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A POSTERIORI ERROR ESTIMATES FOR DISCONTINUOUS GALERKIN TIME-STEPPING METHOD FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY PARABOLIC EQUATIONS*

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Abstract. In this paper, we examine the discontinuous Galerkin (DG) finite element approximation to convex distributed optimal control problems governed by linear parabolic equations, where the discontinuous finite element method is used for the time discretization and the conforming finite element method is used for the space discretization. We derive a posteriori error estimates for both the state and the control approximation, assuming only that the underlying mesh in space is nondegenerate. For problems with control constraints of obstacle type, which are the kind most frequently met in applications, further improved error estimates are obtained.

Key words. optimal control, a posteriori error analysis, finite element approximation, discontinuous Galerkin method

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1. Introduction. Optimal control or design is crucial to many engineering applications. Efficient numerical methods are essential to successful applications of optimal control. Nowadays, the finite element method seems to be the most widely used numerical method in computing optimal control problems, and the relevant literature is extensive. Some recent progress in this area has been made in, for example, [40, 41, 43]. Systematic introduction of the finite element method for PDEs and optimal control problems can be found in, for example, [10, 40, 43]. For instance, there have been extensive theoretical studies for finite element approximation of various optimal control problems; see [3, 15, 16, 18, 19], [20, 21, 22, 23, 24, 25, 26], and [37, 39, 44, 45]. For optimal control problems governed by linear elliptic or parabolic state equations, a priori error estimates of finite element approximation were established long ago; see, for example, [15, 18, 26, 37]. Furthermore a priori error estimates have been also established for some important flow control problems; see, e.g., [19, 20]. A priori error estimates have also been obtained for a class of state constrained con-

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trol problems in [44], although the state equation is assumed to be linear. In [32], the linear assumption has been removed by reformulating the control problem as an abstract optimization problem in some Banach spaces and then applying nonsmooth analysis. In fact, the state equation there can be a variational inequality.

In this paper, we examine an important class of finite element algorithms for a convex distributed optimal control problem governed by a linear parabolic equation, where the discontinuous polynomial base is used in time discretization and the conforming finite element method is used in space discretization. We present an a posteriori error analysis for this approximation.

Adaptive finite element approximation is among the most important means to boost the accuracy and efficiency of the finite element discretization. It ensures a higher density of nodes in certain areas of the given domain, where the solution is more difficult to approximate using an a posteriori error indicator. The decision about whether further refinement of meshes is necessary is based on the estimate of the discretization error. If further refinement is to be performed, then the error indicator is used as a guide to show how the refinement might be accomplished most efficiently. The literature in this area is huge. Some of the techniques directly relevant to our work can be found in [1, 5, 33, 36, 46]. It is our belief that adaptive finite element enhancement is one of the future directions to pursue in developing sophisticated numerical methods for optimal design problems.

Although adaptive finite element approximation is widely used in numerical simulations, it has not yet been *fully* utilized in optimal design. Initial attempts in this aspect have only been reported recently for some design problems (see, e.g., [2, 4, 38, 42]), and only a posteriori error indicators of a heuristic nature are used in most applications. For instance, in some existing work on adaptive finite element approximation of optimal design, the mesh refinement is guided by a posteriori error estimators based on a posteriori error estimates *solely* from the state equation for a fixed control. Thus error information from the approximation of the control (design) is not utilized. This strategy was found to be inefficient in recent numerical experiments (see [7, 27]). Although these methods may work well in some particular applications, they cannot be applied confidently in general. It is unlikely that the potential power of adaptive finite element approximation has been fully utilized due to the lack of more sophisticated a posteriori error indicators.

It is not straightforward to rigorously derive suitable a posteriori error estimators for general optimal control problems. In particular, it seems difficult to apply gradient recovery techniques since the control is normally not differentiable. Recovering approximation in function values is in general difficult. For a similar reason, it also seems difficult to apply the local solution strategy.

Very recently, some error indicators of residual type were developed in [6, 7, 27, 30, 34, 35, 36]. These error estimators are based on a posteriori estimation of the discretization error for the state *and* the control (design).

