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A NEW MIXED FINITE ELEMENT FORMULATION AND THE MAC METHOD FOR THE STOKES EQUATIONS*

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Abstract. A new mixed finite element method is formulated for the Stokes equations, in which the two components of the velocity and the pressure are defined on different meshes. First-order error estimates are obtained for both the velocity and the pressure. Also, the well-known MAC method is derived from the resulting finite element method.

Key words. mixed finite element method, MAC scheme, Stokes equations

AMS subject classifications. 35A35, 35A40, 65N99

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1. Introduction. When the low-order finite element method is applied to the Navier–Stokes equations, some special treatments are usually needed in order to keep the scheme stable. This topic has attracted the attention of many authors [1, 2, 7, 11, 12]. In the paper [8], we presented an economical finite element scheme for the Navier–Stokes equations by using three different quadrangulations for constructing three finite-dimensional subspaces of the pressure p and the velocity components u_1 and u_2. In this paper we extend the idea in [8] to construct a new mixed finite element scheme for Stokes equations, which is more efficient than the scheme given in the paper [8] in the sense that the degree of freedom is reduced. The optimal error estimate of the new scheme is obtained.

It is well known that the marker and cell method (MAC) is one of the simplest and most effective numerical schemes for solving Stokes equations and Navier–Stokes equations. The MAC scheme was introduced by V. I. Lebedev [9] and B. J. Daly et al. [5] in the middle of sixties and has been widely used in engineering applications. The MAC method is outside of the framework of the finite element method and is characterized by the following fact: the MAC method decomposes the fluid into squares or rectangular cells and discretizes the pressure at the center of the cell. Furthermore, it discretizes the first component of the velocity at the midpoint of the vertical sides of the cells and the second component at the midpoint of the horizontal sides of the cells. It does not discretize the two components of the velocity at the same points. S. Choudhury and R. A. Nicolaides [3], R. A. Nicolaides [10], and R. A. Nicolaides and X. Wu [11] analyze the MAC method and the related method based on the covolume method framework. In the paper [6] by V. Girault and H. Lopez the MAC method is interpreted as a mixed finite element method of the "velocity-vorticity" variational formulation of the Navier–Stokes equations coupled with the quadrature formula. Moreover, the error estimates for the MAC scheme are given. In this paper we find the natural connection between the MAC scheme and the new mixed finite element scheme of the variational formulation with primitive variables of

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the Stokes equations. Furthermore, the optimal error estimates for the MAC scheme are obtained. The natural connection brings many advantages, such as setting the framework of the finite element method for the MAC scheme, so many mathematical tools are available to analyze the MAC scheme. In fact, this connection is a bridge for extending the MAC scheme to high-order approximations and three-dimensional cases.

In section 2, we give a new formulation of the mixed finite element method for the Stokes equations and give the error estimates for the numerical solutions. In section 3, we discuss the relations between the mixed finite element formulation and the well-known MAC scheme.

2. A new mixed finite element formulation for the Stokes equations. Let Ω be a rectangular domain on the plane with a boundary Γ. We consider the steady state Stokes equations governing the flow of an incompressible viscous fluid in Ω,

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \tag{1}$$

$$\text{div } \mathbf{u} = 0, \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u}|_\Gamma = 0, \tag{3}$$

where ν is the viscosity, $\mathbf{u} = (u_1, u_2)$ is the velocity, p is the pressure, and $\mathbf{f}$ is the given force. Let $H^n(\Omega)$ and $H_0^1(\Omega)$ be the standard Sobolev spaces equipped with the norm $\|\cdot\|_{n,\Omega}$ and let

$$\mathbf{V} \equiv H_0^1(\Omega) \times H_0^1(\Omega),$$
$$M \equiv \left\{ q \in L^2(\Omega) \text{ and } \int_\Omega q dx = 0 \right\}.$$

Then the boundary value problem (1)–(3) is reduced to the following equivalent variational problem:

find $\mathbf{u} \in \mathbf{V}$ and $p \in M$, such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}, \tag{4}$$

$$b(\mathbf{u}, q) = 0 \qquad \forall q \in M, \tag{5}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \nu \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v} dx,$$
$$b(\mathbf{v}, q) = -\int_\Omega q \text{div } \mathbf{v} dx,$$
$$(\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx.$$

For the variational problem (4), (5) we have the following theorem.

