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PARALLEL IN TIME ALGORITHM WITH SPECTRAL-SUBDOMAIN ENHANCEMENT FOR VOLTERRA INTEGRAL EQUATIONS*

XIANJUAN LI†, TAO TANG‡, AND CHUANJU XU§

Abstract. This paper proposes a parallel in time (also called time parareal) method to solve Volterra integral equations of the second kind. The parallel in time approach follows the spirit of the domain decomposition that consists of breaking the domain of computation into subdomains and solving iteratively the subproblems in a parallel way. To obtain a high order of accuracy, a spectral collocation accuracy enhancement in subdomains will be employed. Our main contributions in this work are twofold: (i) A time parareal method is designed for the integral equations, which to our knowledge is the first of its kind. The new method is an iterative process combining a coarse prediction in the whole domain with fine corrections in subdomains by using spectral approximation, leading to an algorithm of very high accuracy. (ii) A rigorous convergence analysis of the overall method is provided. The numerical experiment confirms that the overall computational cost is considerably reduced while the desired spectral rate of convergence can be obtained.

Key words. time parareal, spectral collocation, Volterra integral equations

AMS subject classifications. 35Q99, 35R35, 65M12, 65M70

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1. Introduction.

We consider the linear Volterra integral equations (VIEs) of the second kind,

$$u(t) - \int_0^t K(t,s)u(s)\mathrm{d}s = g(t) \qquad \forall t \in I := [0,T], \tag{1.1}$$

where g and K are sufficiently smooth in I and $I \times I$, respectively, and $K(t,t) \neq 0$ for all $t \in I$. Under these assumptions, smooth solution $u(t)$ to (1.1) exists and is unique; see, e.g., [3].

The presence of the integral in (1.1) makes the problem *globally* time-dependent. This means that the solution at time t_k depends on the solutions at all previous time $t < t_k$. Consequently, it may require large storage if the solution time is large or high accuracy of approximations is needed. This may become more serious if partial integro-differential equations are considered. To handle this, we will design our method by breaking the domain into subdomains (as in the domain decomposition approach) and then solving iteratively the subproblems in a parallel way. More precisely, we divide the time interval $[0,T]$ into N equispaced subintervals and then break the original problem into a series of independent problems on the small subintervals.

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†College of Mathematics and Computer Science, Fuzhou University, Fuzhou, 350108 China (xjli_math@yahoo.com.cn). The research of this author was partially supported by NSF of China (11201077), NSF of Fujian Province (2012J01007), and a start-up fund of Fuzhou University (0460022456).

‡Department of Mathematics and Institute for Computational and Theoretical Studies, Hong Kong Baptist University, Kowloon Tong, Hong Kong (ttang@math.hkbu.edu.hk). This author was partially supported by Hong Kong Research Grant Council GIF grants and Hong Kong Baptist University FRG grants.

§Corresponding author. School of Mathematical Sciences, Xiamen University, 361005 Xiamen, China (cjxu@xmu.edu.cn). This author was partially supported by NSF of China (grants 11071203 and 91130002).

These independent problems are solved by a fine approximation which can be implemented in a parallel way, together with some coarse grid approximations which have to be implemented in a sequential way.

The parallel in time algorithm for a model ordinary differential equation (ODE) was initially introduced by Lions, Maday, and Turinici [14] for solving evolution problems in parallel. It can be interpreted as a predictor-corrector scheme [1, 2], which involves a prediction step based on a coarse approximation and a correction step computed in parallel based on a fine approximation. Even though the time direction seems intrinsically sequential, the combination of a coarse and a fine solution procedure has been proven to allow for more rapid (convergent) solutions if parallel architectures are available. The parallel in time algorithm has received considerable attention over the years, especially in the communities of domain decomposition methods [6], fluid and structure problems [7], Navier–Stokes equations [8], and quantum control problems [15]. For the parallel in time method based on the finite difference scheme, the convergence analysis for an ODE problem was given in [9].

In our algorithm, we also build in a recent spectral method approach to obtain exponential rate of convergence; see [5, 19, 20]. The main advantage of using high order methods for integral equations is their low storage requirement with the desired precision; this advantage makes high order methods attractive. Among the high order methods, spectral methods are known to be very useful for their exponential rate of convergence, which is also demonstrated for solving VIEs [18, 13]. However, a drawback of the spectral method is also well known, i.e., the matrix associated with the spectral method is full and the computational cost grows more quickly than that of a low order method. Thus for long integration it is desirable to combine the spectral method with domain decomposition techniques to avoid using a single high-degree polynomial.

It is noted that there has been great interest in studying the parabolic integro-differential equations, in particular, the study of discontinuous-Galerkin methods; see, e.g., [10, 17]. These studies are very relevant to the present study and it will be interesting to extend the present study to parabolic integro-differential equations. Another class of relevant problems is about space-time fractional diffusion equations, which have been extensively studied [11, 12]. The problems mentioned above also contain a weakly singular kernel in the memory term, which can provide extra difficulties in convergence analysis; see, e.g., the recent work on the spectral methods for VIEs with a weakly singular kernel [5, 13].

This paper is the first attempt to approximate solutions of the VIEs by the parallel in time method. In addition, this article will provide a full convergence analysis for the proposed method. We will also demonstrate numerically the efficiency and the convergence of the proposed method on some sample problems.

This paper is organized as follows. In section 2, we construct the parallel in time method based on the spectral collocation scheme for the underlying equation and outline the main results. Some lemmas useful for the convergence analysis are provided in section 3. The convergence analysis for the proposed method in L^∞ under some assumptions is given in section 4. Numerical experiments are carried out in section 5. In the final section we analyze the parallelism efficiency of the overall algorithm and describe some implementation details.

2. Outline of the parallel in time method and the main results. The time interval $I = [0, T]$ is first partitioned into N subintervals, determined by the grid points $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ with $t_n = n\Delta t, \Delta t = T/N$. We denote this partition by $I = \cup_{n=1}^{N} I_n$ with $I_n = [t_{n-1}, t_n]$.

2.1. Outline of the method. In the obvious way, problem (1.1) is equivalent to the following system of N integral equations:

$$u_n(t) - K_{nn}u_n(t) = g(t) + \sum_{j=1}^{n-1} K_{nj}u_j(t) \quad \text{on } I_n = [t_{n-1}, t_n],\ 1 \le n \le N, \tag{2.1}$$

with $u_j = u|_{I_j}$,

$$K_{nn}u_n(t) := \int_{t_{n-1}}^{t} K(t,s)u_n(s)ds \quad \text{and}$$

$$K_{nj}u_j(t) := \int_{t_{j-1}}^{t_j} K(t,s)u_j(s)ds, \quad 1 \le j \le n-1. \tag{2.2}$$

The solution of (2.1) can be expressed through the solution operators $\mathcal{S}_n$ as follows:

$$u_n(t) = \mathcal{S}_n(t; u_1, \ldots, u_{n-1}) \quad \text{for } t \in I_n. \tag{2.3}$$

Likewise, at the discrete level, we have the solution operators $\mathcal{F}_n$ for the collocation method to be described later:

$$U_n(t) = \mathcal{F}_n(t; U_1, \ldots, U_{n-1}) \quad \text{for } t \in I_n \tag{2.4}$$

with U_n a polynomial of degree M. The main idea of the proposed parallel in time method is to take a second family of discrete solution operators, $\mathcal{G}_n$, defined in the same way using polynomials of a lower degree $\tilde{M} < M$, and to set up the iteration

$$U_n^k(t) = \mathcal{G}_n(t; U_1^k, \ldots, U_{n-1}^k) + \mathcal{C}_n(t; U_1^{k-1}, \ldots, U_{n-1}^{k-1}), \quad k \ge 1, \tag{2.5}$$

with the initial value

$$U_n^0 = \mathcal{G}_n(t; U_1^0, \ldots, U_{n-1}^0),$$

where the correction term in (2.5) is defined by

$$\mathcal{C}_n(t; U_1^{k-1}, \ldots, U_{n-1}^{k-1}) := \mathcal{F}_n(t; U_1^{k-1}, \ldots, U_{n-1}^{k-1}) - \mathcal{G}_n(t; U_1^{k-1}, \ldots, U_{n-1}^{k-1}). \tag{2.6}$$

Here, the key is that the $\mathcal{C}_n$ can be computed in parallel for $1 \le n \le N$. The term $\mathcal{G}_n$ in (2.5) must be computed sequentially, but this is relatively cheap if $\tilde{M}$ is small compared to M. Thus, provided the iteration converges rapidly, we can use multiple processors to obtain a high-accuracy solution in a small multiple of the time needed to compute a low-accuracy (sequential) solution.

The way of presenting the parallel in time method places it in the category of the predictor-corrector scheme, where the predictor is $\mathcal{G}_n(t; U_1^k, \ldots, U_{n-1}^k)$, while the corrector is $\mathcal{C}_n(t; U_1^{k-1}, \ldots, U_{n-1}^{k-1})$.

2.2. Outline of the main results. The main result of this paper will be the theoretical proof of the error estimate

$$\|u_n - U_n^k\|_\infty = O(M^{3/4-m} + (\tilde{M}/2)^{(3/4-m)(k+1)}), \tag{2.7}$$

which will be confirmed by a number of numerical experiments. Roughly speaking, the numerical approximation U_n^k is accurate to the exact solution with an error $O(M^{3/4-m})$ if $M \approx (\tilde{M}/2)^{k+1}$.

We emphasize that the analysis covers a fully discrete scheme in which quadratures are used to approximate the integrals that occur in the coefficients and right-hand side of the resulting linear system that must be solved at the nth interval to compute U_n^k.

