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AN INTERIOR POINT CONTINUOUS PATH-FOLLOWING TRAJECTORY FOR LINEAR PROGRAMMING

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ABSTRACT. In this paper, an interior point continuous path-following trajectory is proposed for linear programming. The descent direction in our continuous trajectory can be viewed as some combination of the affine scaling direction and the centering direction for linear programming. A key component in our interior point continuous path-following trajectory is an ordinary differential equation (ODE) system. Various properties including the convergence in the limit for the solution of this ODE system are analyzed and discussed in detail. Several illustrative examples are also provided to demonstrate the numerical behavior of this continuous trajectory.

1. Introduction.

In this paper, we consider the following linear programming problem

$$\begin{array}{ll} \min & c^T x \\ s.t. & Ax = b,\ x \geq 0, \end{array} \tag{1}$$

where $x \in R^n$, $A \in R^{m \times n}$, $b \in R^m$, and $c \in R^n$. Our goal is to establish a continuous path starting from any interior feasible point and converging to an optimal solution of (1). Different from the iterative methods, the main idea in the continuous trajectory approach is to convert the optimization problem (1) into finding an equilibrium point of the following ordinary differential equation (ODE) system:

$$\frac{dx}{dt} = f(x,t), \quad t \geq t_0,\ x(t_0) = x_0, \tag{2}$$

where variable t is a scalar, $I \subset R$ denotes the maximal interval of existence of t for the ODE system (2), vector function $f : D = J \times I \subseteq R^n \times R \to R^n$ is a mapping defined on the product of a convex set J of R^n and I. The vector function $x(t)$ is a solution of the ODE system (2) on interval $I \subset R$. In the literature, there has been some non-interior point research work on ODE systems for optimization problems, see [3, 11, 5, 8, 9, 10, 14, 13].

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In addition, the neural network approach for optimization problems has been very active since 1980's, see the review paper [22] for more references.

Since the introduction of the interior point method by Karmarkar [21] in 1984, lots of research has been made both theoretically and numerically for solving the problem (1). These methods can be grouped into the following three main categories:

a) Affine scaling methods, originally due to Dikin [15], were further studied by Adler *et al*. [2], Barnes [4], Megiddo and Shub [24], Monma and Morton [25], Monterio *et al*. [27], Vanderbei *et al*. [40], Sun [35, 36].
b) Path-following methods, consisting of short-step and long-step methods, were studied by Gill *et al*. [17], Monterio and Adler [26], Roos [31], Roos and Vial [32], Gonzaga [18, 20], and so on.
c) Potential reduction methods, were studied by Gonzaga [18, 19], Freund [16], Monterio [29], and so on.

Interior point continuous trajectories for linear programming have been studied by Bayer and Lagarias [6, 7], Megiddo and Shub [24], Adler and Monteiro [26], Witzgall *et al*. [41], Monterio [29], Liao [23], and Qian and Liao [30]. In particular, [24] analyzed continuous trajectories for linear programming in the framework of primal-dual complementarity relationship. [29] analyzed the continuous trajectories corresponding to a polynomial potential reduction (PR) algorithm and showed that all PR trajectories converge to the unique center of the optimal face of the given optimization problem. Liao [23] studied a dual affine scaling continuous trajectory for (1). While Qian and Liao [30] discussed the primal affine scaling continuous trajectory for convex programming. In this paper, we propose an interior point continuous trajectory which belongs to the continuous path-following approach. The descent direction in our continuous trajectory can be viewed as some combination of the affine scaling direction and the centering direction. Our continuous trajectory can be represented by an ODE system in the form of (2). An in-depth and detailed investigation on the behavior of this continuous trajectory will be conducted, in particular, we will prove that this continuous trajectory converges to an optimal solution of the problem (1) in the limit for any interior feasible point.

The rest of this paper is organized as follows. In Section 2, some definitions and basic properties for linear programming are presented. In Section 3, our interior point continuous path-following trajectory, which can be represented by an ODE system, is introduced. Various properties for the solution of this ODE system are explored. In particular, we prove that for any starting interior feasible point, the solution of this ODE system will converge to an optimal solution of the problem (1) in the limit. Several illustrative examples are presented in Section 4. Finally, some concluding remarks are drawn in Section 5.

2. **Preliminaries and definitions.** In this section, some definitions and basic properties for the problem (1) will be presented.

Let $X = diag(x)$ denote the diagonal matrix with the components of x on the diagonal and $e \in R^n$ be the column vector of all one's. The pair of the primal linear programming and its dual are

$$
\begin{array}{llll}
(P) & \min & c^T x \\
 & s.t. & Ax = b, \\
 & & x \geq 0.
\end{array}
\qquad
\begin{array}{llll}
(D) & \max & b^T y \\
 & s.t. & A^T y + s = c, \\
 & & s \geq 0.
\end{array}
$$

Denoted by $\mathscr{P}$ and $\mathscr{D}$ are the feasible set of problems (P) and (D), respectively,

$$\mathscr{P} = \{x : Ax = b, x \geq 0\}, \quad \mathscr{D} = \{(y,s) : A^T y + s = c, s \geq 0\}.$$

The interiors of $\mathscr{P}$ and $\mathscr{D}$ are denoted by $\mathscr{P}^+$ and $\mathscr{D}^+$,

$$\mathscr{P}^+ = \{x : Ax = b, x > 0\}, \quad \mathscr{D}^+ = \{(y,s) : A^T y + s = c, s > 0\}.$$

The affine hulls of $\mathscr{P}$ and $\mathscr{D}$ are $\mathscr{P}_a$ and $\mathscr{D}_a$, respectively,

$$\mathscr{P}_a = \{x : x \in R^n, Ax = b\}, \quad \mathscr{D}_a = \{(y,s) : (y,s) \in R^m \times R^n, A^T y + s = c\}.$$

2.1. **Duality results and the central path.** Finding the optimal solutions of (P) and (D) is equivalent of solving the following equations:

$$Xs = 0, \tag{3a}$$
$$Ax = b, \quad x \geq 0, \tag{3b}$$
$$c - A^T y = s, \quad s \geq 0. \tag{3c}$$

In general, it is very hard to solve (3), because the first equation of (3) is nonlinear. By relaxing the right-hand side of (3a) by μe, $\mu > 0$, we can arrive at the following new system:

$$\begin{cases} Xs = \mu e, \\ Ax = b, \quad x \geq 0, \\ c - A^T y = s, \quad s \geq 0. \end{cases} \tag{4}$$

From the implicit function theorem, it can be easily verified that for any $\mu > 0$, there exists a unique (x, y, s) in (4). Let $(x(\mu), y(\mu), s(\mu))$ denote the unique solution of (4). These solutions are called the μ-centers of (P) and (D). The path that is formed by the set of all μ-centers is called the central path. In Section 3, we will propose a continuous path-following trajectory by applying Newton's method to solve this system.