THEOREM 1. *For given* $\mathbf{f} \in H^{-1}(\Omega)^2$*, the variational problem* (4), (5) *has a unique solution* $(\mathbf{u}, p) \in \mathbf{V} \times M$*. Furthermore, if* $\mathbf{f} \in L^2(\Omega)^2$*, then* $(\mathbf{u}, p) \in (\mathbf{V} \cap H^2(\Omega)^2) \times (M \cap H^1(\Omega))$*, and*

$$\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \leq C_0 \|\mathbf{f}\|_{0,\Omega}. \qquad \square \tag{6}$$

For simplicity we assume that the domain Ω is a unit square, but the finite element method discussed below can be easily generalized to include the case in which the

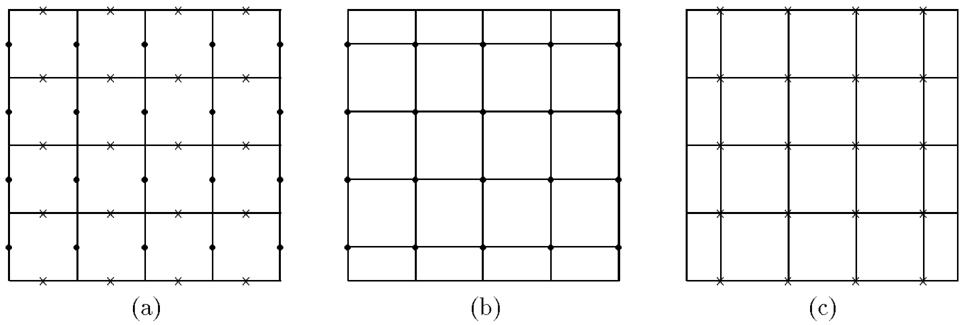

FIG. 1. *Quadrangulations:* (a) $\mathcal{J}_h$, (b) $\mathcal{J}_h^1$, (c) $\mathcal{J}_h^2$.

domain Ω is rectangular. Let N be a given integer and $h = 1/N$. We shall construct the finite-dimensional subspaces of M and $\mathbf{V}$ by introducing three different quadrangulations $\mathcal{J}_h$, $\mathcal{J}_h^1$, $\mathcal{J}_h^2$ of Ω. First we divide Ω into equal squares

$$T_{ij} = \left\{(x^1, x^2) : \; x^1_{i-1} \leq x^1 \leq x^1_i, x^2_{j-1} \leq x^2 \leq x^2_j\right\}, \qquad i, j = 1, \dots, N,$$

where $x^1_i = ih$ and $x^2_j = jh$. The corresponding quadrangulation is denoted by $\mathcal{J}_h$. Then for all $T_{ij} \in \mathcal{J}_h$ we connect all the midpoints of the vertical sides of T_{ij} by straight line segments if the midpoints have a distance h, and extend the resulting mesh to the boundary Γ. Then Ω is divided into squares and rectangles, and the corresponding quadrangulation is denoted by $\mathcal{J}_h^1$. Similarly, for all $T_{ij} \in \mathcal{J}_h$ we connect all the midpoints of the horizontal sides of T_{ij} by straight line segments if the midpoints have a distance h, and extend the resulting mesh to the boundary Γ. Then we obtained the third quadrangulation of Ω, which is denoted by $\mathcal{J}_h^2$ (see Figure 1).

Corresponding to the quadrangulation $\mathcal{J}_h$, let

$$M_h = \left\{q_h : \; q_h|_T = \text{constant } \forall T \in \mathcal{J}_h \text{ and } \int_\Omega q_h dx = 0\right\}.$$

M_h is a subspace of M. Furthermore, using the quadrangulation $\mathcal{J}_h^1$ and $\mathcal{J}_h^2$, we construct two subspaces of $H^1_0(\Omega)$. Set

$$S_h^1 = \{v_h \in C^{(0)}(\bar{\Omega}) : \; v_h|_{T^1} \in Q_1(T^1) \; \forall T^1 \in \mathcal{J}_h^1, \text{ and } v_h|_\Gamma = 0\},$$
$$S_h^2 = \{v_h \in C^{(0)}(\bar{\Omega}) : \; v_h|_{T^2} \in Q_1(T^2) \; \forall T^2 \in \mathcal{J}_h^2, \text{ and } v_h|_\Gamma = 0\},$$

where Q_1 denotes the space of all polynomials of degree ≤ 1 with respect to each of the two variables x^1 and x^2. Let

$$\mathbf{V}_h = S_h^1 \times S_h^2;$$

obviously, $\mathbf{V}_h$ is a subspace of $\mathbf{V}$. Using the subspaces $\mathbf{V}_h$ and M_h instead of $\mathbf{V}$ and M in the variational problem (4) and (5) we obtain the discrete problem:

find $(\mathbf{u_h}, p_h) \in \mathbf{V}_h \times M_h$, such that

$$a(\mathbf{u_h}, \mathbf{v_h}) + b(\mathbf{v_h}, p_h) = (\mathbf{f}, \mathbf{v_h}) \qquad \forall \mathbf{v_h} \in \mathbf{V}_h, \tag{7}$$

