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INTEGRAL-ALGEBRAIC EQUATIONS: THEORY OF COLLOCATION METHODS I*

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Abstract. Our analysis of collocation solutions for general systems of linear integral-algebraic equations (IAEs) is based on the notions of the tractability index and the ν-smoothing property of a Volterra integral operator. These are used to decouple the given IAE system into the inherent system of regular second-kind Volterra integral equations (VIEs) and a system of first-kind VIEs. This decoupling is then used to derive the optimal convergence properties of piecewise polynomial collocation solutions. Numerical examples illustrate the theoretical results.

Key words. integral-algebraic equations, tractability index, index-μ tractability, ν-smoothing Volterra operator, collocation solutions, optimal order of convergence

AMS subject classifications. 65R20, 65L80, 45D05

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1. Introduction.

We consider the general system of linear integral-algebraic equations (IAEs)

$$B(t)x(t) + \int_0^t K(t,s)x(s)ds = g(t), \qquad t \in I := [0,T], \tag{1.1}$$

where the matrices $B,\ K \in \mathbb{R}^{d\times d}$ and the function $g \in \mathbb{R}^d$ ($d \geq 2$) are continuous. We assume that the matrix $B = B(t)$ is singular on I and has constant $\text{rank}(B) = r_0 > 0$ and that $B(0)x(0) = g(0)$.

As in differential-algebraic equations (DAEs) (see, for example, [6], [7], [8], [10], [16], [3], [1], [21], [17], [19], [20], [12], [14]), the concept (and definition) of the *index* of an IAE system is the key to the theoretical and numerical analysis of (1.1). For DAEs there exist different definitions of an index (differentiability index, perturbation index, tractability index, strangeness index); they and their connections are now well understood. Of these the *tractability index* appears to have a number of important advantages (compare the survey papers [18] and [15]).

For IAEs there is as yet no comprehensive study of the index for general systems of such equations, as reflected in the papers [5] (differential or perturbation index), [4] (for IAEs with kernel of convolution type), and [9] (for IAEs of special Hessenberg form).

It is the aim of the present paper to fill this gap, by introducing the tractability index for linear IAE systems of the form (1.1). Owing to the ill-posed nature of

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first-kind Volterra integral equations (VIEs) this is closely related to the concept of ν-smoothing of a Volterra integral operator [13]. These two concepts will form the basis for the decoupling of a given IAE system into a system of inherent regular *second-kind* VIEs and a system of *first-kind* VIEs, and for the analysis of the optimal order of convergence of collocation solutions for (1.1).

In section 2 we present our definition of the tractability index for (1.1) and use it for the decoupling of the IAE system. Various examples of index-1 and index-2 systems illustrate its application. Section 3 deals with collocation solutions for (1.1) and their optimal convergence properties. In section 4 we present a selection of numerical examples. Concluding remarks are presented in section 5.

2. The tractability index of an IAE system.

2.1. Definitions. Before introducing the definition of the tractability index of a system of IAEs, we will consider two examples of (1.1) with $d = 2$.

Example 2.1. Let

$$B(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad K(t,s) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}.$$

The corresponding system of IAEs is

$$\text{(2.1)} \qquad \begin{cases} y(t) + z(t) = g_1(t), \\ \int_0^t z(s)ds = g_2(t), \end{cases} \quad t \in I := [0, T],$$

where $x(t) := (y(t), z(t))^T$. Differentiation of the second equation of (2.1) yields $z(t) = g_2'(t)$. Substituting this into the first equation of (2.1), we obtain $y(t) = g_1(t) - g_2'(t)$, which is related to the first derivative of $g_2(t)$.

Example 2.2. The choice

$$B(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad K(t,s) = \begin{bmatrix} 0 & 0 \\ 0 & t-s \end{bmatrix}, \qquad g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$$

describes the IAE system

$$\text{(2.2)} \qquad \begin{cases} y(t) + z(t) = g_1(t), \\ \int_0^t (t-s)z(s)ds = g_2(t), \end{cases} \quad t \in I := [0, T].$$

Differentiating the second equation of (2.2), we find $\int_0^t z(s)ds = g_2'(t)$, and a further differentiation step then leads to $z(t) = g_2''(t)$. Substitution of this into the first equation of (2.2) gives $y(t) = g_1(t) - g_2''(t)$, which is now related to the second derivative of $g_2(t)$.

These two simple examples reveal that the kernel K seriously impacts on the solution, and hence on the *differential index* (see [5]), of the equation. As we will show below, it will also have a similar impact on the *tractability index* of an IAE. In order to state the definition of the tractability index for systems of IAEs (1.1), we require the definition of the notion of ν-smoothing of the Volterra integral operator $\mathcal{V}: C(I, \mathbb{R}^d) \to C(I, \mathbb{R}^d)$ in (1.1),

$$\text{(2.3)} \qquad (\mathcal{V}x)(t) := \int_0^t K(t,s)x(s)ds, \qquad t \in I.$$

For the case $d = 1$, the definition of ν-smoothing is due to Lamm [13].

Definition 2.1 (see [13]). *The Volterra operator* (2.3) *with* $d = 1$ *is said to be* ν*-smoothing for a given integer* $\nu \geq 1$ *if its kernel* $K(t,s)$ *has the following properties:*

(a) $\frac{\partial^j K(t,s)}{\partial t^j}|_{s=t} = 0, t \in I, j = 0, 1, \ldots, \nu - 2;$

(b) $\frac{\partial^{\nu-1} K(t,s)}{\partial t^{\nu-1}}|_{s=t} = k_\nu \neq 0, t \in I;$

(c) $\frac{\partial^\nu K}{\partial t^\nu} \in C(D)$ $(D := \{(t,s) : 0 \le s \le t \le T\})$.

If $\frac{\partial^j K(t,s)}{\partial t^j}|_{s=t} = 0$ *for* $t \in I$ *and* all $j \in \mathbb{N}$*, then* $\mathcal{V}$ *is an* infinitely smoothing *Volterra operator.*

The first-kind VIE $\mathcal{V}y = f$ *is called a* ν-smoothing problem *if* $\mathcal{V}$ *is a* ν*-smoothing operator and* $f \in C^\nu(I)$.

We now extend the above definition of ν-smoothing to Volterra integral operators $\mathcal{V}$ with $d \ge 2$.

Definition 2.2. *The Volterra integral operator* $\mathcal{V}$ *in* (1.1) *corresponding to the kernel matrix* $K(t,s) = [\,{}_{(p,q=1,\ldots,d)}^{K_{pq}(t,s)}\,]$*, with* $d \ge 2$*, said to be* ν-smoothing *if there exist integers* $\nu_{pq} \ge 1$ *with* $\nu := \max_{1 \le p,q \le d}\{\nu_{pq}\}$ *such that the following hold:*

(a) $\frac{\partial^j K_{pq}(t,s)}{\partial t^j}|_{s=t} = 0, t \in I, j = 0, 1, \ldots, \nu_{pq} - 2;$

(b) $\frac{\partial^{\nu_{pq}-1} K_{pq}(t,s)}{\partial t^{\nu_{pq}-1}}|_{s=t} \neq 0, t \in I;$

(c) $\frac{\partial^{\nu_{pq}} K_{pq}(t,s)}{\partial t^{\nu_{pq}}} \in C(D)$.

We set $\nu_{pq} := 0$ *when* $K_{pq}(t,s) \equiv 0$.

A first-kind VIE $\mathcal{V}x = f$ *is called a* ν*-smoothing problem if* $\mathcal{V}$ *is a* ν*-smoothing operator and* $f \in C^\nu(I)$.

For a $(\nu+1)$-smoothing problem of the form (1.1) we now introduce the concept of *index-*μ *tractability* (cf. Definition 2.3 below). In order to motivate this notion of index, we first look at possible structural properties of systems of IAEs (1.1).

Let $i \ge 0$ be an integer, $K^i, K_i, B_i \in \mathbb{R}^{d\times d}$, and denote by $(K^i)_{pq}, (K_i)_{pq}$ the (p,q)-element of the matrices K^i, K_i, respectively. Consider the following chain of matrix functions:

$$K^0(t,s) := K(t,s), \quad K_0 := K := K(t,t), \quad B_0 := B, \quad B_1 = B_1(t) := B_0 + K_0Q_0.$$

If $(K_i)_{pq}(t,t) \neq 0$ $(i \ge 0)$, define $(K^{i+1})_{pq}(t,s) := 0$; otherwise $(K^{i+1})_{pq}(t,s) := \frac{\partial^{i+1}((K^i)_{pq}(t,s))}{\partial t^{i+1}}$. We set $K_{i+1} = K_{i+1}(t,t) := (K^{i+1}_{pq}(t,s)|_{s=t}\ (p,q = 1,2,\ldots,d))$ and

$$B_{i+2} = B_{i+2}(t) := B_{i+1} + \sum_{l=0}^{i+1} K_l \left(\prod_{j=0}^{i-l} P_j\right) Q_{i-l+1}, \quad 0 \le i \le \nu - 1;$$

$$B_{i+2} = B_{i+2}(t) := B_{i+1} + \sum_{l=0}^{\nu} K_l \left(\prod_{j=0}^{i-l} P_j\right) Q_{i-l+1}, \quad i \ge \nu,$$

with $\prod_{j=0}^{-1} P_j = 1$. Here, $Q_0 = Q_0(t)$ denotes a projector onto $\ker B_0$, while for $j \ge 1$, $Q_j = Q_j(t)$ is a projector into $\ker B_j$ with $Q_jQ_k = 0$ for $k < j$, and $P_j = P_j(t) := I - Q_j$.

Definition 2.3. *Assume that the Volterra integral operator describing the IAE system* (1.1) *is* $(\nu+1)$*-smoothing with* $\nu \ge 0$*. Then* (1.1) *is said to be index-*μ *tractable if all matrices* $B_j,\ t \in I,\ j = 0, \ldots, \mu - 1,$ *defined above, are singular with smooth null space and* B_μ *remains nonsingular at all points in* I.

Remark. We note that the definition of the tractability index μ does not depend on the choice of the projectors Q_j, in analogy to the tractability index for DAEs;

compare Chapters 1 and 2 in [14]. The proof of that lemma is readily adapted to IAEs (1.1).

Example 2.3. We illustrate this definition of the tractability index by returning to the IAE systems introduced in Examples 2.1 and 2.2.

It is easy to see that the IAE system (2.1) is a 1-smoothing problem. Since $B_0 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $K_0 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, we can take $Q_0 = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right]$. We then find that the matrix

$$B_1 = B_0 + K_0 Q_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

is nonsingular, and so the tractability index of the IAE system (2.1) is 1. It can easily be verified that the differential index is also 1.

The IAE system (2.2) of Example 2.2 is an example of a 2-smoothing problem. It is described by the matrices $B_0 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right], K_0 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and thus we can take $Q_0 = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right]$. The corresponding matrix

$$B_1 = B_0 + K_0 Q_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is singular. We now obtain $Q_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], K_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and hence

$$\begin{aligned} B_2 &= B_1 + K_0 P_0 Q_1 + K_1 Q_0 \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned}$$

Since B_2 is nonsingular, the tractability index is 2. We observe that here again the differential index is also 2.