3. **The interior point continuous path-following trajectory.**

In this section, we propose an interior point continuous path-following trajectory for the problem (1).

3.1. **The derivation of the trajectory.** First, let us state some assumptions.

Assumption 3.1. (a) There exists a point in $\mathscr{P}^+$. (b) The set of optimal solutions of the problem (1) is nonempty and bounded. (c) The matrix A has full row rank and c is not in the range space of A^T.

The above assumptions are standard in the literature.

Proposition 1. [28] *Assume that the strictly feasible set for (P) is nonempty, then the following conditions are equivalent:*

(a) The strictly feasible set for (D) is nonempty.
(b) The set of optimal solutions of (P) is nonempty and bounded.
(c) For any feasible point $\bar{x}$, the level set of $c^T \bar{x}$ is bounded.

We denote

$$Q = A^T (AX^2 A^T)^{-1} A,$$

for simplicity, then $\|XQX\| \leq 1$, and $P_{AX} = I_n - XQX$. Let us define

$$\bar{\chi} = \sup\{\|QX^2\| : X > 0\} = \sup\{\|A^T (AX^2 A^T)^{-1} AX^2\| : X > 0\}.$$

Stewart [34] and Todd [37] independently proved that this quantity is always finite.

Next, we derive our interior point continuous trajectory in more detail. Consider the following logarithmic barrier function $\phi(x)$:

$$\phi(x) = c^T x - \mu \sum_{j=1}^{n} \ln x_j. \tag{5}$$

Let $x \in \mathscr{P}^+$ and $\mu > 0$. The gradient of $\phi(x)$ and the Hessian matrix of $\phi(x)$ at x are denoted by $g(x)$ and $H(x)$ respectively, in particular

$$g(x) = c - \mu X^{-1}e \quad \text{and} \quad H(x) = \mu X^{-2}.$$

We denote the Newton step by Δx at x. The next step $x + \Delta x$ should be positive, and satisfies $A(x + \Delta x) = b$. Then Δx must satisfy $A\Delta x = 0$. We define the Newton step Δx at x to be the solution of

$$\begin{array}{ll} \min\limits_{\Delta x} & \Delta x^T g(x) + \frac{1}{2}\Delta x^T H(x)\Delta x \\ s.t. & A\Delta x = 0. \end{array}$$

Instantly, we get the solution Δx :

$$\Delta x = -\frac{1}{\mu} X P_{AX}(Xc - \mu e). \tag{6}$$

The projector P_{AX} is the projection onto the null space of AX and is given by

$$P_{AX} = I_n - XA^T(AX^2A^T)^{-1}AX. \tag{7}$$

Adopting the direction Δx in (6), we can establish our interior point continuous path-following trajectory $x(t)$ as the solution of

$$\frac{dx(t)}{dt} = f_\mu(x) = -XP_{AX}\{Xc - \mu(x)e\},\ x(t_0) = x_0 \in \mathscr{P}^+, \tag{8}$$

where

$$\mu(x) = \alpha(x)\beta(x)$$

with

$$\alpha(x) = \frac{\|XP_{AX}Xc\|}{|e^T P_{AX}Xc|\|XP_{AX}e\| + 2}, \qquad \beta(x) = \max(0, -e^T P_{AX}Xc). \tag{9}$$

From the ODE system (8), it is easy to check that $\frac{dc^T x}{dt} \le 0\ \forall t \ge t_0$. Thus, from Assumption 3.1 and Proposition 1, we have that if $x(t)$ is a solution of (8), $x(t)$ is bounded. As a result, it is easy to see that $\mu(x(t))$ is also bounded.

Lemma 3.2. *Let $x(t)$ be a solution of* (8)*. Then $x(t) > 0$ on its maximal existence interval* $[t_0,\ \alpha)$.

Proof: Since the right-hand side of (8) is continuous in $R^n_+ = \{x : x \in R^n, x > 0\}$, the Cauchy-Peano theorem ensures that there exists a solution $x(t)$ of the dynamical system (8) on its maximal existence interval $[t_0,\ \alpha)$, which is continuous on t.

Similar to the proof of Lemma 3.2 in [23], it can be shown that $x(t) > 0\ \forall t \ge t_0$. □

Lemma 3.3. *Let $x(t)$ be a solution of* (8)*. Then $Ax(t) = b\ \forall t \ge t_0$.*

Proof: From

$$x(t) = x(t_0) + \int_{t_0}^{t} \frac{dx}{d\tau} d\tau = x(t_0) - \int_{t_0}^{t} XP_{AX}(Xc - \mu(x)e)d\tau,$$

it follows that

$$Ax(t) = Ax(t_0) - \int_{t_0}^{t} AXP_{AX}(Xc - \mu(x)e)d\tau = b.$$

□

Lemmas 3.2 and 3.3 ensure that the solution $x(t)$ of the system (8) always stays in $\mathscr{P}^+$. The following lemma shows that the right-hand side of (8) is Lipschitz continuous on any bounded positive set: $\mathbf{D} = \{x \in R^n_+ : x_i \le M\}$, where M is a positive number.

Lemma 3.4. *Let $\mathbf{D} = \{x \in R^n_+ : x_i \le M\}$ with $M > 0$. Then $P_{AX}Xc$, $XP_{AX}Xc$, and $\mu(x)XP_{AX}e$ are all Lipschitz continuous in* $\mathbf{D}$.

Proof: For any $x \in \mathbf{D}$ and $i = 1, \cdots, n$, we have

$$\frac{\partial P_{AX}}{\partial x_i} = -e_i e_i^T QX - XQe_i e_i^T - XA^T \frac{\partial (AX^2A^T)^{-1}}{\partial x_i} AX, \tag{10}$$

where e_i is the ith column of matrix I_n. Since

$$(AX^2A^T)(AX^2A^T)^{-1} = I_m,$$

we have

$$2x_i Ae_i e_i^T A^T (AX^2A^T)^{-1} + (AX^2A^T)\frac{\partial (AX^2A^T)^{-1}}{\partial x_i} = 0,$$

which implies that

$$\frac{\partial (AX^2A^T)^{-1}}{\partial x_i} = -2x_i (AX^2A^T)^{-1} Ae_i e_i^T A^T (AX^2A^T)^{-1}. \tag{11}$$

Equations (10) and (11) imply

$$\frac{\partial P_{AX}}{\partial x_i} = -e_i e_i^T QX - XQe_i e_i^T + 2x_i XQe_i e_i^T QX.$$