2.3. The operators $\mathcal{F}_n$ and $\mathcal{G}_n$. We begin by discussing the operator $\mathcal{F}_n$ defined in (2.4). In this work, we will use a *spectral collocation method* to define this approximation. Define $\mathcal{P}_M(I_n)$ as the polynomial space of degree less than or equal to M with $I_n = [t_{n-1}, t_n]$. Denote by $L_M(x)$ the Legendre polynomial of degree M. Let x_i be the points of the Legendre–Gauss (LG) quadrature formula, defined by $L_{M+1}(x_i) = 0, i = 0, \ldots, M$, arranged by increasing order. The associated weights of the LG quadrature formula are denoted by ω_i, $0 \leq i \leq M$. Then the following identity is well known:

$$\int_{-1}^{1} \varphi(x)\mathrm{d}x = \sum_{i=0}^{M} \varphi(x_i)\omega_i \quad \forall \varphi \in \mathcal{P}_{2M+1}(-1,1).$$

The discrete L^2-inner product associated to the LG quadrature is denoted by

$$(\varphi, \psi)_M := \sum_{i=0}^{M} \varphi(x_i)\psi(x_i)\omega_i. \tag{2.8}$$

Furthermore, define the LG-points $\{\xi_n^i\}_{i=0}^M$ on the element $I_n = [t_{n-1}, t_n]$, i.e.,

$$\xi_n^i = \frac{t_n - t_{n-1}}{2} x_i + \frac{t_n + t_{n-1}}{2}, \quad 0 \leq i \leq M, \quad 1 \leq n \leq N,$$

and the corresponding weights $\rho_n^i = \Delta t \omega_i / 2$. Then it holds that

$$\int_{I_n} \varphi(x)\mathrm{d}x = \sum_{i=0}^{M} \varphi(\xi_n^i)\rho_n^i \quad \forall \varphi \in \mathcal{P}_{2M+1}(I_n), \ \ 1 \leq n \leq N.$$

We now use the Legendre-collocation method to determine the operator $\mathcal{F}_n$ in (2.4). More precisely, we want to find $U_n(t) \in \mathcal{P}_M(I_n)$ with $1 \leq n \leq N$ such that for all $0 \leq i \leq M$,

$$U_n(\xi_n^i) - \int_{t_{n-1}}^{\xi_n^i} K(\xi_n^i, s)U_n(s)\mathrm{d}s = g(\xi_n^i) + \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} K(\xi_n^i, s)U_j(s)\mathrm{d}s, \ \ 1 \leq n \leq N. \tag{2.9}$$

In the implementation, the integral terms on the left-hand sides of (2.9) are evaluated by using the following Gauss quadrature:

$$\int_{t_{n-1}}^{\xi_n^i} K(\xi_n^i, s)U_n(s)\mathrm{d}s = \int_{-1}^{1} \bar{K}(\xi_n^i, s_n^i)U_n(s_n^i)\mathrm{d}x \approx \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right)_M, \tag{2.10}$$

where $s_n^i := s_n^i(x) := \frac{\xi_n^i - t_{n-1}}{2} x + \frac{\xi_n^i + t_{n-1}}{2}, -1 < x < 1$, and

$$\bar{K}(\xi_n^i, s_n^i) := \left(\frac{\xi_n^i - t_{n-1}}{2}\right) K(\xi_n^i, s_n^i). \tag{2.11}$$

The right-hand sides of (2.9) are computed in a similar way:

$$\int_{t_{j-1}}^{t_j} K(\xi_n^i, s)U_j(s)\mathrm{d}s = \frac{\Delta t}{2} \int_{-1}^{1} K(\xi_n^i, s_j)U_j(s_j)\mathrm{d}x \approx \frac{\Delta t}{2} \left(K(\xi_n^i, s_j), U_j(s_j)\right)_M, \tag{2.12}$$

where

$$s_j = s_j(x) = \frac{\Delta t}{2}x + \frac{t_j + t_{j-1}}{2}, \quad -1 < x < 1. \tag{2.13}$$

With the help of the above numerical integrations, (2.9) is further approximated by

(2.14)

$0 \le i \le M$,

$$U_n(\xi_n^i) - \big(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\big)_M = g(\xi_n^i) + \frac{\triangle t}{2}\sum_{j=1}^{n-1}\big(K(\xi_n^i, s_j), U_j(s_j)\big)_M, \quad 1 \le n \le N.$$

The above linear system defines the fine operator $\mathcal{F}_n$ in (2.4).

The operator $\mathcal{G}_n$ is defined by the same spectral collocation method but with the degree $\tilde{M}$, which is much less than M.

Let $\{\eta_n^i\}_{i=0}^{\tilde{M}}$ be a set of LG points on the element $[t_{n-1}, t_n]$ corresponding to the weight $\{\varrho_n^i\}_{i=0}^{\tilde{M}}$. Then the coarse operator $\mathcal{G}_n(t, V_1, \dots, V_{n-1})$ in (2.5) consists in finding $V_n(t) \in \mathcal{P}_{\tilde{M}}(I_n)$ such that for all $0 \le i \le \tilde{M}$

(2.15)

$$V_n(\eta_n^i) - \big(\bar{K}(\eta_n^i, \tau_n^i), V_n(\tau_n^i)\big)_{\tilde{M}} = g(\eta_n^i) + \frac{\Delta t}{2}\sum_{j=1}^{n-1}\big(K(\eta_n^i, s_j), V_j(s_j)\big)_{\tilde{M}}, \quad 1 \le n \le N,$$

where $\tau_n^i := \tau_n^i(x) := \frac{\eta_n^i - t_{n-1}}{2}x + \frac{\eta_n^i + t_{n-1}}{2}$, $-1 < x < 1$, and $\bar{K}$ and s_j are respectively defined in (2.11) and (2.13).

3. Useful lemmas. We first introduce some notation. Let Λ be an arbitrary bounded interval. For nonnegative integer m, $H^m(\Lambda)$ stands for the standard Sobolev space equipped with the norm and seminorm $\|v\|_{m,\Lambda}$ and $|v|_{k,\Lambda}, 0 \le k \le m$. Particularly $L^2(\Lambda) = H^0(\Lambda)$, equipped with the standard L^2-inner product and norm. Similarly, the norm of the space $W^{m,\infty}(\Lambda)$ is denoted by $\|v\|_{W^{m,\infty}(\Lambda)}$ or $\|v\|_{\infty,\Lambda}$ if $m = 0$. The error analysis needs the following seminorm:

$$|v|_{H^{m;M}(\Lambda)} = \left(\sum_{k=\min(m,M+1)}^{m} \|\partial_x^{(k)} v\|_\Lambda^2\right)^{\frac{1}{2}}.$$

Hereafter, in cases where no confusion would arise, the domain symbol Λ may be dropped from the notation. We denote by c generic positive constants independent of the discretization parameter but which may depend on the kernel function $K(\cdot,\cdot)$ or T.

We further introduce two approximation operators. First, we define the Lagrange interpolation operator $\mathcal{I}_\Lambda^M : \mathcal{C}(\Lambda) \to \mathcal{P}_M(\Lambda)$ by the following: for all $v \in \mathcal{C}(\Lambda), \mathcal{I}_\Lambda^M v \in \mathcal{P}_M(\Lambda)$, such that

$$\mathcal{I}_\Lambda^M v(z_i) = v(z_i), \quad 0 \le i \le M,$$

where $\{z_i\}_{i=0}^M$ is the set of LG points on the interval Λ. This polynomial can be expressed as

$$\mathcal{I}_\Lambda^M v(x) = \sum_{i=0}^{M} v(z_i) h_\Lambda^i(x),$$

where h_Λ^i is the Lagrange interpolation basis function associated with $\{z_i\}_{i=0}^M$. Particularly, we use $\mathcal{I}_n^M$ (resp., $h_n^i(x)$) to replace $\mathcal{I}_\Lambda^M$ (resp., $h_\Lambda^i(x)$) when $\Lambda = I_n$.

Then for all $v \in H^m(\Lambda), m \geq 1$, the following optimal error estimates hold (see, e.g., [4, 18]):

$$\|v - \mathcal{I}_\Lambda^M v\| \lesssim M^{-m}|v|_{H^{m;M}}, \tag{3.1}$$

$$\|v - \mathcal{I}_\Lambda^M v\|_\infty \lesssim M^{\frac{3}{4}-m}|v|_{H^{m;M}}. \tag{3.2}$$

For the discrete L^2-inner product defined in (2.8), it holds the following error estimate [4]: for all $\phi \in \mathcal{P}_M(\Lambda)$,

$$|(v,\phi) - (v,\phi)_M| \lesssim M^{-m}|v|_{H^{m;M}}\|\phi\|, \quad v \in H^m(\Lambda), \quad m \geq 1. \tag{3.3}$$

The following result gives the Lebesgue constant for the Lagrange interpolation polynomials associated with the LG-points.

Lemma 3.1 (p. 329, equation (9) in [16]). *Let $\{h_\Lambda^j(x)\}_{j=0}^M$ be the Lagrange interpolation basis associated with the $M+1$ LG-points on the interval Λ. Then*

$$\|I_\Lambda^M\|_\infty := \max_{x\in\Lambda} \sum_{j=0}^M |h_\Lambda^j(x)| \lesssim \sqrt{M}. \tag{3.4}$$

Using the standard Gronwall inequality, we obtain the following estimate for the VIEs.

Lemma 3.2. *Consider the following Volterra equation:*

$$u(t) = v(t) + \int_0^t K(t,s)u(s)\mathrm{d}s. \tag{3.5}$$

If $v \in W^{m,\infty}$ and $K(t,s)$ is sufficiently smooth, then

$$\|u\|_{W^{m,\infty}} \leq c2^m \, \|v\|_{W^{m,\infty}} \,\|. \tag{3.6}$$

Proof. It is known (see, e.g., [3]) that there exists a smooth *resolvent* kernel $R(t,s)$ depending only on $K(t,s)$ such that the solution of (3.6) satisfies

$$u(t) = v(t) + \int_0^t R(t,s)v(s)ds.$$

Using the transformation $\tau = s/t$ leads to

$$u(t) = v(t) + \int_0^1 tR(t,t\tau)v(t\tau)d\tau.$$

Differentiating the above equation with respect to t m-times gives

$$u^{(m)}(t) = v^{(m)}(t) + \sum_{j=0}^m C_m^j \int_0^1 \partial_t^{(m-j)}(tR(t,t\tau))v^{(j)}(t\tau)\tau^j \, d\tau,$$

which leads to

$$\|u^{(m)}\| \leq \|v^{(m)}\| + c\sum_{j=0}^m C_m^j \int_0^1 \tau^j d\tau \|v^{(m)}\| \leq c2^m\|v^{(m)}\|,$$

where the constant c depends on the regularity of R or equivalently the regularity of K. This completes the proof of the lemma. ∎

We also need the following discrete Gronwall lemma (see, e.g., [3]).

