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OPTIMAL SUPERCONVERGENCE RESULTS FOR DELAY INTEGRO-DIFFERENTIAL EQUATIONS OF PANTOGRAPH TYPE*

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Abstract. We analyze the optimal (global and local) orders of superconvergence of collocation solutions u_h on uniform meshes I_h for delay Volterra integro-differential equations with proportional delay functions given by $\theta(t) = qt$ $(0 < q < 1,\ t \in [0,T])$. In particular, we show that if u_h is a continuous piecewise polynomial of degree $m \geq 2$, and if collocation is at the Gauss (–Legendre) points, then the (optimal) order of local superconvergence on I_h is $p^* = m+2$. It turns out that the same order p^* holds for nonlinear (strictly increasing) delay functions vanishing at $t = 0$. However, on judiciously chosen geometric meshes, collocation at the Gauss points yields the order $2m - \varepsilon_N$, where $\varepsilon_N \to 0$ as the number N of mesh points tends to infinity. Optimal local superconvergence results for the pantograph delay differential equation are obtained as special cases of our general analysis.

Key words. Volterra integro-differential equation, vanishing delays, proportional delays, pantograph equation, collocation solutions, optimal order of superconvergence

AMS subject classifications. 65R20, 34K06, 34K28

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1. Introduction.

It is well known that collocation in the space of continuous piecewise polynomials of degree $m \geq 1$ for first-order delay differential and integro-differential equations with *nonvanishing delays* leads to (optimal) $\mathcal{O}(h^{2m})$-superconvergence at the mesh points of a suitably chosen (constrained) mesh I_h if the collocation points are the Gauss (–Legendre) points (see, e.g., Bellen [3], Bellen and Zennaro [7], Brunner [8, 11], and Chapter 4 of the monograph [12]). For the prototype of a functional differential equation with *vanishing delay*, the pantograph equation,

$$y'(t) = a(t)y(t) + b(t)y(qt), \quad t \in I := [0,T] \quad (0 < q < 1) \tag{1.1}$$

(first analyzed by Fox et al. [18] and Kato and McLeod [29]; see also the survey paper [23] by Iserles); this high order of local superconvergence can be attained on special *geometric meshes* (Bellen [4]). The same is true for its generalization, the pantograph Volterra integro-differential equation

$$y'(t) = a(t)y(t) + b(t)y(qt) + g(t) + \int_0^t K_0(t,s)y(s)ds + \int_0^{qt} K_1(t,s)y(s)ds, \quad t \in I \tag{1.2}$$

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(Bellen et al. [5]).

On *uniform meshes* the problem regarding optimal (local) superconvergence of collocation solutions for pantograph-type functional differential equations, and in particular for the pantograph equation itself, has remained open (compare the survey paper [10]). An indication that the "classical" superconvergence results might not remain valid for pantograph-type differential (and integral) equations was given in Brunner [9], Takama, Muroya, and Ishiwata [33], Ishiwata [28], and Muroya, Ishiwata, and Brunner [32]: it was shown in these papers that the collocation solution corresponding to the m Gauss points does not exhibit the (local) order $2m+1$ at $t = t_1 = h$, in contrast to ordinary differential equations or delay differential equations with constant delay.

It is the aim of the present paper to show, employing techniques rather different from those in [13], that for $m > 2$ the attainable order of superconvergence at the points of a uniform mesh $I_h = \{t_n = nh :\ 0 \leq n \leq N\ (t_N = T)\}$ (with $N \geq 2$) cannot exceed $p^* := m + 2$, and that this optimal value p^* can be attained for *any* $q \in (0,1)$ and all $m \geq 2$. This is in sharp contrast to the optimal local superconvergence result in (iterated) collocation solutions for pantograph-type Volterra *integral* equations: Brunner and Hu [13] have shown that the optimal local order $p^* = m+2$ can be attained *only* when $q = 1/2$ and m is even.

2. Volterra integro-differential equations of pantograph type.

2.1. Regularity and representation of solutions. Consider the general linear pantograph Volterra integro-differential equation

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I := [0,T], \tag{2.1}$$

with initial condition $y(0) = y_0$ and delay function $\theta(t) := t - \tau(t)$ satisfying $\theta(0) = 0$ and $\theta(t) < t$ when $t \in (0,T]$ (see also (D1)–(D3) below). The Volterra integral operators $\mathcal{V}$ and $\mathcal{V}_\theta$ are defined by

$$(\mathcal{V}y)(t) := \int_0^t K_0(t,s)y(s)ds, \quad K_0 \in C(D), \tag{2.2}$$

$$(\mathcal{V}_\theta y)(t) := \int_0^{\theta(t)} K_1(t,s)y(s)ds, \quad K_1 \in C(D_\theta), \tag{2.3}$$

where $D := \{(t,s) :\ 0 \leq s \leq t \leq T\}$. We set

$$D_\theta^{(k)} := \{(t,s) :\ 0 \leq s \leq \theta^k(t),\ t \in I\} \quad (k \geq 1) \quad \text{and} \quad D_\theta := D_\theta^{(1)}.$$

$\theta(t) = qt$ $(0 < q < 1)$, and we will usually write $D_\theta^{(k)} = D_q^{(k)}$.

The delay function $\theta(t) = t - \tau(t)$ is subject to the following conditions:

(D1) $\theta \in C^d(I)$, with $d \geq 1$, and $\theta(0) = 0$;

(D2) $\theta(t) \leq q_1 t$ $(t \in I)$, with $q_1 < 1$; and

(D3) $\min_{t \in I} \theta'(t) =: q_0 > 0$, with $q_0 \leq q_1$.

Equation (2.1) includes two important special cases: the *pantograph equation* with variable coefficients, (1.1) (which corresponds to $\theta(t) = qt = t - (1-q)t$), and the delay Volterra integro-differential equation with "pure delay,"

$$y'(t) = b(t)y(\theta(t)) + g(t) + (\mathcal{V}_\theta y)(t), \quad t \in I, \tag{2.4}$$

which exhibits all the essential (quantitative) properties of (2.1) on which the subsequent analysis will focus.

It is readily verified that the initial-value problem for (2.1) is equivalent to the delay integral equation

$$y(t) = g_0(t) + \int_0^t H_0(t,s)y(s)ds + \int_0^{\theta(t)} H_1(t,s)y(s)ds, \quad t \in I, \tag{2.5}$$

where we have set

$$g_0(t) := y_0 + \int_0^t g(s)ds, \tag{2.6}$$

$$H_0(t,s) := a(s) + \int_s^t K_0(v,s)dv, \tag{2.7}$$

$$H_1(t,s) := b(\theta^{-1}(s))\theta'(\theta^{-1}(s)) + \int_{\theta^{-1}(s)}^t K_1(v,s)dv. \tag{2.8}$$

In complete analogy with the pantograph equation (1.1), smooth data in (2.1) lead to smooth solutions on I. This is made precise in the following theorem.

THEOREM 2.1. *Assume that, for some $d \geq 0$, the given functions in* (2.1) *satisfy*
(i) $a,\ b,\ g \in C^d(I)$;
(ii) $K_0 \in C^d(D),\ K_1 \in C^d(D_\theta)$; *and*
(iii) (D1)–(D3), *with* $\theta \in C^{d+1}(I)$.
Then for each $y_0 \in \mathbb{R}$ the initial-value problem for (2.1) *has a unique solution* $y \in C^{d+1}(I)$.

