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AN LQP-BASED DECOMPOSITION METHOD FOR SOLVING A CLASS OF VARIATIONAL INEQUALITIES*

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Abstract. The alternating direction method (ADM) is an influential decomposition method for solving a class of variational inequalities with block-separable structures. In the literature, the subproblems of the ADM are usually regularized by quadratic proximal terms to ensure a more stable and attractive numerical performance. In this paper, we propose to apply the logarithmic-quadratic proximal (LQP) terms to regularize the ADM subproblems, and thus develop an LQP-based decomposition method for solving a class of variational inequalities. Global convergence of the new method is proved under standard assumptions.

Key words. variational inequality, alternating direction method, logarithmic-quadratic proximal method, system of nonlinear equations, complementarity problem

AMS subject classifications. 47J20, 90C25, 90C30

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1. Introduction.

The purpose of a variational inequality (VI) in the finite-dimensional space $\mathcal{R}^s$, denoted by VI(Ω, F), is to find a vector $u^* \in \Omega$ such that

$$(u - u^*)^T F(u^*) \geq 0 \quad \forall u \in \Omega, \tag{1.1}$$

where Ω is a nonempty closed convex subset of $\mathcal{R}^s$ and F is a mapping from $\mathcal{R}^s$ into itself. Inspired by wide applications (see, e.g., [3, 6, 11, 12, 14, 24]), we concentrate on a special case of VI(Ω, F) with the block-separable structure

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad F(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix} \tag{1.2}$$

and

$$\Omega = \left\{(x, y) \mid x \in \mathcal{R}^n_+,\ y \in \mathcal{R}^m_+,\ Ax + By = b\right\}, \tag{1.3}$$

where $A \in \mathcal{R}^{l\times n}$ and $B \in \mathcal{R}^{l\times m}$ are given matrices, $b \in \mathcal{R}^l$ is a given vector, and $f : \mathcal{R}^n_+ \to \mathcal{R}^n$ and $g : \mathcal{R}^m_+ \to \mathcal{R}^m$ are continuous and monotone operators. Throughout, we assume that the solution set of VI (1.1)–(1.3), denoted by Ω^*, is nonempty.

With consideration of the favorable separable structure of (1.1)–(1.3), decomposition methods are of particular interest in the literature. An influential decomposition method for solving (1.1)–(1.3) is the alternating direction method (ADM), which was presented originally in [17] and has received intensive attention, especially in the context of convex programming and VIs; see, e.g., [9, 11, 13, 14, 16, 17, 18, 19, 22]. More

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specifically, for solving (1.1)–(1.3), the iterative scheme of the ADM generates the new iterate $w^{k+1} := (x^{k+1}, y^{k+1}, \lambda^{k+1}) \in \mathcal{R}^n_+ \times \mathcal{R}^m_+ \times \mathcal{R}^l$ by the tasks

$$0 \leq x^{k+1} \perp \left\{ f(x^{k+1}) - A^T[\lambda^k - H(Ax^{k+1} + By^k - b)] \right\} \geq 0, \tag{1.4}$$

$$0 \leq y^{k+1} \perp \left\{ g(y^{k+1}) - B^T[\lambda^k - H(Ax^{k+1} + By^{k+1} - b)] \right\} \geq 0, \tag{1.5}$$

$$\lambda^{k+1} = \lambda^k - H(Ax^{k+1} + By^{k+1} - b), \tag{1.6}$$

where $H \in \mathcal{R}^{l\times l}$ is a matrix of penalty parameters for violating the linear constraint in (1.3). It is easy to see that the decomposition treatment of the ADM makes it possible to take advantage of the particular properties of the individual functions f and g separately. We refer the reader to, e.g., [10, 15, 20], for the close relevance of the ADM to some operator splitting algorithms which were developed in [7, 25] for solving partial differential equations.