LEMMA 3.3. *Assume that ϕ_n, p_n, and k_n are three sequences, k_n is nonnegative, and they satisfy*

$$\begin{cases} \phi_0 \le g_0, \\ \phi_n \le g_0 + \sum\limits_{s=0}^{n-1} p_s + \sum\limits_{s=0}^{n-1} k_s \phi_s, \quad n \ge 1. \end{cases}$$

Then it holds that

$$\begin{cases} \phi_1 \le g_0(1+k_0) + p_0, \\ \phi_n \le g_0 \prod\limits_{s=0}^{n-1}(1+k_s) + \sum\limits_{s=0}^{n-2} p_s \prod\limits_{\tau=s+1}^{n-1}(1+k_r) + p_{n-1}, \quad n \ge 2. \end{cases}$$

Moreover, if $g_0 \ge 0$ and $p_n \ge 0$ for all $n \ge 0$, then it holds that

$$\phi_n \le \left(g_0 + \sum_{s=0}^{n-1} p_s \right) \exp\left(\sum_{s=0}^{n-1} k_s \right), \quad n \ge 1.$$

4. Stability and convergence analysis.

This section is devoted to the stability and convergence analysis of the proposed parallel in time scheme.

4.1. Stability.

LEMMA 4.1. *For $1 \le n \le N$, let $\mathcal{F}_n$ be the fine approximation operator defined by the spectral collocation scheme* (2.14), *$\mathcal{G}_n$ defined by* (2.15), *$g_M = \mathcal{I}_n^M g$, and $\{\psi_i\}_{i=1}^{n-1}$ a polynomial sequence. Then for sufficiently large M and $\tilde{M}$ we have*

$$\|\mathcal{F}_n(t;\psi_1,\dots,\psi_{n-1})\|_\infty \le c\Big(\|g_M\|_\infty + \Delta t \sum_{j=1}^{n-1} \|\psi_j\|_\infty\Big), \tag{4.1}$$

$$\|\mathcal{G}_n(t;\psi_1,\dots,\psi_{n-1})\|_\infty \le c\Big(\|g_{\tilde{M}}\|_\infty + \Delta t \sum_{j=1}^{n-1} \|\psi_j\|_\infty\Big), \quad n \ge 1, \tag{4.2}$$

where it is understood that $\mathcal{F}_1(t;\psi_0) := \mathcal{F}_1(t)$ and $\mathcal{G}_1(t;\psi_0) := \mathcal{G}_1(t)$.

Proof. Since the only difference between $\mathcal{F}_n$ and $\mathcal{G}_n$ is the polynomial degree, we only need to prove (4.1). Let $\psi_n := \mathcal{F}_n(t;\psi_1,\dots,\psi_{n-1})$. It follows from the spectral collocation scheme (2.14) that

$$\psi_n(\xi_n^i) = g(\xi_n^i) + \left(\bar{K}(\xi_n^i, s_n^i), \psi_n(s_n^i)\right)_M + \frac{\Delta t}{2} \sum_{j=0}^{n-1} \left(K(\xi_n^i, s_j), \psi_j(s_j)\right)_M, \ 0 \le i \le M, \tag{4.3}$$

which can be further reorganized as

$$\psi_n(\xi_n^i) = g(\xi_n^i) + \left(\bar{K}(\xi_n^i, s_n^i), \psi_n(s_n^i)\right) + A_n^i + \frac{\Delta t}{2} \sum_{j=1}^{n-1} \left(K(\xi_n^i, s_j), \psi_j(s_j)\right) + \sum_{j=1}^{n-1} A_j^i, \tag{4.4}$$

where

$$\begin{aligned} A_n^i &= -\left(\bar{K}(\xi_n^i, s_n^i), \psi_n(s_n^i)\right) + \left(\bar{K}(\xi_n^i, s_n^i), \psi_n(s_n^i)\right)_M, \\ A_j^i &= -\frac{\Delta t}{2}\left(K(\xi_n^i, s_j), \psi_j(s_j)\right) + \frac{\Delta t}{2}\left(K(\xi_n^i, s_j), \psi_j(s_j)\right)_M, \quad 1 \le j \le n-1. \end{aligned}$$

Multiplying both sides of (4.4) by the Lagrange basis function h_n^i and summing up from $i=0$ to $i=M$, we obtain

$$\begin{aligned}\psi_n(t) = \mathcal{I}_n^M g + \mathcal{I}_n^M \int_{t_{n-1}}^t K(t,s)\psi_n(s)\mathrm{d}s + \sum_{i=0}^M A_n^i h_n^i(t) \\ + \sum_{j=1}^{n-1}\left[\frac{\Delta t}{2}\mathcal{I}_n^M\left(K(t,s_j),\psi_j(s_j)\right) + \sum_{i=0}^M A_j^i h_j^i(t)\right].\end{aligned} \tag{4.5}$$

Consequently, we have

$$\psi_n(t) = \int_{t_{n-1}}^t K(t,s)\psi_n(s)\mathrm{d}s + \mathcal{I}_n^M g + \sum_{j=1}^n J_j + \sum_{j=1}^n I_j + \sum_{j=1}^{n-1} R_j, \tag{4.6}$$

where

$$\begin{aligned}
J_j &= \sum_{i=0}^M A_j^i h_j^i, \quad 1 \le j \le n,\\
I_n &= \mathcal{I}_n^M\left(\bar{K}(t,s_n^i),\psi_n(s_n^i)\right) - \left(\bar{K}(t,s_n^i),\psi_n(s_n^i)\right),\\
I_j &= \frac{\Delta t}{2}\mathcal{I}_n^M\left(K(t,s_j),\psi_j(s_j)\right) - \frac{\Delta t}{2}\left(K(t,s_j),\psi_j(s_j)\right), \quad 1 \le j \le n-1,\\
R_j &= \frac{\Delta t}{2}\left(K(t,s_j),\psi_j(s_j)\right), \quad 1 \le j \le n-1.
\end{aligned}$$

It follows from the standard Gronwall inequality for (4.6) that

$$\|\psi_n\|_\infty \le c\left(\|g_M\|_\infty + \sum_{j=1}^n \|J_j\|_\infty + \sum_{j=1}^n \|I_j\|_\infty + \sum_{j=1}^{n-1}\|R_j\|_\infty\right). \tag{4.7}$$

We now estimate the right-hand side of (4.7) term by term. First, we estimate the terms J_j. Using the error estimate (3.3) for the LG quadrature, we have

$$\begin{aligned}\max_{0\le i\le M}|A_n^i| &= \max_{0\le i\le M}\left|\left(\bar{K}(\xi_n^i,s_n^i),\psi_n(s_n^i)\right)_M - \int_{t_{n-1}}^{\xi_n^i} K(\xi_n^i,s)\psi_n(s)\mathrm{d}s\right|\\
&\le c\Delta t M^{-1}\max_{0\le i\le M}|K(\xi_n^i,s_n^i)|_{H^{1;M}}\max_{0\le i\le M}\|\psi_n(s_n^i)\|\\
&\le c_1\Delta t M^{-1}\|\psi_n\|_\infty,\end{aligned} \tag{4.8}$$

where c_1 depends on $|K(\xi_n^i,s_n^i(\cdot))|_{H^{1;M}}$. Similarly,

$$\max_{0\le i\le M}|A_j^i| \le c_1\Delta t M^{-1}\|\psi_j\|_\infty, \quad 1\le j\le n-1. \tag{4.9}$$

Hence, combining (4.8) and (4.9) with Lemma 3.1 gives

$$\|J_n\|_\infty = \left\|\sum_{i=0}^M A_n^i h_n^i\right\|_\infty \le \max_{0\le i\le M}|A_n^i|\max_{x\in I_n}\sum_{i=0}^M|h_n^i| \le c_1\Delta t M^{-\frac{1}{2}}\|\psi_n\|_\infty \le \frac{1}{3c}\|\psi_n\|_\infty$$

for sufficiently small Δt or large M. Following the same lines leads to

$$\|J_j\|_\infty \le c_1\Delta t M^{-\frac{1}{2}}\|\psi_j\|_\infty, \quad 1\le j\le n-1. \tag{4.10}$$

We now estimate the second term I_j. For $j = n$, applying (3.2) leads to

$$\|I_n\|_\infty = \left\|(\mathcal{I}_n^M - \mathcal{I})\int_{t_{n-1}}^t K(t,s)\psi_n(s)\mathrm{d}s\right\|_\infty \le c_1 M^{-\frac{1}{4}} \left|\int_{t_{n-1}}^t K(t,s)\psi_n(s)\mathrm{d}s\right|_{H^{1;M}}$$

$$\le c_1 M^{-\frac{1}{4}}\|\psi_n\|_\infty \le \frac{1}{3c}\|\psi_n\|_\infty. \tag{4.11}$$

For $1 \le j \le n-1$, in virtue of (3.2), we derive

$$\|I_j\|_\infty = \left\|(\mathcal{I}_n^M - \mathcal{I})\int_{t_{j-1}}^{t_j} K(t,s)\psi_j(s)\mathrm{d}s\right\|_\infty \le c_1 M^{\frac{3}{4}-1} \left|\int_{t_{j-1}}^{t_j} K(t,s)\psi_j(s)\mathrm{d}s\right|_{H^{1;M}}$$
$$\le c_1 \Delta t M^{-\frac{1}{4}}\left\|\psi_j\right\|_\infty.$$

It remains to estimate R_j. By a direct calculation, we have

$$\|R_j\|_\infty = \frac{\Delta t}{2}\left\|(K(t,s_j),\psi_j(s_j))\right\|_\infty \le c\Delta t\|\psi_j\|_\infty, \quad 1 \le j \le n-1. \tag{4.12}$$

Finally, combining (4.7)–(4.12) gives

$$\|\psi_n\|_\infty \le c\|g_M\|_\infty + \frac{1}{3}\|\psi_n\|_\infty + c\Delta t M^{-\frac{1}{2}}\sum_{j=1}^{n-1}\|\psi_j\|_\infty$$
$$+\ \frac{1}{3}\|\psi_n\|_\infty + c\Delta t M^{-\frac{1}{4}}\sum_{j=1}^{n-1}\left\|\psi_j\right\|_\infty + c\Delta t\sum_{j=1}^{n-1}\|\psi_j\|_\infty. \tag{4.13}$$

Now a simple rearrangement leads to (4.1). ∎

We are now ready to state and prove one of the main results of this paper.