The proof of this existence and regularity result (which generalizes Theorems 5.16 and 5.18 in Brunner [12]) follows readily from the Picard iteration process applied to the delay integral equation (2.5). For the delay Volterra integro-differential equation (2.4)—on which our subsequent analysis will focus—we obtain the following theorem on the representation of solutions. This result is an obvious generalization of the one for $\theta(t) = qt$ $(0 < q < 1)$ derived by Chambers [17]; in fact, this result is already implicitly contained in the 1914 paper [1] by Andreoli.

THEOREM 2.2. *Under the assumptions of Theorem* 2.1, *the unique solution of the initial-value problem for the delay Volterra integro-differential equation* (2.4) *can be represented in the form*

$$y(t) = g_0(t) + \sum_{k=1}^{\infty} \int_0^{\theta^k(t)} H_k(t,s)g_0(s)ds, \quad t \in I, \tag{2.9}$$

where $\theta^k := \underbrace{\theta \circ \cdots \circ \theta}_{k}$. *The iterated kernels $H_k(t,s)$ of the kernel $H_1(t,s)$ defined in* (2.8) *are determined recursively by*

$$H_{k+1}(t,s) := \int_{\theta^{-k}(s)}^{\theta(t)} H_1(t,v)H_k(v,s)dv \tag{2.10}$$

$$= \int_{\theta^{-1}(s)}^{\theta^k(t)} H_k(t,v)H_1(v,s)dv, \quad (t,s) \in D_\theta^{(k+1)}\ (k \geq 1). \tag{2.11}$$

An alternative form of the solution representation (2.9) *is*

$$y(t) = \left(1 + \sum_{k=1}^{\infty} \tilde{H}_k(t,0)\right) y_0 + \int_0^t g(s)ds + \sum_{k=1}^{\infty} \int_0^{\theta^k(t)} \tilde{H}_k(t,s)g(s)ds, \tag{2.12}$$

with

$$\tilde{H}_k(t,s) := \int_s^{\theta^k(t)} H_k(t,v)dv, \quad (t,s) \in D_\theta^{(k)}. \tag{2.13}$$

Proof. The only new ingredient in Theorem 2.2 is the alternative expression (2.11) for the iterated kernel $H_{k+1}(t,s)$. It is readily verified by induction, using an obvious change in the order of integration (which is based on assumption (D3) for θ). We will illustrate this by considering $k = 2$. By (2.10),

$$H_3(t,s) = \int_{\theta^{-2}(s)}^{\theta(t)} H_1(t,v)H_2(v,s)dv,$$

and hence,

$$\begin{aligned} H_3(t,s) &= \int_{\theta^{-2}(s)}^{\theta(t)} H_1(t,v)\left(\int_{\theta^{-1}(s)}^{\theta(v)} H_1(v,z)H_1(z,s)dz\right)dv \\ &= \int_{\theta^{-1}(s)}^{\theta^2(t)} \left(\int_{\theta^{-1}(z)}^{\theta(t)} H_1(t,v)H_1(v,z)dv\right)H_1(z,s)dz \\ &= \int_{\theta^{-1}(s)}^{\theta^2(t)} H_2(t,z)H_1(z,s)dz, \quad (t,s) \in D_\theta^{(3)}. \end{aligned}$$

The completion of the induction argument is now clear. The representation (2.12) follows from (2.9) and the definition (2.6) of $g_0(t)$. ∎

COROLLARY 2.3. *Let* $\theta(t) = qt$, $0 < q < 1$, *and define*

$$\beta := q^{-1}\max\{|b(t)| : t \in I\}, \quad \bar{K}_1 := \max\{|K_1(t,s)| : (t,s) \in D_q\}.$$

Then the iterated kernels $\{H_k(t,s)\}$ *corresponding to the kernel* $K_1(t,s)$ *in* (2.4), (2.8) *are bounded by*

$$|H_k(t,s)| \le \frac{(\beta + \bar{K}_1T)^k}{(k-1)!} q^{(k-1)(k-2)/2}[qt - q^{-(k-1)}s]^{k-1}, \quad (t,s) \in D_q^{(k)} \ (k \ge 1). \tag{2.14}$$

Moreover, the kernels $\{\tilde{H}_k(t,s)\}$ *defined in* (2.13) *satisfy*

$$|\tilde{H}_k(t,s)| \le \frac{(\beta + \bar{K}_1T)^k}{k!} q^{k(k-1)/2}[qt - q^{-(k-1)}s]^k, \quad (t,s) \in D_q^{(k)} \ (k \ge 1). \tag{2.15}$$

The estimates (2.14) follow directly from the expression (2.11) for the iterated kernels, using an inductive argument and integration by parts. Hence, the estimates for the integrated kernels $\tilde{H}_k(t,s)$ are an obvious consequence of (2.13).

COROLLARY 2.4. *If* $K_1(t,s) \equiv 0$, *then the iterated kernels associated with the delay integral equation* (2.5) *corresponding to the pantograph equation* (1.1) *satisfy*

$$|H_k(t,s)| \le \frac{\beta^k q^{(k-1)(k-2)/2}}{(k-1)!}[qt - q^{-(k-1)}s]^{k-1}, \quad (t,s) \in D_q^{(k)} \quad (k \ge 1). \tag{2.16}$$

For the sake of completeness we conclude this section by observing that the results of Theorem 2.2 on the representation of the solution to (2.4) are easily extended to the general pantograph equation (2.1), by applying Picard iteration to its equivalent delay integral equation (2.5). The precise result is given in the following theorem.

THEOREM 2.5. *If the given functions in the general pantograph Volterra integro-differential equation* (2.1) *are continuous on their respective domains, and if the delay function θ is subject to* (D1)–(D3), *then its unique solution $y \in C^1(I)$ has the representation*

$$y(t) = g_0(t) + \int_0^t \sum_{k=1}^{\infty} H_{0,k}(t,s) g_0(s) ds + \sum_{k=1}^{\infty} \int_0^{\theta^k(t)} H_{1,k}(t,s) g_0(s) ds + \mathcal{M}(t), \quad t \in I, \tag{2.17}$$

where

$$\mathcal{M}(t) := \sum_{k=1}^{\infty} \int_0^{\theta^k(t)} H_k^{(0,1)}(t,s) g_0(s) ds, \tag{2.18}$$

and $\theta^0(t) := t$. The kernels $H_{0,k}(t,s)$ and $H_{1,k}(t,s)$ are the iterated kernels associated with the kernels $H_0(t,s)$ and $H_1(t,s)$ defined in (2.7) *and* (2.8), *and g_0 is given by* (2.6). *The infinite series in* (2.17) *and* (2.18) *converge absolutely and uniformly on I, and we have $H_k^{(0,1)}(t,s) \equiv 0$ $(k \geq 1)$ when $H_0(t,s) \equiv 0$ or $H_1(t,s) \equiv 0$.*

Proof. As indicated before, the representation (2.17) follows readily from Picard iteration applied to the integral equation (2.5); here, the following lemma (an obvious extension of Dirichlet's formula regarding the change in the order of integration in integrals with variable limits of integration) plays a key role. ∎

