}\right\rangle-\left\langle\frac{\partial f}{\partial p^i},\frac{\partial g}{\partial q^i}\right\rangle\right).$$

We can prove the following assertions:

Theorem 4.1: The functions $\{F_m^{(i)}\}$, $0\leqslant i\leqslant 2$, $m\geqslant 0$, are in involution in pairs, $\{F_m^{(i)},F_l^{(j)}\}=0$, $0\leqslant i,\ j\leqslant 2$, for any $m,l\geqslant 0$.

Theorem 4.2: The $3N$ 1-forms $dF_l^{(i)}$, $1\leqslant l\leqslant N$, $0\leqslant i\leqslant 2$, are linearly independent. The proof of the above two theorems is given in the Appendix.

A direct calculation shows that the Hamiltonian function H of the system (3.12) can be rewritten as follows:

$$
\begin{aligned}
H=&\gamma_0F_1^{(0)}+\gamma_1F_1^{(1)}+\gamma_2F_1^{(2)}+\gamma_3(\beta_1^2F_0^{(0)}-\beta_1F_0^{(1)}+F_0^{(2)})(\beta_2^2F_0^{(0)}\beta_2F_0^{(1)}+F_0^{(2)})\\
&+\gamma_4(\beta_1^2F_0^{(0)}-\beta_1F_0^{(1)}+F_0^{(2)})(\beta_3^2F_0^{(0)}-\beta_3F_0^{(1)}\\
&+F_0^{(2)})+\gamma_5(\beta_2^2F_0^{(0)}-\beta_2F_0^{(1)}+F_0^{(2)})(\beta_3^2F_0^{(0)}-\beta_3F_0^{(1)}+F_0^{(2)}),
\end{aligned}
\tag{4.13}
$$

where

$$\gamma_0=\frac{\alpha_1(\beta_2-\beta_3)\beta_1^2+\alpha_2(\beta_3-\beta_1)\beta_2^2+\alpha_3(\beta_1-\beta_2)\beta_3^2}{(\beta_1-\beta_2)(\beta_2-\beta_3)(\beta_1-\beta_3)},$$

$$\gamma_1=-\frac{\alpha_1(\beta_2-\beta_3)\beta_1+\alpha_2(\beta_3-\beta_1)\beta_2+\alpha_3(\beta_1-\beta_2)\beta_3}{(\beta_1-\beta_2)(\beta_2-\beta_3)(\beta_1-\beta_3)},$$

$$\gamma_2=\frac{\alpha_1(\beta_2-\beta_3)+\alpha_2(\beta_3-\beta_1)+\alpha_3(\beta_1-\beta_2)}{(\beta_1-\beta_2)(\beta_2-\beta_3)(\beta_1-\beta_3)},\quad \gamma_3=\frac{\alpha_2-\alpha_1}{(\beta_1-\beta_2)^3(\beta_1-\beta_3)(\beta_2-\beta_3)},$$

$$\gamma_4=\frac{\alpha_1-\alpha_3}{(\beta_1-\beta_3)^3(\beta_1-\beta_2)(\beta_2-\beta_3)},\quad \gamma_5=\frac{\alpha_3-\alpha_2}{(\beta_2-\beta_3)^3(\beta_1-\beta_2)(\beta_1-\beta_3)}.$$

Hence the integrability of Eq. (3.12) is established resorting to Theorems 4.1 and 4.2.

Theorem 4.3: The finite-dimensional Hamiltonian system (3.12) is completely integrable in the Liouville sense.

V. THE INVOLUTIVE REPRESENTATION OF SOLUTIONS

In this section, we shall give the involutive representation of solutions of the AKNS soliton hierarchy. We first introduce the Lenard gradients $g_l^{(i)}$ defined recursively by

$$Kg_{l-1}^{(i)}=Jg_l^{(i)},\quad g_l^{(i)}|_{u=0}=0,\quad 1\leqslant i\leqslant 3,\quad l\geqslant 1,\tag{5.1}$$

with

$$g_0^{(1)}=\left(\frac{u_{21}}{\alpha_1-\alpha_2},\frac{u_{12}}{\alpha_1-\alpha_2},\frac{u_{31}}{\alpha_1-\alpha_3},\frac{u_{13}}{\alpha_1-\alpha_3},0,0\right)^T,\tag{5.2}$$

$$g_0^{(2)}=\left(\frac{u_{21}}{\alpha_2-\alpha_1},\frac{u_{12}}{\alpha_2-\alpha_1},0,0,\frac{u_{32}}{\alpha_2-\alpha_3},\frac{u_{23}}{\alpha_2-\alpha_3}\right)^T,\tag{5.3}$$

$$g_0^{(3)}=\left(0,0,\frac{u_{31}}{\alpha_3-\alpha_1},\frac{u_{13}}{\alpha_3-\alpha_1},\frac{u_{32}}{\alpha_3-\alpha_2},\frac{u_{23}}{\alpha_3-\alpha_2}\right)^T.\tag{5.4}$$

The corresponding mth order vector is represented by

$$X_m(u,\omega)=J(\omega_1g_m^{(1)}+\omega_2g_m^{(2)}+\omega_3g_m^{(3)}),\quad \omega=(\omega_1,\omega_2,\omega_3),\tag{5.5}$$

from which it is easy to see that $X_m=X_m(u,\beta)$, $\beta=(\beta_1,\beta_2,\beta_3)$. Now we consider the canonical system of the $\bar{F}_m^{(1)}$-flow with $\bar{F}_m^{(1)}=-F_m^{(1)}$,

$$\frac{\partial q^i}{\partial t_m}=\frac{\partial\bar{F}_m^{(1)}}{\partial p^i},\quad \frac{\partial p^i}{\partial t_m}=-\frac{\partial\bar{F}_m^{(1)}}{\partial q^i},\quad 1\leqslant i\leqslant 3,\quad m\geqslant 1.\tag{5.6}$$

Then the systems (3.12) and (5.6) are compatible and their Hamiltonian phase flows $g_H^x, g_{\bar{F}_m}^{t_m}$ commute,[26] which imply that there exists the involutive solution[26,9] of the consistent system of Eqs. (3.12) and (5.6), represented by

$$q^i(x,t_m)=q_H^x g_{\bar{F}_m}^{t_m} q^i(0,0),\ \ p^i(x,t_m)=g_H^x g_{\bar{F}_m}^{t_m} p^i(0,0),\ \ 1\leqslant i\leqslant 3.$$

Here $q^i(0,0),p^i(0,0)$, $1\leqslant i\leqslant 3$ are the given initial values.

Theorem 5.1: Let $\lambda_1,\ldots,\lambda_N$ be N distinct parameters. If $(q^i(x,t_1),p^i(x,t_1))$, $1\leqslant i\leqslant 3$, is an involutive solution of the system of Eqs. (3.12) and (5.6) with $m=1$, then

$$u_{ij}(x,t_1)=\frac{\alpha_i-\alpha_j}{\beta_i-\beta_j}\langle q^i(x,t_1),p^j(x,t_1)\rangle,\ \ 1\leqslant i,\ j\leqslant 3,\ i\neq j, \tag{5.7}$$

solve the three-wave interaction equations

$$u_{ijt_1}=\frac{\beta_i-\beta_j}{\alpha_i-\alpha_j}u_{ijx}+\sum_{\substack{k=1\\k\neq i,j}}^{3}\left(\frac{\beta_k-\beta_i}{\alpha_k-\alpha_i}-\frac{\beta_k-\beta_j}{\alpha_k-\alpha_j}\right)u_{ik}u_{kj}+a_{ij}u_{ij},\ \ i\neq j,\ 1\leqslant i,\ j\leqslant 3, \tag{5.8}$$

where $a_{ij}=\langle q^i(0,0),p^i(0,0)\rangle-\langle q^j(0,0),p^j(0,0)\rangle$, $i\neq j$, $1\leqslant i$, $j\leqslant 3$, are constants independent of x,t.