Notice that

$$x_i e_i e_i^T = Xe_i e_i^T = e_i e_i^T X,$$

we have

$$\begin{aligned} \frac{\partial P_{AX}Xc}{\partial x_i} &= \frac{\partial P_{AX}}{\partial x_i} Xc + c_i P_{AX} e_i \\ &= -e_i e_i^T QX^2 c + 2x_i XQe_i e_i^T QX^2 c + 2c_i P_{AX} e_i - c_i e_i. \end{aligned} \tag{12}$$

Since $\|QX^2\| \le \bar{\chi}$ and $\|XQX\| \le 1$, we have

$$\begin{aligned} \|\frac{\partial P_{AX}Xc}{\partial x_i}\| &\le \|QX^2\|\|c\| + 2x_i \|XQX\|\|X^{-1}e_i e_i^T\|\|QX^2\|\|c\| \\ &\quad +2|c_i|\|P_{AX}\| + |c_i| \\ &\le 3(\bar{\chi}+1)\|c\| \equiv L_1. \end{aligned} \tag{13}$$

From Lemma 4.1.9 in [12], we have for any $\bar{x},\ \hat{x} \in \mathbf{D}$,

$$\|P_{A\bar{X}}\bar{X}c - P_{A\hat{X}}\hat{X}c\| = \|\int_0^1 \frac{\partial P_{AX}Xc}{\partial x}\Big|_{x=\hat{x}+t(\bar{x}-\hat{x})} (\bar{x}-\hat{x})dt\| \le \sqrt{n}\cdot L_1 \cdot \|\bar{x}-\hat{x}\|, \tag{14}$$

that is, $P_{AX}Xc$ is Lipschitz continuous in $\mathbf{D}$.

For $XP_{AX}Xc$, from (12), we have

$$\begin{aligned} \frac{\partial XP_{AX}Xc}{\partial x_i} &= e_i e_i^T P_{AX}Xc + X\frac{\partial P_{AX}Xc}{\partial x_i} \\ &= 2e_i e_i^T P_{AX}Xc + 2x_i X^2 Qe_i e_i^T QX^2 c + 2c_i XP_{AX}e_i - 2c_i x_i e_i. \end{aligned}$$

It follows that

$$\begin{aligned} \|\frac{\partial XP_{AX}Xc}{\partial x_i}\| &\le 2\|P_{AX}\|\|X\|\|c\| + 2x_i\|X^2Q\|\|QX^2\|\|c\| + 2|c_i|\|X\|\|P_{AX}\| + 2x_i|c_i| \\ &\le 2(M + M\bar{\chi}^2 + M + M)\|c\| \\ &= 2(3M + M\bar{\chi}^2)\|c\| \equiv L_2. \end{aligned}$$

Following the same argument as that for (14), we have that $XP_{AX}Xc$ is also Lipschitz continuous in $\mathbf{D}$.

For $\mu(x)XP_{AX}e$, from the above discussions, we have that

$$\begin{aligned}\frac{\partial XP_{AX}e}{\partial x_i} &= X\frac{\partial P_{AX}e}{\partial x_i} + e_ie_i^TP_{AX}e\\ &= X(-e_ie_i^TQX - XQe_ie_i^T + 2x_iXQe_ie_i^TQX)e + e_ie_i^TP_{AX}e\\ &= (-e_ie_i^TXQX - X^2Qe_ie_i^T + 2x_iX^2Qe_ie_i^TQX)e + e_ie_i^TP_{AX}e.\end{aligned}$$

$$\begin{aligned}\|\frac{\partial XP_{AX}e}{\partial x_i}\| &\le \|XQX\|\|e\| + \|X^2Q\|\|e\| + 2x_i\|X^2Q\|\|e_ie_i^TX^{-1}\|\|XQX\|\|e\| + \|P_{AX}e\|\\ &\le (3\bar{\chi}+2)\|e\|\\ &\le (3\bar{\chi}+2)\sqrt{n} \equiv L_3.\end{aligned}$$

From Lemma 4.1.9 in [12], we have for $\bar{x},\ \hat{x} \in \mathbf{D}$,

$$\|\bar{X}P_{A\bar{X}}e - \hat{X}P_{A\hat{X}}e\| = \|\int_0^1 \frac{\partial XP_{AX}e}{\partial x}\Big|_{x=\bar{x}+t(\hat{x}-\bar{x})}(\hat{x}-\bar{x})dt \le L_3\cdot\|\hat{x}-\bar{x}\|,$$

that is, $XP_{AX}e$ is Lipschitz continuous in $\mathbf{D}$. From the above arguments and the definition of $\mu(x)$, it is straight forward to show that $\mu(x)$ is Lipschitz continuous in $\mathbf{D}$ with Lipschitz constant M_2. So, for $\bar{x},\ \hat{x} \in \mathbf{D}$, we have

$$\begin{aligned}\|\mu(\bar{x})\bar{X}P_{A\bar{X}}e - \mu(\hat{x})\hat{X}P_{A\hat{X}}e\| &\le \|\mu(\bar{x})\|\|\bar{X}P_{A\bar{X}}e - \hat{X}P_{A\hat{X}}e\| + \|\mu(\bar{x}) - \mu(\hat{x})\|\|\hat{X}P_{A\hat{X}}e\|\\ &\le M_1\|\bar{X}P_{A\bar{X}}e - \hat{X}P_{A\hat{X}}e\| + M_2\|\mu(\bar{x}) - \mu(\hat{x})\|\end{aligned}$$

$\mu(x)XP_{AX}e$ is Lipschitz continuous about x in $\mathbf{D}$. □

The result of Lemma 3.4 is important in ensuring the existence of the solution of the ODE (8) for all $t \ge t_0$. The result in the following Theorem 3.6 ensures that $\lim_{t\to\infty} P_{AX}Xc = 0$ as $t \to \infty$. But first, the following lemma is needed.

Lemma 3.5. *(**Barbalat's Lemma**)* [33] *If the differentiable function $f(t)$ has a finite limit as $t \to +\infty$, and $\dot{f}$ is uniformly continuous, then $\dot{f} \to 0$ as $t \to +\infty$.*

Now we show that the solution of (8) exists for all $t \ge t_0$ and $\lim_{t\to\infty} P_{AX}Xc = 0$ as $t \to \infty$.