LEMMA 2.6. *Let k and ℓ be given nonnegative integers, and assume that the delay function θ is subject to the hypotheses* (D1)–(D3). *Then for any function $\phi \in C(D_\theta^{(k+\ell)})$,*

$$\int_0^{\theta^k(t)} \left(\int_0^{\theta^\ell(s)} \phi(s,v) dv \right) ds = \int_0^{\theta^{k+\ell}(t)} \left(\int_{\theta^{-\ell}(v)}^{\theta^k(t)} \phi(s,v) ds \right) dv, \quad (t,s) \in D_\theta^{(k+\ell)}. \tag{2.19}$$

We leave the details of the proof of Theorem 2.5 to the reader but illustrate the basic result underlying its induction argument, thus revealing the structure of the kernels $H_k^{(0,1)}(t,s)$ in (2.18). Setting $y_0(t) := g_0(t)$ and $H_{0,1}(t,s) := H_0(t,s)$, $H_{1,1}(t,s) := H_1(t,s)$, the first two iterates obtained by Picard iteration for (2.5) are, respectively,

$$y_1(t) = g_0(t) + \int_0^t H_{0,1}(t,s) g_0(s) ds + \int_0^{\theta(t)} H_{1,1}(t,s) g_0(s) ds,$$

and hence, by Lemma 2.6,

$$\begin{aligned} y_2(t) &= g_0(t) + \int_0^t H_{0,1}(t,s) y_1(s) ds + \int_0^{\theta(t)} H_{1,1}(t,s) y_1(s) ds \\ &= g_0(t) + \int_0^t H_{0,1}(t,s) \bigg(g_0(s) + \int_0^s H_{0,1}(s,v) g_0(v) dv \\ &\qquad + \int_0^{\theta(s)} H_{1,1}(s,v) g_0(v) dv \bigg) ds \end{aligned}$$

$$+ \int_0^{\theta(t)} H_{1,1}(t,s) \left(g_0(s) + \int_0^s H_{0,1}(s,v)g_0(v)dv + \int_0^{\theta(s)} H_{1,1}(s,v)g_0(v)dv \right) ds$$
$$= g_0(t) + \int_0^t H_{0,1}(t,s)g_0(s)ds + \int_0^t H_{0,2}(t,s)g_0(s)ds$$
$$+ \int_0^{\theta(t)} H_{1,1}(t,s)g_0(s)ds + \int_0^{\theta^2(t)} H_{1,2}(t,s)g_0(s)ds + \int_0^{\theta(t)} H_2^{(0,1)}(t,s)g_0(s)ds.$$

Here, $H_{0,2}(t,s)$ and $H_{1,2}(t,s)$ are the first nontrivial iterated kernels for the given kernels $H_0(t,s)$ and $H_1(t,s)$:

$$H_{0,2}(t,s) := \int_s^t H_{0,1}(t,v)H_{0,k-1}(v,s)dv,$$
$$H_{1,2}(t,s) := \int_{\theta^{-(k-1)}(s)}^{\theta(t)} H_{1,1}(t,v)H_{1,k-1}(v,s)dv$$

(cf. (2.10)), and the "mixed" kernel $H_2^{(0,1)}(t,s)$ has the form

$$H_2^{(0,1)}(t,s) := \int_{\theta^{-1}(s)}^t H_{0,1}(t,v)H_{1,1}(v,s)dv + \int_v^{\theta(t)} H_{1,1}(t,v)H_{0,1}(v,s)dv.$$

2.2. Collocation methods in piecewise polynomial spaces. As we have seen in section 2.1 (Theorem 2.1), the analytical solution to the pantograph integro-differential equation (2.1) with smooth data is smooth on the entire interval $I := [0,T]$, in contrast to solutions to such functional equations with nonvanishing delays. Thus, the mesh underlying the piecewise polynomial space in which the approximation u_h to the exact solution will be sought will be chosen as

$$I_h := \{t_n : \ 0 = t_0 < t_1 < \cdots < t_N = T\} \quad (\text{with } \ t_n = t_n^{(N)}), \tag{2.20}$$

and we set $h_n := t_{n+1} - t_n, \ h := \max\{h_n : \ 0 \le n \le N-1\}$. For given I_h and $m \ge 1$, the collocation solution u_h to (2.1) will be an element of the space

$$S_m^{(0)}(I_h) := \{v \in C^0(I) : \ v|_{[t_n,t_{n+1}]} \in \pi_m \ (0 \le n \le N-1)\} \tag{2.21}$$

of continuous (real) piecewise polynomials of degree not exceeding m; u_h is determined by the collocation equation

$$u_h'(t) = a(t)u_h(t) + b(t)u_h(\theta(t)) + g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in X_h, \ \ u_h(0) = y_0. \tag{2.22}$$

Here,

$$X_h := \{t_n + c_i h_n : \ 0 < c_1 < \cdots < c_m \le 1 \ (0 \le n \le N-1)\} \tag{2.23}$$

is the set of collocation points corresponding to given collocation parameters $\{c_i\}$. For continuous data there exists $\bar{h} > 0$ so that (2.19) has a unique solution $u_h \in S_m^{(0)}(I_h)$ for all meshes I_h with $h \in (0, \bar{h})$ (see, e.g., [12, section 5.5.1]).

It is also known [12, section 5.5.2] that, for arbitrarily chosen collocation parameters $\{c_i\}$, the collocation error tends to zero uniformly, as $h \to 0$. More precisely, we have the following global convergence result (which we state for further reference); its proof can be found in [12, Ch. 5].

THEOREM 2.7. *Assume that* $a,\ b,\ g,\ K_0,\ K_1$ *are in* C^m *on their respective domains, and let* θ *be subject to the conditions* (D1)–(D3), *with* $d \geq m+1$ *(cf. section* 2.1*). Then for* $h \in (0, \bar{h})$ *the collocation solution* $u_h \in S_m^{(0)}(I_h)$ *for* (2.1) *satisfies*

$$\|y^{(\nu)} - u_h^{(\nu)}\|_\infty \leq C_\nu h^m \quad (\nu = 0, 1), \tag{2.24}$$

where the constants C_ν *depend on* $\{c_i\}$ *and on* q *but not on* h.

The collocation solution $u_h \in S_m^{(0)}(I_h)$ for (2.1) induces the *defect* (or residual) δ_h defined by

$$\delta_h(t) := -u_h'(t) + a(t)u_h(t) + b(t)u_h(\theta(t)) + g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in I, \tag{2.25}$$

with $\delta_h(t) = 0$ for all $t \in X_h$. It inherits (piecewise, on the subintervals $[t_n, t_{n+1}]$) the regularity of the given functions in (2.1). Since $\delta_h(t)$ can also be written as

$$\delta_h(t) = e_h'(t) - a(t)e_h(t) - b(t)e_h(\theta(t)) - (\mathcal{V}e_h)(t) - (\mathcal{V}_\theta e_h)(t), \quad t \in I, \tag{2.26}$$

where $e_h := y - u_h$, the following result is an immediate consequence of Theorem 2.7.

COROLLARY 2.8. *Under the assumptions of Theorem* 2.7 *we have*

$$\|\delta_h\|_\infty \leq D_0 h^m$$

for all $h \in (0, \bar{h})$, *where the constant* D_0 *does not depend on* h.