Proof: Using Eqs. (4.10), (4.11a), and (4.12a), we have

$$\langle q^1,p^1\rangle=\frac{F_0^{(2)}-\beta_1F_0^{(1)}+\beta_1^2F_0^{(0)}}{(\beta_1-\beta_2)(\beta_1-\beta_3)},\quad \langle q^2,p^2\rangle=\frac{-F_0^{(2)}+\beta_2F_0^{(1)}-\beta_2^2F_0^{(0)}}{(\beta_1-\beta_2)(\beta_2-\beta_3)},$$

$$\langle q^3,p^3\rangle=\frac{F_0^{(2)}-\beta_3F_0^{(1)}+\beta_3^2F_0^{(0)}}{(\beta_1-\beta_3)(\beta_2-\beta_3)},$$

which imply that $\langle q^i,p^i\rangle=\langle q^i(0,0),p^i(0,0)\rangle$, $1\leqslant i\leqslant 3$, are constants. Noticing Eqs. (3.12) and (5.6) with $m=1$, through direct calculations we get

$$\langle q^i,p^j\rangle_x=(\alpha_i-\alpha_j)\langle\Lambda q^i,p^j\rangle-\frac{\alpha_i-\alpha_j}{\beta_i-\beta_j}a_{ij}\langle q^i,p^j\rangle+\sum_{\substack{k=1\\k\neq i,j}}^{3}\left(\frac{\alpha_k-\alpha_i}{\beta_k-\beta_i}-\frac{\alpha_k-\alpha_j}{\beta_k-\beta_j}\right)\langle q^i,p^k\rangle\langle q^k,p^j\rangle,$$

$$\langle q^i,p^j\rangle_{t_1}=(\beta_i-\beta_j)\langle\Lambda q^i,p^j\rangle,\ \ i\neq j,1\leqslant i,j\leqslant 3.$$

Combining these results together yield the desired three-wave interaction equations (5.8).

Generally, we have the following fact:

Theorem 5.2: Let $\lambda_1,\ldots,\lambda_N$ be N distinct parameters. If $(q^i(x,t_m),p^i(x,t_m))$, $1\leqslant i\leqslant 3$, is an involutive solution of the system of Eqs. (3.12) and (5.6), then

$$u_{ij}(x,t_m)=\frac{\alpha_i-\alpha_j}{\beta_i-\beta_j}\langle q^i(x,t_m),p^j(x,t_m)\rangle,\ \ 1\leqslant i,j\leqslant 3,i\neq j, \tag{5.9}$$

satisfy the soliton equations

$$u_{t_m}=X_m(u,\beta)+X_{m-1}(u,\sigma^{(0)})+\cdots+X_0(u,\sigma^{(m-1)}),\ \ m\geqslant 0, \tag{5.10}$$

with suitably chosen constant vectors $\sigma^{(l)}=(\sigma_1^{(l)},\sigma_2^{(l)},\sigma_3^{(l)})$, $0\leqslant l\leqslant m-1$.

Proof: By using Eqs. (5.9) and (5.6), a direct calculation gives

$$u_{t_m}=J\sum_{l=1}^{N}\lambda_l^m\nabla\lambda_l. \tag{5.11}$$

Operating with the operator $(J^{-1}K)^m$ upon the constraint $G_0=\Sigma_{l=1}^{N}\nabla\lambda_l$, we have

$$G_m+\sum_{l=0}^{m-1}\left(\sigma_1^{(l)}g_{m-l-1}^{(1)}+\sigma_2^{(l)}g_{m-l-1}^{(2)}+\sigma_3^{(l)}g_{m-l-1}^{(3)}\right)=\sum_{l=1}^{N}\lambda_l^m\nabla\lambda_l, \tag{5.12}$$

where $\sigma_i^{(l)}$, $0\leqslant l\leqslant m-1$, $1\leqslant i\leqslant 3$, are integral constants. Substituting Eq. (5.12) into Eq. (5.11) and noticing Eq. (5.5), we obtain Eq. (5.10). The proof is finished.

VI. SUMMARY AND CONCLUSIONS

The procedure for the nonlinearization of the $n\times n$ matrix spectral problem and its adjoint spectral problem has been described briefly. To illustrate the general principles, the nonlinearization of the 3×3 AKNS matrix spectral problem and its adjoint spectral problem associated with the three-wave interaction equations is discussed in detail. The characteristic polynomial of solution matrix of the stationary zero-curvature equation is used to generate involutive system of enough conserved integrals of the resulting finite-dimensional Hamiltonian system. This scheme is general, which is suitable for the other systems. It is interesting that the canonical equations of the $F_m^{(1)}$-flow (up to a constant factor) given by $\mathcal{F}_\lambda^{(1)}$ of Eq. (4.1) are exactly the nonlinearized temporal parts of the Lax pairs and adjoint Lax pairs for soliton hierarchy related to the 3×3 AKNS matrix spectral problem. The solutions of the three-wave interaction equations are reduced to solving the two compatible systems of ordinary differential equations. Two generators of involutive systems of conserved integrals are introduced, from which the functional independence of conserved integrals is rigorously proved (see Appendix). We point out that the method used here is general, which is suitable for the cases of $n\times n$ matrix spectral problems. Similar results will be left to a future publication. Moreover, we may also consider construction of action-angle variables for the finite dimensional integrable system and further give the finite-band solutions for the three-wave interaction equations, which will be discussed in other papers.

APPENDIX:

1. The proof of theorem 4.1

In order to prove theorem 4.1, We first introduce the notations

$$I_\lambda^{(1)}=(\beta_2+\beta_3)\overset{11}{Q}_\lambda+(\beta_1+\beta_3)\overset{22}{Q}_\lambda+(\beta_1+\beta_2)\overset{33}{Q}_\lambda, \tag{A1}$$

$$T_\lambda^{(1)}=\sum_{1\leqslant i<j\leqslant 3}\left(\overset{ii}{Q}_\lambda\overset{jj}{Q}_\lambda-\overset{ij}{Q}_\lambda\overset{ji}{Q}_\lambda\right), \tag{A2}$$

$$I_\lambda^{(2)}=\beta_2\beta_3\overset{11}{Q}_\lambda+\beta_1\beta_3\overset{22}{Q}_\lambda+\beta_1\beta_2\overset{33}{Q}_\lambda, \tag{A3}$$

$$T_\lambda^{(2)}=\sum_{1\leqslant i<j\leqslant 3}\beta_{6-i-j}\left(\overset{ii}{Q}_\lambda\overset{jj}{Q}_\lambda-\overset{ij}{Q}_\lambda\overset{ji}{Q}_\lambda\right), \tag{A4}$$

$$R_\lambda=\begin{vmatrix} \overset{11}{Q}_\lambda & \overset{12}{Q}_\lambda & \overset{13}{Q}_\lambda \\ \overset{21}{Q}_\lambda & \overset{22}{Q}_\lambda & \overset{23}{Q}_\lambda \\ \overset{31}{Q}_\lambda & \overset{32}{Q}_\lambda & \overset{33}{Q}_\lambda \end{vmatrix}, \tag{A5}$$

and prove the following several assertions:

Lemma A.1: Let

$$I_\lambda(\sigma)=\sigma_1\overset{11}{Q}_\lambda+\sigma_2\overset{22}{Q}_\lambda+\sigma_3\overset{33}{Q}_\lambda,\quad \sigma=(\sigma_1,\sigma_2,\sigma_3)\in C^3. \tag{A6}$$

Then we have

$$\{I_\mu(\sigma),I_\lambda(\tau)\}=0,\quad \tau=(\tau_1,\tau_2,\tau_3)\in C^3, \tag{A7}$$

$$\begin{aligned}(\mu-\lambda)\{I_\mu(\sigma),R_\lambda\}=&(\sigma_1-\sigma_2)[\overset{12}{Q}_\mu(\overset{23}{Q}_\lambda\overset{31}{Q}_\lambda-\overset{21}{Q}_\lambda\overset{33}{Q}_\lambda)-\overset{21}{Q}_\mu(\overset{13}{Q}_\lambda\overset{32}{Q}_\lambda-\overset{12}{Q}_\lambda\overset{33}{Q}_\lambda)]+(\sigma_1-\sigma_3)\\ &\times[\overset{13}{Q}_\mu(\overset{21}{Q}_\lambda\overset{32}{Q}_\lambda-\overset{22}{Q}_\lambda\overset{31}{Q}_\lambda)-\overset{31}{Q}_\mu(\overset{12}{Q}_\lambda\overset{23}{Q}_\lambda-\overset{13}{Q}_\lambda\overset{22}{Q}_\lambda)]+(\sigma_2-\sigma_3)\\ &\times[\overset{23}{Q}_\mu(\overset{12}{Q}_\lambda\overset{31}{Q}_\lambda-\overset{11}{Q}_\lambda\overset{32}{Q}_\lambda)-\overset{32}{Q}_\mu(\overset{13}{Q}_\lambda\overset{21}{Q}_\lambda-\overset{11}{Q}_\lambda\overset{23}{Q}_\lambda)],\quad \forall\lambda,\mu\in C.\end{aligned} \tag{A8}$$