THEOREM 4.1 (stability). *The iteration scheme* (2.5)–(2.6) *is stable in the sense that the solution* U_n^k *satisfies*

$$\|U_n^k\|_\infty \le c\|g\|_1 \quad \forall k \ge 0, \;\; 1 \le n \le N. \tag{4.14}$$

Proof. For $k = 0$, it follows from the initial value $U_n^0 = \mathcal{G}_n(t; U_1^0, \ldots, U_{n-1}^0)$ and Lemma 4.1 that

$$\|U_n^0\|_\infty = \left\|\mathcal{G}_n\left(t; U_1^0, \ldots, U_{n-1}^0\right)\right\|_\infty \le c_0\Big(\|g_{\tilde{M}}\|_\infty + \Delta t\sum_{j=1}^{n-1}\|U_j^0\|_\infty\Big). \tag{4.15}$$

By applying the discrete Gronwall lemma, Lemma 3.3, we get

$$\|U_n^0\|_\infty \le c_0 e^{c_0(n-1)\Delta t}\|g_{\tilde{M}}\|_\infty \le c_0 e^{c_0 T}\|g\|_1. \tag{4.16}$$

For $k \ge 1$, according to the iteration scheme (2.5)–(2.6), we have

$$\|U_n^k\|_\infty \le \left\|\mathcal{G}_n\left(t; U_1^k, U_2^k, \ldots, U_{n-1}^k\right)\right\|_\infty$$
$$+\left\|\mathcal{F}_n\left(t; U_1^{k-1}, U_2^{k-1}, \ldots, U_{n-1}^{k-1}\right)\right\|_\infty + \left\|\mathcal{G}_n\left(t; U_1^{k-1}, U_2^{k-1}, \ldots, U_{n-1}^{k-1}\right)\right\|_\infty. \tag{4.17}$$

Applying Lemma 4.1 to the right-hand side of (4.17) yields

$$\|U_n^k\|_\infty \le c\Big(\|g_{\tilde{M}}\|_\infty + \Delta t\sum_{j=1}^{n-1}\|U_j^k\|_\infty\Big) + c\Big(\|g_{\tilde{M}}\|_\infty + \|g_M\|_\infty + \Delta t\sum_{j=1}^{n-1}\|U_j^{k-1}\|_\infty\Big)$$

$$\le c_0\Delta t\sum_{j=1}^{n-1}\|U_j^k\|_\infty + c\Big(\|g\|_1 + \Delta t\sum_{j=1}^{n-1}\|U_j^{k-1}\|_\infty\Big). \tag{4.18}$$

In the following, we will derive the inequality

$$\|U_n^k\|_\infty \le c_0 \sum_{l=0}^{k} \frac{e^{c_0(l+1)(n-1)\Delta t}}{l!} \|g\|_1 \quad \forall k \ge 0, 1 \le n \le N, \tag{4.19}$$

where c_0 is a constant independent of k. We do this by induction. First, it follows from (4.16) that (4.19) is true for $k = 0$. We now show that if (4.19) holds for a given k, then it also holds for $k+1$. It follows from (4.18) (replace k by $k+1$) and the discrete Gronwall lemma, Lemma 3.3, that

$$\|U_n^{k+1}\|_\infty \le c_0 e^{c_0(n-1)\Delta t} \left(\|g\|_1 + \Delta t \sum_{j=1}^{n-1} \|U_j^k\|_\infty \right). \tag{4.20}$$

Then using the induction assumption (4.19) gives

$$\begin{aligned}
\|U_n^{k+1}\|_\infty &\le c_0 \|g\|_1 e^{c_0(n-1)\Delta t} \left(1 + c_0 \Delta t \sum_{l=0}^{k} \frac{e^{c_0(l+1)(n-2)\Delta t}}{l!} + \cdots + c_0 \Delta t \sum_{l=0}^{k} \frac{1}{l!} \right) \\
&= c_0 \|g\|_1 e^{c_0(n-1)\Delta t} \left(1 + c_0 \Delta t \sum_{l=0}^{k} \frac{1}{l!} \frac{e^{c_0(l+1)(n-1)\Delta t} - 1}{e^{c_0(l+1)\Delta t} - 1} \right) \\
&\le c_0 \|g\|_1 e^{c_0(n-1)\Delta t} \left(1 + \sum_{l=0}^{k} \frac{1}{(l+1)!} (e^{c_0(l+1)(n-1)\Delta t} - 1) \right) \\
&\le c_0 \|g\|_1 e^{c_0(n-1)\Delta t} \left(1 + \sum_{l=0}^{k} \frac{1}{(l+1)!} e^{c_0(l+1)(n-1)\Delta t} \right) \\
&\le c_0 \|g\|_1 \sum_{l=0}^{k+1} \frac{e^{c_0(l+1)(n-1)\Delta t}}{l!}.
\end{aligned}$$

This implies that (4.19) also holds for k being replaced by $k+1$. Finally, by noticing the fact that

$$\sum_{l=0}^{\infty} \frac{e^{c_0(l+1)(n-1)\Delta t}}{l!} \le e^{c_0(n-1)\Delta t} e^{e^{c_0(n-1)\Delta t}} \le e^{c_0 T} e^{e^{c_0 T}}, \quad 1 \le n \le N, \tag{4.21}$$

the desired result (4.14) follows from (4.19). ∎

Remark. It is seen from (4.19) and (4.21) that the generic constant c in (4.14) has a rapid, double-exponential growth in T. Theoretically, this may become very large. On the other hand, our numerical experiments (see section 5) can take $T = 100$ without any apparent problem. It may be possible to obtain a better generic constant in (4.14), even by assuming something more on the kernel function $K(t, s)$. We believe that other proof techniques are needed to improve this double-exponential growth constant, and this is certainly an interesting theoretical challenge.

4.2. Convergence. We now provide an estimate for the fine approximation operator $\mathcal{F}_n$.

LEMMA 4.2. *Let u_n be the solution of* (2.1) *and U_n be the solution of* (2.4). *If $u \in H^m(I), m \ge 1$, then we have*

$$\|u_n - U_n\|_\infty \le c \left(M^{\frac{3}{4}-m} |u|_{H^{m;M}} + M^{\frac{1}{2}-m} \|u\|_\infty \right), \quad 1 \le n \le N, \tag{4.22}$$

provided the number of collocation points M is sufficiently large.

Proof. Replace t in (2.1) by ξ_n^i. Subtracting the resulting equation from (2.14) gives

$$(4.23)\quad u_n(\xi_n^i) - U_n(\xi_n^i) = \left(\bar{K}(\xi_n^i, s_n^i), u_n(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right)_M + \frac{\Delta t}{2}\sum_{j=1}^{n-1}\Big[\left(K(\xi_n^i, s_j), u_j(s_j)\right) - \left(K(\xi_n^i, s_j), U_j(s_j)\right)_M\Big].$$

It can be further reorganized as

$$u_n(\xi_n^i) - U_n(\xi_n^i) = \left(\bar{K}(\xi_n^i, s_n^i), e_n(s_n^i)\right) + A_n^i + \frac{\Delta t}{2}\sum_{j=1}^{n-1}\Big[\left(K(\xi_n^i, s_j), e_j(s_j)\right) + A_j^i\Big],$$

where $e_j := u_j - U_j$,

$$A_n^i = \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right)_M,$$
$$A_j^i = \frac{\Delta t}{2}\left(K(\xi_n^i, s_j), U_j(s_j)\right) - \frac{\Delta t}{2}\left(K(\xi_n^i, s_j), U_j(s_j)\right)_M, \quad 1 \le j \le n-1.$$

Following the same procedure as in the proof of Lemma 4.1 gives

$$\mathcal{I}_n^M u_n - U_n = \mathcal{I}_n^M\left(\bar{K}(\xi_n^i, s_n^i), e_n(s_n^i)\right) + \sum_{i=0}^{M} A_n^i h_n^i + \sum_{j=1}^{n-1}\left[\frac{\Delta t}{2}\mathcal{I}_n^M\left(K(\xi_n^i, s_j), e_j(s_j)\right) + \sum_{i=0}^{M} A_j^i h_j^i\right].$$

Consequently,

$$u_n - U_n = \int_{t_{n-1}}^{\xi_n^i} K(\xi_n^i, s)e_n(s)\mathrm{d}s + u_n - \mathcal{I}_n^M u_n + \sum_{j=1}^{n} J_j + \sum_{j=1}^{n} I_j + \sum_{j=1}^{n-1} R_j,$$

where

$$J_j = \sum_{i=0}^{M} A_j^i h_j^i, \quad 1 \le j \le n; \quad R_j = \frac{\Delta t}{2}\left(K(\xi_n^i, s_j), e_j\right), \quad 1 \le j \le n-1,$$
$$I_n = \mathcal{I}_n^M\left(\bar{K}(\xi_n^i, s_n^i), e_n(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), e_n(s_n^i)\right),$$
$$I_j = \frac{\Delta t}{2}\mathcal{I}_n^M\left(K(\xi_n^i, s_j), e_j(s_j)\right) - \frac{\Delta t}{2}\left(K(\xi_n^i, s_j), e_j(s_j)\right), \quad 1 \le j \le n-1.$$

Applying the Gronwall lemma, Lemma 3.3, gives

$$(4.24)\quad \|u_n - U_n\|_\infty \le c\left(\left\|\mathcal{I}_n^M u_n - u_n\right\|_\infty + \sum_{j=1}^{n}\|J_j\|_\infty + \sum_{j=1}^{n}\|I_j\|_\infty + \sum_{j=1}^{n-1}\|R_j\|_\infty\right).$$

We now estimate the right-hand side of (4.24) term by term. First, it follows from the inequality (3.2) that

$$(4.25)\quad \left\|\mathcal{I}_n^M u_n - u_n\right\|_\infty \le cM^{\frac{3}{4}-m}|u_n|_{H^{m;M}}.$$