3. Optimal global superconvergence on I.

THEOREM 3.1. *Assume the following:*

(i) *The given functions in the pantograph Volterra integro-differential equation* (2.1) *satisfy, for* κ *specified in* (3.1), $a,\ b,\ g \in C^{m+\kappa}(I)$, $K_0 \in C^{m+\kappa}(D)$, $K_1 \in C^{m+\kappa}(D_\theta)$, *and* $\theta(t) = qt$ $(0 < q < 1)$.

(ii) $u_h \in S_m^{(0)}(I_h)$ *is the collocation solution, with respect to the uniform mesh* I_h *and collocation parameters* $\{c_i\}$, *to* (2.1).

(iii) *The collocation parameters are such that, for some* κ *with* $1 \leq \kappa \leq m$,

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s - c_i) ds = 0, \quad \nu = 0, \ldots, \kappa - 1; \tag{3.1}$$

that is, the interpolatory m*-point quadrature based on the abscissas given by the collocation parameters* $\{c_i\}$ *has degree of precision* $m + \kappa - 1$.

Then u_h *is globally superconvergent:*

$$\|y - u_h\|_\infty \leq C h^{m+1},$$

where in general the order $m+1$ *cannot be replaced by* $m+2$. *The constant* C *depends on* q *and on* $\{c_i\}$ *but not on* h.

Proof. For ease of exposition we will give the proof for (2.4); it is readily adapted (using the representation (2.17) in Theorem 2.5) to the general pantograph integro-differential equation (2.1). It follows from (2.4) and (2.25) (with $a = 0$, $\mathcal{V} = 0$) that the collocation error $e_h := y - u_h$ solves the initial-value problem

$$e_h'(t) = b(t)e_h(qt) + \delta_h(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in I, \quad e_h(0) = 0. \tag{3.2}$$

Hence, by (2.12) of Theorem 2.2 (where the roles of y and g are now assumed by e_h and δ_h, respectively),

$$e_h(t) = \int_0^t \delta_h(s)ds + \sum_{k=1}^{\infty} \int_0^{q^k t} \tilde{H}_k(t,s)\delta_h(s)ds, \quad t \in I. \tag{3.3}$$

Let now $t = t_n + vh$ ($v \in [0,1]$), and define

$$\begin{aligned} q_{k,n}(v) &:= \lfloor q^k(n+v) \rfloor, \quad \gamma_{k,n} = \gamma_{k,n}(v) := q^k n - q_{k,n}, \\ k_n^*(q) &:= \max\{k : \ q_{k,n}(v) \geq 1\}. \end{aligned} \tag{3.4}$$

Thus, (3.3) can be written in the form

$$e_h(t) = h \sum_{\ell=0}^{n-1} \int_0^1 \delta_h(t_\ell + sh)ds + h \int_0^v \delta_h(t_n + sh)ds + S_n^I(v) + S_n^{II}(v), \tag{3.5}$$

where we have set

$$S_n^I(v) := \sum_{k=1}^{k_n^*(q)} \int_0^{t_{q_{k,n}}} \tilde{H}_k(t,s)\delta_h(s)ds = h \sum_{k=1}^{k_n^*(q)} \left(\sum_{\ell=0}^{q_{k,n}-1} \int_0^1 \tilde{H}_k(t, t_\ell + sh)\delta_h(t_\ell + sh)ds \right) \tag{3.6}$$

and

$$S_n^{II}(v) := h \sum_{k=1}^{\infty} \int_0^{\gamma_{n,k}} \tilde{H}_k(t, t_{q_{k,n}} + sh)\delta_h(t_{q_{k,n}} + sh)ds. \tag{3.7}$$

In order to derive the (optimal) order estimate for $S_n^I(v)$, we first observe that, using m-point interpolatory quadrature formulas, with abscissas based on the m collocation parameters $\{c_i\}$ and with $E_{n,\ell}(v)$ and $\tilde{E}_{n,\ell}^{(k)}(v)$ denoting, respectively, the resulting quadrature errors for the integrals with integrands $\delta_h(t_\ell + sh)$ and those with integrands $\tilde{H}_k(t, t_\ell + sh)\delta_h(t_\ell + sh)$, and observing that on each subinterval $[t_n, t_{n+1}]$ the defect δ_h is in $C^{m+\kappa}$, we may write

$$|S_n^I(v)| \leq h \sum_{k=1}^{k_n^*(q)} \left(\sum_{\ell=0}^{q_{k,n}-1} |\tilde{E}_{n,\ell}^{(k)}(v)| \right). \tag{3.8}$$

By assumption (3.1) on the collocation parameters $\{c_i\}$, these quadrature formulas have degree of precision $m + \kappa - 1$, and hence, by the regularity of the integrands on the subintervals $[t_n, t_{n+1}]$,

$$|E_{n,\ell}(v)| \leq Q_m h^{m+\kappa}, \quad |\tilde{E}_{n,\ell}^{(k)}(v)| \leq \tilde{Q}_m h^{m+\kappa}.$$

We therefore obtain

$$|S_n^I(v)| \leq h \cdot \tilde{Q}_m h^{m+\kappa} \sum_{k=1}^{k_n^*(q)} q_{k,n}(v) \leq Nh \cdot \tilde{Q}_m h^{m+\kappa} \frac{q}{1-q} = T\tilde{Q}_m h^{m+\kappa} \frac{q}{1-q} \tag{3.9}$$

($0 \leq n \leq N-1$), since, by (3.4), $q_{k,n} \leq q^k n \leq q^k N$. □

Consider now $S_n^{II}(v)$: it follows from (3.7), Corollary 2.8, and (2.15) that

$$|S_n^{II}(v)| \le h\|\delta_h\|_\infty \sum_{k=1}^{\infty} \frac{(\beta+\bar{K}_1T)^k}{k!} q^{k(k-1)/2} \int_0^{\gamma_{k,n}} [qt - q^{-(k-1)}s]^k ds$$

$$\le h\|\delta_h\|_\infty \sum_{k=1}^{\infty} \frac{(\beta+\bar{K}_1T)^k T^{k+1}}{(k+1)!} q^{k(k+3)/2} =: D_0\tilde{B}(q)h^{m+1}, \quad v \in [0,1] \tag{3.10}$$

$(0 \le n \le N-1)$. Thus, (3.5) together with (3.9) and (3.10) lead to the (optimal) global superconvergence estimate

$$|e_h(t)| \le Ch^{m+1}, \quad \text{with} \quad C := (TQ_m + D_0) + (T\tilde{Q}_m q/(1-q) + D_0\tilde{B}(q)),$$

which holds uniformly for $t \in I$, and for any κ between 1 and m in (3.1).

Owing to the smoothness of the exact solution of the pantograph integro-differential equation (2.1), the optimal global convergence estimate in Theorem 3.1 is not surprising: it agrees with the one for classical and constant-delay Volterra integral equations. However, the picture changes completely for the optimal order of the collocation solution for (2.1) at the mesh points of a uniform mesh, as shown in the following section.