Example 2.4. Consider the IAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \int_0^t \begin{bmatrix} K_{11}(t,s) & K_{12}(t,s) \\ K_{21}(t,s) & 0 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix},$$

where $K_{11}(t,t), K_{12}(t,t), K_{21}(t,t) \neq 0$. The underlying Volterra integral operator is 1-smoothing, and $B_0 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], K_0 = \left[\begin{smallmatrix} K_{11}(t,t) & K_{12}(t,t) \\ K_{21}(t,t) & 0 \end{smallmatrix}\right]$. Thus we can take $Q_0 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. The corresponding matrix

$$\begin{aligned} B_1 &= B_0 + K_0 Q_0 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K_{11}(t,t) & K_{12}(t,t) \\ K_{21}(t,t) & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & K_{12}(t,t) \\ 0 & 0 \end{bmatrix} \end{aligned}$$

is singular. Since $Q_1 = \left[\begin{smallmatrix} 1 & 0 \\ -K_{12}^{-1}(t,t) & 0 \end{smallmatrix}\right]$, we find that

$$\begin{aligned} B_2 &= B_1 + K_0 P_0 Q_1 \\ &= \begin{bmatrix} 1 & K_{12}(t,t) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K_{11}(t,t) & K_{12}(t,t) \\ K_{21}(t,t) & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -K_{12}^{-1}(t,t) & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 + K_{11}(t,t) & K_{12}(t,t) \\ K_{21}(t,t) & 0 \end{bmatrix}. \end{aligned}$$

Owing to our assumptions on $K_{11}(t,t)$, $K_{12}(t,t)$, and $K_{21}(t,t)$, B_2 is nonsingular, and thus the tractability index is 2. We observe that here again the differential index is also 2.

Remark. Since $\operatorname{rank}(B(t)) = r_0 > 0$, there exist two nonsingular matrices $E(t)$ and $F(t)$ such that $E(t)B(t)F(t) = \left[\begin{smallmatrix} I_{r_0} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Let $\left[\begin{smallmatrix} y(t) \\ z(t) \end{smallmatrix}\right] := F^{-1}(t)x(t)$,

$$\tilde{K}(t,s) := \begin{bmatrix} \tilde{K}_{11}(t,s) & \tilde{K}_{12}(t,s) \\ \tilde{K}_{21}(t,s) & \tilde{K}_{22}(t,s) \end{bmatrix} := E(t)K(t,s)F(s),$$

and $\tilde{g}(t) := E(t)g(t)$. We thus obtain

$$\begin{bmatrix} I_{r_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + \int_0^t \tilde{K}(t,s) \begin{bmatrix} y(s) \\ z(s) \end{bmatrix} = \tilde{g}(t).$$

It is obvious that the differential index is 1 if and only if $\tilde{K}_{22}(t,t)$ is nonsingular. Taking $Q_0 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I_{d-r_0} \end{smallmatrix}\right]$, it follows that

$$B_1 = \begin{bmatrix} I_{r_0} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{21} & \tilde{K}_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I_{d-r_0} \end{bmatrix} = \begin{bmatrix} I_{r_0} & \tilde{K}_{12} \\ 0 & \tilde{K}_{22} \end{bmatrix}.$$

Therefore the tractability index is 1 if and only if $\tilde{K}_{22}(t,t)$ is nonsingular. This implies that if $\mu = 1$, the differential index equals the tractability index.

2.2. Decoupling of index-1 IAEs. Define

$$V_0 = V_0(t) := \int_0^t K(t,s)P_0(s)x(s)ds \qquad (j \geq 0),$$
$$W_0 = W_0(t) := \int_0^t K(t,s)Q_0(s)x(s)ds.$$

Omitting the argument t, (1.1) becomes $B(P_0x) + V_0 + W_0 = g$, and this can be written in the form

$$(B + KQ_0)(P_0x + Q_0x) + V_0 + W_0 - KQ_0x = g. \tag{2.4}$$

THEOREM 2.4. *Let* (1.1) *be index*-1 *tractable. Assume that* $P_0(\cdot)B_1^{-1}(\cdot)K(\cdot,\cdot) \in C^l(D)$, $Q_0(\cdot)B_1^{-1}(\cdot)K(\cdot,\cdot) \in C^{l+1}(D)$, $P_0(\cdot)B_1^{-1}(\cdot)g(\cdot) \in C^l(I)$, *and* $Q_0(\cdot)B_1^{-1}(\cdot)g(\cdot) \in C^{l+1}(I)$. *Then* (1.1) *has a unique solution* $(u(t), v(t))^T$ *with* $u, v \in C^l(I)$, *and there exist functions* $A_1, A_2, A_3, A_4 \in C^l(D)$ *such that the solution can be represented in the form*

$$u(t) = P_0B_1^{-1}g(t) + \int_0^t A_1(t,s)P_0(s)B_1^{-1}(s)g(s)ds + \int_0^t A_2(t,s)(Q_0(s)B_1^{-1}(s)g(s))'ds, \tag{2.5}$$

$$v(t) = -Q_0B_1^{-1}KP_0B_1^{-1}g(t) + (Q_0B_1^{-1}g(t))' + \int_0^t A_3(t,s)P_0(s)B_1^{-1}(s)g(s)ds + \int_0^t A_4(t,s)(Q_0(s)B_1^{-1}(s)g(s))'ds. \tag{2.6}$$

Proof. Since (1.1) is index-1 tractable, B_1 is nonsingular. Thus, multiplication of (2.4) by B_1^{-1} leads to

$$P_0x + Q_0x + B_1^{-1}V_0 + B_1^{-1}W_0 - B_1^{-1}KQ_0x = B_1^{-1}g. \tag{2.7}$$

If we now multiply (2.7) by P_0 and Q_0, respectively, we have

$$P_0x + P_0B_1^{-1}V_0 + P_0B_1^{-1}W_0 - P_0B_1^{-1}KQ_0x = P_0B_1^{-1}g, \tag{2.8}$$

$$Q_0x + Q_0B_1^{-1}V_0 + Q_0B_1^{-1}W_0 - Q_0B_1^{-1}KQ_0x = Q_0B_1^{-1}g. \tag{2.9}$$

Let $u(t) := P_0x(t)$ and $v(t) := Q_0x(t)$. Hence,

$$\begin{aligned} u(t) &+ P_0B_1^{-1}\int_0^t K(t,s)u(s)ds + P_0B_1^{-1}\int_0^t K(t,s)v(s)ds \\ &- P_0B_1^{-1}Kv(t) = P_0B_1^{-1}g(t), \end{aligned} \tag{2.10}$$

$$\begin{aligned} v(t) &+ Q_0B_1^{-1}\int_0^t K(t,s)u(s)ds + Q_0B_1^{-1}\int_0^t K(t,s)v(s)ds \\ &- Q_0B_1^{-1}Kv(t) = Q_0B_1^{-1}g(t). \end{aligned} \tag{2.11}$$

Since

$$\begin{aligned} P_0B_1^{-1}Kv(t) &= P_0B_1^{-1}KQ_0v(t) = P_0B_1^{-1}(B + KQ_0)Q_0v(t) = P_0Q_0v(t) = 0, \\ v(t) - Q_0B_1^{-1}Kv(t) &= Q_0v(t) - Q_0B_1^{-1}KQ_0v(t) = Q_0B_1^{-1}BQ_0v(t) = 0, \end{aligned}$$

equations (2.10) and (2.11) become

$$u(t) + P_0B_1^{-1}\int_0^t K(t,s)u(s)ds + P_0B_1^{-1}\int_0^t K(t,s)v(s)ds = P_0B_1^{-1}g(t), \tag{2.12}$$

$$Q_0B_1^{-1}\int_0^t K(t,s)u(s)ds + Q_0B_1^{-1}\int_0^t K(t,s)v(s)ds = Q_0B_1^{-1}g(t). \tag{2.13}$$

Furthermore, $Q_0B_1^{-1}Kv = Q_0B_1^{-1}KQ_0v = Q_0B_1^{-1}(KQ_0 + B)v = Q_0v = v$. Therefore, (1.1) can be decomposed into (2.12) and (2.13), and the nonnull space component $u(t)$ is given by the solution of the so-called "inherent regular" second-kind VIE (2.12), and the null space component $v(t)$ is given by the solution of the first-kind VIE (2.13). By Theorem 2.1.9 of [2, p. 66], the proof is complete. □

2.3. Decoupling of index-μ IAEs. Assuming that the IAE system (1.1) is index-μ tractable and 1-smoothing, we have

$$\begin{aligned} &B_\mu(P_{\mu-1}P_{\mu-2}\cdots P_0P_0P_1\cdots P_{\mu-1}x + P_{\mu-1}P_{\mu-2}\cdots P_0P_0P_1\cdots P_{\mu-2}Q_{\mu-1}x \\ &\quad + P_{\mu-1}P_{\mu-2}\cdots P_0P_0P_1\cdots P_{\mu-3}Q_{\mu-2}x + \cdots + P_{\mu-1}P_{\mu-2}\cdots P_0P_0Q_1x \\ &\quad + P_{\mu-1}P_{\mu-2}\cdots P_1Q_0x + \cdots + P_{\mu-1}Q_{\mu-2}x + Q_{\mu-1}x) + V_{\mu-1} + W_{\mu-1} + \cdots \\ &+ W_1 + W_0 - KQ_0x - KP_0Q_1x - \cdots - KP_0P_1\cdots P_{\mu-2}Q_{\mu-1}x = g. \end{aligned} \tag{2.14}$$

Define $\Pi_i := P_0P_1\cdots P_i$ $(i \geq 0)$. For $j = 1, 2, \ldots, \mu-1$, we find that $Q_j\Pi_{\mu-1} = Q_jP_0P_1\cdots P_{\mu-1} = Q_j(I - Q_0)P_1\cdots P_{\mu-1} = Q_jP_1\cdots P_{\mu-1} = \cdots = Q_jP_j\cdots P_{\mu-1} = 0$, and therefore

$$\begin{aligned} P_{\mu-1}P_{\mu-2}\cdots P_0\Pi_{\mu-1} &= (I - Q_{\mu-1} - Q_{\mu-2})P_{\mu-3}\cdots P_0\Pi_{\mu-1} = \cdots \\ &= (I - Q_{\mu-1} - Q_{\mu-2} - \cdots - Q_1)\Pi_{\mu-1} = \Pi_{\mu-1} - (Q_{\mu-1} + Q_{\mu-2} + \cdots + Q_1)\Pi_{\mu-1} \\ &= \Pi_{\mu-1}. \end{aligned}$$

Multiplication of (2.14) by B_μ^{-1} yields

$$\Pi_{\mu-1}x + \sum_{i=0}^{\mu-2} P_{\mu-1}P_{\mu-2}\cdots P_0\Pi_iQ_{i+1}x + \sum_{i=0}^{\mu-2} P_{\mu-1}P_{\mu-2}\cdots P_{i+1}Q_ix \tag{2.15}$$

$$+ Q_{\mu-1}x + B_\mu^{-1}V_{\mu-1} + B_\mu^{-1}\sum_{i=0}^{\mu-1} W_i - \sum_{i=0}^{\mu-2} B_\mu^{-1}K\Pi_i Q_{i+1}x - B_\mu^{-1}KQ_0x = B_\mu^{-1}g.$$

