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CONCERNING GAUSSIAN–CHEBYSHEV QUADRATURE ERRORS*

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Abstract. We introduce a method for estimating errors in the Gaussian–Chebyshev quadrature formulas for functions of low order continuity. In particular, we improve an error bound obtained by Johnson and Riess.

1. Notations and results. Let T_n be the Chebyshev polynomials of the first kind. Then the zeros of T_n are $x_{n,k} = \cos[(2k-1)\pi/2n]$, $k = 1, \cdots, n$. The Gaussian–Chebyshev quadrature for a function f on the interval $[-1, 1]$ is given by

$$\int_{-1}^{1} (1 - x^2)^{-1/2} f(x)\, dx = \frac{\pi}{n} \sum_{k=1}^{n} f(x_{n,k}) + R_n(f).$$

In this note, we give a method to obtain the following estimates of $|R_n(f)|$, considering functions f of low order continuity [2]. If h is a function of bounded variation on $[-1, 1]$, we let $T(h)$ denote the total variation of h on $[-1, 1]$.

If f is a function of bounded variation on $[-1, 1]$, *then*

$$|R_n(f)| \leqq \pi T(f)/2n. \tag{1}$$

If f is absolutely continuous on $[-1, 1]$, *then*

$$nR_n(f) \to 0. \tag{2}$$

If f is differentiable such that $F(x) = (1 - x^2)^{1/2} f'(x)$ is of bounded variation on $[-1, 1]$, *then*

$$|R_n(f)| \leqq \pi^2 T(F)/8n^2. \tag{3}$$

If f is twice differentiable and f'' is bounded and almost everywhere continuous on $[-1, 1]$, *then*

$$n^2 R_n(f) \to 0. \tag{4}$$

Now suppose that $F(x) = (1 - x^2)^{1/2} f'(x)$ is of bounded variation on $[-1, 1]$. Let $|F(x)| \leqq P$, $-1 \leqq x \leqq 1$, and let C be the number of intervals of $[-1, 1]$ in each of which F is monotone. Then Riess and Johnson [3] proved that

$$|R_n(f)| \leqq CP\pi^2/6n^2.$$

We note that this error bound becomes infinite when $C = \infty$, while the error bound in (3) above is always finite. Furthermore, if $m \leqq F(x) \leqq M$, it is easy to see that $T(F) \leqq C(M - m)$, so that as a corollary of (3), we have

$$|R_n(f)| \leqq \pi^2 C(M - m)/8n^2. \tag{3'}$$

In particular, if $F \geqq 0$ or if $F \leqq 0$, then

$$|R_n(f)| \leqq CP\pi^2/8n^2. \tag{3''}$$

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2. Proofs of the above results. For $0 \leqq t \leqq 1$, we let χ_t denote the characteristic function of the closed interval $[t, 1]$, and for each positive integer n, we write

$$s_n = \sum_{k=1}^{n} \chi_{(2k-1)/2n}$$

and $v_n(t) = s_n(t) - nt$. Then $v_n(0) = v_n(1) = 0$, v_n lies between $-1/2$ and $1/2$ and is linear with an exception of n unit jumps at the points $(2k-1)/2n$, $k = 1, \cdots, n$. The v_n is a "saw-tooth" function (cf. [1]). It is clear that if g is Riemann integrable on $[0, 1]$, then

$$\int_0^1 g(x)\,dx = \frac{1}{n}\sum_{k=1}^{n} g\left(\frac{2k-1}{2n}\right) - \frac{1}{n}\int_0^1 g(x)\,dv_n(x).$$

Now, let $g(x) = f(\cos \pi x)$. Then

$$\int_{-1}^{1} (1 - x^2)^{-1/2} f(x)\,dx = \pi \int_0^1 g(x)\,dx$$

and

$$\frac{\pi}{n}\sum_{k=1}^{n} f(x_{n,k}) = \frac{\pi}{n}\sum_{k=1}^{n} g\left(\frac{2k-1}{2n}\right),$$

so that we have

$$R_n(f) = -\frac{\pi}{n}\int_0^1 g(x)\,dv_n(x).$$

(i) Suppose that f is a function of bounded variation on $[-1, 1]$. Then

$$R_n(f) = \frac{\pi}{n}\int_0^1 v_n(x)\,dg(x) = -\frac{\pi}{n}\int_{-1}^{1} v_n\left(\frac{\arccos x}{\pi}\right) df(x).$$