$$b(\mathbf{u_h}, q_h) = 0 \qquad \forall q_h \in M_h. \tag{8}$$

For the triangulation $\mathcal{J}_h$, we divided the edges of all squares into two sets. The set containing all vertical edges is denoted by L_V, and the set containing all horizontal edges is denoted by L_H. We define the operator I_h: $\mathbf{V} \to \mathbf{V}_h$ by

$$I_h\mathbf{u} = (I_h^1 u_1, I_h^2 u_2) \in S_h^1 \times S_h^2 \text{ satisfying}$$

$$\int_l I_h^1 u_1 ds = \int_l u_1 ds \qquad \forall l \in L_V, \tag{9}$$

$$\int_l I_h^2 u_2 ds = \int_l u_2 ds \qquad \forall l \in L_H. \tag{10}$$

It is straightforward to check that for any $\mathbf{u} \in \mathbf{V}$, $I_h\mathbf{u} \in \mathbf{V}_h$ is uniquely determined by (9) and (10).

LEMMA 1. *The following results hold.*

(i) *For any* $\mathbf{u} \in \mathbf{V}$

$$\int_\Omega q_h \operatorname{div}(\mathbf{u} - I_h\mathbf{u}) dx = 0 \qquad \forall q_h \in M_h. \tag{11}$$

(ii) *There exist two constants* C_1 *and* C_2 *independent of* h*, such that*

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} \le \|\mathbf{u} - I_h\mathbf{u}\|_{1,\Omega} \le C_1 h |\mathbf{u}|_{2,\Omega}, \tag{12}$$

$$\inf_{q_h \in M_h} \|p - q_h\|_{0,\Omega} \le C_2 h |p|_{1,\Omega}. \tag{13}$$

(iii) *There is a constant* C_3 *independent of* h*, such that*

$$\|I_h\mathbf{u}\|_{1,\Omega} \le C_3 \|\mathbf{u}\|_{1,\Omega} \qquad \forall \mathbf{u} \in \mathbf{V}. \tag{14}$$

Proof. The equalities (11), (12), and (13) follow from (9), (10), and the approximation theory. Let $\mathbf{w}_h$ be the orthogonal projection of $\mathbf{u}$ on $\mathbf{V}_h$ for the scalar product of $\mathbf{V}$:

$$a(\mathbf{w}_h - \mathbf{u}, \mathbf{z}_h) = 0 \qquad \forall \mathbf{z}_h \in \mathbf{V}_h. \tag{15}$$

We set $\mathbf{e}_h = I_h\mathbf{u} - \mathbf{w}_h$, $\mathbf{e} = \mathbf{u} - \mathbf{w}_h$; then

$$|I_h\mathbf{u}|_{1,\Omega} \le |\mathbf{w}_h|_{1,\Omega} + |\mathbf{e}_h|_{1,\Omega} \le |\mathbf{u}|_{1,\Omega} + |\mathbf{e}_h|_{1,\Omega}.$$

Let $\Omega_i \subset \Omega$ denote the subdomains $\{(x^1, x^2), x_i^1 \le x^1 \le x_{i+1}^1, 0 \le x^2 \le 1, i = 0, 1, \dots, N-1\}$ (see Figure 2), and let $T_{ij}^1 \subset \Omega_i, j = 0, 1, \dots, N$ denote the squares (rectangles) of $\mathcal{J}_h^1$. Then for the first component of $\mathbf{e}_h$, we have

$$\left|e_h^1\right|_{1,\Omega_i}^2 = \sum_{j=0}^{N} \int_{T_{ij}^1} \left(\left(\frac{\partial e_h^1}{\partial x^1} \right)^2 + \left(\frac{\partial e_h^1}{\partial x^2} \right)^2 \right) dx^1 dx^2. \tag{16}$$

Let $\tilde{\mathbf{E}}_i$ denote the vector whose components are the node values of e_h^1 in the domain Ω_i. Writing (16) in matrix form we obtain

$$\left|e_h^1\right|_{1,\Omega_i}^2 = (\tilde{\mathbf{E}}_i)^T A_i (\tilde{\mathbf{E}}_i), \tag{17}$$

where A_i is a symmetric matrix. Decomposing A_i as $A_i = P_i^T P_i$, we can write (17) as

$$\left|e_h^1\right|_{1,\Omega_i}^2 = \left\|P_i \tilde{\mathbf{E}}_i\right\|_2^2. \tag{18}$$