Similarly, if we multiply (2.15) by $\Pi_{\mu-1}$, $Q_iP_{i+1}\cdots P_{\mu-1}$ ($i = 0, 1, \ldots, \mu-2$), and $Q_{\mu-1}$, respectively, we obtain

$$\begin{aligned}(2.16)\quad & \Pi_{\mu-1}\Pi_{\mu-1}x + \sum_{i=0}^{\mu-2}\Pi_{\mu-1}P_{\mu-1}P_{\mu-2}\cdots P_0\Pi_iQ_{i+1}x \\ & + \sum_{i=0}^{\mu-2}\Pi_{\mu-1}P_{\mu-1}P_{\mu-2}\cdots P_{i+1}Q_ix + \Pi_{\mu-1}B_\mu^{-1}V_{\mu-1} + \Pi_{\mu-1}B_\mu^{-1}\sum_{i=0}^{\mu-1}W_i \\ & - \sum_{i=0}^{\mu-2}\Pi_{\mu-1}B_\mu^{-1}K\Pi_iQ_{i+1}x - \Pi_{\mu-1}B_\mu^{-1}KQ_0x = \Pi_{\mu-1}B_\mu^{-1}g,\end{aligned}$$

$$\begin{aligned}(2.17)\quad & Q_iP_{i+1}\cdots P_{\mu-1}\Pi_{\mu-1}x + \sum_{i=0}^{\mu-2}Q_iP_{i+1}\cdots P_{\mu-1}P_{\mu-1}P_{\mu-2}\ldots P_0\Pi_iQ_{i+1}x \\ & + \sum_{i=0}^{\mu-2}Q_iP_{i+1}\cdots P_{\mu-1}P_{\mu-1}P_{\mu-2}\cdots P_{i+1}Q_ix + Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}V_{\mu-1} \\ & + Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}\sum_{i=0}^{\mu-1}W_i - \sum_{i=0}^{\mu-2}Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}K\Pi_iQ_{i+1}x \\ & - Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}KQ_0x = Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}g,\end{aligned}$$

$$\begin{aligned}(2.18)\quad & Q_{\mu-1}\Pi_{\mu-1}x + Q_{\mu-1}x + Q_{\mu-1}B_\mu^{-1}V_{\mu-1} + Q_{\mu-1}B_\mu^{-1}\sum_{i=0}^{\mu-1}W_i \\ & - \sum_{i=0}^{\mu-2}Q_{\mu-1}B_\mu^{-1}K\Pi_iQ_{i+1}x - Q_{\mu-1}B_\mu^{-1}KQ_0x = Q_{\mu-1}B_\mu^{-1}g.\end{aligned}$$

The proof of the results contained in the following lemma is straightforward.

LEMMA 2.5. *Let* (1.1) *be index-*μ *and* 1*-smoothing. Then for* $i = 0, 1, \ldots, \mu-2$, *we have*

$$\Pi_{\mu-1}\Pi_{\mu-1} = \Pi_{\mu-1}, \Pi_{\mu-1}P_{\mu-1}\cdots P_1P_0\Pi_iQ_{i+1} = 0, \quad \Pi_{\mu-1}P_{\mu-1}P_{\mu-2}\cdots P_{i+1}Q_i = 0,$$
$$\Pi_{\mu-1}B_\mu^{-1}K\Pi_iQ_{i+1} = 0, \quad \Pi_{\mu-1}B_\mu^{-1}KQ_0 = 0, \quad Q_iP_{i+1}\cdots P_{\mu-1}\Pi_{\mu-1} = 0,$$
$$\sum_{j=0}^{\mu-2}Q_iP_{i+1}\cdots P_{\mu-1}P_{\mu-1}P_{\mu-2}\cdots P_0\Pi_jQ_{j+1} = -Q_iQ_{i+1} - \sum_{j=i+1}^{\mu-2}Q_iP_{i+1}\cdots P_jQ_{j+1},$$
$$\sum_{j=0}^{\mu-2}Q_iP_{i+1}\cdots P_{\mu-1}P_{\mu-1}P_{\mu-2}\cdots P_{j+1}Q_j = Q_i,$$
$$\sum_{j=0}^{\mu-2}Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}K\Pi_jQ_{j+1} + Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}KQ_0 = Q_i,$$
$$Q_{\mu-1}\Pi_{\mu-1} = 0, \quad Q_{\mu-1}B_\mu^{-1}K\Pi_iQ_{i+1} = 0\ (i = 0, 1, \ldots, \mu-3),$$
$$Q_{\mu-1}B_\mu^{-1}K\Pi_{\mu-2}Q_{\mu-1} = Q_{\mu-1}, \quad Q_{\mu-1}B_\mu^{-1}KQ_0 = 0.$$

It follows from Lemma 2.5 that (2.16)–(2.18) can be rewritten as

$$\Pi_{\mu-1}x + \Pi_{\mu-1}B_\mu^{-1}V_{\mu-1} + \Pi_{\mu-1}B_\mu^{-1}\sum_{j=0}^{\mu-1}W_j = \Pi_{\mu-1}B_\mu^{-1}g, \tag{2.19}$$

$$\begin{aligned} &-Q_iQ_{i+1}x - \sum_{j=i+1}^{\mu-2}Q_iP_{i+1}\cdots P_jQ_{j+1}x + Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}V_{\mu-1} \\ &\quad + Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}\sum_{j=0}^{\mu-1}W_j = Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}g, \end{aligned} \tag{2.20}$$

$$Q_{\mu-1}B_\mu^{-1}V_{\mu-1} + Q_{\mu-1}B_\mu^{-1}\sum_{j=0}^{\mu-1}W_j = Q_{\mu-1}B_\mu^{-1}g. \tag{2.21}$$

Multiply (2.20) ($i \geq 1$) and (2.21) by Π_{i-1} and $\Pi_{\mu-2}$, respectively. Then (2.20) and (2.21) become

$$\begin{aligned} &-\Pi_{i-1}Q_iQ_{i+1}x - \sum_{j=i+1}^{\mu-2}\Pi_{i-1}Q_iP_{i+1}\cdots P_jQ_{j+1}x \\ &\quad + \Pi_{i-1}Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}V_{\mu-1} + \Pi_{i-1}Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}\sum_{j=0}^{\mu-1}W_j \\ &\quad = \Pi_{i-1}Q_iP_{i+1}\cdots P_{\mu-1}B_\mu^{-1}g, \qquad i \geq 1, \end{aligned} \tag{2.22}$$

$$\Pi_{\mu-2}Q_{\mu-1}B_\mu^{-1}V_{\mu-1} + \Pi_{\mu-2}Q_{\mu-1}B_\mu^{-1}\sum_{j=0}^{\mu-1}W_j = \Pi_{\mu-2}Q_{\mu-1}B_\mu^{-1}g. \tag{2.23}$$

Setting $v_0 := Q_0x$, $v_i := \Pi_{i-1}Q_ix$ ($i = 1, 2, \ldots, \mu-1$), and $u := \Pi_{\mu-1}x$, we have

$$\begin{aligned} &V_{\mu-1} = \int_0^t K(t,s)u(s)ds, \qquad W_i = \int_0^t K(t,s)v_i(s)ds \quad (i = 0, 1, \ldots, \mu-1), \\ &\Pi_{i-1}Q_iQ_{i+1}x = \Pi_{i-1}Q_iQ_{i+1}\Pi_iQ_{i+1}x = \Pi_{i-1}Q_iQ_{i+1}v_{i+1}, \\ &\Pi_{i-1}Q_iP_{i+1}\ldots P_jQ_{j+1}x = \Pi_{i-1}Q_iP_{i+1}\cdots P_jQ_{j+1}\Pi_jQ_{j+1}x \\ &\qquad = \Pi_{i-1}Q_iP_{i+1}\cdots P_jQ_{j+1}v_{j+1}. \end{aligned}$$

It follows that

$$u + \Pi_{\mu-1}B_\mu^{-1}V_{\mu-1} + \Pi_{\mu-1}B_\mu^{-1}\sum_{j=0}^{\mu-1}W_j = \Pi_{\mu-1}B_\mu^{-1}g, \tag{2.24}$$

$$\begin{aligned} &-Q_0Q_1v_1 - \sum_{j=1}^{\mu-2}Q_0P_1\cdots P_jQ_{j+1}v_{j+1} + Q_0P_1\cdots P_{\mu-1}B_\mu^{-1}V_{\mu-1} \\ &\quad + Q_0P_1\cdots P_{\mu-1}B_\mu^{-1}\sum_{j=0}^{\mu-1}W_j = Q_0P_1\cdots P_{\mu-1}B_\mu^{-1}g, \end{aligned} \tag{2.25}$$

$$-\Pi_{i-1}Q_iQ_{i+1}v_{i+1} - \sum_{j=i+1}^{\mu-2}\Pi_{i-1}Q_iP_{i+1}\cdots P_jQ_{j+1}v_{j+1} \tag{2.26}$$

$$+ \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} B_\mu^{-1} V_{\mu-1}$$
$$+ \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} B_\mu^{-1} \sum_{j=0}^{\mu-1} W_j = \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} B_\mu^{-1} g, \ i \geq 1,$$

$$\Pi_{\mu-2} Q_{\mu-1} B_\mu^{-1} V_{\mu-1} + \Pi_{\mu-2} Q_{\mu-1} B_\mu^{-1} \sum_{j=0}^{\mu-1} W_j = \Pi_{\mu-2} Q_{\mu-1} B_\mu^{-1} g. \tag{2.27}$$

We have thus established the following result.

THEOREM 2.6. *Assume that the IAE system* (1.1) *is index-μ tractable and* 1-*smoothing. Then* (1.1) *can be decoupled into a regular part,*

$$u(t) + \Pi_{\mu-1} B_\mu^{-1} V_{\mu-1} + \Pi_{\mu-1} B_\mu^{-1} \sum_{j=0}^{\mu-1} W_j = \Pi_{\mu-1} B_\mu^{-1} g(t),$$

and a subsystem

$$\begin{bmatrix} 0 & -Q_0Q_1 & -Q_0P_1Q_2 & \dots & -Q_0P_1\cdots P_{\mu-3}Q_{\mu-2} & -Q_0P_1\cdots P_{\mu-2}Q_{\mu-1} \\ 0 & 0 & -\Pi_0Q_1Q_2 & \dots & -\Pi_0Q_1P_2\cdots P_{\mu-3}Q_{\mu-2} & -\Pi_0Q_1P_2\cdots P_{\mu-2}Q_{\mu-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\Pi_{\mu-4}Q_{\mu-3}Q_{\mu-2} & -\Pi_{\mu-4}Q_{\mu-3}P_{\mu-2}Q_{\mu-1} \\ 0 & 0 & 0 & \dots & 0 & -\Pi_{\mu-3}Q_{\mu-2}Q_{\mu-1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_0(t) \\ v_1(t) \\ \dots \\ v_{\mu-3}(t) \\ v_{\mu-2}(t) \\ v_{\mu-1}(t) \end{bmatrix}$$
$$+ \int_0^t \begin{bmatrix} Q_0P_1\cdots P_{\mu-1}B_\mu^{-1}K(t,s) & \dots & Q_0P_1\cdots P_{\mu-1}B_\mu^{-1}K(t,s) \\ \Pi_0Q_1P_2\cdots P_{\mu-1}B_\mu^{-1}K(t,s) & \dots & \Pi_0Q_1P_2\cdots P_{\mu-1}B_\mu^{-1}K(t,s) \\ \dots & \dots & \dots \\ \Pi_{\mu-4}Q_{\mu-3}P_{\mu-2}P_{\mu-1}B_\mu^{-1}K(t,s) & \dots & \Pi_{\mu-4}Q_{\mu-3}P_{\mu-2}P_{\mu-1}B_\mu^{-1}K(t,s) \\ \Pi_{\mu-3}Q_{\mu-2}P_{\mu-1}B_\mu^{-1}K(t,s) & \dots & \Pi_{\mu-3}Q_{\mu-2}P_{\mu-1}B_\mu^{-1}K(t,s) \\ \Pi_{\mu-2}Q_{\mu-1}B_\mu^{-1}K(t,s) & \dots & \Pi_{\mu-2}Q_{\mu-1}B_\mu^{-1}K(t,s) \end{bmatrix} \cdot \begin{bmatrix} v_0(s) \\ v_1(s) \\ \dots \\ v_{\mu-3}(s) \\ v_{\mu-2}(s) \\ v_{\mu-1}(s) \end{bmatrix} ds$$
$$+ \begin{bmatrix} Q_0P_1\cdots P_{\mu-1}B_\mu^{-1}\int_0^t K(t,s)u(s)ds \\ \Pi_0Q_1P_2\cdots P_{\mu-1}B_\mu^{-1}\int_0^t K(t,s)u(s)ds \\ \dots \\ \Pi_{\mu-4}Q_{\mu-3}P_{\mu-2}P_{\mu-1}B_\mu^{-1}\int_0^t K(t,s)u(s)ds \\ \Pi_{\mu-3}Q_{\mu-2}P_{\mu-1}B_\mu^{-1}\int_0^t K(t,s)u(s)ds \\ \Pi_{\mu-2}Q_{\mu-1}B_\mu^{-1}\int_0^t K(t,s)u(s)ds \end{bmatrix} = \begin{bmatrix} Q_0P_1\cdots P_{\mu-1}B_\mu^{-1}g(t) \\ \Pi_0Q_1P_2\cdots P_{\mu-1}B_\mu^{-1}g(t) \\ \dots \\ \Pi_{\mu-4}Q_{\mu-3}P_{\mu-2}P_{\mu-1}B_\mu^{-1}g(t) \\ \Pi_{\mu-3}Q_{\mu-2}P_{\mu-1}B_\mu^{-1}g(t) \\ \Pi_{\mu-2}Q_{\mu-1}B_\mu^{-1}g(t) \end{bmatrix}.$$