Proof: By using the definition of the Poisson bracket, we have

$$\{\overset{ii}{Q}_\mu,\overset{ii}{Q}_\lambda\}=\left\langle\frac{\partial\overset{ii}{Q}_\mu}{\partial q^i},\frac{\partial\overset{ii}{Q}_\lambda}{\partial p^i}\right\rangle-\left\langle\frac{\partial\overset{ii}{Q}_\mu}{\partial p^i},\frac{\partial\overset{ii}{Q}_\lambda}{\partial q^i}\right\rangle\quad \{\overset{ii}{Q}_\mu,\overset{jj}{Q}_\lambda\}=0,\quad (i\neq j),$$

which together with the equalities

$$\frac{\partial\overset{ij}{Q}_\lambda}{\partial q^k}=\delta_{ik}(\lambda-\Lambda)^{-1}p^j,\quad \frac{\partial\overset{ij}{Q}_\lambda}{\partial p^k}=\delta_{jk}(\lambda-\Lambda)^{-1}q^i,\quad 1\leq i,j,k\leq 3 \tag{A9}$$

leads to

$$\{\overset{ii}{Q}_\mu,\overset{jj}{Q}_\lambda\}=0,\quad 1\leq i,j\leq 3. \tag{A10}$$

Resorting to Eq. (A10) and the bilinear property of the Poisson bracket, a direct calculation shows that Eq. (A7) holds. Let

$$f_1(\mu,\lambda)=(\mu-\lambda)\sum_{k=1}^{3}\sigma_k\left(\left\langle\frac{\partial\overset{kk}{Q}_\mu}{\partial q^k},\frac{\partial R_\lambda}{\partial p^k}\right\rangle-\left\langle\frac{\partial\overset{kk}{Q}_\mu}{\partial p^k},\frac{\partial R_\lambda}{\partial q^k}\right\rangle\right). \tag{A11}$$

It is easy to see that

$$(\mu-\lambda)\{I_\mu(\sigma),R_\lambda\}=f_1(\mu,\lambda). \tag{A12}$$

In the following calculations of the *Mathematica*, we usually write $\overset{ij}{Q}_\lambda = Q_{\lambda,ij}$ for the sake of convenience. By using the *Mathematica*, we can verify Eq. (A8),

$$a_{11}[\lambda\_]:=Q_{\lambda,22}Q_{\lambda,33}-Q_{\lambda,23}Q_{\lambda,32};\quad a_{12}[\lambda\_]:=Q_{\lambda,23}Q_{\lambda,31}-Q_{\lambda,21}Q_{\lambda,33};$$

$$a_{13}[\lambda\_]:=Q_{\lambda,21}Q_{\lambda,32}-Q_{\lambda,22}Q_{\lambda,31};\quad a_{21}[\lambda\_]:=Q_{\lambda,13}Q_{\lambda,32}-Q_{\lambda,12}Q_{\lambda,33};$$

$$a_{22}[\lambda\_]:=Q_{\lambda,11}Q_{\lambda,33}-Q_{\lambda,13}Q_{\lambda,31};\quad a_{23}[\lambda\_]:=Q_{\lambda,12}Q_{\lambda,31}-Q_{\lambda,11}Q_{\lambda,32};$$

$$a_{31}[\lambda\_]:=Q_{\lambda,12}Q_{\lambda,23}-Q_{\lambda,13}Q_{\lambda,22};\quad a_{32}[\lambda\_]:=Q_{\lambda,13}Q_{\lambda,21}-Q_{\lambda,11}Q_{\lambda,23};$$

$$a_{33}[\lambda\_]:=Q_{\lambda,11}Q_{\lambda,22}-Q_{\lambda,12}Q_{\lambda,21};$$

$$\begin{aligned} f_1[\mu\_,\lambda\_]:=\sigma_1(&a_{21}[\lambda](Q_{\lambda,21}-Q_{\mu,21})+a_{31}[\lambda](Q_{\lambda,31}-Q_{\mu,31})-a_{12}[\lambda](Q_{\lambda,12}-Q_{\mu,12})-a_{13}[\lambda]\\ &\times(Q_{\lambda,13}-Q_{\mu,12}))+\sigma_2(a_{12}[\lambda](Q_{\lambda,12}-Q_{\mu,12})+a_{32}[\lambda](Q_{\lambda,32}-Q_{\mu,32})-a_{21}[\lambda]\\ &\times(Q_{\lambda,21}-Q_{\mu,21})-a_{23}[\lambda](Q_{\lambda,23}-Q_{\mu,23}))+\sigma_3(a_{13}[\lambda](Q_{\lambda,13}-Q_{\mu,13})+a_{23}[\lambda]\\ &\times(Q_{\lambda,23}-Q_{\mu,23})-a_{31}[\lambda]Q_{\lambda,31}-Q_{\mu,31})-a_{32}[\lambda](Q_{\lambda,32}-Q_{\mu,32}));\end{aligned}$$

$$\begin{aligned} g_1=&(\sigma_1-\sigma_2)(a_{12}[\lambda]Q_{\mu,12}-a_{21}[\lambda]Q_{\mu,21})+(\sigma_1-\sigma_3)(a_{13}[\lambda]Q_{\mu,13}-a_{31}[\lambda]Q_{\mu,31})\\ &+(\sigma_2-\sigma_3)(a_{23}[\lambda]Q_{\mu,23}-a_{32}[\lambda]Q_{\mu,32});\end{aligned}$$

$$\text{Simplify}[f_1[\mu,\lambda]-g_1]$$

$$\text{Out}[1]=0.$$

Here the expression of $f_1[\mu\_,\lambda\_]$ can be obtained by substituting Eq. (A5) into Eq. (A11) and using Eq. (A9) and the equality

$$\langle(\mu-\Lambda)^{-1}(\lambda-\Lambda)^{-1}q^i,p^j\rangle=(\mu-\lambda)^{-1}(\overset{ij}{Q}_\lambda-\overset{ij}{Q}_\mu). \tag{A13}$$

Lemma A.2: Let

$$T_\lambda(\sigma)=\sum_{1\leq i<j\leq 3}\sigma_{6-i-j}(\overset{ii}{Q}_\lambda\overset{jj}{Q}_\lambda-\overset{ij}{Q}_\lambda\overset{ji}{Q}_\lambda),\quad \sigma=(\sigma_1,\sigma_2,\sigma_3)\in C^3. \tag{A14}$$

Then we have

$$\begin{aligned}(\mu-\lambda)\{I_\mu(\sigma),T_\lambda(\tau)\}=&\tau_3(\sigma_1-\sigma_2)(\overset{21}{Q}_\mu\overset{12}{Q}_\lambda-\overset{12}{Q}_\mu\overset{21}{Q}_\lambda)+\tau_2(\sigma_1-\sigma_3)\\ &\times(\overset{31}{Q}_\mu\overset{13}{Q}_\lambda-\overset{13}{Q}_\mu\overset{31}{Q}_\lambda)+\tau_1(\sigma_2-\sigma_3)(\overset{32}{Q}_\mu\overset{23}{Q}_\lambda-\overset{23}{Q}_\mu\overset{32}{Q}_\lambda),\end{aligned} \tag{A15}$$

$$\begin{aligned}(\mu-\lambda)\{T_\mu(\tau),T_\lambda(\sigma)\}=&\tau_1\sigma_2W_1(\mu,\lambda)+\tau_2\sigma_1W_1(\lambda,\mu)+\tau_2\sigma_3W_2(\mu,\lambda)\\ &+\tau_3\sigma_2W_2(\lambda,\mu)+\tau_1\sigma_3W_3(\mu,\lambda)+\tau_3\sigma_1W_3(\lambda,\mu),\\ &\forall\lambda,\mu\in C,\tau=(\tau_1,\tau_2,\tau_3)\in C^3,\end{aligned} \tag{A16}$$