Then by using a similar technique as in the proof of Lemma 4.1, we can estimate $\|I_j\|_\infty$ and $\|J_j\|_\infty$ as follows. For $j = n$ and sufficiently large M,

$$\begin{aligned}(4.26)\qquad \|J_n\|_\infty &= \left\|\sum_{i=0}^{M} A_n^i h_n^i\right\|_\infty \\ &\le c_1 \Delta t M^{\frac{1}{2}-m}\Big(\|e_n\|_\infty + \|u_n\|_\infty\Big) \le \frac{1}{3c}\|e_n\|_\infty + c\Delta t M^{\frac{1}{2}-m}\|u_n\|_\infty,\end{aligned}$$

where, as in Lemma 4.1, c_1, and therefore c, depends on $|K(\xi_n^i, s_n^i(\cdot))|_{H^{m;M}}$. Similarly, we have

$$(4.27)\ \|I_n\|_\infty = \left\|\mathcal{I}_n^M\left(\bar K(\xi_n^i, s_n^i), e_n(s_n^i)\right) - \left(\bar K(\xi_n^i, s_n^i), e_n(s_n^i)\right)\right\|_\infty \le \frac{1}{3c}\|e_n\|_\infty.$$

For $1 \le j \le n-1$, we can further have

$$(4.28\text{a})\quad \|J_j\|_\infty = \left\|\sum_{i=0}^{M} A_j^i h_j^i\right\|_\infty \le c\Delta t M^{\frac{1}{2}-m}\Big(\|e_j\|_\infty + \|u_j\|_\infty\Big),$$

$$(4.28\text{b})\quad \|I_j\|_\infty = \frac{\Delta t}{2}\left\|\mathcal{I}_n^M\left(K(\xi_n^i, s_j), e_j(s_j)\right) - \left(K(\xi_n^i, s_j), e_j(s_j)\right)\right\|_\infty \le c\Delta t\|e_j\|_\infty,$$

$$(4.28\text{c})\quad \|R_j\|_\infty = \frac{\Delta t}{2}\left\|(K(t, s_j), e_j(s_j))\right\|_\infty \le c\Delta t\|e_j\|_\infty.$$

Combining (4.24)–(4.28) yields

$$\begin{aligned}\|e_n\|_\infty \le c&\left(M^{\frac{3}{4}-m}|u_n|_{H^{m;M}} + \frac{2}{3c}\|e_n\|_\infty + \Delta t M^{\frac{1}{2}-m}\|u_n\|_\infty\right)\\ &+ c\Delta t M^{\frac{1}{2}-m}\sum_{j=1}^{n-1}\|u_j\|_\infty + c\Delta t\sum_{j=1}^{n-1}\|e_j\|_\infty.\end{aligned}$$

Using the discrete Gronwall lemma gives

$$(4.29)\qquad \|e_n\|_\infty \le c\left(M^{\frac{3}{4}-m}|u|_{H^{m;M}} + n\Delta t M^{\frac{1}{2}-m}\|u\|_\infty\right)e^{c(n-1)\Delta t}.$$

Finally, the lemma is proved by observing that $n\Delta t$ and $e^{c(n-1)\Delta t}$ can be bounded by a constant depending on T. $\square$

By following the same lines as in the proof of Lemma 4.1, we can establish the continuity of the approximation operators $\mathcal{F}_n$ and $\mathcal{G}_n$.

LEMMA 4.3. *For $1 \le n \le N$, both operators $\mathcal{F}_n$ and $\mathcal{G}_n$ are continuous, i.e., for any two polynomial sequences $\{\psi_i\}_{i=1}^{n-1}$ and $\{\varphi_i\}_{i=1}^{n-1}$, we have*

$$(4.30)\quad \|\mathcal{F}_n(t;\psi_1,\ldots,\psi_{n-1}) - \mathcal{F}_n(t;\varphi_1,\ldots,\varphi_{n-1})\|_\infty \le c\Delta t\sum_{j=1}^{n-1}\|\psi_j - \varphi_j\|_\infty;$$

$$(4.31)\quad \|\mathcal{G}_n(t;\psi_1,\ldots,\psi_{n-1}) - \mathcal{G}_n(t,\varphi_1,\ldots,\varphi_{n-1})\|_\infty \le c\Delta t\sum_{j=1}^{n-1}\|\psi_j - \varphi_j\|_\infty.$$

We now define an auxiliary operator $\tilde{\mathcal{S}}_n$. For a sequence $\{\psi_i\}_{i=1}^{n-1}$, we define $\tilde{\mathcal{S}}_n(t;\psi_1,\psi_2,\ldots,\psi_{n-1})$ as the function ψ_n, which is the solution of the following problem:

$$(4.32)\qquad \psi_n(t) - \int_{t_{n-1}}^{t} K(t,s)\psi_n(s)\mathrm{d}s = g(t) + \frac{\Delta t}{2}\sum_{j=1}^{n-1}\left(K(t,s_j),\psi_j(s_j)\right)_M.$$

LEMMA 4.4. *For $1 \le n \le N$, let $\delta\mathcal{F}_n = \tilde{\mathcal{S}}_n - \mathcal{F}_n$, $\delta\mathcal{G}_n = \tilde{\mathcal{S}}_n - \mathcal{G}_n$. Then they are continuous in the sense that they satisfy, for any two sequences $\{\psi_i\}_{i=1}^{n-1}$ and $\{\varphi_i\}_{i=1}^{n-1}$,*

(4.33)
$$|\delta\mathcal{F}_n(t;\psi_1,\dots,\psi_{n-1}) - \delta\mathcal{F}_n(t;\varphi_1,\dots,\varphi_{n-1})| \le c\Delta t 2^m M^{\frac{3}{4}-m} \sum_{j=1}^{n-1} \|\psi_j - \varphi_j\|_\infty;$$

(4.34)
$$|\delta\mathcal{G}_n(t;\psi_1,\dots,\psi_{n-1}) - \delta\mathcal{G}_n(t;\varphi_1,\dots,\varphi_{n-1})| \le c\Delta t 2^m \tilde{M}^{\frac{3}{4}-m} \sum_{j=1}^{n-1} \|\psi_j - \varphi_j\|_\infty,$$

where c depends on $\max_{s\in I} \|K(\cdot,s)\|_{W^{m,\infty}}$, $m \ge 1$.

Proof. Similar to the proof of Lemma 4.1, we only need to prove the first inequality above. Let $q_n(t) = \mathcal{F}_n(t;\psi_1,\dots,\psi_{n-1}), p_n(t) = \mathcal{F}_n(t;\varphi_1,\dots,\varphi_{n-1}), v_n(t) = \tilde{\mathcal{S}}_n(t;\psi_1,\dots,\psi_{n-1}), w_n(t) = \tilde{\mathcal{S}}_n(t;\varphi_1,\dots,\varphi_{n-1})$. Then

$$\delta\mathcal{F}_n(t;\psi_1,\dots,\psi_{n-1}) - \delta\mathcal{F}_n(t;\varphi_1,\dots,\varphi_{n-1}) = (v_n - q_n) - (w_n - p_n). \tag{4.35}$$

Let $e_l = \psi_l - \varphi_l$. From the definitions (4.32) and (2.14), we can verify that

$$\begin{aligned}(v_n - w_n)(\xi_n^i) - (q_n - p_n)(\xi_n^i) \\ = \left(\bar{K}(\xi_n^i, s_n^i), (v_n - w_n)(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), (q_n - p_n)(s_n^i)\right)_M.\end{aligned} \tag{4.36}$$

Let $\Delta_n = (v_n - q_n) - (w_n - p_n)$. By multiplying both sides of (4.36) by h_n^i and summing up from $i = 0$ to $i = M$, we obtain

$$\mathcal{I}_n^M(v_n - w_n) - (q_n - p_n) = \mathcal{I}_n^M\left(\bar{K}(t, s_n^i), \Delta_n(s_n^i)\right) + \sum_{i=0}^{M} A_n^i h_n^i,$$

where

$$A_n^i = \left(\bar{K}(\xi_n^i, s_n^i), (q_n - p_n)(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), (q_n - p_n)(s_n^i)\right)_M.$$

Consequently,

$$\Delta_n(t) = \int_{t_{n-1}}^{t} K(t,s)\Delta_n(s)\mathrm{d}s - \mathcal{I}_n^M(v_n - w_n) + (v_n - w_n) + J_n + I_n, \tag{4.37}$$

where

$$J_n = \sum_{i=0}^{M} A_n^i h_n^i, \quad I_n = \mathcal{I}_n^M\left(\bar{K}(t, s_n^i), \Delta_n(s_n^i)\right) - \left(\bar{K}(t, s_n^i), \Delta_n(s_n^i)\right).$$

Applying the standard Gronwall inequality to (4.37) gives

$$\|\Delta_n\|_\infty \le c\Big(\left\|\mathcal{I}_n^M(v_n - w_n) - (v_n - w_n)\right\|_\infty + \|J_n\|_\infty + \|I_n\|_\infty\Big). \tag{4.38}$$

It follows from the inequality (3.2) that

$$\left\|\mathcal{I}_n^M(v_n - w_n) - (v_n - w_n)\right\|_\infty \le cM^{\frac{3}{4}-m}|v_n - w_n|_{H^{m;M}} \le cM^{\frac{3}{4}-m}\|v_n - w_n\|_{W^{m,\infty}}.$$

Note that

$$v_n - w_n = \left(\bar{K}(t, s_n), (v_n - w_n)(s_n)\right) + \frac{\Delta t}{2} \sum_{j=1}^{n-1} \left(K(t, s_j), e_j(s_j)\right)_M . \tag{4.39}$$

Consequently, it follows from Lemma 3.2 that

$$\begin{aligned} \|v_n - w_n\|_{W^{m,\infty}} &\leq c2^m \sum_{j=1}^{n-1} \| \left(K(t, s_j), e_j(s_j)\right)_M \|_{W^{m,\infty}} \\ &\leq c2^m \Delta t \max_{s\in I} \|K(\cdot, s)\|_{W^{m,\infty}} \sum_{j=1}^{n-1} \|e_j\|_\infty \leq c2^m \Delta t \sum_{j=1}^{n-1} \|e_j\|_\infty. \end{aligned} \tag{4.40}$$