A substantial improvement on the ADM is to combine the classical proximal point algorithm (PPA) [23, 27] with the ADM, and current literature along this direction of research is dominated by the utilization of the quadratic proximal regularization; see, e.g., [5, 9, 19]. In particular, the improved ADM with quadratic proximal regularization developed in [19] needs to solve the following subproblems to generate a new iterate:

$$0 \leq x^{k+1} \perp \left\{ f(x^{k+1}) - A^T[\lambda^k - H(Ax^{k+1} + By^k - b)] + R(x^{k+1} - x^k) \right\} \geq 0, \tag{1.7}$$

$$0 \leq y^{k+1} \perp \left\{ g(y^{k+1}) - B^T[\lambda^k - H(Ax^{k+1} + By^{k+1} - b)] + S(y^{k+1} - y^k) \right\} \geq 0, \tag{1.8}$$

$$\lambda^{k+1} = \lambda^k - H(Ax^{k+1} + By^{k+1} - b), \tag{1.9}$$

where $R(x^{k+1} - x^k)$ and $S(y^{k+1} - y^k)$ are quadratic proximal regularization terms, and the symmetric positive definite matrices $R \in \mathcal{R}^{n\times n}$ and $S \in \mathcal{R}^{m\times m}$ are quadratic proximal parameters. Thus, at each iteration, the ADM with quadratic proximal regularization (1.7)–(1.9) requires us to solve two strongly monotone complementarity problems.

In this paper, we inherit the decomposition framework of the ADM algorithmically, but we regularize the decomposed subproblems with nonquadratic proximal terms. More specifically, we apply the logarithmic-quadratic proximal (LQP) method, which was developed recently in [1, 2], to solve the ADM decomposed subproblems. Thus, an LQP-based decomposition method is developed for solving the structured VI (1.1)–(1.3). The motivation for utilizing the LQP regularization, rather than the conventional quadratic proximal regularization, is that the interior-point property of the LQP method (see section 2 for details) ensures that the LQP-regularized ADM subproblems reduce to systems of equations which are generally easier than the resulting complementarity problems arising in the original ADM or quadratic-proximally regularized ADMs.

The rest of this paper is organized as follows. In section 2, we present the new method and provide some remarks. Some important properties for proving convergence of the new method are proved in section 3. Then the global convergence of the new method is proved in section 4. Finally, some conclusions are made in section 5.

2. The new method.

In this section, we present the LQP-based decomposition method for solving the structured VI (1.1)–(1.3). For this purpose, we first briefly introduce the LQP method.

Let us take the x-related ADM subproblem (1.7) as an example. Instead of using the quadratic proximal term $R(x - x^k)$ in (1.7), the LQP utilizes the nonquadratic proximal regularization term

$$R[(x - x^k) + \mu(x^k - X_k^2 x^{-1})],$$

where $\mu \in (0, 1)$ is a given constant, $X_k = \mathrm{diag}(x_1^k, x_2^k, \ldots, x_n^k)$, and x^{-1} is the vector whose jth element is $1/x_j$. Thus, with the LQP regularization, the complementarity problem (1.7) is substituted by

$$\begin{aligned} 0 \le x^{k+1} \perp \big\{ & f(x^{k+1}) - A^T[\lambda^k - H(Ax^{k+1} + By^k - b)] \\ & + R[(x^{k+1} - x^k) + \mu(x^k - X_k^2(x^{k+1})^{-1})]\big\} \ge 0. \end{aligned} \tag{2.1}$$

As proved in [1, 2], the LQP method guarantees that the new iterate x^{k+1} obtained by solving (2.1) (which has the unique solution) lies in the interior of $\mathcal{R}^n_+$, provided that the previous iterate x^k does. Hence, the complementarity problem (2.1) reduces to the following system of nonlinear equations:

$$f(x^{k+1}) - A^T[\lambda^k - H(Ax^{k+1} + By^k - b)] + R[(x^{k+1} - x^k) + \mu(x^k - X_k^2(x^{k+1})^{-1})] = 0.$$

In addition to the obvious theoretical advantages, the numerical efficiency of the LQP method has been well verified in the literature; see, e.g., [4, 21, 28].

Now, we are ready to present the LQP-based decomposition method for solving VI (1.1)–(1.3). Let $H \in \mathcal{R}^{l\times l}, R \in \mathcal{R}^{n\times n}$, and $S \in \mathcal{R}^{m\times m}$ be symmetric positive definite, where $R = \mathrm{diag}(r_1, r_2, \ldots, r_n)$ and $S = \mathrm{diag}(s_1, s_2, \ldots, s_m)$; and let $X_k = \mathrm{diag}(x_1^k, x_2^k, \ldots, x_n^k)$ and x^{-1} be the vector whose jth element is $1/x_j$; and let $Y_k = \mathrm{diag}(y_1^k, y_2^k, \ldots, y_m^k)$ and y^{-1} be the vector whose jth element is $1/y_j$.