where

$$W_1(\mu,\lambda)=Q_\lambda^{31}(Q_\mu^{13}Q_\mu^{22}-Q_\mu^{12}Q_\mu^{23})+Q_\lambda^{13}(Q_\mu^{32}Q_\mu^{21}-Q_\mu^{22}Q_\mu^{31})$$
$$+Q_\mu^{32}(Q_\lambda^{11}Q_\lambda^{23}-Q_\lambda^{13}Q_\lambda^{21})+Q_\mu^{23}(Q_\lambda^{12}Q_\lambda^{31}-Q_\lambda^{11}Q_\lambda^{32}),$$

$$W_2(\mu,\lambda)=Q_\lambda^{12}(Q_\mu^{21}Q_\mu^{33}-Q_\mu^{23}Q_\mu^{31})+Q_\lambda^{21}(Q_\mu^{13}Q_\mu^{32}-Q_\mu^{12}Q_\mu^{33})$$
$$+Q_\mu^{13}(Q_\lambda^{22}Q_\lambda^{31}-Q_\lambda^{21}Q_\lambda^{32})+Q_\mu^{31}(Q_\lambda^{12}Q_\lambda^{23}-Q_\lambda^{13}Q_\lambda^{22}),$$

$$W_3(\mu,\lambda)=Q_\lambda^{21}(Q_\mu^{12}Q_\mu^{33}-Q_\mu^{13}Q_\mu^{32})+Q_\lambda^{12}(Q_\mu^{23}Q_\mu^{31}-Q_\mu^{21}Q_\mu^{33})$$
$$+Q_\mu^{23}(Q_\lambda^{11}Q_\lambda^{32}-Q_\lambda^{12}Q_\lambda^{31})+Q_\mu^{32}(Q_\lambda^{13}Q_\lambda^{21}-Q_\lambda^{11}Q_\lambda^{23}).$$

Proof: In a way similar to the proof of Eq. (A8), we can verify Eq. (A15). Now we prove Eq. (A16). Let

$$f_2(\mu,\lambda)=(\mu-\lambda)\sum_{k=1}^{3}\left(\left\langle\frac{\partial T_\mu(\tau)}{\partial q^k},\frac{\partial T_\lambda(\sigma)}{\partial p^k}\right\rangle-\left\langle\frac{\partial T_\mu(\tau)}{\partial p^k},\frac{\partial T_\lambda(\sigma)}{\partial q^k}\right\rangle\right). \tag{A17}$$

Then we have

$$(\mu-\lambda)\{T_\mu(\tau),T_\mu(\sigma)\}=f_2(\mu,\lambda). \tag{A18}$$

In the following, we shall prove by using the *Mathematica* that Eq. (A16) holds

$$b_1[\lambda\_]=\sigma_3Q_{\lambda,22}+\sigma_2Q_{\lambda,33};\quad b_2[\lambda\_]=\sigma_3Q_{\lambda,11}+\sigma_1Q_{\lambda,33};$$
$$b_3[\lambda\_]=\sigma_2Q_{\lambda,11}+\sigma_1Q_{\lambda,22};$$
$$c_1[\mu\_]=\tau_3Q_{\mu,22}+\tau_2Q_{\mu,33};\quad c_2[\mu\_]=\tau_3Q_{\mu,11}+\tau_1Q_{\mu,33};$$
$$c_3[\mu\_]=\tau_2Q_{\mu,11}+\tau_1Q_{\mu,22};$$

$$\begin{aligned}
f_2[\mu_,\lambda_]:=&\,c_1[\mu](b_1[\lambda](Q_{\lambda,11}-Q_{\mu,11})-\sigma_3Q_{\lambda,12}(Q_{\lambda,21}-Q_{\mu,21})-\sigma_2Q_{\lambda,13}(Q_{\lambda,31}-Q_{\mu,31}))\\
&-\tau_3Q_{\mu,21}(b_1[\lambda](Q_{\lambda,12}-Q_{\mu,12})-\sigma_3Q_{\lambda,12}(Q_{\lambda,22}-Q_{\mu,22})-\sigma_2Q_{\lambda,13}(Q_{\lambda,32}-Q_{\mu,32}))\\
&-\tau_2Q_{\mu,31}(b_1[\lambda](Q_{\lambda,13}-Q_{\mu,13})-\sigma_3Q_{\lambda,12}(Q_{\lambda,23}-Q_{\mu,23})-\sigma_2Q_{\lambda,13}(Q_{\lambda,33}-Q_{\mu,33}))\\
&+c_2[\mu](b_2[\lambda](Q_{\lambda,22}-Q_{\mu,22})-\sigma_3Q_{\lambda,21}(Q_{\lambda,12}-Q_{\mu,12})-\sigma_1Q_{\lambda,23}(Q_{\lambda,32}-Q_{\mu,32}))\\
&-\tau_3Q_{\mu,12}(b_2[\lambda](Q_{\lambda,21}-Q_{\mu,21})-\sigma_3Q_{\lambda,21}(Q_{\lambda,11}-Q_{\mu,11})-\sigma_1Q_{\lambda,23}(Q_{\lambda,31}-Q_{\mu,31}))\\
&-\tau_1Q_{\mu,32}(b_2[\lambda](Q_{\lambda,23}-Q_{\mu,23})-\sigma_3Q_{\lambda,21}(Q_{\lambda,13}-Q_{\mu,13})-\sigma_1Q_{\lambda,23}(Q_{\lambda,33}-Q_{\mu,33}))\\
&+c_3[\mu](b_3[\lambda](Q_{\lambda,33}-Q_{\mu,33})-\sigma_2Q_{\lambda,31}(Q_{\lambda,13}-Q_{\mu,13})-\sigma_1Q_{\lambda,32}(Q_{\lambda,23}-Q_{\mu,23}))\\
&-\tau_2Q_{\mu,13}(b_3[\lambda](Q_{\lambda,31}-Q_{\mu,31})-\sigma_2Q_{\lambda,31}(Q_{\lambda,11}-Q_{\mu,11})-\sigma_1Q_{\lambda,32}(Q_{\lambda,21}-Q_{\mu,21}))\\
&-\tau_1Q_{\mu,23}(b_3[\lambda](Q_{\lambda,32}-Q_{\mu,32})-\sigma_2Q_{\lambda,31}(Q_{\lambda,12}-Q_{\mu,12})-\sigma_1Q_{\lambda,32}(Q_{\lambda,22}-Q_{\mu,22}))\\
&+b_1[\lambda](c_1[\mu](Q_{\mu,11}-Q_{\lambda,11})-\tau_3Q_{\mu,12}(Q_{\mu,21}-Q_{\lambda,21})-\tau_2Q_{\mu,13}(Q_{\mu,31}-Q_{\lambda,31}))\\
&-\sigma_3Q_{\lambda,21}(c_1[\mu](Q_{\mu,12}-Q_{\lambda,12})-\tau_3Q_{\mu,12}(Q_{\mu,22}-Q_{\lambda,22})-\tau_2Q_{\mu,13}(Q_{\mu,32}-Q_{\lambda,32}))
\end{aligned}$$

$$
\begin{aligned}
&-\sigma_2 Q_{\lambda,31}(c_1[\mu](Q_{\mu,13}-Q_{\lambda,13})-\tau_3 Q_{\mu,12}(Q_{\mu,23}-Q_{\lambda,23})-\tau_2 Q_{\mu,13}(Q_{\mu,33}-Q_{\lambda,33}))\\
&+b_2[\lambda](c_2[\mu](Q_{\mu,22}-Q_{\lambda,22})-\tau_3 Q_{\mu,21}(Q_{\mu,12}-Q_{\lambda,12})-\tau_1 Q_{\mu,23}(Q_{\mu,32}-Q_{\lambda,32}))\\
&-\sigma_3 Q_{\lambda,12}(c_2[\mu](Q_{\mu,21}-Q_{\lambda,21})-\tau_3 Q_{\mu,21}(Q_{\mu,11}-Q_{\lambda,11})-\tau_1 Q_{\mu,23}(Q_{\mu,31}-Q_{\lambda,31}))\\
&-\sigma_1 Q_{\lambda,32}(c_2[\mu](Q_{\mu,23}-Q_{\lambda,23})-\tau_3 Q_{\mu,21}(Q_{\mu,13}-Q_{\lambda,13})-\tau_1 Q_{\mu,23}(Q_{\mu,33}-Q_{\lambda,33}))\\
&+b_3[\lambda](c_3[\mu](Q_{\mu,33}-Q_{\lambda,33})-\tau_2 Q_{\mu,31}(Q_{\mu,13}-Q_{\lambda,13})-\tau_1 Q_{\mu,32}(Q_{\mu,23}-Q_{\lambda,23}))\\
&-\sigma_2 Q_{\lambda,13}(c_3[\mu](Q_{\mu,31}-Q_{\lambda,31})-\tau_2 Q_{\mu,31}(Q_{\mu,11}-Q_{\lambda,11})-\tau_1 Q_{\mu,32}(Q_{\mu,21}-Q_{\lambda,21}))\\
&-\sigma_1 Q_{\lambda,23}(c_3[\mu](Q_{\mu,32}-Q_{\lambda,32})-\tau_2 Q_{\mu,31}(Q_{\mu,12}-Q_{\lambda,12})-\tau_1 Q_{\mu,32}(Q_{\mu,22}\\
&-Q_{\lambda,22}));
\end{aligned}
$$