3. Collocation methods for index-1 IAEs.

3.1. The general collocation equations. In this section, we consider the IAE system (1.1) under the assumption that B_1^{-1} is uniformly bounded on I.

Let

$$I_h := \{t_n := nh : \ n = 0, 1, \dots, N \ (t_N = T)\}$$

be a given mesh on I. The solution x of (1.1) will be approximated by the element x_h of the piecewise polynomial space

$$S_{m-1}^{(-1)}(I_h) := \{v : \ v|_{e_n} \in \pi_{m-1} \ (0 \leq n \leq N-1)\}. \tag{3.1}$$

Here, $e_n := (t_n, t_{n+1}]$. If $w \in \mathbb{R}^d$, the notation $w \in \pi_m$ means that each component of w is a real polynomial of degree not exceeding m. Let X_h be given by

$$X_h := \{t = t_n + c_i h : \ 0 < c_1 < \cdots < c_m \leq 1 \ (0 \leq n \leq N-1)\}.$$

The collocation equation reads as

$$B(t)x_h(t) + \int_0^t K(t,s)x_h(s)ds = g(t), \qquad t \in X_h. \tag{3.2}$$

Because now (1.1) is index-1 tractable, $B_1 = B_1(t_{n,i})$ is nonsingular, and multiplying (3.2) by B_1^{-1}, we have at $t = t_{n,i} := t_n + c_i h$

$$P_0 x_h + Q_0 x_h + B_1^{-1} V_0^h + B_1^{-1} W_0^h - B_1^{-1} K Q_0 x_h = B_1^{-1} g, \tag{3.3}$$

where $V_0^h = V_0^h(t) := \int_0^t K(t,s)P_0(s)x_h(s)ds, W_0^h = W_0^h(t) := \int_0^t K(t,s)Q_0(s)x_h(s)ds$. Multiplying both sides of (3.3) by P_0 and Q_0, respectively, we obtain (in analogy to (2.8) and (2.9))

$$P_0 x_h + P_0 B_1^{-1} V_0^h + P_0 B_1^{-1} W_0^h - P_0 B_1^{-1} K Q_0 x_h = P_0 B_1^{-1} g, \tag{3.4}$$

$$Q_0 x_h + Q_0 B_1^{-1} V_0^h + Q_0 B_1^{-1} W_0^h - Q_0 B_1^{-1} K Q_0 x_h = Q_0 B_1^{-1} g. \tag{3.5}$$

Let $u_h(t) := P_0 x_h(t),\ v_h(t) := Q_0 x_h(t)$. We have at $t = t_{n,i}$

$$\begin{aligned} &u_h(t) + P_0 B_1^{-1} \int_0^t K(t,s)u_h(s)ds + P_0 B_1^{-1} \int_0^t K(t,s)v_h(s)ds - P_0 B_1^{-1} K v_h(t) \\ &= P_0 B_1^{-1} g(t), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &v_h(t) + Q_0 B_1^{-1} \int_0^t K(t,s)u_h(s)ds + Q_0 B_1^{-1} \int_0^t K(t,s)v_h(s)ds - Q_0 B_1^{-1} K v_h(t) \\ &= Q_0 B_1^{-1} g(t). \end{aligned} \tag{3.7}$$

Since

$$P_0 B_1^{-1} K v_h(t) = P_0 B_1^{-1} K Q_0 v_h(t) = P_0 B_1^{-1}(K Q_0 + B) Q_0 v_h(t) = P_0 Q_0 v_h(t) = 0$$

and

$$v_h(t) - Q_0 B_1^{-1} K v_h(t) = Q_0 v_h(t) - Q_0 B_1^{-1} K Q_0 v_h(t) = Q_0 B_1^{-1} B Q_0 v_h(t) = 0,$$

where, at $t = t_{n,i}$, equations (3.6) and (3.7) become

$$u_h(t) + P_0 B_1^{-1} \int_0^t K(t,s)u_h(s)ds + P_0 B_1^{-1} \int_0^t K(t,s)v_h(s)ds = P_0 B_1^{-1} g(t), \tag{3.8}$$

and

$$Q_0 B_1^{-1} \int_0^t K(t,s)u_h(s)ds + Q_0 B_1^{-1} \int_0^t K(t,s)v_h(s)ds = Q_0 B_1^{-1} g(t), \tag{3.9}$$

then (3.2) can be decoupled into (3.8) and (3.9), which is equivalent to applying collocation methods to the decoupled equations (2.12) and (2.13). So, in the following, we use (2.12) and (2.13) as the basis for our collocation methods.

Setting $U_{n,i} := u_h(t_{n,i})$ and $V_{n,i} := v_h(t_{n,i})$, we can write

$$u_h(t_n + sh) = \sum_{j=1}^m L_j(s) U_{n,j},\ v_h(t_n + sh) = \sum_{j=1}^m L_j(s) V_{n,j} \qquad (s \in (0,1]). \tag{3.10}$$

By (3.8) and (3.9), we have

$$
\begin{aligned}
(3.11)\qquad & U_{n,i} + hP_0(t_{n,i})B_1^{-1}(t_{n,i})\sum_{j=1}^{m}\int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(s)dsU_{n,j} \\
&+ hP_0(t_{n,i})B_1^{-1}(t_{n,i})\sum_{j=1}^{m}\int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(s)dsV_{n,j} \\
&= -hP_0(t_{n,i})B_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\sum_{j=1}^{m}\int_0^{1} K(t_{n,i}, t_l + sh)L_j(s)dsU_{l,j} \\
&- hP_0(t_{n,i})B_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\sum_{j=1}^{m}\int_0^{1} K(t_{n,i}, t_l + sh)L_j(s)dsV_{l,j} \\
&+ P_0(t_{n,i})B_1^{-1}(t_{n,i})g(t_{n,i}),
\end{aligned}
$$

$$
\begin{aligned}
(3.12)\qquad & hQ_0(t_{n,i})B_1^{-1}(t_{n,i})\sum_{j=1}^{m}\int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(s)dsU_{n,j} \\
&+ hQ_0(t_{n,i})B_1^{-1}(t_{n,i})\sum_{j=1}^{m}\int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(s)dsV_{n,j} \\
&= -hQ_0(t_{n,i})B_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\sum_{j=1}^{m}\int_0^{1} K(t_{n,i}, t_l + sh)L_j(s)dsU_{l,j} \\
&- hQ_0(t_{n,i})B_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\sum_{j=1}^{m}\int_0^{1} K(t_{n,i}, t_l + sh)L_j(s)dsV_{l,j} \\
&+ Q_0(t_{n,i})B_1^{-1}(t_{n,i})g(t_{n,i}).
\end{aligned}
$$

Employing the notation

$$
\begin{aligned}
&M_n := \left(\int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(s)ds\ \ (i, j = 1, 2, \ldots, m)\right), \\
&B_{1,n}^{-1} := \operatorname{diag}(B_1^{-1}(t_{n,i})),\ \ P_{0,n} := \operatorname{diag}(P_0(t_{n,i})),\ \ Q_{0,n} := \operatorname{diag}(Q_0(t_{n,i}))\ \ (i = 1, \ldots, m), \\
&M^{n,l} := \left(\int_0^{1} K(t_{n,i}, t_l + sh)L_j(s)ds\ \ (i, j = 1, 2, \ldots, m)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&K^n := \operatorname{diag}(K(t_{n,i})\ (i = 1, \ldots, m)),\ g_n := (g(t_{n,1}), \ldots, g(t_{n,m}))^T, \\
&U_n := (U_{n,1}, \ldots, U_{n,m})^T, \\
&V_n := (V_{n,1}, \ldots, V_{n,m})^T,\ A := (a_{i,j})\ (i, j = 1, 2, \ldots, m)),\ a_{i,j} := \int_0^{c_i} L_j(s)ds, \\
&\bar{M}_n := \left(\int_0^{c_i} K(t_{n,i}, t_n + sh)Q_0(t_n + sh)L_j(s)ds\ \ (i, j = 1, 2, \ldots, m)\right),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\begin{bmatrix} I_m \otimes I_d + hP_{0,n}B_{1,n}^{-1}M_n & hP_{0,n}B_{1,n}^{-1}M_n \\ hQ_{0,n}B_{1,n}^{-1}M_n & hQ_{0,n}B_{1,n}^{-1}\bar{M}_nQ_{0,n} \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix} \\
&= h\sum_{l=0}^{n-1} \begin{bmatrix} -hP_{0,n}B_{1,n}^{-1}M^{n,l} & -hP_{0,n}B_{1,n}^{-1}M^{n,l} \\ -hQ_{0,n}B_{1,n}^{-1}M^{n,l} & -hQ_{0,n}B_{1,n}^{-1}M^{n,l} \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix} + \begin{bmatrix} P_{0,n}B_{1,n}^{-1}g_n \\ Q_{0,n}B_{1,n}^{-1}g_n \end{bmatrix}.
\end{aligned}
$$

Observing that

$$
\begin{aligned}
&Q_0(t_{n,i})B_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(s)dsV_{n,j}\\
&= \left[Q_0(t_{n,j})B_1^{-1}(t_{n,j})K(t_{n,j}, t_{n,j}) + O(h)\right] V_{n,j}a_{i,j}\\
&= \left[Q_0(t_{n,j})B_1^{-1}(t_{n,j})\left[K(t_{n,j}, t_{n,j})Q_0(t_{n,j}) + B(t_{n,i})\right] + O(h)\right] V_{n,j}a_{i,j}\\
&= \left[Q_0(t_{n,j}) + O(h)\right] V_{n,j}a_{i,j} = (I + O(h))V_{n,j}a_{i,j},
\end{aligned}
$$

the above system for U_n and V_n can be written as

$$
\begin{aligned}
&\begin{bmatrix} I_m \otimes I_d + hP_{0,n}B_{1,n}^{-1}M_n & hP_{0,n}B_{1,n}^{-1}M_n \\ hQ_{0,n}B_{1,n}^{-1}M_n & h(A\otimes I_d) + O(h^2) \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix}\\
&= h\sum_{l=0}^{n-1} \begin{bmatrix} -hP_{0,n}B_{1,n}^{-1}M_l & -hP_{0,n}B_{1,n}^{-1}M_l \\ -hQ_{0,n}B_{1,n}^{-1}M_l & -hQ_{0,n}B_{1,n}^{-1}M_l \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix} + \begin{bmatrix} P_{0,n}B_{1,n}^{-1}g_n \\ Q_{0,n}B_{1,n}^{-1}g_n \end{bmatrix}.
\end{aligned}
$$

Therefore, the determinant of the above coefficient matrix is $h^{dm}\det(A)^d\det(I_d)^m + O(h^{2dm}) = h^{dm}\det(A)^d + O(h^{2dm})$, and so there exists a unique solution $(U_n, V_n)^T$ for sufficiently small h.