An LQP-based decomposition method for VI (1.1)–(1.3).

Step 0. Let $\mu \in (0, 1)$ and $w^0 = (x^0, y^0, \lambda^0) \in \mathcal{R}^n_{++} \times \mathcal{R}^m_{++} \times \mathcal{R}^l$. Set $k = 0$.

Step 1. Solve the following system of nonlinear equations to obtain x^{k+1}:

$$f(x) - A^T[\lambda^k - H(Ax + By^k - b)] + R[(x - x^k) + \mu(x^k - X_k^2 x^{-1})] = 0. \tag{2.2}$$

Step 2. Solve the following system of nonlinear equations to obtain y^{k+1}:

$$g(y) - B^T[\lambda^k - H(Ax^{k+1} + By - b)] + S[(y - y^k) + \mu(y^k - Y_k^2 y^{-1})] = 0. \tag{2.3}$$

Step 3. Update the Lagrange multiplier

$$\lambda^{k+1} = \lambda^k - H(Ax^{k+1} + By^{k+1} - b). \tag{2.4}$$

Remark 2.1. It is well known (see, e.g., [3, 16, 17]) that when we attach a Lagrange multiplier vector $\lambda \in \mathcal{R}^l$ to the linear constraint $Ax + By = b$, VI (1.1)–(1.3) has the following equivalent form:

$$w = (x, y, \lambda) \in \mathcal{W}, \quad \begin{cases} (x' - x)^T \big(f(x) - A^T\lambda\big) \ge 0 \\ (y' - y)^T \big(g(y) - B^T\lambda\big) \ge 0 \\ Ax + By - b = 0 \end{cases} \quad \forall\, w' \in \mathcal{W}, \tag{2.5}$$

where

$$\mathcal{W} = \mathcal{R}^n_+ \times \mathcal{R}^m_+ \times \mathcal{R}^l. \tag{2.6}$$

We denote the solution set of (2.5)–(2.6) by $\mathcal{W}^*$. Obviously, for all $(x^*, y^*) \in \Omega^*$, there exists a vector $\lambda^* \in \mathcal{R}^l$ such that $w^* := (x^*, y^*, \lambda^*)$ is a solution of (2.5)–(2.6). Hence, $\mathcal{W}^*$ is nonempty under the assumption that Ω^* is nonempty.

It follows immediately from [8] that solving (2.5)–(2.6) is equivalent to finding a zero point of

$$e(w) := \begin{pmatrix} e_x(w) \\ e_y(w) \\ e_\lambda(w) \end{pmatrix} = \begin{pmatrix} x - P_{\mathcal{R}^n_+}[x - (f(x) - A^T\lambda)] \\ y - P_{\mathcal{R}^m_+}[y - (g(y) - B^T\lambda)] \\ Ax + By - b \end{pmatrix}, \tag{2.7}$$

where $P_{\mathcal{V}}[v]$ denotes the projection of v onto $\mathcal{V}$ in the Euclidean norm. Therefore, we can use $\|e(w^k)\|_\infty < \varepsilon$ as the stopping criterion.

Remark 2.2. Subject to certain criteria (see, e.g., [19]), the parameter matrices (R, S, H) in the proposed algorithm can be adjusted self-adaptively to accelerate convergence empirically, and the convergence of the new method with dynamically varied parameters can also be proved without difficulty. But for a simpler exposition of the main result, we limit our theoretical discussion to the case when these parameter matrices are constant matrices.

Remark 2.3. In the literature on ADMs, the step of updating the Lagrange multiplier (see (1.6) and (1.9)) has been generalized to

Step 3′. $\lambda^{k+1} = \lambda^k - \gamma H(Ax^{k+1} + By^{k+1} - b)$, where $\gamma \in (0, \frac{1+\sqrt{5}}{2})$ for the original ADM in [18] and $\gamma \in (0, 2)$ for some proximal ADMs in [9, 10]. For the proposed LQP-based decomposition method, the convergence is also ensured if we replace Step 3 by Step 3′ and allow $\gamma \in (0, \frac{1+\sqrt{5}}{2})$. Since the proof differs slightly from the convergence analysis to be presented, we omit it.