4. Optimal local superconvergence on uniform meshes. It was shown in Brunner and Hu [13] that for Volterra *integral* equations of pantograph type, the optimal order of local superconvergence at the points of a uniform mesh cannot exceed $p^* = m+2$, and that the optimal value is attained only when $q = 1/2$ and m is even. For Volterra *integro-differential* equations of pantograph type, we obtain the same value of p^* but it is now attained for all $q \in (0,1)$ and all $m \ge 2$. This result, stated in the following theorem, provides the (affirmative) answer to a conjecture in [11, section 4] and [12, section 5.5.2]. We will comment on the reason for this difference in the optimal local superconvergence orders following the proof of Theorem 4.1.

THEOREM 4.1. *Assume the following:*

(i) *The given functions in* (2.1) *satisfy* $a,\ b,\ g \in C^{m+\kappa}(I)$, $K_0 \in C^{m+\kappa}(D)$, *and* $K_1 \in C^{m+\kappa}(D_\theta)$ *for some* κ *with* $1 \le \kappa \le m$.

(ii) $\theta(t) = qt$ $(0 < q < 1)$.

(iii) $u_h \in S_m^{(0)}(I_h)$ *is the collocation solution to* (2.1) *on a uniform mesh* I_h *and corresponding to collocation parameters* $\{c_i\}$ *satisfying the orthogonality conditions* (3.1) *with* $1 \le \kappa \le m$.

Then for any $q \in (0,1)$ *and any* $m \ge 2$,

$$\max\{|y(t) - u_h(t)| : t \in I_h\} \le C^* h^{m+2}, \tag{4.1}$$

and the exponent $m+2$ *cannot, in general, be replaced by* $m+3$.

Proof. The starting point is again the error equation (3.2): for the representation of its solution we now use (2.9), where the roles of g in (2.6) and those of y are assumed by δ_h and e_h, respectively. For a given mesh point $t = t_n$ $(n = 1,\dots,N)$ we set, in analogy to (3.4),

$$q_{k,n} := \lfloor q^k n \rfloor, \quad \gamma_{k,n} := q^k n - q_{k,n}, \quad k_n^*(q) := \max\{k : q_{k,n} \ge 1\}. \tag{4.2}$$

Thus, the counterpart of (3.5) is given by

$$e_h(t_n) = \int_0^{t_n} \delta_h(s)ds + S_n^I + S_n^{II}, \tag{4.3}$$

where

$$
\begin{aligned}
(4.4)\qquad S_n^I &:= \sum_{k=1}^{k_n^*(q)} \int_0^{t_{q_{k,n}}} H_k(t_n,s)\left(\int_0^s \delta_h(v)dv\right) ds \\
&= h\sum_{k=1}^{k_n^*(q)} \sum_{\ell=0}^{q_{k,n}-1} H_k(t_n,t_\ell+sh)\Bigg(h\sum_{\nu=0}^{\ell-1}\int_0^1 \delta_h(t_\nu+vh)dv \\
&\qquad\qquad + h\int_0^s \delta_h(t_\ell+vh)dv\Bigg) ds
\end{aligned}
$$

and

$$
\begin{aligned}
(4.5)\qquad S_n^{II} &:= h\sum_{k=1}^{\infty} \int_0^{\gamma_{k,n}} H_k(t_n,t_{q_{k,n}}+sh)\left(\int_0^{t_{q_{k,n}}+sh} \delta_h(v)dv\right) ds \\
&= h\sum_{k=1}^{\infty} \int_0^{\gamma_{k,n}} H_k(t_n,t_{q_{k,n}}+sh)\Bigg(h\sum_{\ell=0}^{q_{k,n}-1}\int_0^1 \delta_h(t_\ell+vh)dv \\
&\qquad\qquad + h\int_0^s \delta_h(t_{q_{k,n}}+vh)dv\Bigg) ds.
\end{aligned}
$$

The techniques for estimating $|S_n^I|$ and $|S_n^{II}|$ closely parallel the ones used in section 3 (cf. (3.8) and (3.9)). Observe first that in the upper bound for $|S_n^I|$ we have

$$
h\sum_{\nu=0}^{\ell-1}\left|\int_0^1 \delta_h(t_\nu+vh)dv\right| + h\left|\int_0^s \delta_h(t_\ell+vh)dv\right| \le TQ_m h^{m+\kappa} + h\cdot D_0 h^m =: \tilde{D}_0 h^{m+1}.
$$

Hence, by (2.14) of Corollary 2.3 and by observing the factor h in front of the first summation sign in the second line of (4.4) we are led to

$$
\begin{aligned}
|S_n^I| &\le h\cdot \tilde{D}_0 h^{m+1}\cdot \sum_{k=1}^{k_n^*(q)} \sum_{\ell=0}^{q_{k,n}-1} \int_0^1 |H_k(t_n,t_\ell+sh)|ds \\
(4.6)\qquad &\le \tilde{D}_0 h^{m+2}\sum_{k=1}^{k_n^*(q)} \frac{(\beta+\bar{K}_1 T)^k T^{k-1}}{(k-1)!} q^{(k-1)(3k-4)/2} =: C_0 h^{m+2}.
\end{aligned}
$$

Similarly, (4.5) together with (2.14) and Corollary 2.8 allow us to obtain the estimate

$$
(4.7)\qquad |S_n^{II}| \le C_1 h^{m+2} \quad (1\le n\le N).
$$

Hence, (4.3) and the two estimates (4.6) and (4.7) yield the desired optimal superconvergence result on I_h,

$$
|e_h(t_n)| \le (C_0+C_1)h^{m+2} =: C^* h^{m+2}, \quad n=1,\dots,N;
$$

as the above analysis shows, the power h^{m+2} cannot be replaced by h^{m+3}, except in trivial cases. ∎

The above result answers the question regarding the optimal order of local superconvergence for the pantograph delay differential equation (1.1).

COROLLARY 4.2. *The collocation solution $u_h \in S_m^{(0)}(I_h)$ $(m \geq 2)$ for the pantograph delay differential equation* (1.1)*, with uniform mesh I_h and collocation points based on the Gauss points $\{c_i\}$, has the optimal local superconvergence order $p^* = m+2$ on I_h:*

$$\max\{|y(t) - u_h(t)| : \ t \in I_h\} \leq Ch^{m+2}.$$

Remark. As we have indicated before, the above optimal local superconvergence result differs sharply from the one corresponding to the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ (the space of piecewise polynomials of degree $m-1 \geq 0$ that are allowed to possess finite jumps at the mesh points) and its iterate $u_h^{it|}$ for the pantograph integral equation

$$u(t) = g(t) + \int_0^{qt} K_1(t,s)u(s)ds, \quad t \in [0,T]. \tag{4.8}$$

Here, the iterated collocation error e_h^{it} $(= e_h - \delta_h)$ at $t = t_n$ has the representation

$$e_h^{it}(t_n) = \sum_{k=1}^{\infty} \int_0^{q^k t_n} H_k(t_n, s)\delta_h(s)ds, \quad n = 1, \ldots, N,$$

which, recalling (4.2), can be written in a form analogous to (4.3), namely, $e_h^{it}(t_n) = \hat{S}_n^I + \hat{S}_n^{II}$. However, the terms $\hat{S}_n^I$ and $\hat{S}_n^{II}$ corresponding to S_n^I and S_n^{II} (cf. (4.4) and (4.5)) no longer contain integrals of the defect function δ_h. In particular, we now have

$$\hat{S}_n^{II} = h \sum_{k=1}^{\infty} \int_0^{\gamma_{k,n}} H_k(t_n, t_{q_{k,n}} + sh)\delta_h(s)ds,$$

where, as before, $\|\delta_h\|_\infty = \mathcal{O}(h^m)$. This implies that if collocation is at the Gauss points, then $\hat{S}_n^{II} = \mathcal{O}(h^{m+2})$ $(n = 1, \ldots, N)$ can now be attained only for special values of q and m, namely, when $q = 1/2$ and m is even. Details can be found in Brunner and Hu [13, p. 1940].

