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## CONCERNING GAUSSIAN-Chebyshev QUADRATURE ERRORS\*

CHARLES K. CHUI†

**Abstract.** We introduce a method for estimating errors in the Gaussian-Chebyshev quadrature formulas for functions of low order continuity. In particular, we improve an error bound obtained by Johnson and Riess.

**1. Notations and results.** Let  $T_n$  be the Chebyshev polynomials of the first kind. Then the zeros of  $T_n$  are  $x_{n,k} = \cos [(2k - 1)\pi/2n]$ ,  $k = 1, \dots, n$ . The Gaussian-Chebyshev quadrature for a function  $f$  on the interval  $[-1, 1]$  is given by

$$\int_{-1}^1 (1 - x^2)^{-1/2} f(x) dx = \frac{\pi}{n} \sum_{k=1}^n f(x_{n,k}) + R_n(f).$$

In this note, we give a method to obtain the following estimates of  $|R_n(f)|$ , considering functions  $f$  of low order continuity [2]. If  $h$  is a function of bounded variation on  $[-1, 1]$ , we let  $T(h)$  denote the total variation of  $h$  on  $[-1, 1]$ .

*If  $f$  is a function of bounded variation on  $[-1, 1]$ , then*

$$(1) \quad |R_n(f)| \leq \pi T(f)/2n.$$

*If  $f$  is absolutely continuous on  $[-1, 1]$ , then*

$$(2) \quad nR_n(f) \rightarrow 0.$$

*If  $f$  is differentiable such that  $F(x) = (1 - x^2)^{1/2}f'(x)$  is of bounded variation on  $[-1, 1]$ , then*

$$(3) \quad |R_n(f)| \leq \pi^2 T(F)/8n^2.$$

*If  $f$  is twice differentiable and  $f''$  is bounded and almost everywhere continuous on  $[-1, 1]$ , then*

$$(4) \quad n^2 R_n(f) \rightarrow 0.$$

Now suppose that  $F(x) = (1 - x^2)^{1/2}f'(x)$  is of bounded variation on  $[-1, 1]$ . Let  $|F(x)| \leq P$ ,  $-1 \leq x \leq 1$ , and let  $C$  be the number of intervals of  $[-1, 1]$  in each of which  $F$  is monotone. Then Riess and Johnson [3] proved that

$$|R_n(f)| \leq CP\pi^2/6n^2.$$

We note that this error bound becomes infinite when  $C = \infty$ , while the error bound in (3) above is always finite. Furthermore, if  $m \leq F(x) \leq M$ , it is easy to see that  $T(F) \leq C(M - m)$ , so that as a corollary of (3), we have

$$(3') \quad |R_n(f)| \leq \pi^2 C(M - m)/8n^2.$$

In particular, if  $F \geq 0$  or if  $F \leq 0$ , then

$$(3'') \quad |R_n(f)| \leq CP\pi^2/8n^2.$$

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**2. Proofs of the above results.** For  $0 \leqq t \leqq 1$ , we let  $\chi_t$  denote the characteristic function of the closed interval  $[t, 1]$ , and for each positive integer  $n$ , we write

$$s_n = \sum_{k=1}^n \chi_{(2k-1)/2n}$$

and  $v_n(t) = s_n(t) - nt$ . Then  $v_n(0) = v_n(1) = 0$ ,  $v_n$  lies between  $-1/2$  and  $1/2$  and is linear with an exception of  $n$  unit jumps at the points  $(2k - 1)/2n, k = 1, \dots, n$ . The  $v_n$  is a ‘‘saw-tooth’’ function (cf. [1]). It is clear that if  $g$  is Riemann integrable on  $[0, 1]$ , then

$$\int_0^1 g(x) dx = \frac{1}{n} \sum_{k=1}^n g\left(\frac{2k-1}{2n}\right) - \frac{1}{n} \int_0^1 g(x) dv_n(x).$$

Now, let  $g(x) = f(\cos \pi x)$ . Then

$$\int_{-1}^1 (1 - x^2)^{-1/2} f(x) dx = \pi \int_0^1 g(x) dx$$

and

$$\frac{\pi}{n} \sum_{k=1}^n f(x_{n,k}) = \frac{\pi}{n} \sum_{k=1}^n g\left(\frac{2k-1}{2n}\right),$$

so that we have

$$R_n(f) = -\frac{\pi}{n} \int_0^1 g(x) dv_n(x).$$

(i) Suppose that  $f$  is a function of bounded variation on  $[-1, 1]$ . Then

$$R_n(f) = \frac{\pi}{n} \int_0^1 v_n(x) dg(x) = -\frac{\pi}{n} \int_{-1}^1 v_n\left(\frac{\arccos x}{\pi}\right) df(x).$$