3. Some contractive properties. The following lemma plays an important role in the convergence analysis of the proposed method.

LEMMA 3.1. *Let* $P := \operatorname{diag}(p_1, p_2, \dots, p_t) \in \mathcal{R}^{t\times t}$ *be a positive definite diagonal matrix, and let* $q(u) \in \mathcal{R}^t$ *be a monotone mapping of* u *with respect to* $\mathcal{R}^t_+$. *Let* $\mu \in (0, 1)$. *For a given* $\bar{u} > 0$, *let* $\bar{U} := \operatorname{diag}(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_t)$. *Then, the equation*

$$q(u) + P[(u - \bar{u}) + \mu(\bar{u} - \bar{U}^2 u^{-1})] = 0 \tag{3.1}$$

has the unique positive solution u. *In addition, for this positive solution* $u > 0$ *and any* $v \geq 0$, *we have*

$$(v - u)^T q(u) \geq \frac{1+\mu}{2}\left(\|u - v\|_P^2 - \|\bar{u} - v\|_P^2\right) + \frac{1-\mu}{2}\|\bar{u} - u\|_P^2. \tag{3.2}$$

Moreover, we have

$$(v - u)^T q(u) \geq (\bar{u} - u)^T P[(1+\mu)v - (\mu\bar{u} + u)]. \tag{3.3}$$

Proof. The proof of the first assertion is a straightforward conclusion of Proposition 1 in [2], and hence it is omitted. We prove the second assertion. Note the obvious fact that for $u > 0$ and $\bar{u} > 0$, we have

$$(\bar{u}_i)^2/u_i \geq 2\bar{u}_i - u_i, \quad i = 1, \dots, t.$$

It follows from this and (3.1) that for $i = 1, \dots, t$,

$$\begin{aligned} (v_i - u_i)q_i &= p_i\left\{(u_i - v_i)\left[u_i - (1-\mu)\bar{u}_i - \mu(\bar{u}_i)^2/u_i\right]\right\} \\ &\geq p_i\left\{(u_i)^2 - (1-\mu)u_i\bar{u}_i - \mu(\bar{u}_i)^2 - u_iv_i + (1-\mu)\bar{u}_iv_i + \mu v_i(2\bar{u}_i - u_i)\right\} \\ &= p_i\left\{(u_i)^2 - (1-\mu)u_i\bar{u}_i - \mu(\bar{u}_i)^2 - (1+\mu)u_iv_i + (1+\mu)\bar{u}_iv_i\right\} \\ &= p_i\left\{\frac{1+\mu}{2}\left[(u_i - v_i)^2 - (\bar{u}_i - v_i)^2\right] + \frac{1-\mu}{2}(\bar{u}_i - u_i)^2\right\}. \end{aligned}$$

Hence, (3.2) holds.

To prove (3.3), we notice that

$$\begin{aligned}
&\frac{1+\mu}{2}\left(\|u-v\|_P^2-\|\bar{u}-v\|_P^2\right)+\frac{1-\mu}{2}\|\bar{u}-u\|_P^2\\
&=(1+\mu)v^TP\bar{u}-(1+\mu)v^TPu-(1-\mu)u^TP\bar{u}-\mu\|\bar{u}\|_P^2+\|u\|_P^2\\
&=(1+\mu)v^TP(\bar{u}-u)-(\bar{u}-u)^TP(\mu\bar{u}+u)\\
&=(\bar{u}-u)^TP[(1+\mu)v-(\mu\bar{u}+u)].
\end{aligned}$$

Thus, (3.3) follows immediately from the above equation and (3.2). The proof is completed. ▯

Based on Lemma 3.1, we have the following conclusions with respect to the proposed method.