3.2. Convergence analysis. Familiar estimates of the interpolation error (see section 1.8 of [2]) imply that the exact solution $u(t), v(t)$ satisfies

$$u(t_n + sh) = \sum_{j=1}^{m} L_j(s)u(t_n + c_jh) + h^m S_{m,n}^1(s), \tag{3.13}$$

$$v(t_n + sh) = \sum_{j=1}^{m} L_j(s)v(t_n + c_jh) + h^m S_{m,n}^2(s), \tag{3.14}$$

where the Peano remainder terms and the Peano kernel are given by

$$S_{m,n}^1(s) := \int_0^1 K_m(s,\nu)u^{(m)}(t_n + \nu h)d\nu, \quad S_{m,n}^2(s) := \int_0^1 K_m(s,\nu)v^{(m)}(t_n + \nu h)d\nu,$$

and $K_m(s,\nu) := \frac{1}{(m-1)!}\left\{(s-\nu)_+^{m-1} - \sum_{k=1}^m L_k(s)(c_k-\nu)_+^{m-1}\right\},\ s \in (0,1]$.

THEOREM 3.1. *Assume that the following hold:*

(a) *The given functions in* (1.1) *satisfy the conditions of Theorem* 2.4 *for* $x \in C^l(I)$ *with* $l \geq m$.

(b) $(u_h, v_h)^T$ *is the collocation solution to* $(u, v)^T$ *of* (1.1) *with* $u_h, v_h \in S_{m-1}^{(-1)}(I_h)$.

(c) $\bar{h} > 0$ *is such that, for any* $h \in (0, \bar{h})$, *each of the linear systems* (3.12) *has a unique solution.*

Then for all uniform meshes I_h *with* $h \in (0, \bar{h})$ *the collocation solution* $(u_h, v_h)^T$ *converges uniformly on* I *to the solution* $(u, v)^T$ *of* (1.1) *if and only if the collocation parameters satisfy the condition*

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^m \frac{1-c_i}{c_i} \leq 1.$$

For $h \in (0, \bar{h})$ *and arbitrary* X_h *with* $0 < c_1 < \cdots < c_m \leq 1$, *the attainable global order of convergence is then described by*

$$\|u - u_h\|_\infty := \max_{t\in I}\|u(t) - u_h(t)\| \leq \bar{C}h^m$$

and

$$\|v - v_h\|_\infty := \max_{t \in I} \|v(t) - v_h(t)\| \le \tilde{C} \begin{cases} h^m & \text{if } -1 \le \rho_m < 1, \\ h^{m-1} & \text{if } \rho_m = 1. \end{cases}$$

The constants $\bar{C}$ and $\tilde{C}$ depend on the collocation parameters $\{c_i\}$ but are independent of h, and the exponent m of h cannot in general be replaced by $m+1$.

Proof. Let $e_h(t_n + sh) := u(t_n + sh) - u_h(t_n + sh)$ and $\tilde{e}_h(t_n + sh) := v(t_n + sh) - v_h(t_n + sh)$. Then by (3.10), (3.13), and (3.14), we obtain the expressions

$$e_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) e_h(t_{n,j}) + h^m S^1_{m,n}(s), \qquad s \in (0, 1], \tag{3.15}$$

$$\tilde{e}_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) \tilde{e}_h(t_{n,j}) + h^m S^2_{m,n}(s), \qquad s \in (0, 1]. \tag{3.16}$$

By (2.12) and (3.8), we have at $t_{n,i}$

(3.17)

$$\begin{aligned} &e_h(t_{n,i}) + hP_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{j=1}^{m} \int_0^{c_i} K(t_{n,i}, t_n + sh) L_j(s) ds \, [e_h(t_{n,j}) + \tilde{e}_h(t_{n,j})] \\ &= -hP_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \sum_{j=1}^{m} \int_0^1 K(t_{n,i}, t_l + sh) L_j(s) ds \, [e_h(t_{l,j}) + \tilde{e}_h(t_{l,j})] \\ &\quad + h^m \rho_{n,i}, \end{aligned}$$

and similarly, by (2.13) and (3.7),

$$\begin{aligned} &Q_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{j=1}^{m} \int_0^{c_i} K(t_{n,i}, t_n + sh) L_j(s) ds \, [e_h(t_{n,j}) + \tilde{e}_h(t_{n,j})] \\ &\quad = - Q_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \sum_{j=1}^{m} \int_0^1 K(t_{n,i}, t_l + sh) L_j(s) ds \, [e_h(t_{l,j}) + \tilde{e}_h(t_{l,j})] \\ &\qquad + h^{m-1} \sigma_{n,i}, \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} \rho_{n,i} :=& - hP_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{j=1}^{m} \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[S^1_{m,n}(s) + S^2_{m,n}(s)\right] ds \\ & - hP_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \sum_{j=1}^{m} \int_0^1 K(t_{n,i}, t_l + sh) \left[S^1_{m,n}(s) + S^2_{m,n}(s)\right] ds, \\ \sigma_{n,i} :=& - hQ_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{j=1}^{m} \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[S^1_{m,n}(s) + S^2_{m,n}(s)\right] ds \\ & - hQ_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \sum_{j=1}^{m} \int_0^1 K(t_{n,i}, t_l + sh) \left[S^1_{m,n}(s) + S^2_{m,n}(s)\right] ds. \end{aligned}$$

In the following, we distinguish between the two cases $c_m = 1$ and $c_m < 1$.

Case I: $c_m = 1$. Since

$$Q_0(t_{n-1,m}) = Q_0(t_{n,i}) - c_i h Q_0'(\cdot), \qquad B_1^{-1}(t_{n-1,m}) = B_1^{-1}(t_{n,i}) - c_i h (B_1^{-1})'(\cdot),$$

$$K(t_{n-1,m}, t_{n-1} + sh) = K(t_{n,i}, t_{n-1} + sh) - c_i h \partial_1 K(\cdot, t_{n-1} + sh),$$

we see that, by (3.18),

(3.19)

$$\begin{aligned}
&Q_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{j=1}^{m} \int_0^{c_i} K(t_{n,i}, t_n + sh) L_j(s) ds \, [e_h(t_{n,j}) + \tilde{e}_h(t_{n,j})] \\
&= Q_0(t_{n-1,m})B_1^{-1}(t_{n-1,m}) \sum_{j=1}^{m} \int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh) L_j(s) ds \\
&\quad \cdot [e_h(t_{n-1,j}) + \tilde{e}_h(t_{n-1,j})] \\
&- Q_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{j=1}^{m} \int_0^{1} K(t_{n,i}, t_{n-1} + sh) L_j(s) ds \, [e_h(t_{n-1,j}) + \tilde{e}_h(t_{n-1,j})] \\
&- c_i h Q_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{l=0}^{n-2} \sum_{j=1}^{m} \int_0^{1} \partial_1 K(\cdot, t_l + sh) L_j(s) ds \, [e_h(t_{l,j}) + \tilde{e}_h(t_{l,j})] \\
&+ h\bar{Q}_{n,i} \sum_{l=0}^{n-2} \sum_{j=1}^{m} \int_0^{1} K(t_{n-1,m}, t_l + sh) L_j(s) ds \, [e_h(t_{l,j}) + \tilde{e}_h(t_{l,j})] \\
&+ h^{m-1}(\sigma_{n,i} - \sigma_{n-1,m}),
\end{aligned}$$

with obvious meaning of $\bar{Q}_{n,i}$. It therefore follows from (3.17) and (3.19) that

$$\begin{aligned}
&\begin{bmatrix} I_m \otimes I_d + hP_{0,n}B_{1,n}^{-1}M_n & hP_{0,n}B_{1,n}^{-1}M_n \\ Q_{0,n}B_{1,n}^{-1}M_n & (A \otimes I_d) + O(h) \end{bmatrix} \begin{bmatrix} E_n \\ \tilde{E}_n \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ Q_{0,n}B_{1,n}^{-1}K^n e(e_m^T A - b^T) \otimes I_d & e(e_m^T A - b^T) \otimes I_d \end{bmatrix} \begin{bmatrix} E_{n-1} \\ \tilde{E}_{n-1} \end{bmatrix} \\
&+ h \sum_{l=0}^{n-1} \begin{bmatrix} -hP_{0,n}B_{1,n}^{-1}M^{n,l} & -hP_{0,n}B_{1,n}^{-1}M^{n,l} \\ \bar{M}^{n,l} & \bar{M}^{n,l} \end{bmatrix} \begin{bmatrix} E_l \\ \tilde{E}_l \end{bmatrix} + h^m \begin{bmatrix} \rho_n \\ \sigma_n \end{bmatrix},
\end{aligned}$$

again with obvious meaning of $\bar{M}^{n,l}$, and with

$$E_n := (e_h(t_{n,1}), \ldots, e_h(t_{n,m}))^T, \qquad \tilde{E}_n := (\tilde{e}_h(t_{n,1}), \ldots, \tilde{e}_h(t_{n,m}))^T,$$

$$\rho_n := (\rho_{n,1}, \ldots, \rho_{n,m})^T, \qquad \sigma_n := \frac{1}{h}(\sigma_{n,1} - \sigma_{n-1,m}, \ldots, \sigma_{n,m} - \sigma_{n-1,m})^T.$$

Since $c_m = 1$ we have $e_m^T A - b^T = (a_{m1} - b_1, \ldots, a_{mm} - b_m) = (0, \ldots, 0)^T$, and hence

(3.20)

$$\begin{aligned}
&\begin{bmatrix} I_m \otimes I_d + hP_{0,n}B_{1,n}^{-1}M_n & hP_{0,n}B_{1,n}^{-1}M_n \\ Q_{0,n}B_{1,n}^{-1}M_n & (A \otimes I_d) + O(h) \end{bmatrix} \begin{bmatrix} E_n \\ \tilde{E}_n \end{bmatrix} \\
&= h \sum_{l=0}^{n-1} \begin{bmatrix} -hP_{0,n}B_{1,n}^{-1}M^{n,l} & -hP_{0,n}B_{1,n}^{-1}M^{n,l} \\ \bar{M}^{n,l} & \bar{M}^{n,l} \end{bmatrix} \begin{bmatrix} E_l \\ \tilde{E}_l \end{bmatrix} + \begin{bmatrix} h^m \rho_n \\ h^m \sigma_n \end{bmatrix}.
\end{aligned}$$