Theorem 3.6. *Let $x(t)$ be the solution of* (8)*. Then $x(t)$ is well defined and unique in $[t_0,\infty)$, and $\lim_{t\to\infty} P_{AX}Xc = 0$.*

Proof: First, from Lemma 3.2, we get that the solution $x(t)$ of (8) stays in $\mathscr{P}^+$ on its maximal existence interval $[t_0,\ \alpha)$. Furthermore, we know

$$\frac{dc^Tx}{dt} = \begin{cases} -\|P_{AX}Xc\|^2 & \text{if } e^TP_{AX}Xc \ge 0,\\ -\|P_{AX}Xc\|^2 - \alpha(x)(e^TP_{AX}Xc)^2 & \text{if } e^TP_{AX}Xc < 0.\end{cases} \tag{15}$$

For all cases of (15), we get

$$\frac{dc^Tx}{dt} \le 0.$$

Thus $c^Tx(t)$ is decreasing along the trajectory space, so $x(t)$ is bounded (the bound may depend on x_0) for any $t \ge t_0$ from Assumption 3.1 and Proposition 1. So there exists a unique solution $x(t)$ of (8) in $[t_0,+\infty)$ followed from Lemma 3.4, the Cauchy-Peano theorem and Picard-Lindelöf theorem.

In addition, from Assumption 3.1 and Proposition 1, we have that if $x(t)$ is a solution of (8), $x(t)$ is bounded. From Lemma 3.4, $P_{AX}Xc$ and $XP_{AX}Xc$ are Lipschitz continuous. Thus, it is straightforward to verify that $\|P_{AX}Xc\|^2$ and $\alpha(x)(e^T P_{AX}Xc)^2$ are also Lipschitz continuous. Then from (15), it is easy to see that $\frac{dc^T x}{dt}$ is uniformly continuous in t. Thus Lemma 3.5 ensures

$$\lim_{t\to\infty} P_{AX}Xc = 0.$$

$\square$

From Theorem 3.6, we can see that the right-hand side of (8) will converge to zero as $t \to +\infty$. The next lemma shows that if the right-hand side of (8) equals to zero, i.e., $f_\mu(x) = 0$, the points satisfying $f_\mu(x) = 0$ lie on the primal central path.

Lemma 3.7. *For a point $x \in \mathscr{P}^+$, $f_\mu(x) = 0$ if and only if $Xs = \mu(x) \cdot e$ for some $(y,s) \in \mathscr{D}^+$.*

Proof: The following equivalences are straightforward.

$$\begin{aligned} f_\mu(x) = 0 &\Longleftrightarrow P_{AX}[Xc - \mu(x)e] = 0 \\ &\Longleftrightarrow Xc - \mu(x)e \in \text{range}(XA^T) \\ &\Longleftrightarrow c - \mu(x)X^{-1}e \in \text{range}(A^T) \\ &\Longleftrightarrow Xs = \mu(x)e \qquad \text{for some } (y,s) \in \mathscr{D}^+. \end{aligned} \tag{16}$$

$\square$

The next lemma shows that the right-hand side of (8) does not vanish in finite time.

Lemma 3.8. *Under Assumption 3.1, let $x(t)$ be the solution of (8). If $\|f_\mu(x)|_{t=t_0}\| \neq 0$, then $\|f_\mu(x)\| \neq 0$ for any $t \geq t_0$.*

Proof: Assume, by contradiction, that there exists a finite time, say $\bar{t} > 0$, such that $f_\mu(x) = 0$. By Lemma 3.7, we get that $Xs = \mu(x)e$ for some $(y,s) \in \mathscr{D}^+$. From Lemma 3.2, we have $x(t) > 0 \; \forall t \geq t_0$.

Case 1: $\mu(x) = 0$. Since $f_\mu(x) = 0$, we obtain that $P_{AX}Xc = \mu(x)P_{AX}e = 0$. Then, we have

$$c = A^T(AX^2A^T)^{-1}AX^2c.$$

Let us define $y_\varepsilon = (AX^2A^T)^{-1}AX^2c$, then $c = A^T y_\varepsilon$, this contradicts with Assumption 3.1.

Case 2: $\mu(x) > 0$. From the definition of $\mu(x)$, we get $e^T P_{AX}Xc < 0$. Since $f_\mu(x) = 0$, we obtain that

$$P_{AX}Xc = \mu(x)P_{AX}e. \tag{17}$$

Multiplying both sides of (17) from the left by e^T, it follows that

$$e^T P_{AX}Xc = \mu(x)e^T P_{AX}e. \tag{18}$$

Hence, the right-hand side of (18) is negative while the left-hand side is nonnegative. So, we get a contradiciton.

From the above two cases, the lemma is proved. $\square$

3.2. **Convergence analysis of (8).** In this section, we will study and verify the global convergence of the solution trajectory $x(t)$ of the ODE system (8). First, let us state some basic properties for an ODE system. Consider the following ODE system:

$$\frac{dx}{dt} = g(t)f(x), \quad x(t_0) = x_0, \tag{19}$$

where $g : (\alpha, \beta) \to R$ is continuous. A solution of (19) is a differentiable path for all t in the open interval $I \subseteq (\alpha, \beta)$. The ODE system (19) is called autonomous if $g(t) \equiv 1$. In this case, (19) becomes:

$$\frac{dx}{dt} = f(x), \quad x(t_0) = x_0. \tag{20}$$

Proposition 2. [29] *Let $\psi : (\alpha^-, \alpha^+) \to U$ and $x : (\omega^-, \omega^+) \to U$ denote the maximal solutions of ODEs (19) and (20), respectively. Assume that $g(t) > 0$ for all $t \in (\alpha, \beta)$ and let $\gamma : (\alpha, \beta) \to R$ be the function defined by $\gamma(t) \equiv t^0 + \int_{t_0}^{t} g(s)ds$ for all $t \in (\alpha, \beta)$. Then we have (a) $(\alpha^-, \alpha^+) = \{t \in (\alpha, \beta); \omega^- < \gamma(t) < \omega^+\}$, and (b) $\psi(t) = x(\gamma(t))$ for all $t \in (\alpha^-, \alpha^+)$.*

In Proposition 2, both α^+ and ω^+ can be extended to $+\infty$. Next, we show that $\lim_{t\to\infty} x(t)$ exists, where $x(t)$ is the solution of (8). First, let us introduce two important results.

Theorem 3.9. [38, 39] *There exists a positive constant $\Delta(A, c)$ which is determined from A and c such that*

$$\Gamma(x) \equiv \frac{\|P_{AX}Xc\|^2}{\|c\| \cdot \|XP_{AX}Xc\|} \geq \Delta > 0 \qquad \forall x \in \mathscr{P}^+. \tag{21}$$

Theorem 3.10. [1] *Let $E(\cdot)$ be a real analytic function and let $x(t)$ be a $\mathscr{C}^1$ curve in R^n, with $\dot{x} = \frac{dx(t)}{dt}$ denoting its time derivative. Assume that there exists a $\delta > 0$ and a real τ such that for $t > \tau$, $x(t)$ satisfies the angle condition*

$$\frac{dE(t)}{dt} \equiv \langle \nabla E(x(t)), \dot{x}(t) \rangle \leq -\delta \cdot \|\nabla E(x(t))\| \cdot \|\dot{x}(t)\| \tag{22}$$

and a weak decrease condition

$$[\frac{d}{dt}E(x(t)) = 0] \Rightarrow [\dot{x}(t) = 0]. \tag{23}$$

Then, either $\lim_{t\to\infty} x(t) = \infty$ or there exists $x^* \in R^n$ such that $\lim_{t\to\infty} x(t) = x^*$.