COROLLARY 3.2. *Let* $w^{k+1}=(x^{k+1},y^{k+1},\lambda^{k+1})$ *be generated by* (2.2)–(2.4) *from a given* $w^k=(x^k,y^k,\lambda^k)$. *Let*

$$f_k(x^{k+1}):=f(x^{k+1})-A^T[\lambda^k-H(Ax^{k+1}+By^k-b)] \tag{3.4}$$

and

$$g_k(y^{k+1}):=g(y^{k+1})-B^T[\lambda^k-H(Ax^{k+1}+By^{k+1}-b)]. \tag{3.5}$$

Then, for any $w=(x,y,\lambda)\in\mathcal{W}$, *we have*

$$(x-x^{k+1})^Tf_k(x^{k+1})\ge\frac{1+\mu}{2}\left(\|x^{k+1}-x\|_R^2-\|x^k-x\|_R^2\right)+\frac{1-\mu}{2}\|x^k-x^{k+1}\|_R^2, \tag{3.6}$$

$$(x-x^{k+1})^Tf_k(x^{k+1})\ge(x^k-x^{k+1})^TR\left\{(1+\mu)x-(\mu x^k+x^{k+1})\right\}, \tag{3.7}$$

$$(y-y^{k+1})^Tg_k(y^{k+1})\ge\frac{1+\mu}{2}\left(\|y^{k+1}-y\|_S^2-\|y^k-y\|_S^2\right)+\frac{1-\mu}{2}\|y^k-y^{k+1}\|_S^2, \tag{3.8}$$

and

$$(y-y^{k+1})^Tg_k(y^{k+1})\ge(y^k-y^{k+1})^TS\left\{(1+\mu)y-(\mu y^k+y^{k+1})\right\}. \tag{3.9}$$

Proof. Applying Lemma 3.1 to Step 1 of the proposed method (namely, by setting $P=R,\bar{u}=x^k,u=x^{k+1},q(\cdot)=f_k(\cdot)$, and $v=x$ in (3.2) and (3.3), respectively), we have (3.6) and (3.7) immediately. Analogously, applying Lemma 3.1 to Step 2 of the proposed method by setting $P=S,\bar{u}=y^k,u=y^{k+1},q(\cdot)=g_k(\cdot)$, and $v=y$ in (3.2) and (3.3), respectively, we can prove (3.8) and (3.9) easily. ▯

LEMMA 3.3. *Let* x^{k+1} *and* y^{k+1} *be generated by* (2.2)–(2.3) *from a given* $w^k=(x^k,y^k,\lambda^k)$. *Then, for any* $w^*=(x^*,y^*,\lambda^*)\in\mathcal{W}^*$, *we have*

$$\begin{aligned}
&(\lambda^k-\lambda^*)^T(Ax^{k+1}+By^{k+1}-b)\\
&\quad\ge\frac{1+\mu}{2}\left(\|x^{k+1}-x^*\|_R^2-\|x^k-x^*\|_R^2\right)+\frac{1-\mu}{2}\|x^k-x^{k+1}\|_R^2\\
&\qquad+\frac{1+\mu}{2}\left(\|y^{k+1}-y^*\|_S^2-\|y^k-y^*\|_S^2\right)+\frac{1-\mu}{2}\|y^k-y^{k+1}\|_S^2\\
&\qquad+\frac{1}{2}\left(\|B(y^{k+1}-y^*)\|_H^2-\|B(y^k-y^*)\|_H^2\right)+\frac{1}{2}\|B(y^k-y^{k+1})\|_H^2\\
&\qquad+\|Ax^{k+1}+By^{k+1}-b\|_H^2+(Ax^{k+1}+By^{k+1}-b)^TH(By^k-By^{k+1}).
\end{aligned} \tag{3.10}$$

Proof. It follows from Lemma 3.1 that $x^{k+1} \in \mathcal{R}^n_{++}$ and $y^{k+1} \in \mathcal{R}^m_{++}$. In addition, since $w^* \in \mathcal{W}^*$, we have

$$(x^{k+1} - x^*)^T (f(x^*) - A^T \lambda^*) \geq 0, \tag{3.11}$$

$$(y^{k+1} - y^*)^T (g(y^*) - B^T \lambda^*) \geq 0, \tag{3.12}$$

and

$$Ax^* + By^* - b = 0.$$

On the other hand, setting $x = x^*$ in (3.6) and using

$$f_k(x^{k+1}) \overset{(3.4)}{=} f(x^{k+1}) - A^T[\lambda^k - H(Ax^{k+1} + By^{k+1} - b) - H(By^k - By^{k+1})],$$

we have

$$\begin{aligned}(x^* - x^{k+1})^T \big\{ f(x^{k+1}) - A^T \big[\lambda^k - H(Ax^{k+1} + By^{k+1} - b) - H(By^k - By^{k+1})\big] \big\} \\ \geq \frac{1+\mu}{2} \left(\|x^{k+1} - x^*\|_R^2 - \|x^k - x^*\|_R^2 \right) + \frac{1-\mu}{2} \|x^k - x^{k+1}\|_R^2.\end{aligned} \tag{3.13}$$