Using Lemma 4.3 gives

$$\|J_n\|_\infty = \left\| \sum_{i=0}^{M} A_n^i h_n^i \right\|_\infty \leq c\Delta t M^{\frac{1}{2}-m} \|q_n - p_n\|_\infty \leq c\Delta t^2 M^{\frac{1}{2}-m} \sum_{j=1}^{n-1} \|e_j\|_\infty. \tag{4.41}$$

Moreover, using the same technique as used in the proof of Lemma 4.1 yields

$$\|I_n\|_\infty = \left\|\mathcal{I}_n^M \left(\bar{K}(t, s_n^i), \Delta_n(s_n^i)\right) - \left(\bar{K}(t, s_n^i), \Delta_n(s_n^i)\right)\right\|_\infty \leq \frac{1}{3c}\|\Delta_n\|_\infty. \tag{4.42}$$

Combining (4.38)–(4.42), we conclude that

$$\|\Delta_n\|_\infty \leq c\Delta t 2^m M^{\frac{3}{4}-m} \sum_{j=1}^{n-1} \|e_j\|_\infty.$$

The proof is then complete. ∎

THEOREM 4.2 (convergence). *For $1 \leq n \leq N$, let u_n be the solution of* (2.1) *and U_n^k be the solution of iteration scheme* (2.5)–(2.6). *If* $\max_{s\in I} \|K(\cdot, s)\|_{W^{m,\infty}} \leq c$, *$u \in H^m(I), m \geq 1$, then*

$$\|u_n - U_n^k\|_\infty \leq c\Big(M^{\frac{3}{4}-m} + (\tilde{M}/2)^{(\frac{3}{4}-m)(k+1)}\Big)\Big(|u|_{H^{m;M}} + \|u\|_\infty\Big), \quad k = 0, 1, \ldots, \tag{4.43}$$

where c depends on T and $\max_{s\in I} \|K(\cdot, s)\|_{W^{m,\infty}}$.

Proof. We first prove that (4.43) is true for $k = 0$. By construction, we have $U_n^0 = \mathcal{G}_n(t; U_1^0, \ldots, U_{n-1}^0)$. Applying Lemma 4.2 to the above coarse solution gives

$$\begin{aligned} \|u_n - U_n^0\|_\infty &\leq c_1 \left(\tilde{M}^{\frac{3}{4}-m}|u|_{H^{m;M}} + \tilde{M}^{\frac{1}{2}-m}\|u\|_\infty\right) \leq c_2\tilde{M}^{\frac{3}{4}-m}\Big(|u|_{H^{m;M}} + \|u\|_\infty\Big) \\ &\leq c(\tilde{M}/2)^{\frac{3}{4}-m}\Big(|u|_{H^{m;M}} + \|u\|_\infty\Big). \end{aligned} \tag{4.44}$$

This proves (4.43) for $k = 0$. We now prove (4.43) for $k \geq 1$. It follows from (2.4) and (2.5) that

$$\begin{aligned} \left|U_n^k - U_n\right| \leq &\left|\mathcal{G}_n\left(t; U_1^k, \ldots, U_{n-1}^k\right) - \mathcal{G}_n\left(t; U_1, \ldots, U_{n-1}\right)\right| \\ &+ \left|\delta\mathcal{G}_n\left(t; U_1^{k-1}, \ldots, U_{n-1}^{k-1}\right) - \delta\mathcal{G}_n(t; U_1, \ldots, U_{n-1})\right| \\ &+ \left|\delta\mathcal{F}_n\left(t; U_1^{k-1}, \ldots, U_{n-1}^{k-1}\right) - \delta\mathcal{F}_n(t; U_1, \ldots, U_{n-1})\right|, \end{aligned}$$

where $\delta\mathcal{F}_n = \tilde{\mathcal{S}}_n - \mathcal{F}_n$ and $\delta\mathcal{G}_n = \tilde{\mathcal{S}}_n - \mathcal{G}_n$ with $\tilde{\mathcal{S}}_n$ defined by (4.32). Using Lemmas 4.3 and 4.4 gives

$$\|U_n^k - U_n\|_\infty \le c\Delta t \sum_{j=1}^{n-1} \|U_j^k - U_j\|_\infty + c\varepsilon\Delta t \sum_{j=1}^{n-1} \|U_j^{k-1} - U_j\|_\infty, \tag{4.45}$$

where $\varepsilon = 2^m \tilde{M}^{\frac{3}{4}-m}$, and we have used the fact that $M^{\frac{3}{4}-m} \le \tilde{M}^{\frac{3}{4}-m}$. Next we will derive the inequality

$$\|U_n^k - U_n\|_\infty \le \frac{c}{k!} e^{c(k+1)(n-1)\Delta t} \varepsilon^{k+1} (|u|_{H^{m;M}} + \|u\|_\infty) \ \forall k \ge 1, \tag{4.46}$$

where c is a constant independent of k. We do this by induction. For $k = 1$, using the Gronwall inequality for (4.45) gives

$$\begin{aligned} \|U_n^1 - U_n\|_\infty &\le c\Delta t \varepsilon e^{c(n-1)\Delta t} \sum_{j=1}^{n-1} \|U_j^0 - U_j\|_\infty \\ &\le c\Delta t \varepsilon e^{c(n-1)\Delta t} \sum_{j=1}^{n-1} (\|U_j^0 - u_j\|_\infty + \|u_j - U_j\|_\infty). \end{aligned}$$

Using (4.44) and (4.29) yields

$$\begin{aligned} \|U_n^1 - U_n\|_\infty &\le c^2 \Delta t \varepsilon^2 e^{c(n-1)\Delta t} \sum_{j=1}^{n-1} e^{c(j-1)\Delta t} (|u|_{H^{m;M}} + \|u\|_\infty) \\ &\le c^2 \Delta t \varepsilon^2 e^{c(n-1)\Delta t} \frac{e^{c(n-1)\Delta t} - 1}{e^{c\Delta t} - 1} (|u|_{H^{m;M}} + \|u\|_\infty) \\ &\le c e^{2c(n-1)\Delta t} \varepsilon^2 (|u|_{H^{m;M}} + \|u\|_\infty). \end{aligned}$$

This means (4.46) holds for $k = 1$. Now assuming (4.46) holds for a given k, we want to prove that it also holds for $k+1$. Again, replacing k in (4.45) by $k+1$ and using the Gronwall inequality for the resulting inequality yield

$$\|U_n^{k+1} - U_n\|_\infty \le c\Delta t \varepsilon e^{c(n-1)\Delta t} \sum_{j=1}^{n-1} \|U_j^k - U_j\|_\infty. \tag{4.47}$$

Using the induction assumption for index k, i.e., (4.46), gives

$$\begin{aligned} \|U_n^{k+1} - U_n\|_\infty &\le \frac{c^2}{k!} \Delta t \varepsilon e^{c(n-1)\Delta t} \sum_{i=1}^{n-1} \varepsilon^{k+1} e^{c(k+1)(i-1)\Delta t} (|u|_{H^{m;M}} + \|u\|_\infty) \\ &\le \frac{c^2}{k!} \Delta t e^{c(n-1)\Delta t} \frac{e^{c(k+1)(n-1)\Delta t} - 1}{e^{c(k+1)\Delta t} - 1} \varepsilon^{k+2} (|u|_{H^{m;M}} + \|u\|_\infty) \\ &\le \frac{c}{(k+1)!} e^{c(k+2)(n-1)\Delta t} \varepsilon^{k+2} (|u|_{H^{m;M}} + \|u\|_\infty). \end{aligned}$$

Thus we have proved (4.46) for all $k \ge 1$. Then as in the proof of Theorem 4.1, the boundedness of $e^{c(k+1)(n-1)\Delta t}/k!$ implies that

$$\|U_n^k - U_n\|_\infty \le c\varepsilon^{k+1(}|u|_{H^{m;M}} + \|u\|_\infty) \le c(\tilde{M}/2)^{(\frac{3}{4}-m)(k+1)} (|u|_{H^{m;M}} + \|u\|_\infty) \ \forall k \ge 1. \tag{4.48}$$

Finally, combining the above estimate with Lemma 4.2 leads to the desired result (4.43). ∎

5. Numerical results. We first give some implementation details of the parallel in time scheme (2.5)–(2.6). The overall algorithm is described below:

ALGORITHM A1.

- Initialization ($k = 0$): $\{U_1^0(t), \dots, U_N^0(t)\}$, given by solving $\mathcal{G}_n(t; U_1^0, \dots, U_{n-1}^0)$ *successively* for $n = 1, \dots, N$.
- At step k: Suppose known $\{U_1^{k-1}(t), \dots, U_N^{k-1}(t)\}$. For $n = 1, \dots, N$
  1. solve the fine problem $\mathcal{F}_n(t; U_1^{k-1}, \dots, U_{n-1}^{k-1})$ *simultaneously*;
  2. solve the coarse problem $\mathcal{G}_n(t; U_1^k, \dots, U_{n-1}^k)$ *successively*;
  3. $\mathcal{G}_n(t; U_1^{k-1}, \dots, U_{n-1}^{k-1})$ are available from the previous step.
- Update: Use (2.5)–(2.6) to update $\{U_1^k(t), \dots, U_N^k(t)\}$.

Obviously, the most expensive part of this algorithm is to solve the fine problem $\mathcal{F}_n(t; U_1^{k-1}, \dots, U_{n-1}^{k-1})$. However, this can be handled by parallel realization. Next we describe the matrix form of the linear system associated with $\mathcal{F}_n(t; U_1^{k-1}, \dots, U_{n-1}^{k-1})$, defined over the subinterval I_n.