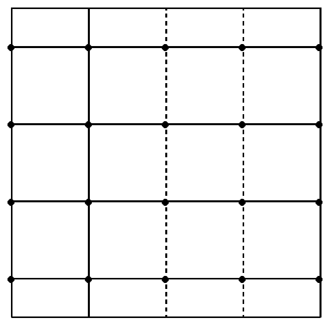

FIG. 2. *Subdomain Ω_i, bounded by the dash lines.*

By using the fact that

$$\int_l e_h^1 ds = \int_l e^1 ds \qquad \forall l \in L_V, \tag{19}$$

we get

$$B_i \tilde{\mathbf{E}}_i = \frac{1}{h}\mathbf{E}_i, \tag{20}$$

where B_i is a 2×2 block diagonal matrix and the two identical block diagonals are a tridiagonal matrix with $\{5, 6, 6, \ldots, 6, 5\}$ as its main diagonal and $\{1, 1, \ldots, 1\}$ as the two subdiagonals, and $\mathbf{E}_i$ is a vector whose components are the integrals of the right-hand side of (19). Using (20) we get

$$\begin{aligned} \left|e_h^1\right|_{1,\Omega_i}^2 &= \frac{1}{h^2}\left\|P_i B_i^{-1}\mathbf{E}_i\right\|_2^2 \\ &\le \frac{1}{h^2}\max(\lambda_{P_i^T P})\max(\lambda_{(B_i^{-1})^T B_i^{-1}})\|\mathbf{E}_i\|_2^2 \\ &\le \frac{1}{h^2}\max(\lambda_{A_i})(\min(\lambda_{B_i}))^{-2}\|\mathbf{E}_i\|_2^2, \end{aligned}$$

where $\max(\lambda_{P_i^T P})$, $\max(\lambda_{(B_i^{-1})^T B_i^{-1}})$, and $\max(\lambda_{A_i})$ are the largest eigenvalues of the corresponding matrices, and $\min(\lambda_{B_i})$ is the smallest eigenvalue of B_i. It is easy to see that

$$|\lambda_{A_i}| \le \frac{8}{3}, \qquad |\lambda_{B_i}| \ge 4.$$

Therefore, we have

$$\left|e_h^1\right|_{1,\Omega_i}^2 \le \frac{1}{6h^2}\|\mathbf{E}_i\|_2^2.$$

On the other hand, using the Hölder inequality we get

$$\|\mathbf{E}_i\|_2^2 \le h \int_{\partial\Omega_i} (e^1)^2 ds,$$

where e^1 is the first component of $\mathbf{e}$. By mapping Ω_i into the unit square and using the trace theorem we obtain

$$\|\mathbf{E}_i\|_2^2 \le C\left(\left\|e^1\right\|_{0,\Omega_i}^2 + h^2\left|e^1\right|_{1,\Omega_i}^2\right).$$

Thus, we have

$$\left|e_h^1\right|_{1,\Omega} \le C\left(h^{-2}\left\|e^1\right\|_{0,\Omega}^2 + \left|e^1\right|_{1,\Omega}^2\right)^{1/2}.$$

Similarly, for the second component we have

$$\left|e_h^2\right|_{1,\Omega} \le C\left(h^{-2}\left\|e^2\right\|_{0,\Omega}^2 + \left|e^2\right|_{1,\Omega}^2\right)^{1/2}.$$

Therefore,

$$|\mathbf{e}_h|_{1,\Omega} \le C(h^{-2}\|\mathbf{e}\|_{0,\Omega}^2 + |\mathbf{e}|_{1,\Omega}^2)^{1/2}.$$

By using the classical duality argument,

$$\left\|e^i\right\|_{0,\Omega} = \sup_{g\in L^2(\Omega)} \frac{(e^i, g)}{\|g\|_{0,\Omega}}, \qquad i = 1, 2,$$

we find

$$\|\mathbf{e}\|_{0,\Omega} \le Ch|\mathbf{e}|_{1,\Omega}.$$

Finally, from (15) we have

$$|\mathbf{e}|_{1,\Omega} \le |\mathbf{u}|_{1,\Omega};$$

this finishes the proof of the lemma. □

For the bilinear forms $a(\mathbf{u}, \mathbf{v})$ and $b(\mathbf{v}, q)$ we have the standard results.

LEMMA 2. *$a(\mathbf{u}, \mathbf{v})$ is a bounded and coercive bilinear form on $\mathbf{V} \times \mathbf{V}$ and $b(\mathbf{v}, q)$ is a bounded bilinear form on $\mathbf{V} \times M$. Namely,*

(i) *there are two constants $A > 0$ and $\alpha > 0$ such that*

$$|a(\mathbf{u}, \mathbf{v})| \le A\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}, \tag{21}$$

$$a(\mathbf{v}, \mathbf{v}) \ge \alpha\|\mathbf{v}\|_{1,\Omega}^2 \quad \forall \mathbf{v} \in \mathbf{V}; \tag{22}$$

(ii) *there is a constant $B > 0$ such that*

$$|b(\mathbf{v}, q)| \le B\|q\|_{0,\Omega}\|\mathbf{v}\|_{1,\Omega} \quad \forall q \in M, \mathbf{v} \in \mathbf{V}. \tag{23}$$ □

For the subspaces $\mathbf{V}_h$ and M_h, the Ladyzhenskaya–Babuška–Brezzi (L-B-B) condition holds, and we have the following lemma.