5. Equations with nonlinear vanishing delays vanishing at $t = 0$. We now turn to the superconvergence analysis of collocation solutions for pantograph-type Volterra integro-differential equations (2.1) where delay function given by $\theta(t) = t - \tau(t)$ is nonlinear and satisfies the hypotheses (D1)–(D3). Since (D2) implies the inequalities

$$\theta^k(t) \leq q_1^k t \quad (k \geq 1) \quad \text{and} \quad \theta^{-1}(s) \geq q_1^{-1} s \tag{5.1}$$

(cf. (2.11)), they suggest that our global and local superconvergence results of Theorems 3.1 and 4.1 will remain valid for such nonlinear vanishing delays. The basis for the proofs is given by the following generalization of Corollary 2.3.

LEMMA 5.1. *Let θ be subject to the hypotheses* (D1)–(D3) *of section* 2*, and define*

$$\beta := \max\{|b(\theta^{-1}(t))|\theta'(\theta^{-1}(t)) : \ t \in I\}, \quad \bar{K}_1 := \max\{|K_1(t,s)| : \ (t,s) \in D_\theta\}.$$

The iterated kernels associated with the delay integral equation (2.5) *corresponding to the pantograph integro-differential equation* (2.4) *satisfy*

$$|H_k(t,s)| \leq \frac{(\beta + \bar{K}_1 T)^k}{(k-1)!} q_1^{(k-1)(k-2)/2}[q_1 t - q_1^{-(k-1)} s]^{k-1}, \tag{5.2}$$
$$(t,s) \in D_\theta^{(k)} \ \ (k \geq 1).$$

Moreover, the kernels $\{\tilde{H}_k(t,s)\}$ *defined in* (2.13) *satisfy*

$$|\tilde{H}_k(t,s)| \leq \frac{(\beta+\bar{K}_1 T)^k}{k!} q_1^{k(k-1)/2}[q_1 t - q_1^{-(k-1)} s]^k, \quad (t,s) \in D_\theta^{(k)} \ \ (k \geq 1). \tag{5.3}$$

The proof is an immediate consequence of (5.1) (which is based on the condition (D2) for the delay function θ)) and the result of Corollary 2.3, where the role of q is now assumed by q_1.

THEOREM 5.2. *Assume the following:*

(i) *The given functions* a, b, g, K_0, *and* K_1 *in the general Volterra pantograph integro-differential equation* (2.1) *possess continuous derivatives of order* $m+\kappa$ *for some* κ *with* $1 \leq \kappa \leq m$ *on their respective domains.*

(ii) *The delay function* θ *is subject to the hypotheses* (D1)–(D3) *of section* 2, *with* $d \geq m+\kappa+1$.

(iii) $u_h \in S_m^{(0)}(I_h)$ *is the collocation solution, with respect to a uniform mesh* I_h *with sufficiently small mesh diameter* $h>0$, *to the initial-value problem for* (2.1).

(iv) *The collocation parameters* $\{c_i\}$ *in* (2.21) *satisfy* (3.1) *for some* κ *with* $1 \leq \kappa \leq m$.

Then the following hold:

(a) u_h *is globally superconvergent on* I, *with optimal order described by*

$$\|y-u_h\|_\infty \leq C h^{m+1}. \tag{5.4}$$

(b) *For any* $q_1 \in (0,1)$ *(cf. hypothesis* (D3)*) and any* $m \geq 2$, u_h *is locally superconvergent at the mesh points*

$$\max\{|y(t)-u_h(t)| : \ t \in I_h\} \leq C^* h^{m+2}, \tag{5.5}$$

where in general the order $p^* := m+2$ *cannot be replaced by* $m+3$.

Proof. We will prove the local superconvergence estimate (5.5); the global estimate (5.4) can be established along very similar lines, as is already suggested by the proof of Theorem 3.1.

By (2.1) and (2.20) the collocation error satisfies the initial-value problem

$$e_h'(t) = a(t)e_h(t) + b(t)e_h(\theta(t)) + \delta_h(t) + (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in I,$$
$$e_h(0) = 0,$$

where $\delta_h(t)=0$ on the set X_h of collocation points. Hence, it follows from the solution representation (2.9) in Theorem 2.5 (with e_h and δ_h replacing y and g, respectively) that, at $t=t_n$ $(n=1,\ldots,N)$,

$$e_h(t_n) = \int_0^{t_n} \delta_h(s)ds + \sum_{k=0}^{\infty} \int_0^{\theta^k(t_n)} H_k(t_n,s)\left(\int_0^s \delta_h(v)dv\right)ds$$

$$= \int_0^{t_n} \delta_h(s)ds + \int_0^{t_n} H_0(t,s)(t_n,s)\left(\int_0^s \delta_h(v)dv\right)ds \tag{5.6}$$

$$+\sum_{k=1}^{\infty} \int_0^{\theta^k(t_n)} H_k(t_n,s)\left(\int_0^s \delta_h(v)dv\right)ds. \tag{5.7}$$

For given n and $k \geq 1$ let, in analogy to (4.2), the (unique) $q_{k,n} \in \mathbb{N}$ be such that

$$\theta^k(t_n) \in [t_{q_{k,n}}, t_{q_{k,n}+1}), \tag{5.8}$$

and define

$$\gamma_{k,n} := (\theta^k(t_n) - t_{q_{k,n}})/h, \quad k_n^* := \max\{k :\ q_{k,n} \geq 1\}. \tag{5.9}$$

We thus may write, in analogy to (4.3)–(4.5),

$$e_h(t_n) = h\sum_{\ell=0}^{n-1}\int_0^1 \delta_h(t_\ell + sh)ds + h\sum_{\ell=0}^{n-1}\int_0^1 H_0(t_n, t_\ell + sh)\left(\int_0^{t_\ell+sh}\delta_h(v)dv\right)ds$$

$$+ S_n^I + S_n^{II} \quad (n = 1, \ldots, N), \tag{5.10}$$

where

$$S_n^I := \sum_{k=1}^{k_n^*}\int_0^{t_{q_{k,n}}} H_k(t_n, s)\left(\int_0^s \delta_h(v)dv\right)ds \tag{5.11}$$

and

$$S_n^{II} := h\sum_{k=1}^{\infty}\int_0^{\gamma_{k,n}} H_k(t_n, t_{q_{k,n}} + sh)\left(\int_0^{t_{q_{k,n}}+sh}\delta_h(v)dv\right)ds. \tag{5.12}$$

(Recall that by Lemma 5.1 the above infinite series converge uniformly for all $n = 1, \ldots, N$.)