Since  $|v_n| \leqq 1/2$ , we have

$$|R_n(f)| \leqq \pi T(f)/2n.$$

(ii) Suppose that  $f$  is absolutely continuous on  $[-1, 1]$ . Then  $f'$  is Lebesgue integrable on  $[-1, 1]$ , so that  $g'(t) = -\pi \sin \pi t f(\cos \pi t)$  is Lebesgue integrable on  $[0, 1]$ . Hence, by a proof similar to that of the Riemann–Lebesgue theorem (cf. [1]), we see that

$$nR_n(f) = -\pi \int_0^1 g(x) dv_n(x) = \pi \int_0^1 v_n(x)g'(x) dx \rightarrow 0.$$

(iii) Suppose that  $f$  is differentiable such that  $F(x) = (1 - x^2)^{1/2}f'(x)$  is of bounded variation on  $[-1, 1]$ . Let

$$u_n(x) = \int_0^x v_n(t) dt.$$

Then  $u_n(k/n) = u_n(0) = 0$  for  $k = 1, \dots, n$ , and each  $u_n$  is periodic with period  $1/n$  and

$$\sup_{0 \leqq x \leqq 1} |u_n(x)| = \int_0^{1/2n} -v_n(x) dx = \frac{1}{8n}.$$

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Hence, we have

$$\begin{aligned} nR_n(f)/\pi &= \int_0^1 v_n(x)g'(x) dx \\ &= - \int_0^1 u_n(x) dg'(x) = \pi \int_{-1}^1 u_n\left(\frac{\arccos x}{\pi}\right) dF(x), \end{aligned}$$

so that

$$|R_n(f)| \leq \pi^2 T(F)/8n^2.$$

(iv) Suppose that  $f$  is twice differentiable and that  $f''$  is bounded and almost everywhere continuous on  $[-1, 1]$ . Then by Lebesgue's theorem,

$$g''(x) = \pi^2 \sin^2 \pi x f''(\cos \pi x) - \pi^2 \cos \pi x f'(\cos \pi x)$$

is Riemann integrable on  $[0, 1]$ . Now,

$$\begin{aligned} n^2 R_n(f) &= -\pi n \int_0^1 u_n(x)g''(x) dx \\ &= -\pi n \sum_{k=1}^n \int_{(k-1)/n}^{k/n} u_n(x)g''(x) dx \\ &= -\pi n \sum_{k=1}^n \int_0^{1/n} u_n\left(t + \frac{k-1}{n}\right)g''\left(t + \frac{k-1}{n}\right) dt. \end{aligned}$$

But  $u_n(t + (k - 1)/n) = u_n(t)$  for each  $k = 1, \dots, n$ . Hence,

$$\begin{aligned} n^2 R_n(f) &= -\pi n \int_0^{1/n} u_n(t) \sum_{k=1}^n g''\left(t + \frac{k-1}{n}\right) dt \\ &= -\pi n \int_0^1 u_n(x/n)S_n(x) dx, \end{aligned}$$

where

$$S_n(x) = \frac{1}{n} \sum_{k=1}^n g''\left(\frac{x+k-1}{n}\right).$$

Let

$$w(x) = \begin{cases} x^2/2 & \text{if } 0 \leq x < 1/2, \\ (1-x)^2/2 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Then we have

$$\begin{aligned} u_n(x/n) &= \int_0^{x/n} v_n(t) dt \\ &= \begin{cases} \int_0^{x/n} -nt dt & \text{if } 0 \leq x < 1/2, \\ -\frac{1}{8n} + \int_{1/2n}^{x/n} (1-nt) dt & \text{if } 1/2 \leq x \leq 1 \end{cases} \\ &= -w(t)/n. \end{aligned}$$

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Hence,

$$n^2 R_n(f) = \pi \int_0^1 w(x) S_n(x) dx.$$

But  $g''$  is Riemann integrable on  $[0, 1]$ , so that we have

$$S_n(x) \rightarrow \int_0^1 g''(t) dt = g'(1) - g'(0) = 0$$

for each  $x \in [0, 1]$ . By the bounded convergence theorem, we conclude that

$$n^2 R_n(f) \rightarrow 0.$$

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