LEMMA 3. *There is a constant $\beta_0 > 0$ independent of h, such that*

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \ge \beta_0\|q_h\|_{0,\Omega} \quad \forall q_h \in M_h. \tag{24}$$

Proof. For the spaces $\mathbf{V}$ and M, the L-B-B condition holds. Namely, there is a constant $\beta > 0$, such that

$$\sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,\Omega}} \ge \beta\|q\|_{0,\Omega} \quad \forall q \in M. \tag{25}$$

For any $\mathbf{v} \in \mathbf{V}$, from (11) we have

$$b(\mathbf{v}, q_h) = b(I_h\mathbf{v}, q_h) \quad \forall q_h \in M_h.$$

Using inequality (14) we obtain

$$\frac{b(\mathbf{v}, q_h)}{\|\mathbf{v}\|_{1,\Omega}} \le C_3 \frac{b(I_h\mathbf{v}, q_h)}{\|I_h\mathbf{v}\|_{1,\Omega}} \le C_3 \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \qquad \forall \mathbf{v} \in \mathbf{V}, q_h \in M_h.$$

Applying (25) we have

$$\beta \|q_h\|_{0,\Omega} \le C_3 \sup_{\mathbf{v}_h \in \mathbf{V}} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \qquad \forall q_h \in M_h.$$

Taking $\beta_0 = \beta/C_3$ we get (24). ∎

The existence and uniqueness of the discrete solution can be obtained by applying Theorem 1.1 in Chapter II of [7].

THEOREM 2. *The discrete problem* (7) *and* (8) *has a unique solution* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$.

Furthermore, we have the following error estimate.

THEOREM 3. *Suppose that* $\mathbf{f} \in L^2(\Omega)^2$*; the following error estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \le Ch\|\mathbf{f}\|_{0,\Omega}, \tag{26}$$

where C *is a constant independent of* h*;* $(\mathbf{u}, p)$ *is the solution of the variational problem* (4)–(5)*; and* $(\mathbf{u}_h, p_h)$ *is the solution of discrete problem* (7)–(8).

Proof. Let $\mathbf{z}_h = \mathbf{u}_h - I_h\mathbf{u}$; then, by the inequality (22), we have

$$\|\mathbf{z}_h\|_{1,\Omega}^2 \le \frac{1}{\alpha} a(\mathbf{z}_h, \mathbf{z}_h).$$

On the other hand, we have

$$\begin{aligned} a(\mathbf{z}_h, \mathbf{z}_h) &= a(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - I_h\mathbf{u}) + a(\mathbf{u} - I_h\mathbf{u}, \mathbf{u}_h - I_h\mathbf{u}) \\ &= b(\mathbf{u}_h - I_h\mathbf{u}, p - p_h) + a(\mathbf{u} - I_h\mathbf{u}, \mathbf{u}_h - I_h\mathbf{u}) \\ &= b(\mathbf{u}_h - I_h\mathbf{u}, p - q_h) + a(\mathbf{u} - I_h\mathbf{u}, \mathbf{u}_h - I_h\mathbf{u}) \ \forall q_h \in M_h. \end{aligned}$$

Combining the inequalities (21) and (23) we obtain

$$\|\mathbf{z}_h\|_{1,\Omega} \le \frac{1}{\alpha} \left\{ B \inf_{q_h \in M_h} \|p - q_h\|_{0,\Omega} + A\|\mathbf{u} - I_h\mathbf{u}\|_{1,\Omega} \right\}.$$

From the triangle inequality and the inequalities (6), (12), and (13) we get

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} &\le \|\mathbf{u} - I_h\mathbf{u}\|_{1,\Omega} + \|I_h\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \\ &\le \frac{B}{\alpha} \inf_{q_h \in M} \|p - q_h\|_{0,\Omega} + \left(1 + \frac{A}{\alpha}\right) \|\mathbf{u} - I_h\mathbf{u}\|_{1,\Omega} \\ &\le h \left(C_2 \frac{B}{\alpha} |p|_{1,\Omega} + C_1 \left(1 + \frac{A}{\alpha}\right) |\mathbf{u}|_{2,\Omega} \right) \\ &\le Ch\|\mathbf{f}\|_{0,\Omega}. \end{aligned}$$