The inverse of the coefficient matrix is

$$\begin{bmatrix} I_{md} & 0 \\ -Q_{n,0}B_{1,n}^{-1}M_n(A\otimes I_d)^{-1} & (A\otimes I_d)^{-1} \end{bmatrix} + O(h),$$

implying that (3.20) can be written as

$$\begin{bmatrix} E_n \\ \tilde{E}_n \end{bmatrix} = h\sum_{l=0}^{n-1}\left[\begin{bmatrix} -hP_{0,n}B_{1,n}^{-1}M^{n,l} & -hP_{0,n}B_{1,n}^{-1}M^{n,l} \\ \tilde{M}^{n,l} & \tilde{M}^{n,l} \end{bmatrix} + O(h)\right]\begin{bmatrix} E_l \\ \tilde{E}_l \end{bmatrix}$$
$$+ \begin{bmatrix} h^m\tilde{\rho}_n \\ h^m\tilde{\sigma}_n \end{bmatrix},$$

where the meaning of $\tilde{M}^{n,l}$, $\tilde{\rho}_n$, and $\tilde{\sigma}_n$ is clear. Therefore, there exist constants C_1 and C_2 such that

$$\|E_n\| \le C_1h^m, \qquad \|\tilde{E}_n\| \le C_2h^m.$$

Case II: $c_m < 1$. In this case, (3.20) becomes

(3.21)
$$\begin{bmatrix} E_n \\ \tilde{E}_n \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ (A\otimes I_d)^{-1}Q_{0,n}B_{1,n}^{-1}K^n[M\otimes I_d] & (A\otimes I_d)^{-1}[M\otimes I_d] \end{bmatrix}\begin{bmatrix} E_{n-1} \\ \tilde{E}_{n-1} \end{bmatrix}$$
$$+ h\sum_{l=0}^{n-1}\left(\begin{bmatrix} -hP_{0,n}B_{1,n}^{-1}M^{n,l} & -hP_{0,n}B_{1,n}^{-1}M^{n,l} \\ \tilde{\tilde{M}}^{n,l} & \tilde{\tilde{M}}^{n,l} \end{bmatrix} + O(h)\right)\begin{bmatrix} E_l \\ \tilde{E}_l \end{bmatrix} + h^m\begin{bmatrix} \tilde{\rho}_n \\ \tilde{\sigma}_n \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ (A\otimes I_d)^{-1}Q_{0,n}B_{1,n}^{-1}K^n[M\otimes I_d] & A^{-1}M\otimes I_d \end{bmatrix}\begin{bmatrix} E_{n-1} \\ \tilde{E}_{n-1} \end{bmatrix}$$
$$+ h\sum_{l=0}^{n-1}\left(\begin{bmatrix} -hP_{0,n}B_{1,n}^{-1}M^{n,l} & -hP_{0,n}B_{1,n}^{-1}M^{n,l} \\ \tilde{\tilde{M}}^{n,l} & \tilde{\tilde{M}}^{n,l} \end{bmatrix} + O(h)\right)\begin{bmatrix} E_l \\ \tilde{E}_l \end{bmatrix} + h^m\begin{bmatrix} \tilde{\rho}_n \\ \tilde{\sigma}_n \end{bmatrix},$$

with obvious meaning of $\tilde{\tilde{M}}^{n,l}$ and with $M := e(e_m^TA - b^T)$. Using standard techniques of error estimation for collocation solutions of VIEs (see [2]), there exist constants C_3 and C_4 such that

$$\|E_n\| \le C_3\begin{cases} h^m & \text{if } -1 \le \rho_m < 1, \\ h^{m-1} & \text{if } \rho_m = 1, \end{cases} \qquad \|\tilde{E}_n\| \le C_4\begin{cases} h^m & \text{if } -1 \le \rho_m < 1, \\ h^{m-1} & \text{if } \rho_m = 1. \end{cases}$$

For $\rho_m = 1$ we have, using (3.17),

$$e_h(t_{n,i}) + hP_0(t_{n,i})B_1^{-1}(t_{n,i})\sum_{j=1}^{m}\int_0^{c_i}K(t_{n,i}, t_n+sh)L_j(s)ds\,[e_h(t_{n,j}) + \tilde{e}_h(t_{n,j})]$$
$$= e_h(t_{n-1,i}) + hP_0(t_{n-1,i})B_1^{-1}(t_{n-1,i})$$
$$\cdot \sum_{j=1}^{m}\int_0^{c_i}K(t_{n-1,i}, t_{n-1}+sh)L_j(s)ds\,[e_h(t_{n-1,j}) + \tilde{e}_h(t_{n-1,j})]$$
$$+ e_h(t_{n-1,m}) + hP_0(t_{n-1,m})B_1^{-1}(t_{n-1,m})$$

$$\begin{aligned}
&\cdot \sum_{j=1}^{m} \int_0^{c_m} K(t_{n-1,m}, t_{n-1}+sh)L_j(s)ds\,[e_h(t_{n-1,j})+\tilde{e}_h(t_{n-1,j})]\\
&- e_h(t_{n-2,m}) - hP_0(t_{n-2,m})B_1^{-1}(t_{n-2,m})\\
&\cdot \sum_{j=1}^{m} \int_0^{c_m} K(t_{n-2,m}, t_{n-2}+sh)L_j(s)ds\,[e_h(t_{n-2,j})+\tilde{e}_h(t_{n-2,j})]\\
&- hP_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{j=1}^{m} \int_0^{1} K(t_{n,i}, t_{n-1}+sh)L_j(s)ds\,[e_h(t_{n-1,j})+\tilde{e}_h(t_{n-1,j})]\\
&+ hP_0(t_{n-1,i})B_1^{-1}(t_{n-1,i}) \sum_{j=1}^{m} \int_0^{1} K(t_{n-1,i}, t_{n-2}+sh)L_j(s)ds\,[e_h(t_{n-2,j})+\tilde{e}_h(t_{n-2,j})]\\
&- hP_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{l=0}^{n-2}\sum_{j=1}^{m} \int_0^{1} K(t_{n,i}, t_l+sh)L_j(s)ds\,[e_h(t_{l,j})+\tilde{e}_h(t_{l,j})]\\
&+ hP_0(t_{n-1,i})B_1^{-1}(t_{n-1,i}) \sum_{l=0}^{n-3}\sum_{j=1}^{m} \int_0^{1} K(t_{n-1,i}, t_l+sh)L_j(s)ds\,[e_h(t_{l,j})+\tilde{e}_h(t_{l,j})]\\
&+ hP_0(t_{n-1,m})B_1^{-1}(t_{n-1,m}) \sum_{l=0}^{n-2}\sum_{j=1}^{m} \int_0^{1} K(t_{n-1,m}, t_l+sh)L_j(s)ds\,[e_h(t_{l,j})+\tilde{e}_h(t_{l,j})]\\
&- hP_0(t_{n-2,m})B_1^{-1}(t_{n-2,m}) \sum_{l=0}^{n-3}\sum_{j=1}^{m} \int_0^{1} K(t_{n-2,m}, t_l+sh)L_j(s)ds\,[e_h(t_{l,j})+\tilde{e}_h(t_{l,j})]\\
&+ h^m(\rho_{n,i} - \rho_{n-1,m} - \rho_{n-1,i} + \rho_{n-2,m}).
\end{aligned}$$

In a similar fashion we derive the equation

$$\begin{aligned}
&Q_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{j=1}^{m} \int_0^{c_i} K(t_{n,i}, t_n+sh)L_j(s)ds\,[e_h(t_{n,j})+\tilde{e}_h(t_{n,j})]\\
&= Q_0(t_{n-1,i})B_1^{-1}(t_{n-1,i}) \sum_{j=1}^{m} \int_0^{c_i} K(t_{n-1,i}, t_{n-1}+sh)L_j(s)ds\,[e_h(t_{n-1,j})+\tilde{e}_h(t_{n-1,j})]\\
&+ Q_0(t_{n-1,i})B_1^{-1}(t_{n-1,i}) \sum_{j=1}^{m} \int_0^{1} K(t_{n-1,i}, t_{n-2}+sh)L_j(s)ds\,[e_h(t_{n-2,j})+\tilde{e}_h(t_{n-2,j})]\\
&+ Q_0(t_{n-1,m})B_1^{-1}(t_{n-1,m}) \sum_{j=1}^{m} \int_0^{c_m} K(t_{n-1,m}, t_{n-1}+sh)L_j(s)ds\,[e_h(t_{n-1,j})+\tilde{e}_h(t_{n-1,j})]\\
&- Q_0(t_{n-2,m})B_1^{-1}(t_{n-2,m}) \sum_{j=1}^{m} \int_0^{c_m} K(t_{n-2,m}, t_{n-2}+sh)L_j(s)ds\,[e_h(t_{n-2,j})+\tilde{e}_h(t_{n-2,j})]\\
&- Q_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{j=1}^{m} \int_0^{1} K(t_{n,i}, t_{n-1}+sh)L_j(s)ds\,[e_h(t_{n-1,j})+\tilde{e}_h(t_{n-1,j})]\\
&- Q_0(t_{n,i})B_1^{-1}(t_{n,i}) \sum_{l=0}^{n-2}\sum_{j=1}^{m} \int_0^{1} K(t_{n,i}, t_l+sh)L_j(s)ds\,[e_h(t_{l,j})+\tilde{e}_h(t_{l,j})]
\end{aligned}$$

$$+ Q_0(t_{n-1,m})B_1^{-1}(t_{n-1,m}) \sum_{l=0}^{n-2}\sum_{j=1}^{m} \int_0^1 K(t_{n-1,m}, t_l + sh)L_j(s)ds\,[e_h(t_{l,j}) + \tilde{e}_h(t_{l,j})]$$

$$+ Q_0(t_{n-1,i})B_1^{-1}(t_{n-1,i}) \sum_{l=0}^{n-3}\sum_{j=1}^{m} \int_0^1 K(t_{n-1,i}, t_l + sh)L_j(s)ds\,[e_h(t_{l,j}) + \tilde{e}_h(t_{l,j})]$$

$$- Q_0(t_{n-2,m})B_1^{-1}(t_{n-2,m}) \sum_{l=0}^{n-3}\sum_{j=1}^{m} \int_0^1 K(t_{n-2,m}, t_l + sh)L_j(s)ds\,[e_h(t_{l,j}) + \tilde{e}_h(t_{l,j})]$$

$$+ h^{m-1}(\sigma_{n,i} - \sigma_{n-1,m} - \sigma_{n-1,i} + \sigma_{n-2,m}).$$

In order to see the structure of the above error equations we write them in matrix-vector form:

$$\begin{bmatrix} I_m \otimes I_d + hP_{0,n}B_{1,n}^{-1}M_n & P_{0,n}B_{1,n}^{-1}M_n & 0 & 0 \\ O(h) & (A\otimes I_d) + O(h) & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} E_n \\ h\tilde{E}_n \\ E_{n-1} \\ h\tilde{E}_{n-1} \end{bmatrix}$$

$$= \left(\begin{bmatrix} (I_m + ee_m^T)\otimes I_d & P_{0,n}B_{1,n}^{-1}K^n[(A + ee_m^T A - eb^T)\otimes I_d] & -ee_m^T \otimes I_d \\ 0 & (A + ee_m^T A - eb^T)\otimes I_d & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}\right.$$

$$\left.\begin{matrix} P_{0,n}B_{1,n}^{-1}K^n[(-ee_m^T A + eb^T)\otimes I_d] \\ (eb^T - ee_m^T A)\otimes I_d \\ 0 \\ 0 \end{matrix}\right] + O(h) \Bigg) \begin{bmatrix} E_{n-1} \\ h\tilde{E}_{n-1} \\ E_{n-2} \\ h\tilde{E}_{n-2} \end{bmatrix}$$

$$+ \sum_{l=0}^{n-1} \begin{bmatrix} O(h^2) & O(h^2) & 0 & 0 \\ O(h^2) & O(h^2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_l \\ h\tilde{E}_l \\ E_{l-1} \\ h\tilde{E}_{l-1} \end{bmatrix} + \begin{bmatrix} O(h^{m+1}) \\ O(h^{m+1}) \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} E_n \\ h\tilde{E}_n \\ E_{n-1} \\ h\tilde{E}_{n-1} \end{bmatrix}$$

$$= \left(\begin{bmatrix} (I_m + ee_m^T)\otimes I_d & 0 & -ee_m^T \otimes I_d & 0 \\ 0 & (I_m + M)\otimes I_d & 0 & -M\otimes I_d \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} + O(h) \right) \begin{bmatrix} E_{n-1} \\ h\tilde{E}_{n-1} \\ E_{n-2} \\ h\tilde{E}_{n-2} \end{bmatrix}$$

$$+ \sum_{l=0}^{n-1} \begin{bmatrix} O(h^2) & O(h^2) & 0 & 0 \\ O(h^2) & O(h^2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_l \\ h\tilde{E}_l \\ E_{l-1} \\ h\tilde{E}_{l-1} \end{bmatrix} + \begin{bmatrix} O(h^{m+1}) \\ O(h^{m+1}) \\ 0 \\ 0 \end{bmatrix}.$$