When there is no constraint in a control problem, normally the optimality conditions consist of coupled partial differential equations only. Consequently one may be able to write down the dual system of the *whole* optimality conditions, and then to apply the weighted a posteriori error estimation technique to obtain a posteriori estimators for objective functional approximation error of the control problem; see [6, 7]. Such estimators have indeed been derived for some unconstrained elliptic control problems, and have proved quite efficient in the numerical tests carried out in [6].

However, there frequently exist some constraints for the control in applications. In such cases, the optimality conditions often contain a variational inequality and then have some very different properties. For example, the dual system is generally unknown. Thus it does not seem to be always possible to apply the techniques used in [6, 7] to constrained control problems.

In our work, constrained cases are studied via residual estimation using the norms of energy type. A posteriori error estimators are derived for some constrained control problems governed by elliptic and parabolic equations; see [27, 34, 35, 36].

In recent years, the discontinuous Galerkin (DG) discretization has proved useful in computing time-dependent convection and diffusion equations; see [12, 13, 14] for the DG time-stepping method where only time discretization is discontinuous. It will be simply referred as to the DG method in this paper, although we are aware that there exist several DG discretization schemes in the literature. The DG has proved important in diffusion dominated equations, such as the heat equations, which govern our control problems to be examined in this paper. Furthermore the DG method has been found useful in computing optimal control of diffusion dominated systems; see [40]. However, there is a lack of an a posteriori error analysis for the DG approximation of the control systems, which is vital for further studies of mesh adaptivity of the control problems.

The purpose of this work is to extend the approaches in [12, 27, 34, 35, 36] and to derive a posteriori error estimates for the DG finite element approximation of distributed convex optimal problems governed by linear parabolic equations. Deriving such estimates for the DG finite element scheme is much more involved than for the backward-Euler scheme; see [36]. For example, some approaches applied in [12, 13, 14] have to be essentially modified for our purpose. Furthermore, novel approaches are needed to derive the improved estimates for the control with constraints of obstacle type. Optimal control with obstacle constraints is most frequently met in practical control problems. In fact, the majority of the existing research on constrained control concentrates on this type problem; see [28] and [43], for instance.

The plan of the paper is as follows. In section 2 we shall give a brief review of the finite element method and the discontinuous Galerkin discretization, and then construct the approximation schemes for the optimal control problem. In section 3, a posteriori error bounds are derived for the control problem. In section 4, some applications are discussed. In section 5, improved error estimates are derived for the problem with an obstacle constraint.

Let Ω and Ω_U be bounded open sets in \mathbb{R}^n ($n \leq 3$) with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,q,\Omega}$ and seminorm $|\cdot|_{m,q,\Omega}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and set $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$.

We denote by $L^s(0, T; W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from $(0, T)$ into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(0,T;W^{m,q}(\Omega))} = (\int_0^T \|v\|_{W^{m,q}(\Omega)}^s dt)^{\frac{1}{s}}$ for $s \in [1, \infty)$ and the standard modification for $s = \infty$. Similarly, we define the spaces $H^1(0, T; W^{m,q}(\Omega))$ and $C^l(0, T; W^{m,q}(\Omega))$. The details can be found in [29]. In addition c or C denotes a general positive constant independent of h .

2. Approximation scheme of optimal control problems governed by parabolic equations. In this section we study the finite element and the discontinuous Galerkin approximation of distributed convex optimal control problems, where the state is governed by a parabolic equation. In this paper, we shall take the state

space $W = L^2(0, T; Y)$ with $Y = H_0^1(\Omega)$ and the control space $X = L^2(0, T; U)$ with $U = L^2(\Omega_U)$ to fix the idea. Let B be a linear continuous operator from X to $L^2(0, T; Y')$ and K be a closed convex set in X . We are interested in the following optimal control problem:

$$\min_{u \in K} \int_0^T (g(y) + h(u)) dt$$

subject to

$$\begin{cases} \partial_t y - \operatorname{div}(A \nabla y) = f + Bu, & x \in \Omega, & t \in (0, T], \\ y|_{\partial\Omega} = 0, & & t \in [0, T], \\ y(x, 0) = y_0(x), & x \in \Omega, & \end{cases}$$

where $f \in L^2(0, T; Y')$, $y_0 \in H_0^1(\Omega)$, and

$$A(x) = (a_{ij}(x))_{n \times n} \in (C^\infty(\bar{\Omega}))^{n \times n}$$

such that there is a constant $c > 0$ satisfying

$$(A\xi) \cdot \xi \geq c|\xi|^2 \quad \forall \xi \in \mathbb{R}^n.$$

Let

$$\begin{aligned} a(v, w) &= \int_{\Omega} (A \nabla v) \cdot \nabla w & \forall v, w \in H^1(\Omega), \\ (f_1, f_2) &= \int_{\Omega} f_1 f_2 & \forall f_1, f_2 \in L^2(\Omega), \\ (v, w)_U &= \int_{\Omega_U} vw & \forall v, w \in L^2(\Omega_U). \end{aligned}$$

It follows from the assumptions on A that there are constants c and $C > 0$ such that

$$a(v, v) \geq c\|v\|_{1,\Omega}^2, \quad |a(v, w)| \leq C\|v\|_{1,\Omega}\|w\|_{1,\Omega} \quad \forall v, w \in Y.$$

Then a weak formulation of the convex optimal control problem reads as

$$(1) \quad \min_{u \in K} \int_0^T (g(y) + h(u)) dt,$$

where $y \in W$ is subject to

$$\begin{cases} (\partial_t y, w) + a(y, w) = (f + Bu, w) & \forall w \in Y, t \in (0, T], \\ y(0) = y_0. \end{cases}$$

We assume that g is a convex functional which is continuously differentiable on $L^2(\Omega)$, and h is a strictly convex and continuously differentiable function on U . We further assume that $h(u) \rightarrow +\infty$ as $\|u\|_U \rightarrow \infty$ and that the functional $g(\cdot)$ is bounded below. This setting includes the most widely used quadratic control problem:

$$\min_{u \in K} \left\{ \frac{1}{2} \int_0^T (\|y - z_d\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega_U)}^2) dt \right\},$$

where y, u are defined as above and z_d is a given state. It is well known (see, e.g., [28]) that the control problem (1) has a unique solution (y, u) , and that a pair (y, u) is the solution of (1) if and only if there is a costate $p \in W$ such that the triplet (y, p, u) satisfies the following optimality conditions:

$$(2) \quad \begin{cases} (\partial_t y, w) + a(y, w) = (f + Bu, w) & \forall w \in Y, & y(0) = y_0, \\ -(\partial_t p, q) + a(q, p) = (g'(y), q) & \forall q \in Y, & p(T) = 0, \\ \int_0^T (h'(u) + B^*p, v - u)_U dt \geq 0 & \forall v \in K, \end{cases}$$

where B^* is the adjoint operator of B .

Let us consider the finite element approximation of the control problem (1). Here we consider only n -simplices Lagrange elements.

Let Ω^h be a polygonal approximation to Ω with boundary $\partial\Omega^h$. Let T^h be a partitioning of Ω^h into disjoint regular n -simplex τ , so that $\bar{\Omega}^h = \cup_{\tau \in T^h} \bar{\tau}$. Each element has at most one face on $\partial\Omega^h$, and joint elements $\bar{\tau}$ and $\bar{\tau}'$ have either only one common vertex or a whole edge or face if τ and $\tau' \in T^h$. We further require that $P_i \in \partial\Omega^h$ implies $P_i \in \partial\Omega$, where $\{P_i\}$ ($i = 1, 2, \dots, J$) is the vertex set associated with the triangulation T^h . We assume that Ω is a convex polygon so that $\Omega = \Omega^h$. The convexity assumption is also important to have the H^2 a priori estimate for the dual equations in Lemma 3.4, which is used in deriving our L^2 - L^2 and L^∞ - L^2 a posteriori error estimates, although it is not needed for L^2 - H^1 estimates. Without the convexity assumption, in general the order of our estimates for the state and costate approximation will be lower if $\partial\Omega$ is nonsmooth. We denote by h_τ the maximum diameter of the element τ in T^h .