A glimpse at (4.4) and (4.5) now reveals that the estimates (4.6) and (4.7) carry over to S_n^I and S_n^{II} defined in (5.11) and (5.12), except that now, by (5.2) in Lemma 5.1, q has to be replaced by $q_1 \in (0,1)$, corresponding to the hypothesis (D2) for the nonlinear delay function θ. Thus, employing the arguments used in the proof of Theorem 4.1 allows us to derive the optimal estimate (5.5) of Theorem 5.2.

COROLLARY 5.3. *Assume that the delay function θ satisfies the conditions* (D1)–(D3) *of section* 2*, with $d \geq 2m+1$, and let I_h be a uniform mesh with sufficiently small $h > 0$. If $a,\ b,\ g \in C^{2m}(I)$, then the collocation solution $u_h \in S_m^{(0)}(I_h)$* $(m \geq 2)$ *for the generalized pantograph delay differential equation*

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t), \quad t \in I, \quad y(0) = y_0, \tag{5.13}$$

with collocation points (2.21) *based on the Gauss points $\{c_i\}$, has the optimal local superconvergence order $p^* = m+2$ for all $q \in (0,1)$:*

$$\max\{|y(t) - u_h(t)| :\ t \in I_h\} \leq C^* h^{m+2}.$$

Here, the constant C^* depends on the $\{c_i\}$ and on q_1 but not on h.

Remarks. 1. Recall that the hypothesis (D2) (section 2) requires that $\theta(t) \leq q_1 t,\ t \in I$, for some $q_1 \in (0,1)$. Do the optimal superconvergence estimates (5.4) and (5.5) remain true if the nonlinear delay function θ satisfies only (D1) and (D3) and is such that $\theta'(0) = 1$? Examples of such delay functions are $\theta(t) = q_1\log(1+t),\ \ 0 < q_1 \leq 1$, for which we have $\theta'(t) = q_1\frac{1}{1+t},\ \ t \in [0,T]$, and thus $\theta'(0) = 1$ when $q_1 = 1$; and $\theta(t) = t - t^r,\ \ r \in \mathbb{N}\ (r \geq 2)$. In this case, $\theta'(0) = 1,\ \theta^{(\nu)}(0) = 0\ (\nu = 2, \ldots, r-1)$ (see also [5, section 4]).

The analysis of optimal superconvergence of collocation solutions for (2.1) with delay functions of the above type is yet to be carried out. It appears that the approach in Brunner and Maset [15] will yield the tools to extend the analysis in the present paper to functional differential and integro-differential equations (2.1) containing these more general θ.

2. A related question concerns delay functions θ that satisfy (D1) and (D3) on $[0,T]$ but have the properties that (i) $\theta(0) = 0$ and $\theta(t) < t$ for $t \in (0,t^*)$, (ii) $\theta(t^*) = t^*$ (i.e., $\tau(t^*) = 0$), and (iii) e.g., (D1)–(D3) hold on $[t^*,T]$. The optimal convergence analysis of (2.1) with such "doubly vanishing" delay functions is studied in [15].

6. Optimal local superconvergence on geometric meshes. We shall now show that, in analogy to pantograph-type Volterra integral equations (Brunner and Hu [13]), $\mathcal{O}(h^{2m})$-superconvergence at the mesh points can (almost) be restored if we replace the uniform mesh I_h by a judiciously chosen *geometric mesh* (see also [21] and [14]). To be precise, we shall seek the collocation solution u_h to (2.1) in $S_m^{(0)}(I_h)$, where the underlying mesh I_h is defined by the mesh points

$$t_0 = 0, \ \ t_n = t_n^{(N)} = d^{N-n}T \quad (n = 1, \dots, N) \tag{6.1}$$

for some $d = d(q;N) \in (0,1)$.

THEOREM 6.1. *Let the assumptions of Theorem* 3.1 *hold with $\kappa = m$ (implying, by* (3.1)*, that the $\{c_i\}$ are the m Gauss points). If the mesh I_h corresponds to the points defined by* (6.1)*, with*

$$d = q^{1/r}, \quad r := \left\lfloor \frac{\ln(q)}{\ln[1 - \frac{2m\ln(N)}{(m+2)N}]} \right\rfloor, \tag{6.2}$$

then the estimate

$$\max\{|y(t) - u_h(t)| : \ t \in I_h\} \le CN^{-(2m-\varepsilon_N)} \quad \text{as } \ N \to \infty \tag{6.3}$$

for the collocation solution $u_h \in S_m^{(0)}(I_h)$ $(m \ge 2)$ to (2.1) *holds for any $q \in (0,1)$. Here, $C = C(q)$, and ε_N is defined by*

$$\varepsilon_N := \log_N\left(\frac{(2m \cdot \ln N)^{2m}}{(2m+1)(m+2)^{2m}}\right) \tag{6.4}$$

and has the property $\varepsilon_N \to 0^+$ as $N \to \infty$.

Proof. For ease of exposition we describe the proof of (2.4); the analysis is readily extended to the general pantograph integro-differential equation (2.1) by employing the error representation based on the result (2.17) in Theorem 2.5. By (2.9) of Theorem 2.2 we obtain the error representation

$$e_h(t_n) = \int_0^{t_n} \delta_h(s)ds + \sum_{k=1}^{\infty} \int_0^{\theta^k(t_n)} H_k(t_n,s)\left(\int_0^s \delta_h(\tau)d\tau\right) ds, \quad n = 1, \dots, N$$

(recall also (3.3)). Using integration by parts, this expression can be rewritten as

$$e_h(t_n) = \int_0^{t_n} \delta_h(s)ds + \sum_{k=1}^{\infty} \int_0^{\theta^k(t_n)} \tilde{H}_k(t_n,\tau)\delta_h(\tau)d\tau, \quad n = 1, \dots, N, \tag{6.5}$$

with

$$\tilde{H}_k(t_n,\tau) := \int_\tau^{\theta^k(t_n)} H_k(t_n,s)ds.$$

Resorting to the quadrature argument used in the proof of Theorem 4.1, we find that

$$\left|\int_0^{t_n} \delta_h(s)ds\right| \le CN^{-2m} \quad (n = 1, \ldots, N). \tag{6.6}$$

Thus, the proof reduces to estimating the infinite series in (6.5). It is easy to see that

$$\theta^k(t_n) = q^k t_n = d^{kr} \cdot t_n = d^{N-n+kr}T,$$

with r as defined in (6.2). This, together with the definition (6.1) of t_n, leads to the following results:

(i) For $kr - n \ge -1$, we have $\theta^k(t_n) \le t_1 \le CN^{-\frac{2m}{m+2}}$ (as $N \to +\infty$).

(ii) For $kr - n \le -1$, there holds $\theta^k(t_n) = t_{n-kr} \in I_h$.

Note that (i) follows from Lemma 3.1(i) of [14].