Our strong convergence result can be obtained by using the above two theorems.

Theorem 3.11. *For any $x_0 \in \mathscr{P}^+$, let $x(t)$ be the solution of (8). Then $x(t)$ is convergent as $t \to +\infty$ and its limit $x^*(x_0) \in \mathscr{P}$.*

Proof: We know that the solution of (8) exists and is unique from Lemmas 3.2 and 3.3, and Theorem 3.6.

If $\|f_\mu(x)|_{t=t_0}\| = 0$, by Lemma 3.8, we have $P_{AX}Xc = 0$. Similar to the proof of Case 1 in Lemma 3.8, this contradicts with Assumption 3.1, so $\|f_\mu(x)|_{t=t_0}\| \neq 0$. Again from Lemma 3.8, $\|f_\mu(x)\| \neq 0$ for any $t \geq t_0$. In Theorem 3.10, let us define

$$E(x) = c^T x, \qquad \frac{dx(t)}{dt} = -XP_{AX}\{Xc - \mu(x)e\},$$

where $\mu(x)$ is in (8). So, we can write

$$\frac{dE(x)}{dt} = \frac{dc^T x}{dt} = -\|P_{AX}Xc\|^2 + \alpha(x)\beta(x)e^T P_{AX}Xc. \tag{24}$$

Now, we define

$$\Pi(x) = \frac{\|P_{AX}Xc\|^2 - \alpha(x)\beta(x)e^T P_{AX}Xc}{\|XP_{AX}Xc - \alpha(x)\beta(x)XP_{AX}e\|}. \tag{25}$$

From the numerator of (25), by the definition of $\alpha(x)$ and $\beta(x)$ in (8), we get

$$\|P_{AX}Xc\|^2 - \alpha(x)\beta(x)e^T P_{AX}Xc \geq \|P_{AX}Xc\|^2 + \alpha(x)\beta(x)^2 \geq \|P_{AX}Xc\|^2. \tag{26}$$

From the denominator of (25), we get

$$\|XP_{AX}Xc - \alpha(x)\beta(x)XP_{AX}e\| \leq \|XP_{AX}Xc\| + \alpha(x)\beta(x)\|XP_{AX}e\|. \tag{27}$$

Substituting

$$\alpha(x) = \frac{\|XP_{AX}Xc\|}{|e^T P_{AX}Xc|\|XP_{AX}e\| + 2}$$

into (27), we get

$$\|XP_{AX}Xc - \alpha(x)\beta(x)XP_{AX}e\| \leq 2\|XP_{AX}Xc\|. \tag{28}$$

Using (25), (26), (28), and Theorem 3.9, we obtain

$$\Pi(x) \geq \frac{\|P_{AX}Xc\|^2}{2\|XP_{AX}Xc\|} \geq \frac{\|c\|\Delta}{2} > 0 \qquad \forall x \in \mathscr{P}^+.$$

So, all conditions of Theorem 3.10 are satisfied. In addition, we know that the trajectory $x(t)$ of (8) is bounded for all $t \geq t_0$, hence we have that there exists a point $x^*(x_0) \in \mathscr{P}$ such that

$$\lim_{t\to+\infty} x(t) = x^*(x_0).$$

$\square$

This theorem shows that the solution $x(t)$ of the ODE system (8) converges to a point $x^*(x_0)$. Next, we prove that this $x^*(x_0)$ is an optimal solution of (1).

3.3. **Optimality.** In this section, we will study in more detail about the limit point property of the solution of (8). In addition, we will also introduce the dual variable and dual estimates. Without loss of generality, we will study an equivalent form of the ODE system (8). We consider a new ODE system:

$$\begin{cases} \dfrac{dx(t)}{dt} = \dfrac{1}{h(t)} f_\mu(x) = -\dfrac{1}{h(t)} XP_{AX}\{Xc - \mu(x)e\},\ x(t_0) = x_0 \in \mathscr{P}^+, & \text{(29a)} \\ \dfrac{dh(t)}{dt} = \mu(x) - h(t),\ h(t_0) = 1. & \text{(29b)} \end{cases}$$

Here, the vector field associated with (29a) and (29b) is the new function

$$\Psi_\mu(x,h) = (h^{-1} f_\mu(x), \mu(x) - h),$$

whose domain of the definition is the set $\mathscr{P}^+ \times R^+ = \{t : t \in R, t > 0\}$. We know that $h(t) > 0$ for all t in the definition of (8) if $(x(t), h(t))$ is the solution of (29).

Remark 1. (a) The function $\Psi_\mu(x,h)$ does not vanish in the set $\mathscr{P}^+ \times R^+$. (b) If $(x(t), h(t))$ is the solution of (29), the merit function defined as $\bar{E}(x,h) = E(x) = c^T x$ is a decreasing function of t.

Proposition 3. *Let $\xi : (\omega^-, \omega^+) \to \mathscr{P}^+$ and $(x,h) : (\alpha^-, \alpha^+) \to \mathscr{P}^+ \times R^+$ denote the solutions of (8) and (29), respectively. Then*

(a) $h(t) = e^{-t+t_0} \int_{t_0}^{t} e^{s-t_0} g(s)ds + e^{-t+t_0}$ for all $t \in (\alpha^-, \alpha^+)$, where $g(t) \equiv \mu(x(t))$ for all $t \in (\alpha^-, \alpha^+)$ and $t_0 \geq 0$.

(b) Let $\eta(t) \equiv \int_{t_0}^{t} h(s)^{-1} ds$ for all $t \in (\omega^-, \omega^+)$. Then, $\{\eta(t) : (\alpha^-, \alpha^+)\} \subseteq (\omega^-, \omega^+)$ and $x(t) = \xi(\eta(t))$ for all $t \in (\alpha^-, \alpha^+)$ and $t_0 \geq 0$.
(c) The set $\{x(t) : t \in [t_0, \alpha^+)\} \subseteq \mathscr{P}^+$ and $\{h(t) : t \in [t_0, \alpha^+)\} \subseteq R^+$.

Proof: It is similar to the proof of Proposition 3.1 in [29]. □

Now, let us define the dual estimates associated with the solution of (29).