Owing to the special structure of the first matrix on the right-hand side, it is easy to verify that its eigenvalues are

$$\underbrace{O(h), \ldots, O(h)}_{(d-1)m}, \underbrace{1 + O(h), \ldots, 1 + O(h)}_{m}; 1 + O(h), \ldots, 1 + O(h);$$

$$1+O(h),\dots,1+O(h);\underbrace{O(h),\dots,O(h)}_{(d-1)m},\underbrace{1+O(h),\dots,1+O(h)}_{m}.$$

Thus there exists a positive constant C_5 such that

$$\|E_n\| \le C_5 h^m \quad \text{if } \rho_m = 1.$$

By (3.15) and (3.16), the proof is complete. ∎

Remark. In [11], [2], the authors studied the convergence of IAEs with index 1 and derived an estimation similar to the one in (3.21). They also found that u_h converges to the u-component for any choice of $\{c_i\}$. We point out that this result should be revised, since u_h converges to u if and only if $-1 \le \rho_m \le 1$. The order of convergence is then $\|u-u_h\| \le Ch^m$.

3.3. Global and local superconvergence. Let

(3.22)
$$u_h^{it}(t) := -P_0B_1^{-1}\int_0^t K(t,s)u_h(s)ds - P_0B_1^{-1}\int_0^t K(t,s)v_h(s)ds + P_0B_1^{-1}g(t),\ t\in I,$$

and

(3.23)
$$\delta_h(t) := -u_h(t) - P_0B_1^{-1}\int_0^t K(t,s)u_h(s)ds - P_0B_1^{-1}\int_0^t K(t,s)v_h(s)ds + P_0B_1^{-1}g(t),$$

$$d_h(t) := -Q_0B_1^{-1}\int_0^t K(t,s)u_h(s)ds - Q_0B_1^{-1}\int_0^t K(t,s)v_h(s)ds + Q_0B_1^{-1}g(t), \tag{3.24}$$

for $t\in I$ denote the defects (or residuals) associated with the collocation solution $(u_h,v_h)^T$ to the initial-value problem (1.1). By definition of the collocation solution, the defects δ_h and d_h vanish on the set X_h: $\delta_h(t)=0, d_h(t)=0$ for all $t\in X_h$. We have for $t\in I$

$$e_h(t) + P_0B_1^{-1}\int_0^t K(t,s)e_h(s)ds + P_0B_1^{-1}\int_0^t K(t,s)\tilde{e}_h(s)ds = \delta_h(t), \tag{3.25}$$

$$Q_0B_1^{-1}\int_0^t K(t,s)e_h(s)ds + Q_0B_1^{-1}\int_0^t K(t,s)\tilde{e}_h(s)ds = d_h(t). \tag{3.26}$$

Setting $e^{it}(t) := u(t) - u_h^{it}(t)$, we obtain

$$e_h^{it}(t) := -P_0B_1^{-1}\int_0^t K(t,s)e_h(s)ds - P_0B_1^{-1}\int_0^t K(t,s)\tilde{e}_h(s)ds, \quad t\in I. \tag{3.27}$$

Theorem 2.4 now induces the following theorem.

THEOREM 3.2. *Assume that the given functions in* (1.1) *satisfy the conditions of Theorem* 2.4 *so that $u,v\in C^l(I)$. Then the system of equations* (3.25) *and* (3.26) *has a unique solution $(e_h,\tilde{e}_h)^T$ with $e_h(t),\tilde{e}_h(t)\in C^l((t_n,t_{n+1}])$ $(n\ge 0)$, and there exist functions $D_i\in C^l((t_n,t_{n+1}]), \kappa_{ij}\in C^l(D_n), i,j=1,2$ $(D_n := \{(t,s): t_n < s \le t \le t_{n+1}\})$, so that the solution can be represented in the form*

$$e_h(t) = \delta_h(t) + D_1(t)d_h(t) + \int_0^t \kappa_{11}(t,s)\delta_h(s)ds + \int_0^t \kappa_{12}(t,s)d_h(s)ds, \tag{3.28}$$

$$\tilde{e}_h(t) = -Q_0 B_1^{-1} K \delta_h(t) + D_2(t) d_h(t) + d'_h(t) + \int_0^t \kappa_{21}(t,s)\delta_h(s)ds + \kappa_{22}(t,s)d_h(s)ds, \tag{3.29}$$

$$e_h^{it}(t) = D_1(t)d_h(t) + \int_0^t \kappa_{11}(t,s)\delta_h(s)ds + \int_0^t \kappa_{12}(t,s)d_h(s)ds. \tag{3.30}$$

The following two theorems are direct consequences of Theorem 3.2.

THEOREM 3.3. *Assume that assumptions* (b), (c) *of Theorem* 3.1 *hold, and let* (a) *be replaced by the assumption that* $u, v \in C^l(I)$ *with* $l \geq m+1$. *If* $c_m = 1$ *and the* m *collocation parameters* $\{c_i\}$ *are subject to*

$$-1 \leq \rho_m = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \leq 1$$

and satisfy the orthogonality condition

$$J_0 := \int_0^1 \prod_{i=1}^{m}(s-c_i)ds = 0,$$

then the corresponding collocation solution $(u_h, v_h)^T$ *with* $u_h, v_h \in S_{m-1}^{(-1)}(I_h)$ *satisfies*

$$\max\{|u(t)-u_h(t)| : t \in I\} \leq \bar{C}_2 h^{m+1},$$

$$\max\{|v(t)-v_h(t)| : t \in I\} \leq \tilde{C}_2 \begin{cases} h^m & \text{if } -1 \leq \rho_m < 1, \\ h^{m-1} & \text{if } \rho_m = 1, \end{cases}$$

$$\max\{|u(t)-u_h^{it}(t)| : t \in I\} \leq \hat{C}_2 h^{m+1}.$$

Here, the constants $\bar{C}_2, \tilde{C}_2$, *and* $\hat{C}_2$ *depend only on the collocation parameters* $\{c_i\}$ *but not on* h.

THEOREM 3.4. *Assume the following:*

(a) *The given functions satisfy the conditions of Theorem* 2.4 *so that* $u, v \in C^l(I)$ *with* $l \geq m+\kappa$, *for some* κ *with* $1 \leq \kappa \leq m$ *specified in* (b) *below.*

(b) $c_m = 1$, *and the* m *collocation parameters* $\{c_i\}$ *are subject to*

$$-1 \leq \rho_m = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \leq 1$$

and satisfy the orthogonality conditions

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m}(s-c_i)ds = 0, \quad \nu = 0, 1, \ldots, \kappa-1,$$

with $J_\kappa \neq 0$.

Then, for all meshes I_h *with* $h \in (0, \bar{h})$, *the corresponding collocation solution* $(u_h, v_h)^T$ *with* $u_h, v_h \in S_{m-1}^{(-1)}(I_h)$ *satisfies*

$$\max\{|u(t)-u_h(t)| : t \in I_h\} \leq \bar{C}_3 h^{m+\kappa},$$

$$\max\{|z(t)-z_h(t)| : t \in I_h\} \leq \tilde{C}_3 \begin{cases} h^m & \text{if } -1 \leq \rho_m < 1, \\ h^{m-1} & \text{if } \rho_m = 1, \end{cases}$$

$$\max\{|u(t)-u_h^{it}(t)| : t \in I_h\} \leq \hat{C}_3 h^{m+\kappa},$$

with $\bar{C}_3, \tilde{C}_3$, *and* $\hat{C}_3$ *depending on the collocation parameters* $\{c_i\}$ *but not on* h.

4. Numerical examples. We now present an example to illustrate the foregoing convergence results.

Example 4.1. Consider the IAE system

$$\begin{cases} u(t) = f(t) + \int_0^t [(1+t-s)u(s) + e^{t-s}v(s)]\,ds, \\ \quad 0 = g(t) + \int_0^t \left[u(s) + e^{2t-s}v(s)\right] ds, \end{cases} \tag{4.1}$$

on $I = [0,1]$, with $f(t) = rte^{\beta t} - r(-2e^{\beta t} + e^{\beta t}\beta^2 t - e^{\beta t}\beta + e^{\beta t}\beta t + 2 + \beta + \beta t)/\beta^3 + 1/2\cos t - 1/2\sin t - 1/2e^t$ and $g(t) = -r(e^{\beta t}\beta t - e^{\beta t} + 1)/\beta^2 + 1/2e^t\cos t - 1/2e^t\sin t - 1/2e^{2t}$. It can be verified that the exact solution is $u(t) = rte^{\beta t},\ v(t) = \cos t$.

The kernel of the above Volterra integral operator, $K(t,s) = \left(\begin{smallmatrix} 1+t-s & e^{t-s} \\ 1 & e^{2t-s} \end{smallmatrix}\right)$, is 1-smoothing, and $B(t) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ has rank 1 for all $t \in I$; hence the tractability index is $\mu = 1$.

For the numerical solution of (4.1) we choose $m = 2$ and $m = 3$. For $m = 2$ we use the Gauss collocation parameters, $c_1 = \frac{3-\sqrt{3}}{6},\ c_2 = \frac{3+\sqrt{3}}{6}$; the Radau IIA collocation parameters, $c_1 = \frac{1}{3}$, $c_2 = 1$; and four sets of random collocation parameters, $c_1 = \frac{1}{2}$, $c_2 = 1$; $c_1 = \frac{1}{4}$, $c_2 = \frac{5}{6}$; $c_1 = \frac{1}{3}$, $c_2 = \frac{2}{3}$; $c_1 = \frac{1}{6}$, $c_2 = \frac{1}{2}$, respectively. For $m = 3$ we use the Gauss collocation parameters, $c_1 = \frac{5-\sqrt{15}}{10}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{5+\sqrt{15}}{10}$; the Radau IIA collocation parameters, $c_1 = \frac{4-\sqrt{6}}{10}$, $c_2 = \frac{4+\sqrt{6}}{10}$, $c_3 = 1$; and four sets of random collocation parameters, $c_1 = \frac{1}{2}$, $c_2 = \frac{2}{3}$, $c_3 = 1$; $c_1 = \frac{1}{3}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{2}{3}$; $c_1 = \frac{1}{2}$, $c_2 = \frac{2}{3}$, $c_3 = \frac{8}{9}$; $c_1 = \frac{1}{9}$, $c_2 = \frac{1}{3}$, $c_3 = \frac{1}{2}$, respectively. In Tables 4.1–4.8 we list the absolute errors of the u and v components at $t = 1$ for the six collocation parameters and for $m = 2$ or $m = 3$, and the ratios of the absolute values of the errors of $N = 16$ over those of $N = 32$.