Associated with T^h is a finite dimensional subspace S^h of $C(\bar{\Omega}^h)$ such that $w|_\tau$ are m -order polynomials ($m \geq 1$) for all $w \in S^h$ and $\tau \in T^h$. Let $Y^h = S^h \cap H_0^1(\Omega)$, $W^h = L^2(0, T; Y^h)$; it is easy to see that $Y^h \subset Y$, $W^h \subset W$.

Similarly, we do a partitioning of Ω_U and use the following corresponding notations: T_U^h , τ_U , h_{τ_U} , P_i^U ($i = 1, 2, \dots, J_U$), and $\Omega_U^h = \Omega_U$.

Associated with T_U^h is another finite dimensional subspace U^h of $L^2(\Omega_U^h)$ such that $v|_{\tau_U}$ are m -order polynomials ($m \geq 0$) for all $v \in U^h$ and $\tau_U \in T_U^h$. Here there is no requirement for the continuity. Let $X^h = L^2(0, T; U^h)$. It is easy to see that $U^h \subset U$ and $X^h \subset X$.

Let K^h be an approximation of K . Here we assume that $K^h \subset K$ and $K^h \subset X^h$ for ease of exposition. A nonconforming finite element method will be used later for the problem with the constraint of obstacle type. For more general cases, the readers are referred to [35]. Then a possible semidiscrete finite element approximation of (1) is as follows:

$$(3) \quad \min_{u_h \in K^h} \int_0^T (g(y_h) + h(u_h)) dt$$

with $y_h \in W^h$ subject to

$$\begin{cases} (\partial_t y_h, w) + a(y_h, w) = (f + Bu_h, w) & \forall w \in Y^h, \quad t \in (0, T], \\ y_h(0) = y_0^h, \end{cases}$$

where K^h is a closed convex set in X^h , $y_0^h \in Y^h$ is an approximation of y_0 .

It follows that the control problem (3) has a unique solution (y_h, u_h) and that a pair $(y_h, u_h) \in W^h \times K^h$ is the solution of (3) if and only if there is a costate $p_h \in W^h$

such that the triplet $(y_h, p_h, u_h) \in W^h \times W^h \times K^h$ satisfies the following optimality conditions:

$$(4) \quad \begin{cases} (\partial_t y_h, w) + a(y_h, w) = (f + Bu_h, w) & \forall w \in Y^h, & y_h(0) = y_0^h, \\ -(\partial_t p_h, q) + a(q, p_h) = (g'(y_h), q) & \forall q \in Y^h, & p_h(T) = 0, \\ \int_0^T (h'(u_h) + B^* p_h, v - u_h)_U dt \geq 0 & \forall v \in K^h. \end{cases}$$

The optimality conditions in (4) are the semidiscrete approximation to the problem (1). Now, we are going to consider the fully discrete approximation for the above semidiscrete problem by using the DG method.

Let $0 = t_0 < t_1 < \dots < t_N = T$, $I_k = (t_{k-1}, t_k]$, $\Delta t_k = t_k - t_{k-1}$ ($k = 1, 2, \dots, N$). For $k = 1, 2, \dots, N$, construct the finite element spaces $Y^{h,k} \in H_0^1(\Omega)$ (similar to Y^h) with the mesh $T^{h,k}$, and construct the finite element spaces $U^{h,k} \in L^2(\Omega_U)$ (similar to U^h) with the mesh $T_U^{h,k}$. Let h_{τ^k} ($h_{\tau_U^k}$) denote the maximum diameter of the element τ^k (τ_U^k) in $T^{h,k}$ ($T_U^{h,k}$). To simplify notation, we will regard a discrete quantity Q^k as $Q(t)$ such that $Q(t)|_{I_k} \equiv Q^k$, and we will denote $\tau(t)$, $\tau_U(t)$, $h_\tau(t)$, and $h_{\tau_U}(t)$ by τ , τ_U , h_τ , and h_{τ_U} , respectively. Let