We now assume, without loss of generality, that $n \ge r+1$. Otherwise, the case (i) always occurs for each k, and the corresponding analysis is obvious. We decompose the infinite series in (6.5) into two parts:

$$\sum_{k=1}^{\infty} \int_0^{\theta^k(t_n)} \tilde{H}_k(t_n,\tau)\delta_h(\tau)d\tau = \sum_{k=[\frac{n-1}{r}]+1}^{\infty} \int_0^{\theta^k(t_n)} \tilde{H}_k(t_n,\tau)\delta_h(\tau)d\tau$$

$$+ \sum_{k=1}^{[\frac{n-1}{r}]} \int_0^{t_{n-kr}} \tilde{H}_k(t_n,\tau)\delta_h(\tau)d\tau =: I_1 + I_2. \tag{6.7}$$

Since

$$|\tilde{H}_k(t_n,\tau)| \le C\theta^k(t_n), \quad \tau \in [0,\ \theta^k(t_n)],$$

a standard argument leads to

$$\left|\int_0^{\theta^k(t_n)} \tilde{H}_k(t_n,\tau)\delta_h(\tau)d\tau\right| \le C(\theta^k(t_n))^{m+2} \le C(d^{N-n+kr})^{m+2}.$$

Furthermore, it is easy to verify that

$$|I_1| \le Cd^{(m+2)(N-1)} \le Ct_1^{m+2} \le CN^{-2m}. \tag{6.8}$$

Since the term I_2 in (6.7) can be written in the form

$$I_2 = \sum_{k=1}^{[\frac{n-1}{r}]} \sum_{i=1}^{n-kr} \int_{t_{i-1}}^{t_i} \tilde{H}_k(t_n,\tau)\delta_h(\tau)d\tau,$$

we obtain, observing Corollary 2.3,

$$\left|\int_{t_{i-1}}^{t_i} \tilde{H}_k(t_n,\tau)\delta_h(\tau)d\tau\right| \le C(t_i - t_{i-1})^{2m+1}.$$

It then follows from part (ii) of Lemma 3.1 in [14] that

$$|I_2| \le C(1-d)^{2m+1} \sum_{k=1}^{[\frac{n-1}{r}]} \sum_{i=1}^{n-kr} d^{(N-i)(2m+1)}$$

$$\leq C(1-d)^{2m} \sum_{k=1}^{[\frac{n-1}{r}]} d^{(2m+1)(N-n+kr)}$$
$$\leq C\frac{q^{2m+1}(1-d)^{2m}}{1-q^{2m+1}} \leq CN^{-(2m-\varepsilon_N)},$$

with ε_N given by (6.4). We now substitute (6.8) and the above inequality into (6.7); this yields

$$\left|\sum_{k=1}^{\infty}\int_0^{\theta^k(t_n)} \tilde{H}_k(t_n,\tau)\delta_h(\tau)d\tau\right| \leq CN^{-(2m-\varepsilon_N)}.$$

The result (6.3) in Theorem 6.1 now follows from (6.5), (6.6), and the above estimate. ∎

COROLLARY 6.2. *Let* $a, b, g \in C^{2m}(I)$. *If the solution* y *of the pantograph delay differential equation* (5.13) *is approximated by the collocation solution* $u_h \in S_m^{(0)}(I_h)$ $(m \geq 2)$, *with geometric mesh* I_h *given by* (6.1), (6.2), *and with collocation points corresponding to the Gauss points* $\{c_i\}$, *then*

$$\max\{|y(t)-u_h(t)| :\ t \in I_h\} \leq CN^{-(2m-\varepsilon_N)} \quad \text{as } N \to \infty,$$

with ε_N *defined in* (6.4).

Remark. It was shown in Bellen [4] (see also [7] and [5]) that the optimal local superconvergence order $p^* = 2m$ can be restored if a different type of (quasi-) geometric mesh is employed (see also [31] and [6], where such meshes were introduced for the stability analysis of one-point collocation methods for the pantograph equation). This approach relies, however, on the assumption that a sufficiently accurate initial approximation is known on a "small" initial interval $[0, t_0]$; the quasi-geometric mesh is then generated on $[t_0, T]$. This contrasts the quasi-optimal local superconvergence results of Theorem 6.1 and Corollary 6.2 which do not require such an initial approximation.

7. Concluding remarks. We conclude our presentation by pointing to a number of open problems in the numerical analysis of pantograph-type functional differential equations.

(i) *Higher-order pantograph-type integro-differential equations.* Special cases of the initial-value problem

$$y''(t) = a(t)y(t) + b(t)y(\theta) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I := [0,T], \tag{7.1}$$

with $\theta(t) = qt$ $(0 < q < 1)$, were studied, both theoretically and numerically, by Bélair [2] and by Zhang and Brunner [34]. If (7.1) is rewritten as a system of first-order integro-differential equations and solved by collocation in the piecewise polynomial space $S_m^{(0)}(I_h)$, then the superconvergence analysis of the previous sections can be readily extended to this system. However, if the initial-value problem (7.1) is solved directly, by using the collocation space $S_{m+1}^{(1)}(I_h)$, then the analysis of the optimal order of superconvergence on uniform meshes remains to be carried out.

(ii) *Pantograph equations of neutral type.* The convergence analysis in $S_m^{(0)}(I_h)$ for the neutral-type analogue of (2.4),

$$u'(t) = b(t)u(\theta(t)) + c(t)u'(\theta(t)) + g(t) + \int_0^{\theta(t)} (K_1(t,s)u(s) + K_2(t,s)u'(s))\, ds, \quad t \in [0,T], \tag{7.2}$$

with $\theta(t) = qt$ $(0 < q < 1)$, is much more complex than the one for (2.4) and is the subject of ongoing work. The key difficulty lies in the fact that (7.2) is in essence equivalent to a nonstandard pantograph-type Volterra integral equation of the form

$$u(t) = g_0(t) + \hat{b}(t)u(qt) + \int_0^{qt} \hat{K}(t,s)u(s)ds, \quad t \in I.$$

We note that even for the initial-value problem for the neutral pantograph equation

$$u'(t) = a(t)u(t) + b(t)u(qt) + c(t)u'(qt) + g(t), \quad t \in [0,T],$$

the existence of a unique (exact or collocation) solution is a nontrivial problem (it depends on the "size" of $c(t)$); compare, e.g., [27, 26, 16].

(iii) *Pantograph equations with multiple delays.* The paper [35] by Zhao, Xu, and Qiao contains an analysis of the optimal order of piecewise polynomial collocation solutions at $t = t_1 = h$ for the double pantograph equation (that is, for (1.1) with two proportional delays). Their result generalizes the analogous ones in [9, 33, 28, 32] for (1.1) with constant a and b. It is not yet known if the optimal superconvergence results of Corollaries 4.2 and 5.3 remain valid for delay differential equations with several proportional delays.

(iv) *Asymptotic stability of collocation solutions.* The problem of the asymptotic behavior (asymptotic stability; contractivity) of collocation solutions on *uniform meshes* to pantograph-type (integro-) differential equations remains essentially open. While there is such a result for the special case $u_h \in S_1^{(0)}(I_h)$ $(m = 1)$ and $q = 1/2$ (see Buhmann, Iserles, and Nørsett [16]; compare also Iserles [23, 24], Iserles and Liu [25]), there has been extensive work on asymptotic stability for (1.1) when I_h is a *geometric mesh*. We refer the reader to the papers by Liu [30, 31], Bellen, Guglielmi, and Torelli [6], Guglielmi and Zennaro [20], Huang and Vandewalle [22], and Guglielmi [19] and to the monograph [7] by Bellen and Zennaro.

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