From Tables 4.1–Table 4.8, we can see that the numerical results are consistent with our theoretical analysis.

TABLE 4.1
The absolute errors of u_h for Example 4.1 with $m = 2$, $r = 1$, and $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, 1)$	$(\frac{1}{4}, \frac{5}{6})$	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{6}, \frac{1}{2})$
2^2	2.2249e-03	7.1016e-04	5.4836e-03	1.8326e-03	6.0645e-03	9.8378e-03
2^3	5.1505e-04	8.9986e-05	1.2927e-03	4.0694e-04	1.4800e-03	2.4337e-03
2^4	1.2410e-04	1.1318e-05	3.1200e-04	9.5663e-05	3.6502e-04	6.0440e-04
2^5	3.0473e-05	1.4189e-06	7.6521e-05	2.3178e-05	9.0597e-05	4.2921e+00
Ratio	4.0726e+00	7.9765e+00	4.0773e+00	4.1274e+00	4.0291e+00	-

TABLE 4.2
The absolute errors of v_h for Example 4.1 with $m = 2$, $r = 1$, and $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, 1)$	$(\frac{1}{4}, \frac{5}{6})$	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{6}, \frac{1}{2})$
2^2	7.8060e-02	6.0427e-03	5.8052e-03	2.9911e-02	7.8436e-02	6.2157e+00
2^3	3.9549e-02	1.3346e-03	1.3371e-03	6.6804e-03	3.9639e-02	1.1595e+03
2^4	1.9894e-02	3.1302e-04	3.2041e-04	1.3396e-03	1.9916e-02	1.2481e+08
2^5	9.9757e-03	7.5757e-05	7.8393e-05	2.9321e-04	9.9811e-03	5.0125e+18
Ratio	1.9943e+00	4.1319e+00	4.0873e+00	4.5687e+00	1.9954e+00	-

Table 4.3
The absolute errors of u_h for Example 4.1 with $m = 2$, $r = 10$, and $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, 1)$	$(\frac{1}{4}, \frac{5}{6})$	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{6}, \frac{1}{2})$
2^2	2.2249e-02	7.1016e-03	5.4836e-02	1.8326e-02	6.0645e-02	9.8378e-02
2^3	5.1505e-03	8.9986e-04	1.2927e-02	4.0694e-03	1.4800e-02	2.4337e-02
2^4	1.2410e-03	1.1318e-04	3.1200e-03	9.5663e-04	3.6502e-03	6.0440e-03
2^5	3.0473e-04	1.4189e-05	7.6521e-04	2.3178e-04	9.0597e-04	2.7299e+01
Ratio	4.0726e+00	7.9765e+00	4.0773e+00	4.1274e+00	4.0291e+00	-

Table 4.4
The absolute errors of v_h for Example 4.1 with $m = 2$, $r = 10$, and $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, 1)$	$(\frac{1}{4}, \frac{5}{6})$	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{6}, \frac{1}{2})$
2^2	4.0120e-01	1.9931e-02	2.8146e-02	1.3191e-01	4.0521e-01	3.8144e+01
2^3	2.0575e-01	4.0665e-03	6.4722e-03	2.4993e-02	2.0677e-01	7.5777e+03
2^4	1.0400e-01	9.1534e-04	1.5507e-03	4.1067e-03	1.0426e-01	8.4357e+08
2^5	5.2260e-02	2.1692e-04	3.7940e-04	8.1156e-04	5.2324e-02	3.4472e+19
Ratio	1.9900e+00	4.2197e+00	4.0871e+00	5.0602e+00	1.9925e+00	-

Table 4.5
The absolute errors of u_h for Example 4.1 with $m = 3$, $r = 1$, and $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, \frac{2}{3}, 1)$	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, \frac{2}{3}, \frac{8}{9})$	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$
2^2	1.0827e-04	4.9658e-07	4.5708e-04	2.0846e-04	3.5078e-04	9.7697e-04
2^3	1.2726e-05	1.5840e-08	5.5342e-05	2.6278e-05	4.2034e-05	1.1868e-04
2^4	1.5430e-06	5.0178e-10	6.7883e-06	3.3041e-06	5.1248e-06	1.9082e-04
2^5	1.8997e-07	1.5799e-11	8.3988e-07	4.1437e-07	6.3200e-07	1.1026e+14
Ratio	8.1223e+00	3.1760e+01	8.0824e+00	7.9739e+00	8.1088e+00	-

Table 4.6
The absolute errors of v_h for Example 4.1 with $m = 3$, $r = 1$, and $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, \frac{2}{3}, 1)$	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, \frac{2}{3}, \frac{8}{9})$	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$
2^2	5.0411e-04	2.3164e-04	5.4261e-06	6.1245e-04	7.9138e-05	9.7665e+00
2^3	6.5707e-05	2.7954e-05	6.3653e-08	7.9201e-05	9.3934e-06	9.2868e+04
2^4	8.3073e-06	3.4415e-06	4.5225e-08	9.9925e-06	1.1500e-06	5.3861e+13
2^5	1.0417e-06	4.2714e-07	7.7251e-09	1.2524e-06	1.4246e-07	1.2916e+32
Ratio	7.9744e+00	8.0570e+00	5.8542e+00	7.9789e+00	8.0723e+00	-

Table 4.7
The absolute errors of u_h for Example 4.1 with $m = 3$, $r = 10$, and $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, \frac{2}{3}, 1)$	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, \frac{2}{3}, \frac{8}{9})$	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$
2^2	1.0827e-03	4.9658e-06	4.5708e-03	2.0846e-03	3.5078e-03	9.7697e-03
2^3	1.2726e-04	1.5840e-07	5.5342e-04	2.6278e-04	4.2034e-04	1.1868e-03
2^4	1.5430e-05	5.0178e-09	6.7883e-05	3.3041e-05	5.1248e-05	2.1573e-03
2^5	1.8997e-06	1.5800e-10	8.3988e-06	4.1437e-06	6.3200e-06	2.0970e+13
Ratio	8.1223e+00	3.1758e+01	8.0824e+00	7.9739e+00	8.1088e+00	-

Table 4.8
The absolute errors of v_h for Example 4.1 with $m = 3$, $r = 10$, and $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, \frac{2}{3}, 1)$	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, \frac{2}{3}, \frac{8}{9})$	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$
2^2	6.7405e-03	9.1422e-04	7.3275e-04	8.3899e-03	4.4323e-04	9.4555e+01
2^3	8.7101e-04	9.9042e-05	1.0135e-04	1.0790e-03	6.4222e-05	9.1560e+05
2^4	1.0992e-04	1.1565e-05	1.3172e-05	1.3605e-04	8.5005e-06	5.3505e+14
2^5	1.3781e-05	1.3984e-06	1.6749e-06	1.7055e-05	1.0896e-06	1.2874e+33
Ratio	7.9766e+00	8.2701e+00	7.8643e+00	7.9770e+00	7.8017e+00	-

5. Concluding remarks. In the present analysis of the decoupling of index-μ IAEs, we have assumed that (1.1) is index-μ tractable and ν-smoothing with $\nu = 1$. We shall deal with the case $\nu \geq 2$ in the sequel (Part II) to this paper. There we will also present the extension of our analysis to higher-index IAE systems.

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REFERENCES

[1] K.E. Brenan, S.L. Campbell, and L.R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, Classics in Appl. Math. 14, SIAM, Philadelphia, 1996.

[2] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge University Press, Cambridge, UK, 2004.

[3] S.L. Campbell and C.W. Gear, *The index of general nonlinear DAEs*, Numer. Math., 72 (1995), pp. 173–196.

[4] V.F. Chistyakov, *On some properties of systems of Volterra integral equations of the fourth kind with kernel of convolution type*, Math. Notes, 80 (2006), pp. 109–113.

[5] C.W. Gear, *Differential algebraic equations, indices, and integral algebraic equations*, SIAM J. Numer. Anal., 27 (1990), pp. 1527–1534.

[6] C.W. Gear and L.R. Petzold, *ODE methods for the solution of differential/algebraic systems*, SIAM J. Numer. Anal., 21 (1984), pp. 716–728.

[7] E. Griepentrog and R. März, *Differential-Algebraic Equations and Their Numerical Treatment*, Teubner Texts in Math. 88, B.G. Teubner, Leipzig, 1986.

[8] E. Griepentrog and R. März, *Basic properties of some differential-algebraic equations*, Z. Anal. Anwendungen, 8 (1989), pp. 25–41.

[9] M. Hadizadeh, F. Ghoreishi, and S. Pishbin, *Jacobi spectral solution for integral algebraic equations of index-*2, Appl. Numer. Math., 61 (2011), pp. 131–148.

[10] E. Hairer, Ch. Lubich, and M. Roche, *The Numerical Solution of Differential-Algebraic Systems by Runge-Kutta Methods*, Lecture Notes in Math. 1409, Springer-Verlag, Berlin, Heidelberg, 1989.

[11] J.P. Kauthen, *The numerical solution of integral-algebraic equations of index* 1 *by polynomial spline collocation methods*, Math. Comp., 70 (2000), pp. 1503–1514.

[12] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations: Analysis and Numerical Solution*, EMS Textbooks in Math., European Mathematical Society (EMS), Zürich, 2006.

[13] P.K. Lamm, *A survey of regularization methods for first-kind Volterra equations*, in Surveys on Solution Methods for Inverse Problems, D. Colton, H.W. Engl, H.K. Louis, J.R. McLaughlin, and W. Rundell, eds., Springer, Vienna, 2000, pp. 53–82.

[14] R. Lamour, R. März, and C. Tischendorf, *Differential-Algebraic Equations: A Projector Based Analysis*, Springer, Heidelberg, 2013.

[15] R. Lamour and R. März, *Detecting structures in differential algebraic equations: Computational aspects*, J. Comput. Appl. Math., 236 (2012), pp. 4055–4066.

[16] R. März, *Numerical methods for differential-algebraic equations*, Acta Numer., 1 (1991), pp. 141–198.

[17] R. März, *The index of linear differential-algebraic equations with properly stated leading terms*, Results Math., 42 (2002), pp. 308–338.

[18] R. März, *Differential algebraic equations anew*, Appl. Numer. Math., 42 (2002), pp. 315–335.

[19] R. März, *Solvability of linear differential algebraic equations with properly stated leading terms*, Results Math., 45 (2004), pp. 88–105.

[20] R. März, *Fine decouplings of regular differential algebraic equations*, Results Math., 46 (2004), pp. 57–72.

[21] P.J. Rabier and W.C. Rheinboldt, *Theoretical and numerical analysis of differential-algebraic equations*, in Handbook of Numerical Analysis, Vol. VIII, P.G. Ciarlet and J.L. Lions, eds., North–Holland, Amsterdam, 2002, pp. 183–540.