Let $p_n^k = \mathcal{F}_n(t; U_1^{k-1}, \dots, U_{n-1}^{k-1})$. Then by definition (2.14), $p_n^k \in \mathcal{P}_M(I_n)$ satisfies, for all $0 \le i \le M$,

$$p_n^k(\xi_n^i) - \left(\bar{K}(\xi_n^i, s_n^i), p_n^k(s_n^i)\right)_M = g(\xi_n^i) + \frac{\Delta t}{2} \sum_{j=1}^{n-1} \left(K(\xi_n^i, s_j), U_j^{k-1}(s_j)\right)_M . \tag{5.1}$$

Using the notation developed in the previous sections, we obtain, for all $0 \le i \le M$,

$$p_n^k(\xi_n^i) - \sum_{j=0}^{M} p_n^k(\xi_n^j) \left(\bar{K}(\xi_n^i, s_n^i), h_n^j(s_n^i)\right)_M = g(\xi_n^i) + \frac{\Delta t}{2} \sum_{j=1}^{n-1} \left(K(\xi_n^i, s_j), U_j^{k-1}(s_j)\right)_M , \tag{5.2}$$

which can be rewritten in the matrix form:

$$(\mathbf{I} - \mathbf{A})\mathbf{p}_n^k = \mathbf{f}_n^k, \quad n = 1, \dots, N. \tag{5.3}$$

In our calculations, the linear system (5.3) is solved by the Gauss–Seidel iterative method with the initial guess U_n^{k-1}, obtained at the previous step in Algorithm A1.

The coarse problems $\mathcal{G}_n(t; U_1^k, \dots, U_{n-1}^k)$ can be solved in a similar way. Note that solving the coarse problems is strictly sequential with respect to n, but as $\tilde{M}$ is much smaller than M, the cost of this part is relatively inexpensive.

Below we will present some numerical results obtained by the proposed parallel in time scheme.

Example 5.1. The linear VIE with a regular kernel is

$$u(t) + \int_0^t \sin(\pi(t-s))u(s)\mathrm{d}s = g(t), \qquad 0 \le t \le 100. \tag{5.4}$$

We take

$$g(t) = (1 + 1/2\pi)\sin(\pi t) - (t/2)\cos(\pi t)$$

such that the exact solution $u(t) = \sin(\pi t)$.

The main purpose is to investigate the convergence behavior of numerical solutions with respect to the polynomial degrees M and iteration number k. In Figure 5.1, we plot the L^∞-errors in semilog scale as a function of M with $\tilde{M}$ fixed to 13 and N fixed to 20. That is, the domain is partitioned into 20 subintervals and the coarse

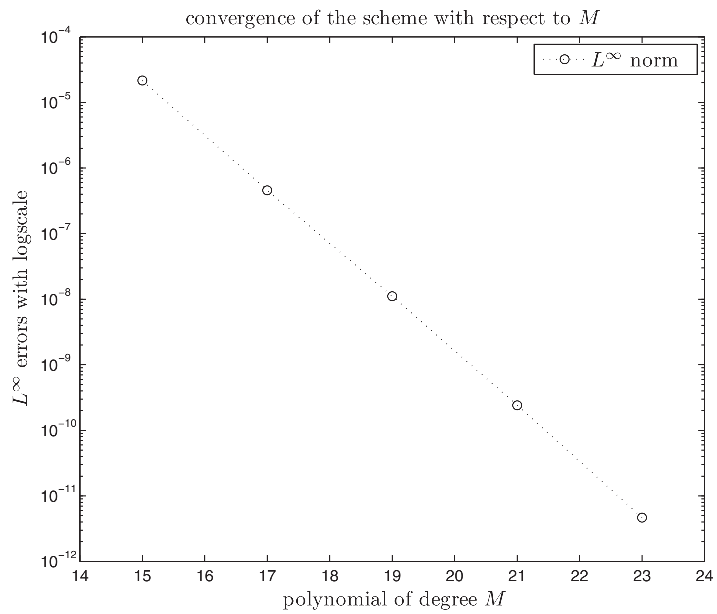

FIG. 5.1. *L^∞-errors versus the degree of freedom for fine approximation M with $\tilde{M} = 13, N = 20$.*

algorithm is solved with 13 collocation points in each subinterval. To separate different error sources, the solution is iterated with sufficiently large number of k so that the error $\tilde{M}^{(\frac{3}{4}-m)(k+1)}$ (see (4.43)) is negligible as compared with the error of the fine resolution. As expected, the errors show an exponential decay, since in this semilog representation one observes that the error variations are linear versus the degrees of polynomial M.

Next we investigate the convergence behavior with respect to the iteration number k, which is more interesting to us. For a reason similar to that mentioned above, we now fix a large enough $M = 25$ and let k vary for different values of $\tilde{M}$. In Figure 5.2, we plot the error decay with increasing iteration number k for several values of $\tilde{M}$. It is observed that the errors decay quickly with increasing k, and for $\tilde{M} = 13$ only six iterations are required to reach machine accuracy.

In the stability analysis (see Lemma 4.1), we have assumed sufficiently large M and $\tilde{M}$. In practical calculations it is interesting to see how large these polynomial degrees really must be to guarantee the stability of the scheme. In Figure 5.3, we present the error behavior as a function of the iteration number k for $M = 13, N = 20, \tilde{M} = 4$, and $T = 20$. It is observed that the error decays to machine accuracy within about 10 iterations. This result shows that even with moderate M and small $\tilde{M}$, the proposed parallel in time scheme remains convergent.

Finally, to test the sharpness of the estimate given in (4.43), we plot the errors with respect to the degree of freedom $\tilde{M}$ for three values $k = 2, 3$, and 4 with $M = 25$ in Figure 5.4. As known for spectral methods to analytical solutions (see, e.g., [4]), we expect from (4.43) an error that behaves like $e^{-c\tilde{M}(k+1)}$ with some constant c. The result presented in Figure 5.4 indeed confirms this prediction, as in this semilog representation the error curves for different k are all straight lines with slopes corresponding to the constant $c = 1.4$.

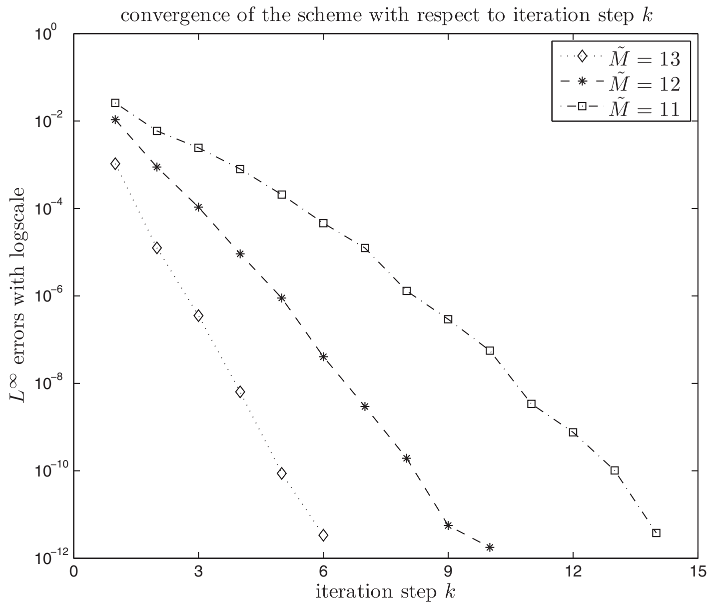

FIG. 5.2. *L^∞-errors versus iteration numbers k with $M = 25, N = 20$, $\tilde{M} = 11, 12, 13$.*

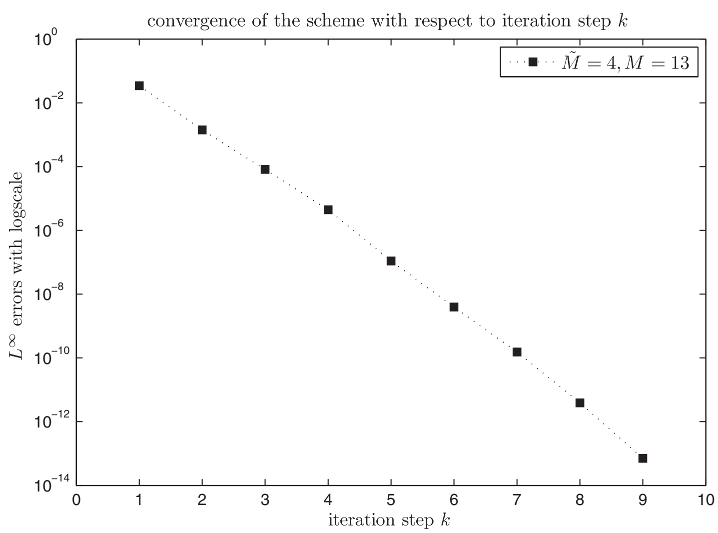

FIG. 5.3. *L^∞-errors versus iteration numbers k with $M = 13, N = 20, \tilde{M} = 4$, and $T = 20$.*

Example 5.2. Consider the VIE with the kernel of exponential form $K(t, s) = (t - s)e^{s-t}$ and the exact solution $u(t) = \frac{1}{4}(2t - 1 + e^{-2t})$.

For this problem with a convolution exponential kernel, we repeat the convergence investigation for the numerical solutions as in the last example. Similar observations have been obtained. For example, L^∞-errors of numerical solutions versus M with $\tilde{M} = 5, N = 20$ are presented in Figure 5.5, where it is observed that convergence remains exponential with respect to M. It is also noted that the growth in t of the

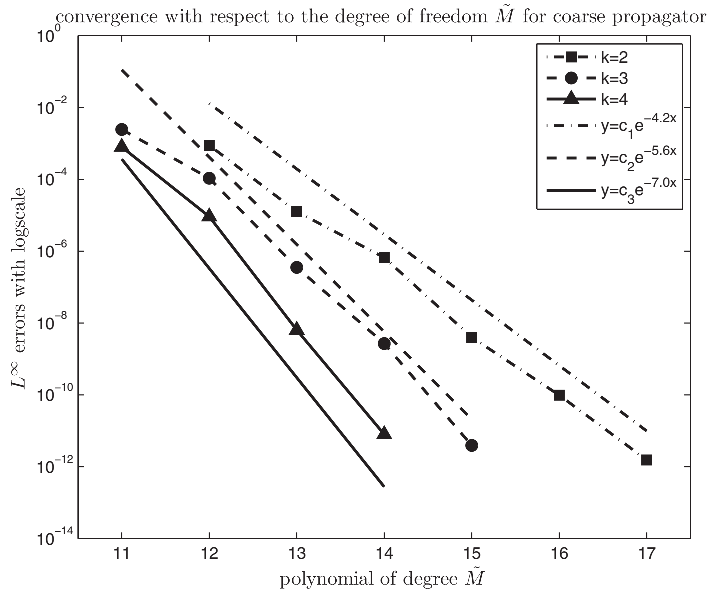

FIG. 5.4. *L^∞-errors versus the degree of freedom in coarse approximation $\tilde{M}$ with $M = 25$, $N = 20$, and $k = 2, 3$, and 4.*