Definition 3.12. The dual estimates $(y_\mu(x), s_\mu(x)) \in \mathscr{D}_a$ at the point $x \in \mathscr{P}_a$ are defined as:

$$\begin{aligned} y_\mu(x) =& (AX^2A^T)^{-1}AX(Xc - \mu(x)e), \\ s_\mu(x) =& c - A^T y_\mu(x). \end{aligned}$$

Next, we study the dual solution curves associated with the solution of (29). Let $(x,h) : (\alpha^-, \alpha^+) \to \mathscr{P}^+ \times R^+$ denote the solution of (29). For a given point $(y^0, s^0) \in \mathscr{D}_a$, let us define the dual solution curves through (y^0, s^0) to be the solution of $(y,s) : (\alpha^-, \alpha^+) \to \mathscr{D}_a$ of the following ODE system:

$$\begin{cases} \frac{dy(t)}{dt} = y_\mu(x) - y, & y(t_0) = y^0, \\ \frac{ds(t)}{dt} = s_\mu(x) - s, & s(t_0) = s^0, \end{cases}$$

whose domain of the definition is the set $\mathscr{D}_a \times (\alpha^-, \alpha^+)$.

Remark 2. The solution $(x,h) : (\alpha^-, \alpha^+) \to \mathscr{P}^+ \times R^+$ of (29) and its associated dual solution curves $(y,s) : (\alpha^-, \alpha^+) \to \mathscr{D}_a$ through $(y^0, s^0) \in \mathscr{D}_a$ satisfy the following relations:

$$\dot{s}(t) + h(t)x(t)^{-2}\dot{x}(t) = \mu(x)x(t)^{-1} - s(t), \tag{30a}$$

$$\dot{h}(t) = \mu(x) - h(t), \tag{30b}$$

$$A\dot{x}(t) = 0, \tag{30c}$$

$$A^T\dot{y}(t) + \dot{s}(t) = 0. \tag{30d}$$

By using the dual solution curves, we can study the limiting behavior of the solution of (29).

Proposition 4. *Let $(x,h) : (\alpha^-, \alpha^+) \to \mathscr{P}^+ \times R^+$ be the solution of (29) and its associated dual solution curve be denoted as $(y,s) : (\alpha^-, \alpha^+) \to \mathscr{D}_a$ through $(y^0, s^0) \in \mathscr{D}_a$. Then for all $t \in (\alpha^-, \alpha^+)$,*

$$s(t) - h(t)x(t)^{-1} = pe^{-t}, \tag{31}$$

where $p = s^0 - (x^0)^{-1} > 0$.

Proof: Let $\Phi(t) = s(t) - h(t)x(t)^{-1}$, $\quad t \in (\alpha^-, \alpha^+)$. From (30a) and (30b), we can obtain

$$\begin{aligned} \frac{d\Phi(t)}{dt} &= \frac{d}{dt}(s(t) - h(t)x(t)^{-1}) \\ &= \dot{s}(t) + h(t)x(t)^{-2}\dot{x}(t) - \dot{h}(t)x(t)^{-1} \\ &= \mu(x)x(t)^{-1} - s(t) - \dot{h}(t)x(t)^{-1} \\ &= h(t)x(t)^{-1} - s(t) \\ &= -\Phi(t). \end{aligned}$$

Here, we have

$$\dot{\Phi}(t) = -\Phi(t), \quad \Phi(t_0) = p.$$

Therefore, the unique solution of this problem is equal to pe^{-t}. So, we get

$$s(t) - h(t)x(t)^{-1} = pe^{-t}.$$

$\square$

From (30a) and (31), we know that $(x(t), y(t), s(t))$ can be regarded as the optimal solutions of some convex optimization problem. The following corollary reveals the relationship between the solution of (29) and this convex optimization problem.

Corollary 1. *Let $(x,h) : (\alpha^-, \alpha^+) \to \mathscr{P}^+ \times R^+$ be the solution of (29) and its associated dual solution curve be denoted as $(y,s) : (\alpha^-, \alpha^+) \to \mathscr{D}_a$ through $(y^0, s^0) \in \mathscr{D}_a$. Then for all $t \in (\alpha^-, \alpha^+)$, $x(t)$ is the (unique) optimal solution of the problem*

$$\begin{aligned} \min \quad & c^T x - e^{-t} p^T x - h(t) \sum_{j=1}^{n} lnx_j \\ s.t. \quad & Ax = b, \\ & x > 0. \end{aligned} \tag{32}$$

Proof: Let us define $\psi(x) = c^T x - e^{-t} p^T x - h(t) \sum_{j=1}^{n} lnx_j$, $\psi(x)$ is strictly convex and differentiable. Thus, the Lagrangian function of (32) is defined as

$$\mathscr{L}(x,y) = \psi(x) - y^T (Ax - b). \tag{33}$$

From the optimality condition of (32), we can write

$$\nabla \mathscr{L}_x(x,y) = 0, \tag{34a}$$
$$Ax = b, \quad x > 0, \tag{34b}$$
$$A^T y + s = c, \quad s \geq 0. \tag{34c}$$

Let $(x(t), y(t), s(t))$ be the unique solution of (34). By simplifying (34a) and Proposition 4, we can have

$$s(t) - h(t)x(t)^{-1} = pe^{-t}, \tag{35a}$$
$$Ax(t) = b, \quad x > 0, \tag{35b}$$
$$A^T y(t) + s(t) = c, \quad s \geq 0, \tag{35c}$$

where $x(t) \in \mathscr{P}^+$ and $((y(t), s(t)) \in \mathscr{D}_a$. Thus the result is proved. $\square$

Proposition 5. *Let $(x,h) : (\alpha^-, \alpha^+) \to \mathscr{P}^+ \times R^+$ be the solution of (29) and its associated dual solution curve be denoted as $(y,s) : (\alpha^-, \alpha^+) \to \mathscr{D}_a$ through $(y^0, s^0) \in \mathscr{D}_a$. Then (a) the set $\{(y(t), s(t)) : t \in [t_0, \alpha^+)\}$ is bounded, and (b) $\lim_{t \to \infty} h(t) = 0$.*

Proof: (a) From Assumption 3.1, we know that $rank(A) = m$ implies that $\{y(t) : [t_0, \alpha^+)\}$ is bounded. By (31), it follows that

$$nh(t) + e^{-t} p^T x(t) = x(t)^T s(t) = c^T x(t) - b^T y(t). \tag{36}$$

This implies

$$\begin{aligned} x_0^T (s(t) - e^{-t} p) &= x_0^T s(t) - e^{-t} p^T x_0 \\ &= c^T x_0 - b^T y(t) - e^{-t} p^T x_0 \\ &= c^T x_0 + nh(t) + e^{-t} p^T x(t) - c^T x(t) - e^{-t} p^T x_0 \end{aligned}$$

for all $t \in (\alpha^-, \alpha^+)$. By Proposition 3, we have that the sets $\{x(t) : t \in [t_0, \alpha^+)\}$ and $\{h(t) : t \in [t_0, \alpha^+)\}$ are bounded. We can get that every term in the last formla is also bounded. So, there exists an $M > 0$ such that

$$\|x_0^T(s(t) - e^{-t}p)\| \leq M \quad \forall t \in [t_0, \alpha^+).$$

Since $x_0 > 0$ and $s(t) - e^{-t}p > 0$ for all $t \in [t_0, \alpha^+)$, we can see that $(s(t) - e^{-t}p)$ is bounded and $s(t) > 0$ is bounded for all $t \in [t_0, \alpha^+)$.