$$W_1[\mu\_,\lambda\_]:=a_{13}[\mu]Q_{\lambda,13}-a_{31}[\mu]Q_{\lambda,31}-a_{32}[\lambda]Q_{\mu,32}+a_{23}[\lambda]Q_{\mu,23};$$

$$W_2[\mu\_,\lambda\_]:=a_{21}[\mu]Q_{\lambda,21}-a_{12}[\mu]Q_{\lambda,12}-a_{13}[\lambda]Q_{\mu,13}+a_{31}[\lambda]Q_{\mu,31};$$

$$W_3[\mu\_,\lambda\_]:=a_{12}[\mu]Q_{\lambda,12}-a_{21}[\mu]Q_{\lambda,21}-a_{23}[\lambda]Q_{\mu,23}+a_{32}[\lambda]Q_{\mu,32};$$

$$
\begin{aligned}
g_2=&\tau_1\sigma_2 W_1[\mu,\lambda]+\tau_2\sigma_1 W_1[\lambda,\mu]+\tau_2\sigma_3 W_2[\mu,\lambda]+\tau_3\sigma_2 W_2[\lambda,\mu]\\
&+\tau_1\sigma_3 W_3[\mu,\lambda]+\tau_3\sigma_1 W_3[\lambda,\mu];
\end{aligned}
$$

$$\text{Simplify}[f_2[\mu,\lambda]-g_2]$$

$$\text{Out}[2]=0.$$

Lemma A.3: Under the same assumption as the Lemma A.2, we have

$$
\begin{aligned}
(\mu-\lambda)\{T_\mu(\sigma),R_\lambda\}=&\sigma_1[Y_1(\mu,\lambda)-Y_2(\mu,\lambda)]+\sigma_2[Y_3(\mu,\lambda)-Y_1(\mu,\lambda)]\\
&+\sigma_3[Y_2(\mu,\lambda)-Y_3(\mu,\lambda)],
\end{aligned}
$$

$$\forall\lambda,\mu\in C. \tag{A19}$$

where

$$Y_1(\mu,\lambda)=(Q^{21}_\mu Q^{33}_\mu-Q^{23}_\mu Q^{31}_\mu)(Q^{13}_\lambda Q^{32}_\lambda-Q^{12}_\lambda Q^{33}_\lambda)+(Q^{13}_\mu Q^{32}_\mu-Q^{12}_\mu Q^{33}_\mu)(Q^{23}_\lambda Q^{31}_\lambda-Q^{21}_\lambda Q^{33}_\lambda),$$

$$Y_2(\mu,\lambda)=(Q^{22}_\mu Q^{13}_\mu-Q^{12}_\mu Q^{23}_\mu)(Q^{21}_\lambda Q^{32}_\lambda-Q^{22}_\lambda Q^{31}_\lambda)+(Q^{21}_\mu Q^{32}_\mu-Q^{22}_\mu Q^{31}_\mu)(Q^{12}_\lambda Q^{23}_\lambda-Q^{13}_\lambda Q^{22}_\lambda),$$

$$Y_3(\mu,\lambda)=(Q^{11}_\mu Q^{32}_\mu-Q^{12}_\mu Q^{31}_\mu)(Q^{13}_\lambda Q^{21}_\lambda-Q^{11}_\lambda Q^{23}_\lambda)+(Q^{13}_\mu Q^{21}_\mu-Q^{11}_\mu Q^{23}_\mu)(Q^{12}_\lambda Q^{31}_\lambda-Q^{11}_\lambda Q^{32}_\lambda).$$

Proof: Let

$$f_3(\mu,\lambda)=(\mu-\lambda)\sum_{k=1}^{3}\left(\left\langle\frac{\partial T(\sigma)_\mu}{\partial q^k},\frac{\partial R_\lambda}{\partial p^k}\right\rangle-\left\langle\frac{\partial T(\sigma)_\mu}{\partial p^k},\frac{\partial R_\lambda}{\partial q^k}\right\rangle\right). \tag{A20}$$

Then we have

$$(\mu-\lambda)\{T(\sigma)_\mu,R_\lambda\}=f_3(\mu,\lambda). \tag{A21}$$

With the help of the *Mathematica*, we can verify Eq. (A19)

$$
\begin{aligned}
f_3[\mu\_,\lambda\_]:=&b_1[\mu](a_{11}[\lambda](Q_{\lambda,11}-Q_{\mu,11})+a_{21}[\lambda](Q_{\lambda,21}-Q_{\mu,21})+a_{31}[\lambda](Q_{\lambda,31}-Q_{\mu,31}))\\
&-\sigma_3 Q_{\mu,21}(a_{11}[\lambda](Q_{\lambda,12}-Q_{\mu,12})+a_{21}[\lambda](Q_{\lambda,22}-Q_{\mu,22})+a_{31}[\lambda](Q_{\lambda,32}-Q_{\mu,32}))\\
&-\sigma_2 Q_{\mu,31}(a_{11}[\lambda](Q_{\lambda,13}-Q_{\mu,13})+a_{21}[\lambda](Q_{\lambda,23}-Q_{\mu,23})+a_{31}[\lambda](Q_{\lambda,33}-Q_{\mu,33}))\\
&-b_1[\mu](a_{11}[\lambda](Q_{\lambda,11}-Q_{\mu,11})+a_{12}[\lambda](Q_{\lambda,12}-Q_{\mu,12})+a_{13}[\lambda](Q_{\lambda,13}-Q_{\mu,13}))\\
&+\sigma_3 Q_{\mu,12}(a_{11}[\lambda](Q_{\lambda,21}-Q_{\mu,21})+a_{12}[\lambda](Q_{\lambda,22}-Q_{\mu,22})+a_{13}[\lambda](Q_{\lambda,23}-Q_{\mu,23}))\\
&+\sigma_2 Q_{\mu,13}(a_{11}[\lambda](Q_{\lambda,31}-Q_{\mu,31})+a_{12}[\lambda](Q_{\lambda,32}-Q_{\mu,32})+a_{13}[\lambda](Q_{\lambda,33}-Q_{\mu,33}))\\
&+b_2[\mu](a_{12}[\lambda](Q_{\lambda,2}-Q_{\mu,12})+a_{22}[\lambda](Q_{\lambda,22}-Q_{\mu,22})+a_{32}[\lambda](Q_{\lambda,32}-Q_{\mu,32}))\\
&-\sigma_3 Q_{\mu,12}(a_{12}[\lambda](Q_{\lambda,11}-Q_{\mu,11})+a_{22}[\lambda](Q_{\lambda,21}-Q_{\mu,21})+a_{32}[\lambda](Q_{\lambda,31}-Q_{\mu,31}))\\
&-\sigma_1 Q_{\mu,32}(a_{12}[\lambda](Q_{\lambda,13}-Q_{\mu,13})+a_{22}[\lambda](Q_{\lambda,23}-Q_{\mu,23})+a_{32}[\lambda](Q_{\lambda,33}-Q_{\mu,33}))\\
&-b_2[\mu](a_{21}[\lambda](Q_{\lambda,21}-Q_{\mu,21})+a_{22}[\lambda](Q_{\lambda,22}-Q_{\mu,22})+a_{23}[\lambda](Q_{\lambda,23}-Q_{\mu,23}))\\
&+\sigma_3 Q_{\mu,21}(a_{21}[\lambda](Q_{\lambda,11}-Q_{\mu,11})+a_{22}[\lambda](Q_{\lambda,12}-Q_{\mu,12})+a_{23}[\lambda](Q_{\lambda,13}-Q_{\mu,13}))\\
&+\sigma_1 Q_{\mu,23}(a_{21}[\lambda](Q_{\lambda,31}-Q_{\mu,31})+a_{22}[\lambda](Q_{\lambda,32}-Q_{\mu,32})+a_{23}[\lambda](Q_{\lambda,33}-Q_{\mu,33}))\\
&+b_3[\mu](a_{13}[\lambda](Q_{\lambda,13}-Q_{\mu,13})+a_{23}[\lambda](Q_{\lambda,23}-Q_{\mu,23})+a_{33}[\lambda](Q_{\lambda,33}-Q_{\mu,33}))\\
&-\sigma_2 Q_{\mu,13}(a_{13}[\lambda](Q_{\lambda,11}-Q_{\mu,11})+a_{23}[\lambda](Q_{\lambda,21}-Q_{\mu,21})+a_{33}[\lambda](Q_{\lambda,31}-Q_{\mu,31}))\\
&-\sigma_1 Q_{\mu,23}(a_{13}[\lambda](Q_{\lambda,12}-Q_{\mu,12})+a_{23}[\lambda](Q_{\lambda,22}-Q_{\mu,22})+a_{33}[\lambda](Q_{\lambda,32}-Q_{\mu,32}))\\
&-b_3[\mu](a_{31}[\lambda](Q_{\lambda,31}-Q_{\mu,31})+a_{32}[\lambda](Q_{\lambda,32}-Q_{\mu,32})+a_{33}[\lambda](Q_{\lambda,33}-Q_{\mu,33}))\\
&+\sigma_2 Q_{\mu,31}(a_{31}[\lambda](Q_{\lambda,11}-Q_{\mu,11})+a_{32}[\lambda](Q_{\lambda,12}-Q_{\mu,12})+a_{33}[\lambda](Q_{\lambda,13}-Q_{\mu,13}))\\
&+\sigma_1 Q_{\mu,32}(a_{31}[\lambda](Q_{\lambda,21}-Q_{\mu,21})+a_{32}[\lambda](Q_{\lambda,22}-Q_{\mu,22})+a_{33}[\lambda](Q_{\lambda,23}-Q_{\mu,23}));
\end{aligned}
$$