Finally, we estimate $\|p - q_h\|_{0,\Omega}$; from the L-B-B condition we have

$$\begin{aligned} \|p_h - q_h\|_{0,\Omega} &\le \frac{1}{\beta_0} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, p_h - q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \\ &\le \frac{1}{\beta_0} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \\ &\le \frac{1}{\beta_0} \left\{ A\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + B\|p - q_h\|_{0,\Omega} \right\}. \end{aligned}$$

Therefore,

$$\|p - p_h\|_{0,\Omega} \le \frac{A}{\beta_0}\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \left(1 + \frac{B}{\beta_0}\right)\|p - q_h\|_{0,\Omega} \quad \forall q_h \in M_h.$$

Since the left-hand side is independent of q_h, we have

$$\begin{aligned}\|p - p_h\|_{0,\Omega} &\le C\left(\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \inf_{q_h \in M_h}\|p - q_h\|_{0,\Omega}\right)\\ &\le Ch\|\mathbf{f}\|_{0,\Omega}.\end{aligned}$$

The error estimate is proved completely. ◻

3. MAC scheme from the variational formulation (7), (8). To obtain the MAC scheme from the variational problem (7), (8), we introduce suitable quadrature formulas for computing the bilinear forms $a(\mathbf{u}_h, \mathbf{v}_h)$ and $b(\mathbf{v}_h, q_h)$. We first define the following interpolating operators:

$$\begin{aligned}&\Pi_h^1 : C^{(0)}(\bar{T}_1) \to Q_1(T_1), \text{ such that}\\ &\quad (\Pi_h^1\phi)(a_i) := \phi(a_i),\ i = 1, 2, 3, 4,\end{aligned}$$

where a_i, $i = 1, 2, 3, 4$, are the four nodes of $T_1 \in \mathcal{J}_h^1$,

$$\begin{aligned}&\Pi_h^2 : C^{(0)}(\bar{T}_2) \to Q_1(T_2), \text{ such that}\\ &\quad (\Pi_h^2\phi)(b_i) := \phi(b_i),\ i = 1, 2, 3, 4,\end{aligned}$$

where b_i, $i = 1, 2, 3, 4$, are the four nodes of $T_2 \in \mathcal{J}_h^2$, and

$$\begin{aligned}&\Pi_h : C^{(0)}(\bar{T}) \to Q_0(T), \text{ such that}\\ &\quad (\Pi_h\phi)(c) = \phi(c),\end{aligned}$$

where c is the center of $T \in \mathcal{J}_h$. Then we introduce the bilinear forms

$$a_h(\mathbf{u}, \mathbf{v}) = \nu\left\{\sum_{T_1 \in \mathcal{J}_h^1}\int_{T_1}\Pi_h^1(\nabla u_1 \cdot \nabla v_1)dx + \sum_{T_2 \in \mathcal{J}_h^2}\int_{T_2}\Pi_h^2(\nabla u_2 \cdot \nabla v_2)dx\right\} \forall \mathbf{u}, \mathbf{v} \in V_h$$

and

$$b_h(\mathbf{v}_h, q_h) = -\sum_{T \in \mathcal{J}_h}\int_T q_h\Pi_h(\mathrm{div}\mathbf{v}_h)dx. \tag{27}$$

We now consider the following simplified variational problem:

$$\text{find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h, \text{ such that}$$

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) = F(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{28}$$

$$b_h(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in M_h, \tag{29}$$

where $F(\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)$. We first discuss the bilinear form $a_h(\mathbf{u}_h, \mathbf{v}_h)$. A direct computation gives us the following lemma.

LEMMA 4. *There exist two constants $M_0 > 0$ and $\alpha_0 > 0$ independent of h such that*

$$|a_h(\mathbf{u}_h, \mathbf{v}_h)| \le M_0\|\mathbf{u}_h\|_{1,\Omega}\|\mathbf{v}_h\|_{1,\Omega}\ \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h. \tag{30}$$

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \ge \alpha_0\|\mathbf{v}_h\|_{1,\Omega}^2 \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{31}$$

◻

Furthermore, for the bilinear form $b_h(q_h, \mathbf{v}_h)$, we introduce the operator $K_h : \mathbf{V} \to \mathbf{V}_h$, such that

$$K_h\mathbf{v} = (K_h^1 v_1, K_h^2 v_2) \in \mathbf{V}_h \ \forall \mathbf{v} \in \mathbf{V}$$

and

$$(K_h^1 v_1)(x_i^1, x_{j+1/2}^2) = \frac{1}{h}\int_{l_{i,j+1/2}} v_1 ds \ \forall l_{i,j+1/2} \in L_V, i = 0, 1, \ldots, N, j = 0, 1, \ldots, N-1,$$
$$(K_h^2 v_2)(x_{i+1/2}^1, x_j^2) = \frac{1}{h}\int_{l_{i+1/2,j}} v_2 ds \ \forall l_{i+1/2,j} \in L_H, i = 0, 1, \ldots, N-1, j = 0, 1, \ldots, N,$$

where $l_{i,j+1/2}$ denote the vertical side of $T \subset \mathcal{J}_h$ with midpoint at $(x_i^1, x_{j+1/2}^2)$, and $l_{i+1/2,j}$ denote the horizontal side of $T \subset \mathcal{J}_h$ with midpoint at $(x_{i+1/2}^1, x_j^2)$. Then we have the following lemma.