(b) From (9) and (3.6), we can have that $\lim_{t\to\infty} \mu(x) = \lim_{t\to\infty} \alpha(x(t))\beta(x(t)) = 0$. Let $\varepsilon > 0$ be given, there exists a $t_1 \geq 0$ such that $\mu(x) \leq \frac{\varepsilon}{2}$ for all $t \geq t_1$. Let $t_2 \geq t_1$ be such that

$$e^{-t+t_0}[\int_{t_0}^{t_1} e^{\nu - t_0}\mu(x(\nu))d\nu + 1] \leq \frac{\varepsilon}{2}$$

for all $t \geq t_2$. Hence, by Proposition 3, we have

$$\begin{aligned}
h(t) &= e^{-t+t_0}[\int_{t_0}^{t} e^{\nu - t_0}\mu(x(\nu))d\nu + 1] \\
&= e^{-t+t_0}[\frac{\varepsilon}{2}\int_{t_1}^{t} e^{\nu - t_0}d\nu + \int_{t_0}^{t_1} e^{\nu - t_0}\mu(x(\nu))d\nu + 1] \\
&\leq \frac{\varepsilon}{2}e^{-t+t_0}[e^{t-t_0} - e^{t_1 - t_0}] + \frac{\varepsilon}{2} \\
&\leq \varepsilon.
\end{aligned}$$

So, the results follow. □

The next theorem will reveal the relationship between the solution of (29) and the optimal solution of the problem (32).

Theorem 3.13. *For any $t > 0$ and $p = s^0 - (x^0)^{-1}$, let $(x(t), y(t), s(t))$ be the solution of (37). Then $x(t)$ is a solution of (29).*

Proof: Let $(x(t), y(t), s(t))$ be the solution of the following system:

$$\begin{aligned}
s - h(t)x^{-1} &= pe^{-t}, \\
Ax = b, &\quad x > 0, \\
A^T y + s &= c.
\end{aligned} \tag{37}$$

It is easy to check that the Jacobian matrix of the above system is nonsingular. From the implicit function theorem, there exists a unique solution $(x(t), y(t), s(t))$ for the above system, in addition $(x(t), y(t), s(t))$ has continuous derivatives. By differentiating (37), we get

$$\begin{aligned}
\dot{s}(t) + h(t)x(t)^{-2}\dot{x}(t) - \dot{h}(t)x(t)^{-1} &= -pe^{-t} \\
A\dot{x}(t) &= 0, \\
A^T\dot{y}(t) + \dot{s}(t) &= 0.
\end{aligned} \tag{38}$$

After some straightforward manipulations and using the equations in (38) and (37), we can get that $x(t)$ is a solution of (29). □

Theorem 3.14. *Let $(x, h) : (t_0, \infty) \to \mathscr{P}^+ \times R^+$ be the solutions of (29). Then*

$$\lim_{t\to\infty} x(t) = x^*,$$

where x^* is an optimal solution of the problem (1).

Proof: From Theorem 3.11, let $\xi(t)$ be the solution of (8), we know that there exists a point x^* such that

$$\lim_{t\to\infty} \xi(t) = x^*.$$

Using Proposition 3, we obtain that $x(t) = \xi(\eta(t)),\ t \in (t_0, \infty)$. Thus, we get

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \xi(\eta(t)) = x^*.$$

By (31), we get

$$x(t)s(t) = pe^{-t}x(t) + h(t). \tag{39}$$

From Proposition 5, there is a subsequence $\{t_n\}$ of t with $\lim_{t_n\to\infty} s(t_n) = s^*,\ s^* \geq 0$, and $\lim_{t_n\to\infty} y(t_n) = y^*$. Then, from Proposition 5, we have $\lim_{t_n\to\infty} h(t_n) = 0$. When $t_n \to \infty$, by taking the limit of both sides of (39), we can have

$$\lim_{t_n\to\infty} x(t_n)s(t_n) = \lim_{t_n\to\infty} pe^{-t_n}x(t_n) + \lim_{t_n\to\infty} h(t_n)$$

and

$$X^*s^* = 0.$$

Using the similar technique as (35b) and (35c), we can obtain

$$Ax^* = b,\ x^* \geq 0, \quad \text{and } A^T y^* + s^* = c,\ s^* \geq 0.$$

By (3), we get that x^* is an optimal solution of the problem (1). □

From Theorem 3.14 and Proposition 3, we can obtain the following result.

Corollary 2. *For any $x_0 \in \mathscr{P}^+$, let $x(t)$ be the solution of (8). Then*

$$\lim_{t\to\infty} x(t) = x^*,$$

where x^* is an optimal solution of the problem (1).

This corollary shows that the continuous path is formed from any initial point $x_0 \in \mathscr{P}^+$ and converges to an optimal solution of the problem (1).

4. Numerical experiments.

In this section, we illustrate some numerical results by using our proposed continuous path-following trajectory. We simulate several small examples to verify the effectiveness of our trajectory and show all these trajectories approaching to the optimal solutions in the limit. All our experiments are carried out on a computer with a Dell Pentium(R) CPU 3.40GHz and 2GB RAM on the MATLAB platform.

Example 4.1.

$$\begin{array}{ll} \min & -4x_1 - 3x_2 \\ s.t. & x_1 + x_2 + x_3 = 40, \\ & 2x_1 + x_2 + x_4 = 60, \\ & x_i \geq 0, \quad i = 1,2,3,4. \end{array}$$

The optimal solution of this problem is $x^* = (20,20,0,0)$. Two feasible starting points $x_0 = (20,10,10,10)$ and $x_0^{'} = (15,15,10,15)$ are used in the test. We use our continuous path-following trajectory to solve this problem and provide the following figures to illustrate the convergence of our trajectory.