$$Y_1=a_{21}[\mu]a_{12}[\lambda]-a_{12}[\mu]a_{21}[\lambda]; \quad Y_2=a_{13}[\mu]a_{31}[\lambda]-a_{31}[\mu]a_{13}[\lambda];$$

$$Y_3=a_{32}[\mu]a_{23}[\lambda]-a_{23}[\mu]a_{32}[\lambda]; \quad g_3=\sigma_1(Y_1-Y_2)+\sigma_2(Y_3-Y_1)+\sigma_3(Y_2-Y_3);$$

$$\text{Simplify}[f_3[\mu,\lambda]-g_3]$$

$$\text{Out}[3]=0.$$

Lemma A.4:

$$\{T_\mu^{(1)},T_\lambda^{(1)}\}=0, \quad \{R_\mu,R_\lambda\}=0, \quad \{T_\mu^{(1)},R_\lambda\}=0, \quad \forall\lambda,\mu\in C. \tag{A22}$$

Proof: By Eqs. (A16) and (A19), we get that

$$\{T_\mu^{(1)},T_\lambda^{(1)}\}=\{T_\mu(\tau),T_\lambda(\sigma)\}|_{\tau=\sigma=(1,1,1)}=0, \quad \{T_\mu^{(1)},R_\lambda\}=\{T_\mu(\sigma),R_\lambda\}|_{\sigma=(1,1,1)}=0.$$

Let

$$f_4(\mu,\lambda)=(\mu-\lambda)\sum_{k=1}^{3}\left(\left\langle\frac{\partial R_\mu}{\partial q^k},\frac{\partial R_\lambda}{\partial p^k}\right\rangle\right). \tag{A23}$$

It is easy to see that

$$(\mu-\lambda)\{R_\mu,R_\lambda\}=f_4(\mu,\lambda)+f_4(\lambda,\mu). \tag{A24}$$

The second expression of Eq. (A22) can be verified by the *Mathematica*,

$$
\begin{aligned}
f_4[\mu\_,\lambda\_]:=&a_{11}[\mu](a_{11}[\lambda](Q_{\lambda,11}-Q_{\mu,11})+a_{11}[\lambda](Q_{\lambda,21}-Q_{\mu,21})+a_{31}[\lambda](Q_{\lambda,31}-Q_{\mu,31}))\\
&+a_{12}[\mu](a_{11}[\lambda](Q_{\lambda,12}-Q_{\mu,12})+a_{21}[\lambda](Q_{\lambda,22}-Q_{\mu,22})+a_{31}[\lambda](Q_{\lambda,32}-Q_{\mu,32}))\\
&+a_{13}[\mu](a_{11}[\lambda](Q_{\lambda,13}-Q_{\mu,13})+a_{21}[\lambda](Q_{\lambda,23}-Q_{\mu,23})+a_{31}[\lambda](Q_{\lambda,33}-Q_{\mu,33}))\\
&+a_{21}[\mu](a_{12}[\lambda](Q_{\lambda,11}-Q_{\mu,11})+a_{22}[\lambda](Q_{\lambda,21}-Q_{\mu,21})+a_{32}[\lambda](Q_{\lambda,31}-Q_{\mu,31}))\\
&+a_{22}[\mu](a_{12}[\lambda](Q_{\lambda,12}-Q_{\mu,12})+a_{22}[\lambda](Q_{\lambda,22}-Q_{\mu,22})+a_{32}[\lambda](Q_{\lambda,32}-Q_{\mu,32}))\\
&+a_{23}[\mu](a_{12}[\lambda](Q_{\lambda,13}-Q_{\mu,13})+a_{22}[\lambda](Q_{\lambda,23}-Q_{\mu,23})+a_{32}[\lambda](Q_{\lambda,33}-Q_{\mu,33}))\\
&+a_{31}[\mu](a_{13}[\lambda](Q_{\lambda,11}-Q_{\mu,11})+a_{23}[\lambda](Q_{\lambda,21}-Q_{\mu,21})+a_{33}[\lambda](Q_{\lambda,31}-Q_{\mu,31}))\\
&+a_{32}[\mu](a_{13}[\lambda](Q_{\lambda,12}-Q_{\mu,12})+a_{23}[\lambda](Q_{\lambda,22}-Q_{\mu,22})+a_{33}[\lambda](Q_{\lambda,32}-Q_{\mu,32}))\\
&+a_{33}[\mu](a_{13}[\lambda](Q_{\lambda,13}-Q_{\mu,13})+a_{23}[\lambda](Q_{\lambda,23}-Q_{\mu,23})+a_{33}[\lambda](Q_{\lambda,33}-Q_{\mu,33}));
\end{aligned}
$$

$$\mathrm{Simplify}[f_4[\mu,\lambda]+f_4[\lambda,\mu]]$$

$$\mathrm{Out}[4]=0.$$

Proposition A.5:

$$\{\mathcal{F}_\mu^{(i)},\mathcal{F}_\lambda^{(j)}\}=0,\quad 1\leqslant i,j\leqslant 3,\forall\lambda,\mu\in C, \tag{A25}$$

with $\mathcal{F}_\lambda^{(1)}=I_\lambda^{(1)}+T_\lambda^{(1)}$, $\mathcal{F}_\lambda^{(2)}=I_\lambda^{(2)}+T_\lambda^{(2)}+R_\lambda$.