LEMMA 5. *For the operator K_h the following results hold.*

(i) *For any $\mathbf{v} \in \mathbf{V}$*

$$b(\mathbf{v}, q_h) = b_h(K_h\mathbf{v}, q_h) \quad \forall q_h \in M_h. \tag{32}$$

(ii) *Suppose that $\mathbf{v} \in \mathbf{V} \cap H^2(\Omega)^2$ and $p \in M \cap H^1(\Omega)$; then*

$$\|\mathbf{v} - K_h\mathbf{v}\|_{1,\Omega} \le Ch|\mathbf{v}|_{2,\Omega}, \tag{33}$$
$$\|p - \Pi_h p\|_{0,\Omega} \le Ch|p|_{1,\Omega}, \tag{34}$$

where C is a constant independent of h.

(iii) *There is a constant C independent of h such that*

$$\|K_h\mathbf{v}\|_{1,\Omega} \le C\|\mathbf{v}\|_{1,\Omega} \quad \forall \mathbf{v} \in \mathbf{V}. \tag{35}$$

Proof. (32), (33), and (34) follow immediately from the definitions of K_h and $b(\cdot,\cdot)$ and the approximation theory. The proof of (35) is similar to the proof of (14). □

Combining (25), (32), and (35) we obtain the following lemma.

LEMMA 6. *There is a constant $\beta_1 > 0$, independent of h such that*

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b_h(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \ge \beta_1 \|q_h\|_{0,\Omega} \ \forall q_h \in M_h. \quad \square \tag{36}$$

From Lemmas 4 and 5 we obtain the following theorem.

THEOREM 4. *For a given function $\mathbf{f} \in L^2(\bar{\Omega})^2$, the problem* (28), (29) *has a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$.* □

Choosing suitable test functions in (28), (29), we obtain the system of finite difference equations, which is equivalent to the MAC scheme. For example, at an interior point a_5, which is the midpoint of a vertical side of $T \in \mathcal{J}_h$ with the numbering of Figure 3, we take $\mathbf{v}_h = (v_h^1, 0) \in \mathbf{V}_h$ in (28) with (see Figure 3(a))

$$v_h^1(a_i) = \begin{cases} 1, & i = 5, \\ 0, & \text{otherwise.} \end{cases}$$

A computation shows that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \nu(4u_{1,5} - u_{1,1} - u_{1,2} - u_{1,3} - u_{1,4}),$$
$$b_h(\mathbf{v}_h, p_h) = h(p_7 - p_6).$$

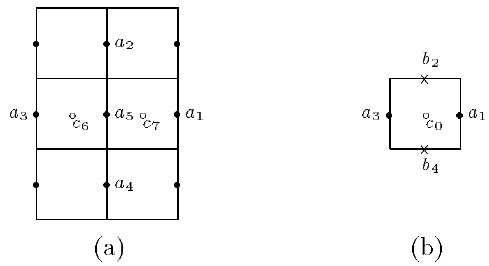

FIG. 3.

Hence we get the finite difference equation at point a_5,

$$-\nu\frac{u_{1,1}+u_{1,2}+u_{1,3}+u_{1,4}-4u_{1,5}}{h^2}+\frac{1}{h}(p_7-p_6)=f_{1,5},$$

where

$$f_{1,5}=\frac{1}{h^2}\int_\Omega f_1 v_1 dx.$$

Similarly, taking q_h with (see Figure 3(b))

$$q_h(c_i)=\begin{cases}1, & i=0,\\ 0, & \text{otherwise,}\end{cases}$$

we have

$$u_{1,1}-u_{1,3}+u_{2,2}-u_{2,4}=0.$$

We now analyze the error between the solution $(\mathbf{u}_h, p_h)$ of problem (28), (29) and the solution $(\mathbf{u}, p)$ of the original problem (4), (5). We take $\mathbf{v}=\mathbf{v}_h\in\mathbf{V}_h$ in (4) and $q=q_h\in M_h$ in (5). Then the equalities (4), (5) can be rewritten as follows:

$$a_h(K_h\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,\Pi_h p)=F(\mathbf{v}_h)+R_1(\mathbf{u},\mathbf{v}_h)+R_2(p,\mathbf{v}_h)\ \forall\mathbf{v}_h\in\mathbf{V}_h, \tag{37}$$

$$b_h(K_h\mathbf{u},q_h)=0, \tag{38}$$

where

$$R_1(\mathbf{u},\mathbf{v}_h)=a_h(K_h\mathbf{u},\mathbf{v}_h)-a(\mathbf{u},\mathbf{v}_h),$$
$$R_2(p,\mathbf{v}_h)=b_h(\mathbf{v}_h,\Pi_h p)-b(\mathbf{v}_h,p).$$