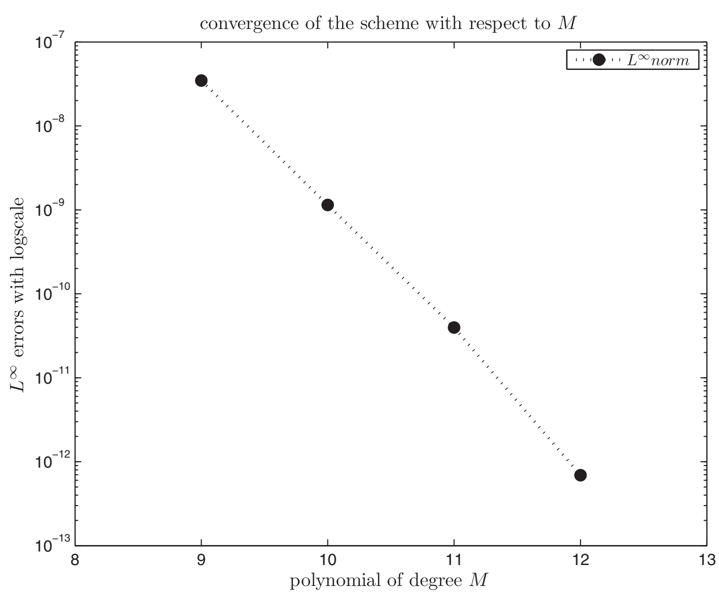

FIG. 5.5. *L^∞-errors versus M with $\tilde{M} = 5, N = 20$.*

exact solution does not affect the convergence rate even for large values of T, as pointed out at the end of section 4.1.

6. Parallelism efficiency. Although not yet tested in a parallel machine, the parallelism efficiency of the proposed scheme can be investigated through a cost estimate. To simplify the cost estimation, we suppose that the interprocessor communication cost in the implementation of the parallel in time scheme is negligible as compared to the overall cost. The parallelism efficiency is demonstrated by a cost comparison between the parallel in time scheme (2.5)–(2.6) and the classical sequential scheme based on the corresponding fine mesh.

First, the classical sequential scheme based on the fine mesh consists of solving the problems $\mathcal{F}_n\,(t; U_1, U_2, \ldots, U_{n-1})$ consecutively for $n = 1, \ldots, N$. The computational complexity is equal to the sum of all the elementary operations in $I_n, n = 1, \ldots, N$. Denote the total computational cost by $\mathcal{C}_{\mathcal{F}}$ and the cost for nth subproblem by $\mathcal{C}_{\mathcal{F}}^n$. Then

$$\mathcal{C}_{\mathcal{F}} = \sum_{n=1}^{N} \mathcal{C}_{\mathcal{F}}^n.$$

Neglecting the cost of evaluating the integral terms on the right-hand sides, the spectral discretization $\mathcal{F}_n$ produces an elementary cost $\mathcal{C}_{\mathcal{F}}^n$ approximately equal to $\mathcal{O}(M^2 M)$, where $\mathcal{O}(M^2)$ is the number of operations needed for the matrix vector multiplication and $\mathcal{O}(M)$ is the estimated iteration number required to achieve the convergence of the iterative method. As a result, the total computational complexity of the sequential fine solutions is

$$\mathcal{C}_{\mathcal{F}} = \mathcal{O}(NM^3). \tag{6.1}$$

If we implement the scheme (2.5)–(2.6) in a parallel architecture with enough processors, then the total computational time corresponds to the cost to solve a sequential set of N coarse subproblems and a fine subproblem in a single processor. The cost of solving the sequential set of N coarse subproblems is estimated to be $\sum_{n=1}^{N} \mathcal{C}_{\mathcal{G}}^n$, where $\mathcal{C}_{\mathcal{G}}^n = \mathcal{O}(\tilde{M}^2 + M\tilde{M})$ is the cost to solve the nth coarse subproblem, with $M\tilde{M}$ being the cost of the fine-to-coarse interpolation. Note that here we only count the number of operations needed for the matrix vector multiplication in the coarse mesh, which is equal to $\mathcal{O}(\tilde{M}^2)$, because the Gauss–Seidel iterative method has been employed to solve the final linear system with the previous solution as the initial guess, and it is found that the convergence was achieved within a few iterations. For the same reason, the cost of solving a single fine subproblem is approximately $\mathcal{O}(M^2)$. Note also that in the implementation of the parallel in time scheme there is a need to interpolate the solution between the fine mesh and coarse mesh, the cost of which is $\mathcal{O}(NM\tilde{M})$. Therefore if K is the number of iterations required to achieve the desired convergence of the parallel in time algorithm, then the total computational complexity is

$$K\left[\mathcal{O}(N\tilde{M}^2 + NM\tilde{M}) + \mathcal{O}(M^2) + \mathcal{O}(NM\tilde{M})\right]. \tag{6.2}$$

Comparing (6.1) with (6.2), we obtain a speedup (i.e., the percentage with respect to the sequential scheme) close to

$$\mathcal{O}\left(\frac{NM^3/K}{N\tilde{M}^2 + NM\tilde{M} + M^2}\right) = \mathcal{O}\left(\frac{M/K}{(\tilde{M}/M)^2 + \tilde{M}/M + 1/N}\right) = \mathcal{O}\left(\frac{M/K}{\tilde{M}/M + 1/N}\right), \tag{6.3}$$

where in the last step we have used the fact $\tilde{M} < M$. This gives us the speedup of the parallel in time algorithm:

$$\mathsf{speedup} = \begin{cases} \mathcal{O}(NM/K) & \text{if } 1/N \geq \tilde{M}/M, \\ \mathcal{O}(M^2/(\tilde{M}K)) & \text{otherwise.} \end{cases} \tag{6.4}$$

Note that in the case that the number of degrees of freedom for the coarse solver is far less than that of the fine solver, e.g., $\tilde{M}N \approx M$, then the speedup using the parallel in time algorithm would be close to $\mathcal{O}(NM/K)$.

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REFERENCES

[1] G. Bal *Parallelization in time of (stochastic) ordinary differential equations*, Preprint, http://www.columbia.edu/ gb2030/PAPERS/paralleltime.pdf, 2013.

[2] G. Bal and Y. Maday, *A parareal time discretization for non-linear PDEs with application to the pricing of an American put*, in Proceedings of the Workshop on Domain Decomposition, Lecture Notes in Comput. Sci. Eng. 23, Springer, Berlin, 2001, pp. 189–201.

[3] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, Cambridge, UK, 2004.

[4] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods: Fundamentals in Single Domains*, Springer, Berlin, 2006.

[5] Y. Chen and T. Tang, *Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel*, Math. Comp., 79 (2010), pp. 147–167.

[6] N. Debit, M. Garbey, R. Hoppe, D. Keyes, Y. Kuznetsov, and J. Periaux, *Domain Decomposition Methods in Science and Engineering*, International Center for Numerical Methods in Engineering (CIMNE), Barcelona, Spain, 2002.

[7] C. Farhat and M. Chandesris, *Time-decomposed parallel time-integrators: Theory and feasibility studies for fluid, structure, and fluid-structure applications*, Internat. J. Numer. Methods Engrg., 58 (2003), pp. 1397–1434.

[8] P. Fischer, F. Hecht, and Y. Maday, *A parareal in time semi-implicit approximation of the Navier-Stokes equations*, in Proceedings of the 15th International Domain Decomposition Conference, R. Kornhuber, R. H. W. Hoppe, J. Péeriaux, O. Pironneau, O. B. Widlund, and J. Xu, eds., Lecture Notes in Comput. Sci. Engrg. 40, Springer, Berlin, 2003, pp. 1397–1434.

[9] M. J. Gander and S. Vandewalle, *Analysis of the parareal time-parallel time-integration method*, Analysis, 29 (2007), pp. 556–578.

[10] S. Larsson, V. Thomee, and L. B. Wahlbin, *Numerical solution of parabolic integro-differential equations by the discontinuous Galerkin method*, Math. Comp., 67 (1998) pp. 45–71.

[11] X. Li and C. Xu, *A space-time spectral method for the time fractional diffusion equation*, SIAM J. Numer. Anal., 47 (2009), pp. 2108–2131.

[12] X. Li and C. Xu, *Existence and Uniqueness of the Weak Solution of the Space-Time Fractional Diffusion Equation and a Spectral Method Approximation*, Commun. Comput. Phys., 8 (2011), pp. 1016–1051.

[13] X.-J. Li and T. Tang, *Convergence analysis of Jacobi spectral collocation methods for Abel-Volterra integral equations of second kind*, Front. Math. China, 7 (2012), pp. 69–84.

[14] J. L. Lions, Y. Maday, and G. Turinici, Résolution d'EDP par un schéma en temps pararéel, C. R. Acad. Sci. Ser. I Math., 332 (2001), pp. 661–668.

[15] Y. Maday and G. Turinici, *Parallel in time algorithms for quantum control: Parareal time discretization scheme*, Internat. J. Quantum Chemistry, 93 (2003), pp. 223–228.

[16] G. Mastroianni and D. Occorsio, *Optimal systems of nodes for Lagrange interpolation on bounded intervals. A survey*, J. Comput. Appl. Math., 134 (2001), pp. 325–341.

[17] K. Mustapha, H. Brunner, H. Mustapha, and D. Schotazu, *An hp-version discontinuous Galerkin method for integro-differential equations of parabolic type*, SIAM J. Numer. Anal., 49 (2011) pp. 1369–1396.

[18] J. SHEN, T. TANG, AND L.-L. WANG, *Spectral Methods: Algorithms, Analysis and Applications*, Springer Ser. Comput. Math., Springer, Berlin, 2011.

[19] T. TANG, X. XU, AND J. CHENG, *On spectral methods for Volterra integral equations and the convergence analysis*, J. Comput. Math, 26 (2008), pp. 825–837.

[20] Z.-Q. XIE, X.-J. LI, AND T. TANG, *Convergence analysis of spectral Galerkin methods for Volterra type integral equations*, J. Sci. Comput., 53 (2012), pp. 414–434.