Proof: It is easy to see that the equalities

$$\mathcal{F}_\lambda^{(0)}=I_\lambda(\sigma)|_{\sigma=(1,1,1)},\quad I_\lambda^{(1)}=I_\lambda(\sigma)|_{\sigma=(\beta_2+\beta_3,\beta_1+\beta_3,\beta_1+\beta_2)},$$

$$I_\lambda^{(2)}=I_\lambda(\sigma)|_{\sigma=(\beta_2\beta_3,\beta_1\beta_3,\beta_1\beta_2)},\quad T_\lambda^{(1)}=T_\lambda(\sigma)|_{\sigma=(1,1,1)},\quad T_\lambda^{(2)}=T_\lambda(\sigma)|_{\sigma=(\beta_1,\beta_2,\beta_3)}.$$

Obviously $\{\mathcal{F}_\mu^{(0)},\mathcal{F}_\lambda^{(0)}\}=0$ from Eq. (A7). Using Eq. (A7) and the symmetry of Eq. (A15), we obtain

$$\{I_\mu^{(i)},I_\lambda^{(i)}\}=0,\quad \{I_\mu^{(i)},T_\lambda^{(i)}\}+\{T_\mu^{(i)},I_\lambda^{(i)}\}=0,\quad i=1,2, \tag{A26}$$

which together with the first expression of Eq. (A22) imply $\{\mathcal{F}_\mu^{(1)},\mathcal{F}_\lambda^{(1)}\}=0$ resorting to the bilinear property of the Poisson bracket. Noticing Eqs. (A8), (A16), and (A19), we get that

$$\{I_\mu^{(2)},R_\lambda\}+\{R_\mu,I_\lambda^{(2)}\}+\{T_\mu^{(2)},T_\lambda^{(2)}\}=0,\quad \{T_\mu^{(2)},R_\lambda\}+\{R_\mu,T_\lambda^{(2)}\}=0, \tag{A27}$$

which together with Eqs. (A26) and (A22) lead up to $\{\mathcal{F}_\mu^{(2)},\mathcal{F}_\lambda^{(2)}\}=0$. In view of Eqs. (A7), (A8), and (A15), we obtain

$$\{\mathcal{F}_\mu^{(0)},I_\lambda^{(i)}\}=0,\quad \{\mathcal{F}_\mu^{(0)},R_\lambda\}=0,\quad \{\mathcal{F}_\mu^{(0)},T^{(i)}\}=0,\quad i=1,2,$$

$$\{I_\mu^{(1)},I_\lambda^{(2)}\}=0,\quad \{I_\mu^{(1)},R_\lambda\}+\{T_\mu^{(1)},T_\lambda^{(2)}\}=0,\quad \{I_\mu^{(1)},T_\lambda^{(2)}\}+\{T_\mu^{(1)},I_\lambda^{(2)}\}=0. \tag{A28}$$

By using Eq. (A28) and the third expression of Eq. (A22), we have

$$\{\mathcal{F}_\mu^{(0)},\mathcal{F}_\lambda^{(i)}\}=0,\quad \{\mathcal{F}_\mu^{(1)},\mathcal{F}_\lambda^{(2)}\}=0,\quad i=1,2.$$

The proof is completed.

Equation (4.9) and Proposition A.5 imply Theorem 4.1.

2. Two generators and the proof of Theorem 4.2

To prove Theorem 4.2, we introduce two generators of involutive systems of conserved integrals

$$\Gamma_l^{(ij)}=\sum_{\substack{k=1\\k\neq l}}^{N}\frac{B_{lk}^{(ij)}}{\lambda_l-\lambda_k},\quad B_{lk}^{(ij)}=(q_l^iq_k^j-q_k^iq_l^j)(p_l^ip_k^j-p_k^ip_l^j),\tag{A29}$$

$$\Upsilon_l=\sum_{\substack{s=1\\s\neq k}}^{N}\sum_{\substack{k=1\\k\neq l}}^{N}\frac{A_{lks}}{(\lambda_l-\lambda_k)(\lambda_k-\lambda_s)},\quad A_{lks}=\begin{vmatrix}q_l^1&q_l^2&q_l^3\\q_k^1&q_k^2&q_k^3\\q_s^1&q_s^2&q_s^3\end{vmatrix}\begin{vmatrix}p_l^1&p_l^2&p_l^3\\p_k^1&p_k^2&p_k^3\\p_s^1&p_s^2&p_s^3\end{vmatrix},\tag{A30}$$

which are two natural generalizations of the 2×2 case.[22]

Proposition A.6:

$$\overset{ii}{Q}_\lambda\overset{jj}{Q}_\lambda-\overset{ij}{Q}_\lambda\overset{ji}{Q}_\lambda=\sum_{l=1}^{N}\frac{\Gamma_l^{(ij)}}{\lambda-\lambda_l},\quad 1\leq i,j\leq 3,\tag{A31}$$

$$R_\lambda=\sum_{l=1}^{N}\frac{\Upsilon_l}{\lambda-\lambda_l}.\tag{A32}$$

Proof: Put the partial-fraction expansion of $\overset{ij}{Q}_\lambda$, i.e.,

$$\overset{ij}{Q}_\lambda=\sum_{l=1}^{N}\frac{q_l^ip_l^j}{\lambda-\lambda_l},\tag{A33}$$

into the right-hand side of Eq. (A31). The left-hand side of Eq. (A31) is obtained through direct calculations on account of

$$\frac{1}{(\lambda-\lambda_l)(\lambda-\lambda_k)}=\frac{1}{\lambda_l-\lambda_k}\left(\frac{1}{\lambda-\lambda_l}-\frac{1}{\lambda-\lambda_k}\right).\tag{A34}$$

Substituting Eq. (A33) into Eq. (A5) yields

$$\begin{aligned}R_\lambda&=\sum_{l=1}^{N}\sum_{k=1}^{N}\sum_{s=1}^{N}\frac{q_l^1q_k^2q_s^3}{(\lambda-\lambda_l)(\lambda-\lambda_k)(\lambda-\lambda_s)}\begin{vmatrix}p_l^1&p_l^2&p_l^3\\p_k^1&p_k^2&p_k^3\\p_s^1&p_s^2&p_s^3\end{vmatrix}\\&=\frac{1}{6}\sum_{l=1}^{N}\sum_{k=1}^{N}\sum_{s=1}^{N}\frac{A_{lks}}{(\lambda-\lambda_l)(\lambda-\lambda_k)(\lambda-\lambda_s)}.\end{aligned}$$

Resorting to Eq. (A34), we get

$$R_\lambda=\frac{1}{6}\sum_l \sum_k \sum_{s\neq k} \frac{A_{lks}}{(\lambda_k-\lambda_s)(\lambda-\lambda_l)}\left(\frac{1}{\lambda-\lambda_k}-\frac{1}{\lambda-\lambda_s}\right)$$
$$=\frac{1}{6}\sum_l \sum_{k\neq l} \sum_{s\neq k,l} \frac{A_{lks}}{\lambda_k-\lambda_s}\left[\frac{1}{\lambda_l-\lambda_k}\left(\frac{1}{\lambda-\lambda_l}-\frac{1}{\lambda-\lambda_k}\right)-\frac{1}{\lambda_l-\lambda_s}\left(\frac{1}{\lambda-\lambda_l}-\frac{1}{\lambda-\lambda_s}\right)\right]$$
$$=\frac{2}{6}\sum_{l=1}^N \frac{\Upsilon_l}{\lambda-\lambda_l}+\frac{2}{6}\sum_l \sum_k \sum_{s\neq k,l} \frac{A_{lks}}{(\lambda_k-\lambda_s)(\lambda_l-\lambda_s)(\lambda-\lambda_s)}$$
$$=\frac{2}{6}\sum_{l=1}^N \frac{\Upsilon_l}{\lambda-\lambda_l}+\frac{2}{6}\sum_l \sum_{k\neq l} \sum_{s\neq k,l} \frac{A_{lks}}{(\lambda-\lambda_s)(\lambda_l-\lambda_k)}\left(\frac{1}{\lambda_k-\lambda_s}-\frac{1}{\lambda_l-\lambda_s}\right)$$
$$=\sum_{l=1}^N \frac{\Upsilon_l}{\lambda-\lambda_l},$$

where it is used that $A_{llk}=A_{lkl}=A_{lkk}=0$, $A_{lks}=A_{lsk}=A_{skl}$. The proof is completed.