Let $\mathbf{e}_h=K_h\mathbf{u}-\mathbf{u}_h$, $\lambda_h=\Pi_h p-p_h$. Then $(\mathbf{e}_h,\lambda_h)$ satisfies

$$a_h(\mathbf{e}_h,\mathbf{v}_h)+b_h(\mathbf{v}_h,\lambda_h)=R_1(\mathbf{u},\mathbf{v}_h)+R_2(p,\mathbf{v}_h)\ \forall\mathbf{v}_h\in\mathbf{V}_h, \tag{39}$$

$$b_h(\mathbf{e}_h,q_h)=0\ \forall q_h\in M_h. \tag{40}$$

For the bilinear forms $R_1(\mathbf{u},\mathbf{v}_h)$ and $R_2(p,\mathbf{v}_h)$, similar to the proof given in [4] we have the following lemma.

LEMMA 7. *There exists a constant C independent of h such that*

$$|R_1(\mathbf{u},\mathbf{v}_h)|\le Ch|\mathbf{u}|_{2,\Omega}\|\mathbf{v}_h\|_{1,\Omega}\ \forall\mathbf{v}_h\in\mathbf{V}_h, \tag{41}$$

and

$$|R_2(p, \mathbf{v}_h)| \le Ch|p|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega} \ \forall \mathbf{v}_h \in \mathbf{V}_h. \qquad \square \tag{42}$$

In (39), taking $\mathbf{v}_h = \mathbf{e}_h$, we obtain

$$\begin{aligned} \|\mathbf{e}_h\|_{1,\Omega}^2 &\le \frac{1}{\alpha_1} a_h(\mathbf{e}_h, \mathbf{e}_h) = \frac{1}{\alpha_1} \{R_1(\mathbf{u}, \mathbf{e}_h) + R_2(p, \mathbf{e}_h)\} \\ &\le \frac{C}{\alpha_1} h(|\mathbf{u}|_{2,\Omega} + |p|_{1,\Omega}) \|\mathbf{e}_h\|_{1,\Omega} \end{aligned}$$

or

$$\|\mathbf{e}_h\|_{1,\Omega} \le \frac{C}{\alpha_1} h(|\mathbf{u}|_{2,\Omega} + |p|_{1,\Omega}). \tag{43}$$

On the other hand,

$$\begin{aligned} |b_h(\mathbf{v}_h, \lambda_h)| &= |R_1(\mathbf{u}, \mathbf{v}_h) + R_2(p, \mathbf{v}_h) - a_h(\mathbf{e}_h, \mathbf{v}_h)| \\ &\le Ch(|\mathbf{u}|_{2,\Omega} + |p|_{1,\Omega}) \|\mathbf{v}_h\|_{1,\Omega} + M_1 \|\mathbf{e}_h\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega}. \end{aligned}$$

Hence,

$$\begin{aligned} \|\lambda_h\|_{0,\Omega} &\le \frac{1}{\beta_1} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b_h(\mathbf{v}_h, \lambda_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \\ &\le Ch(|\mathbf{u}|_{2,\Omega} + |p|_{1,\Omega}) + M_1 \|\mathbf{e}_h\|_{1,\Omega} \\ &\le Ch(|\mathbf{u}|_{2,\Omega} + |p|_{1,\Omega}). \end{aligned} \tag{44}$$

Finally, we obtain the following error estimate.

THEOREM 5. *Assume* $\mathbf{f} \in L^2(\Omega)^2$*; then the following error estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \le Ch\|\mathbf{f}\|_{0,\Omega}. \tag{45}$$

Proof. The error estimate (45) follows immediately from (43), (44), (33), (34), (6), and the triangle inequality. $\square$

4. Conclusions. The formulation proposed in this paper gives a new way to produce the low-order finite element method for the Stokes equations. The well-known MAC method is obtained from the resulting finite element equations combined with quadrature formulas, as is the error estimate for the MAC method. It is possible to extend this method to three-dimensional Stokes equations and the nonlinear Navier–Stokes equations. However, it is not clear whether or not this method can be extended to the problems with unstructured geometry.

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