Adding (3.11) and (3.13), and using the monotonicity of operator f, we get

$$\begin{aligned}(x^{k+1} - x^*)^T \big\{ A^T(\lambda^k - \lambda^*) - A^T H(Ax^{k+1} + By^{k+1} - b) - A^T H(By^k - By^{k+1}) \big\} \\ \geq \frac{1+\mu}{2} \left(\|x^{k+1} - x^*\|_R^2 - \|x^k - x^*\|_R^2 \right) + \frac{1-\mu}{2} \|x^k - x^{k+1}\|_R^2.\end{aligned} \tag{3.14}$$

Using (3.8) and (3.5), we obtain

$$\begin{aligned}(y^* - y^{k+1})^T \left\{ g(y^{k+1}) - B^T[\lambda^k - H(Ax^{k+1} + By^{k+1} - b)] \right\} \\ \geq \frac{1+\mu}{2} \left(\|y^{k+1} - y^*\|_S^2 - \|y^k - y^*\|_S^2 \right) + \frac{1-\mu}{2} \|y^k - y^{k+1}\|_S^2.\end{aligned} \tag{3.15}$$

Adding (3.12) and (3.15), and using the monotonicity of operator g, we have

$$\begin{aligned}(y^{k+1} - y^*)^T \left\{ B^T(\lambda^k - \lambda^*) - B^T H(Ax^{k+1} + By^{k+1} - b) \right\} \\ \geq \frac{1+\mu}{2} \left(\|y^{k+1} - y^*\|_S^2 - \|y^k - y^*\|_S^2 \right) + \frac{1-\mu}{2} \|y^k - y^{k+1}\|_S^2.\end{aligned} \tag{3.16}$$

Combining (3.14) and (3.16) and using $Ax^* + By^* = b$, we get

$$\begin{aligned}(\lambda^k - \lambda^*)^T (Ax^{k+1} + By^{k+1} - b) + (Ax^{k+1} - Ax^*)^T H(By^{k+1} - By^k) \\ \geq \frac{1+\mu}{2} \left(\|x^{k+1} - x^*\|_R^2 - \|x^k - x^*\|_R^2 \right) + \frac{1-\mu}{2} \|x^k - x^{k+1}\|_R^2 \\ + \frac{1+\mu}{2} \left(\|y^{k+1} - y^*\|_S^2 - \|y^k - y^*\|_S^2 \right) + \frac{1-\mu}{2} \|y^k - y^{k+1}\|_S^2 \\ + \|Ax^{k+1} + By^{k+1} - b\|_H^2.\end{aligned} \tag{3.17}$$

Note that the following is an identity:

$$\begin{aligned}(By^{k+1} - By^*)^T H(By^{k+1} - By^k) \\ = \frac{1}{2} \left(\|B(y^{k+1} - y^*)\|_H^2 - \|B(y^k - y^*)\|_H^2 \right) + \frac{1}{2} \|B(y^k - y^{k+1})\|_H^2.\end{aligned} \tag{3.18}$$

Adding (3.17) and (3.18), and using $Ax^* + By^* = b$, we get the assertion (3.10). ∎

Now, we observe the updated form of λ^{k+1} in Step 3. Using (2.4), we have the following identity:

$$2(\lambda^k-\lambda^*)^T(Ax^{k+1}+By^{k+1}-b) = \left(\|\lambda^k-\lambda^*\|^2_{H^{-1}}-\|\lambda^{k+1}-\lambda^*\|^2_{H^{-1}}\right)+\|Ax^{k+1}+By^{k+1}-b\|^2_H, \tag{3.19}$$

which implies the following contractive property.