Proposition A.7: Let $E_1^{(i)},\ldots,E_N^{(i)}$ $(0\leqslant i\leqslant 2)$ defined as follows:

$$E_l^{(0)}=q_l^1p_l^1+q_l^2p_l^2+q_l^3p_l^3, \tag{A35}$$

$$E_l^{(1)}=(\beta_2+\beta_3)q_l^1p_l^1+(\beta_1+\beta_3)q_l^2p_l^2+(\beta_1+\beta_2)q_l^3p_l^3+\Gamma_l^{(12)}+\Gamma_l^{(13)}+\Gamma_l^{(23)}, \tag{A36}$$

$$E_l^{(2)}=\beta_2\beta_3q_l^1p_l^1+\beta_1\beta_3q_l^2p_l^2+\beta_1\beta_2q_l^3p_l^3+\beta_3\Gamma_l^{(12)}+\beta_2\Gamma_l^{(13)}+\beta_1\Gamma_l^{(23)}+\Upsilon_l. \tag{A37}$$

Then

$$F_m^{(i)}=\sum_{l=1}^N \lambda_l^m E_l^{(i)}, \quad 0\leqslant i\leqslant 2. \tag{A38}$$

Proof: By Proposition A.6 and Eq. (A33), it is easy to see that

$$\mathcal{F}_\lambda^{(i)}=\sum_{l=1}^N \frac{E_l^{(i)}}{\lambda-\lambda_l}. \tag{A39}$$

Expanding $(\lambda-\lambda_l)^{-1}$ as a power series in λ^{-1} and substituting into Eq. (A39), we obtain

$$\mathcal{F}_\lambda^{(i)}=\sum_{l=0}^\infty \lambda^{-m-1}\sum_{l=1}^N \lambda_l^m E_l^{(i)},$$

which together with Eq. (4.9) gives $F_m^{(i)}=\sum_{l=1}^N\lambda_l^mE_l^{(i)}$.

Proposition A.8: The $3N$ 1-forms $dE_l^{(i)}$, $1\leqslant l\leqslant N$, $0\leqslant i\leqslant 2$, are linearly independent.

Proof: Suppose that there exist $3N$ constants $c_l^{(i)}$, $1\leqslant l\leqslant N$, $0\leqslant i\leqslant 2$, satisfying

$$\sum_{l=1}^N (c_l^{(0)}dE_l^{(0)}+c_l^{(1)}dE_l^{(1)}+c_l^{(2)}dE_l^{(2)})=0. \tag{A40}$$

Then Eq. (A40) implies

$$\sum_{l=1}^N \left(c_l^{(0)}\frac{\partial E_l^{(0)}}{\partial p^i}+c_l^{(1)}\frac{\partial E_l^{(1)}}{\partial p^i}+c_l^{(2)}\frac{\partial E_l^{(2)}}{\partial p^i}\right)=0, \quad 1\leqslant i\leqslant 3. \tag{A41}$$

In order to deduce all constants $c_l^{(i)}=0$, $1\leqslant l\leqslant N$, $0\leqslant i\leqslant 2$, we demand the following equalities, which can be calculated directly:

$$\frac{\partial^2 E_l^{(0)}}{\partial q^i\partial p^i}=(\underbrace{0,\ldots,0}_{l-1},1,\underbrace{0,\ldots,0}_{N-l})^T,\quad 1\leqslant i\leqslant 3, \tag{A42}$$

$$\frac{\partial^2\Gamma_l^{(ij)}}{\partial q^i\partial p^i}=\left(\frac{q_l^jp_l^j}{\lambda_l-\lambda_1},\ldots,\frac{q_l^jp_l^j}{\lambda_l-\lambda_{l-1}},\sum_{\substack{k=1\\k\neq l}}^{N}\frac{q_k^jp_k^j}{\lambda_l-\lambda_k},\frac{q_l^jp_l^j}{\lambda_l-\lambda_{l+1}},\ldots,\frac{q_l^jp_l^j}{\lambda_l-\lambda_N}\right)^T, \tag{A43a}$$

$$\frac{\partial^2\Gamma_l^{(ij)}}{\partial q^j\partial p^j}=\left(\frac{q_l^ip_l^i}{\lambda_l-\lambda_1},\ldots,\frac{q_l^ip_l^i}{\lambda_l-\lambda_{l-1}},\sum_{\substack{k=1\\k\neq l}}^{N}\frac{q_k^ip_k^i}{\lambda_l-\lambda_k},\frac{q_l^ip_l^i}{\lambda_l-\lambda_{l+1}},\ldots,\frac{q_l^ip_l^i}{\lambda_l-\lambda_N}\right)^T,\quad 1\leqslant i<j\leqslant 3, \tag{A43b}$$

$$\partial^2\Gamma_l^{(ij)}/(\partial q^k\partial p^k)=0,\ \ k\neq i,j. \tag{A43c}$$

Now we introduce the operator

$$D_s^{(j)}=\mathrm{diag}\underbrace{(\partial^2/(\partial q_s^j\partial p_s^j),\ldots,\partial^2/(\partial q_s^j\partial p_s^j))}_{N},\quad 1\leqslant j\leqslant 3, 1\leqslant s\leqslant N.$$

Then we have from Eq. (A43) that

$$D_s^{(j)}\frac{\partial^2\Gamma_l^{(ij)}}{\partial q^i\partial p^i}=\begin{cases}\left(\frac{1}{\lambda_l-\lambda_1},\ldots,\frac{1}{\lambda_l-\lambda_{l-1}},0,\frac{1}{\lambda_l-\lambda_{l+1}},\ldots,\frac{1}{\lambda_l-\lambda_N}\right)^T, & l=s\\ 0,\quad l\neq s, 1\leqslant i<j\leqslant 3,\end{cases} \tag{A44a}$$

$$D_s^{(i)}\frac{\partial^2\Gamma_l^{(1j)}}{\partial q^1\partial p^1}=0,\quad D_s^{(j)}\frac{\partial^2\Gamma_l^{(23)}}{\partial q^1\partial p^1}=0,\quad D_s^{(3)}\frac{\partial^2\Gamma_l^{(1j)}}{\partial q^2\partial p^2}=0,\quad 2\leqslant i,j\leqslant 3, i\neq j. \tag{A44b}$$

Acting with the operator $D_s^{(j)}\partial/(\partial q^i)|_{q=p=0}$, $(1\leqslant i<j\leqslant 3, 1\leqslant s\leqslant N)$, upon Eq. (A41), we get

$$c_s^{(1)}+\beta_1c_s^{(2)}=0,\quad c_s^{(1)}+\beta_2c_s^{(2)}=0,\quad c_s^{(1)}+\beta_3c_s^{(2)}=0,\quad 1\leqslant s\leqslant N, \tag{A45}$$

in view of Eq. (A44) and the following equality:

$$D_s^{(j)}\frac{\partial^2\Upsilon_l}{\partial q^i\partial p^i}\bigg|_{q=p=0}=0,\quad q=(q^1,q^2,q^3),\quad p=(p^1,p^2,p^3),\quad 1\leqslant i,j\leqslant 3, 1\leqslant l,s\leqslant N.$$

Equations (A45) imply $c_s^{(1)}=c_s^{(2)}=0$, $1\leqslant s\leqslant N$. Operating with $\partial/(\partial q^i)$ on Eq. (A41) and noticing $c_l^{(1)}=c_l^{(2)}=0$, we derive $c_l^{(0)}=0$, $1\leqslant l\leqslant N$. Hence the $3N$ 1-forms $dE_l^{(i)}$, $0\leqslant i\leqslant 2$, $1\leqslant l\leqslant N$, are linearly independent.

The proof of Theorem 4.2: Assume that there exist $3N$ constants $b_m^{(i)}$, $1\leqslant m\leqslant N$, $0\leqslant i\leqslant 2$, so that

$$\sum_{m=1}^{N}(b_m^{(0)}dF_m^{(0)}+b_m^{(1)}dF_m^{(1)}+b_m^{(2)}dF_m^{(2)})=0. \tag{A46}$$

Substituting Eq. (A38) into Eq. (A46) and noting the independence of the $dE_l^{(i)}$, $1\leqslant l\leqslant N$, $0\leqslant i\leqslant 2$, we get

$$\sum_{m=1}^{N} b_m^{(i)}\lambda_l^{m-1}=0, \quad 1\leq l\leq N, 0\leq i\leq 2,$$

which implies $b_m^{(i)}=0$, $1\leq m\leq N$, $0\leq i\leq 2$, by utilizing that Vandermonde determinant is not zero. The proof is finished.

ACKNOWLEDGMENTS

Project 19671074 was supported by National Natural Science Foundation of China. One of the authors (X. Geng) would like to thank the Henan Science Foundation Committee of China for financial support. This work was partially supported by Hong Kong RGC/97-98/21.

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