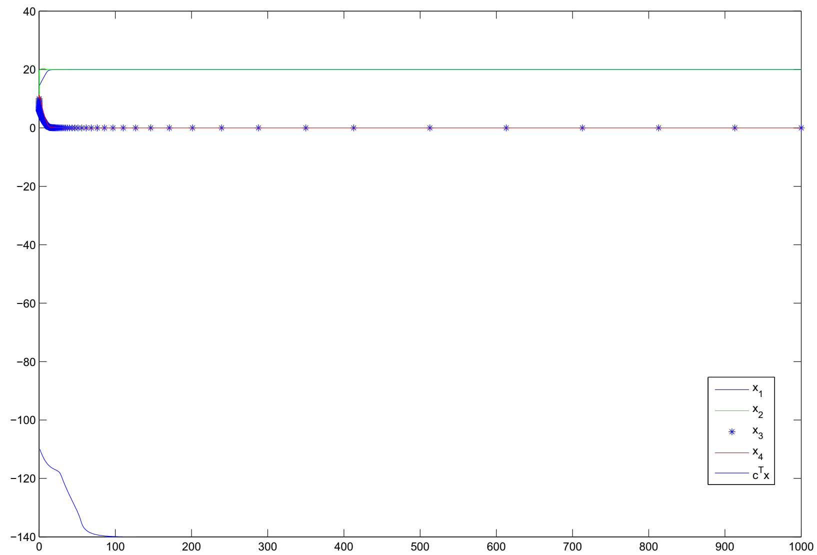

FIGURE 1. Transient behaviors of the continuous path of $x(t)$ and the objective function $c^T x$ in Example 4.1 with starting point x_0.

From Fig. 1 and Fig. 2, we can see that $x(t)$'s converge to the optimal solution x^* in our continuous path-following trajectories in the limit.

The next example has multiple optimal solutions.

Example 4.2.

$$\begin{array}{ll} \min & -x_1 - x_2 - x_3 \\ s.t. & x_1 - x_2 + x_3 \geq -2, \\ & -x_1 + x_2 + x_3 \geq -3, \\ & x_1 + x_2 - x_3 \geq -1, \\ & -x_1 - x_2 - x_3 \geq -4, \\ & x_i \geq 0, \quad i = 1,2,3. \end{array}$$

There are infinitely many optimal solutions for Example 4.2, here we only provide two optimal solutions $x^* = (3.5,0,0.5,6,0,4,0)^T$ and $x^* = (1.5,0,2.5,6,4,0,0)^T$. Two feasible starting points $x_0 = (1,1,1,3,5,2,1)^T$ and $x_0^{'} = (1,1,0.5,2.5,4.5,2.5,1.5)^T$ are used in our test.

Figs. 3 and 4 illustrate the transient behaviors of the solution $x(t)$ of (8) with two different starting points, x_0 and $x_0^{'}$ respectively. The two figures clearly show that $x(t)$'s converge to some optimal solutions of Example 4.2.

5. **Conclusion.** In this paper, an interior point continuous path-following trajectory is proposed for linear programming. Strong convergence of our continuous trajectory with any starting interior feasible point is proved. In addition, the limit of this continuous trajectory is shown to be an optimal solution of the original problem. Our preliminary numerical results clearly show the convergence property of our continuous path-following trajectory.

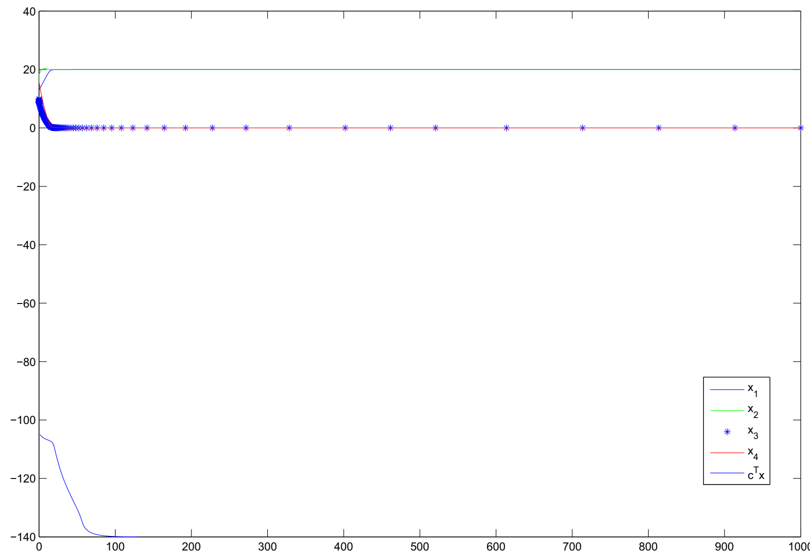

FIGURE 2. Transient behaviors of the continuous path of $x(t)$ and the objective function $c^T x$ in Example 4.1 with starting point $x_0^{'}$.

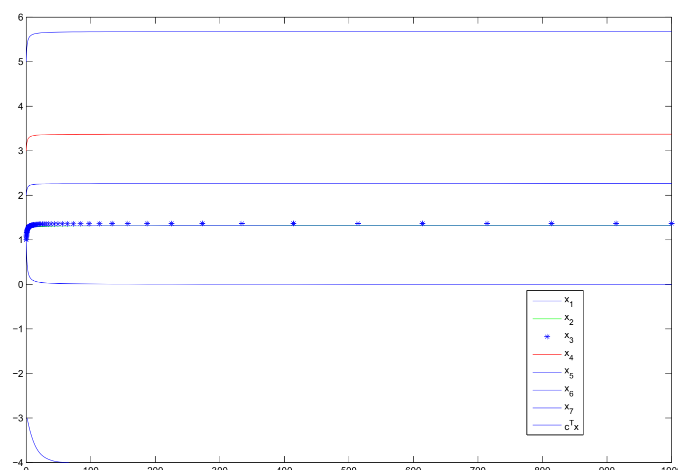

FIGURE 3. Transient behaviors of the continuous path of $x(t)$ and the objective function $c^T x$ in Example 4.2 with starting point x_0.

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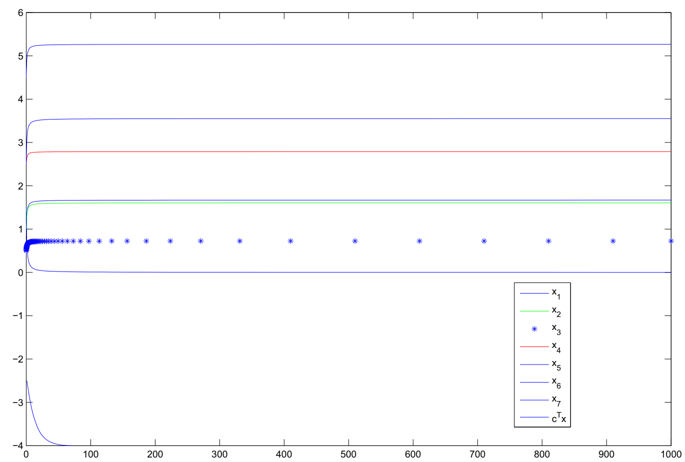

FIGURE 4. Transient behaviors of the continuous path of $x(t)$ and the objective function $c^T x$ in Example 4.2 with starting point x_0'.

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