Lemma 3.4. *Let $w^{k+1}=(x^{k+1},y^{k+1},\lambda^{k+1})$ be generated by (2.2)–(2.4) from a given $w^k=(x^k,y^k,\lambda^k)$. Then for any $w^*=(x^*,y^*,\lambda^*)\in\mathcal{W}^*$ we have*

$$\begin{aligned}\|w^k-w^*\|^2_G \ge \|w^{k+1}-w^*\|^2_G &+ (1-\mu)\left(\|x^k-x^{k+1}\|^2_R+\|y^k-y^{k+1}\|^2_S\right)\\ &+\|Ax^{k+1}+By^k-b\|^2_H,\end{aligned} \tag{3.20}$$

where

$$G=\begin{pmatrix}(1+\mu)R & & \\ & (1+\mu)S+B^THB & \\ & & H^{-1}\end{pmatrix}. \tag{3.21}$$

Proof. Substituting (3.19) in (3.10), we get

$$\begin{aligned}&\left(\|\lambda^k-\lambda^*\|^2_{H^{-1}}-\|\lambda^{k+1}-\lambda^*\|^2_{H^{-1}}\right)+\|Ax^{k+1}+By^{k+1}-b\|^2_H\\ &\quad\ge(1+\mu)\left(\|x^{k+1}-x^*\|^2_R-\|x^k-x^*\|^2_R\right)+(1-\mu)\|x^k-x^{k+1}\|^2_R\\ &\qquad+(1+\mu)\left(\|y^{k+1}-y^*\|^2_S-\|y^k-y^*\|^2_S\right)+(1-\mu)\|y^k-y^{k+1}\|^2_S\\ &\qquad+\left(\|B(y^{k+1}-y^*)\|^2_H-\|B(y^k-y^*)\|^2_H\right)+\|B(y^k-y^{k+1})\|^2_H\\ &\qquad+2\|Ax^{k+1}+By^{k+1}-b\|^2_H+2(Ax^{k+1}+By^{k+1}-b)^TH(By^k-By^{k+1}),\end{aligned}$$

and thus

$$\begin{aligned}&(1+\mu)\left(\|x^k-x^*\|^2_R-\|x^{k+1}-x^*\|^2_R\right)+(1+\mu)\left(\|y^k-y^*\|^2_S-\|y^{k+1}-y^*\|^2_S\right)\\ &\quad+\left(\|\lambda^k-\lambda^*\|^2_{H^{-1}}-\|\lambda^{k+1}-\lambda^*\|^2_{H^{-1}}\right)+\left(\|B(y^k-y^*)\|^2_H-\|B(y^{k+1}-y^*)\|^2_H\right)\\ &\qquad\ge(1-\mu)\|x^k-x^{k+1}\|^2_R+(1-\mu)\|y^k-y^{k+1}\|^2_S+\|B(y^k-y^{k+1})\|^2_H\\ &\qquad\quad+\|Ax^{k+1}+By^{k+1}-b\|^2_H+2(Ax^{k+1}+By^{k+1}-b)^TH(By^k-By^{k+1}).\end{aligned} \tag{3.22}$$

Note that

$$\begin{aligned}&\|B(y^k-y^{k+1})\|^2_H+\|Ax^{k+1}+By^{k+1}-b\|^2_H\\ &\quad+2(Ax^{k+1}+By^{k+1}-b)^TH(By^k-By^{k+1})\\ &\qquad=\|Ax^{k+1}+By^k-b\|^2_H.\end{aligned}$$

Obviously, G defined in (3.21) is a positive definite (block diagonal) matrix. It follows from the above equation and using a compact form of (3.22) that

$$\begin{aligned}\|w^k-w^*\|^2_G \ge \|w^{k+1}-w^*\|^2_G &+ (1-\mu)\left(\|x^k-x^{k+1}\|^2_R+\|y^k-y^{k+1}\|^2_S\right)\\ &+\|Ax^{k+1}+By^k-b\|^2_H.\end{aligned}$$

Assertion (3.20) is proved. ∎

This lemma shows that the sequence $\{w^k\}$ is Fejér monotone with respect to $\mathcal{W}^*$ under the G-norm.