Since $|v_n| \leqq 1/2$, we have

$$|R_n(f)| \leqq \pi T(f)/2n.$$

(ii) Suppose that f is absolutely continuous on $[-1, 1]$. Then f' is Lebesgue integrable on $[-1, 1]$, so that $g'(t) = -\pi \sin \pi t\, f(\cos \pi t)$ is Lebesgue integrable on $[0, 1]$. Hence, by a proof similar to that of the Riemann–Lebesgue theorem (cf. [1]), we see that

$$nR_n(f) = -\pi \int_0^1 g(x)\,dv_n(x) = \pi \int_0^1 v_n(x) g'(x)\,dx \to 0.$$

(iii) Suppose that f is differentiable such that $F(x) = (1 - x^2)^{1/2} f'(x)$ is of bounded variation on $[-1, 1]$. Let

$$u_n(x) = \int_0^x v_n(t)\,dt.$$

Then $u_n(k/n) = u_n(0) = 0$ for $k = 1, \cdots, n$, and each u_n is periodic with period $1/n$ and

$$\sup_{0 \leqq x \leqq 1} |u_n(x)| = \int_0^{1/2n} - v_n(x)\,dx = \frac{1}{8n}\,.$$

Hence, we have

$$nR_n(f)/\pi = \int_0^1 v_n(x)g'(x)\,dx$$
$$= -\int_0^1 u_n(x)\,dg'(x) = \pi \int_{-1}^1 u_n\left(\frac{\arc\cos x}{\pi}\right) dF(x),$$

so that

$$|R_n(f)| \leqq \pi^2 T(F)/8n^2.$$

(iv) Suppose that f is twice differentiable and that f'' is bounded and almost everywhere continuous on $[-1, 1]$. Then by Lebesgue's theorem,

$$g''(x) = \pi^2 \sin^2 \pi x\, f''(\cos \pi x) - \pi^2 \cos \pi x\, f'(\cos \pi x)$$

is Riemann integrable on $[0, 1]$. Now,

$$n^2 R_n(f) = -\pi n \int_0^1 u_n(x)g''(x)\,dx$$
$$= -\pi n \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} u_n(x)g''(x)\,dx$$
$$= -\pi n \sum_{k=1}^{n} \int_0^{1/n} u_n\left(t + \frac{k-1}{n}\right) g''\left(t + \frac{k-1}{n}\right) dt.$$

But $u_n(t + (k-1)/n) = u_n(t)$ for each $k = 1, \cdots, n$. Hence,

$$n^2 R_n(f) = -\pi n \int_0^{1/n} u_n(t) \sum_{k=1}^{n} g''\left(t + \frac{k-1}{n}\right) dt$$
$$= -\pi n \int_0^1 u_n(x/n) S_n(x)\,dx,$$

where

$$S_n(x) = \frac{1}{n} \sum_{k=1}^{n} g''\left(\frac{x + k - 1}{n}\right).$$

Let

$$w(x) = \begin{cases} x^2/2 & \text{if } 0 \leqq x < 1/2, \\ (1-x)^2/2 & \text{if } 1/2 \leqq x \leqq 1. \end{cases}$$

Then we have

$$u_n(x/n) = \int_0^{x/n} v_n(t)\,dt$$
$$= \begin{cases} \displaystyle\int_0^{x/n} -nt\,dt & \text{if } 0 \leqq x < 1/2, \\ \displaystyle -\frac{1}{8n} + \int_{1/2n}^{x/n} (1 - nt)\,dt & \text{if } 1/2 \leqq x \leqq 1 \end{cases}$$
$$= -w(t)/n.$$

Hence,

$$n^2 R_n(f) = \pi \int_0^1 w(x) S_n(x)\, dx.$$

But g'' is Riemann integrable on $[0, 1]$, so that we have

$$S_n(x) \to \int_0^1 g''(t)\, dt = g'(1) - g'(0) = 0$$

for each $x \in [0, 1]$. By the bounded convergence theorem, we conclude that

$$n^2 R_n(f) \to 0.$$

REFERENCES

[1] C. K. Chui, *A convergence theorem for certain Riemann sums*, Canad. Math. Bull., 12 (1969), pp. 523–525.

[2] P. Rabinowitz, *Error bounds in Gaussian integration of functions of low-order continuity*, Math. Comp., 22 (1968), pp. 431–434.

[3] R. D. Riess and L. W. Johnson, *Estimating Gauss–Chebyshev quadrature errors*, this Journal, 6 (1969), pp. 557–559.