4. Convergence. In this section, we establish the convergence of the proposed LQP-based decomposition method.

THEOREM 4.1. *The sequence* $\{w^k\}$ *generated by the proposed method converges to some* w^∞*, which is a solution of* (2.5)–(2.6)*, and thus provides a solution of VI* (1.1)–(1.3).

Proof. It follows from (3.20) that $\{w^k\}$ is bounded and

(4.1)
$$\lim_{k\to\infty} \|x^k - x^{k+1}\| = 0, \quad \lim_{k\to\infty} \|y^k - y^{k+1}\| = 0, \quad \text{and} \quad \lim_{k\to\infty} \|Ax^{k+1} + By^k - b\| = 0.$$

Consequently,

$$\lim_{k\to\infty} \|H^{-1}(\lambda^k - \lambda^{k+1})\| \overset{(2.4)}{=} \lim_{k\to\infty} \|Ax^{k+1} + By^{k+1} - b\| = 0. \tag{4.2}$$

By (3.4), (3.5), and (2.4), we can easily obtain

$$f_k(x^{k+1}) = f(x^{k+1}) - A^T\lambda^{k+1} + A^T HB(y^k - y^{k+1})$$

and

$$g_k(y^{k+1}) = g(y^{k+1}) - B^T\lambda^{k+1}.$$

On the other hand, since $\{w^k\}$ is bounded, so are $\{x^k\}$ and $\{y^k\}$. Then, according to (3.7), (3.9), and (4.1), we have

$$\begin{cases} \liminf_{k\to\infty}(x - x^{k+1})^T \left[f(x^{k+1}) - A^T\lambda^{k+1}\right] \ge 0 & \forall x \in \mathcal{R}^n_+, \\ \liminf_{k\to\infty}(y - y^{k+1})^T \left[g(y^{k+1}) - B^T\lambda^{k+1}\right] \ge 0 & \forall y \in \mathcal{R}^m_+. \end{cases} \tag{4.3}$$

Recall that $\{w^k\}$ is bounded and has at least one cluster point. Let w^∞ be a cluster point of $\{w^k\}$, and let the subsequence $\{w^{k_j}\}$ converge to w^∞. It follows from (4.2) and (4.3) that

$$\begin{cases} \liminf_{j\to\infty}(x - x^{k_j})^T \left[f(x^{k_j}) - A^T\lambda^{k_j}\right] \ge 0 & \forall x \in \mathcal{R}^n_+, \\ \liminf_{j\to\infty}(y - y^{k_j})^T \left[g(y^{k_j}) - B^T\lambda^{k_j}\right] \ge 0 & \forall y \in \mathcal{R}^m_+, \\ \lim_{j\to\infty}(Ax^{k_j} + By^{k_j} - b) = 0, \end{cases} \tag{4.4}$$

and consequently,

$$\begin{cases} (x - x^\infty)^T \left[f(x^\infty) - A^T\lambda^\infty\right] \ge 0 & \forall x \in \mathcal{R}^n_+, \\ (y - y^\infty)^T \left[g(y^\infty) - B^T\lambda^\infty\right] \ge 0 & \forall y \in \mathcal{R}^m_+, \\ Ax^\infty + By^\infty - b = 0. \end{cases}$$

This means that w^∞ is a solution of (2.5)–(2.6). Note that the inequality (3.20) is true for all solution points of (2.5)–(2.6). We have

$$\|w^{k+1} - w^\infty\|_G^2 \le \|w^k - w^\infty\|_G^2 \quad \forall k \ge 0. \tag{4.5}$$

Hence, the sequence $\{w^k\}$ converges to w^∞, which is a solution of (2.5)–(2.6), and (x^∞, y^∞) is a solution of VI (1.1)–(1.3). ∎

5. Conclusions. In this paper, by applying the logarithmic-quadratic proximal (LQP) regularization to the subproblems decomposed by the alternating direction method (ADM), we propose an LQP-based decomposition method for solving a class of variational inequalities with block-separable structures. The novelty of the new method is that the blend of the ADM and the LQP method can yield simple subproblems for iterations. In the future, we will be interested in the comprehensive numerical study of the proposed method. In addition, as pointed out by a referee, developing generalized Newton methods (see, e.g., [26]) for structured variational inequalities via semismooth reformulations deserves intensive research.

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