$$\begin{aligned} W^\delta &= \left\{ w \mid w(x, t)|_{\Omega \times I_k} = \sum_{j=0}^r t^j \varphi_j(x), \quad \varphi_j \in Y^{h,k} \right\}, \quad r \geq 0, \\ X^\delta &= \{v \mid v(x, t)|_{\Omega \times I_k} = \psi(x), \quad \psi \in U^{h,k}\}, \quad K^\delta \subset (X^\delta \cap K), \\ [w]_k &= w_k^+ - w_k^-, \quad w_k^\pm = \lim_{s \rightarrow 0^\pm} w(t_k + s). \end{aligned}$$

The fully discrete approximation scheme is to find $(y_\delta, u_\delta) \in W^\delta \times X^\delta$ such that

$$(5) \quad \min_{u_\delta \in K^\delta} \int_0^T (g(y_\delta) + h(u_\delta)) dt$$

subject to

$$\begin{aligned} \int_0^T ((\partial_t y_\delta, w) + a(y_\delta, w)) dt + \sum_{k=1}^{N-1} ([y_\delta]_k, w_k^+) + ((y_\delta)_0^+ - y_0^h, w_0^+) \\ = \int_0^T (f + Bu_\delta, w) dt \quad \forall w \in W^\delta, \end{aligned}$$

where $y_0^h \in Y^{h,0}$ is the approximation to y_0 . It follows that the control problem (5) has a unique solution (y_δ, u_δ) , and that a pair $(y_\delta, u_\delta) \in W^\delta \times X^\delta$ is the solutions of (5) if and only if there is costate $p_\delta \in W^\delta$ such that the triplet $(y_\delta, p_\delta, u_\delta)$ satisfies

the following optimality conditions:

$$(6) \quad \left\{ \begin{array}{l} \int_0^T ((\partial_t y_\delta, w) + a(y_\delta, w)) dt + \sum_{k=0}^{N-1} ([y_\delta]_k, w_k^+) \\ = \int_0^T (f + Bu_\delta, w) dt \\ \int_0^T (-\partial_t p_\delta, q) + a(p_\delta, q) dt - \sum_{k=1}^N ([p_\delta]_k, q_k^-) \\ = \int_0^T (g'(y_\delta), q) dt \\ \int_0^T (h'(u_\delta) + B^* p_\delta, v - u_\delta)_U dt \geq 0 \end{array} \right. \quad \begin{array}{l} \forall w \in W^\delta, \quad (y_\delta)_0^- = y_0^h, \\ \forall q \in W^\delta, \quad (p_\delta)_N^+ = 0, \\ \forall v \in K^\delta. \end{array}$$

This is a finite dimensional optimization problem and may be solved by existing mathematical programming methods. The above DG approximation of the control problem has been used in practical problems; see [40].

In order to obtain a numerical solution of acceptable accuracy for the optimal control problem, the finite element meshes have to be refined according to a mesh refinement scheme. Adaptive finite element approximation uses a posteriori error indicator to guide the mesh refinement procedure. In the following section we shall derive some a posteriori error estimates for the DG finite element approximation of the optimal control problem governed by parabolic equations, which can be used as such an error indicator in developing adaptive finite element schemes of the control problem.

3. A posteriori error estimates. In this section we derive a posteriori error estimates for the DG finite element approximation of the convex optimal problem governed by a parabolic equation. In general, analysis of the finite element approximation of a control problem governed by parabolic equations is more involved than is that of a control problem governed by elliptic equations. The main complication is due to the fact that the properties of the time variable and its discretization are quite different from those of the space (elliptic) variables. Thus different techniques are needed to handle the two groups of variables, and their interactions.

We now need more assumptions on B and g in deriving our estimates. We essentially assume that B is bounded from $L^2(0, T; L^2(\Omega_U))$ to $L^2(0, T; L^2(\Omega))$ so that differential operators are excluded. To derive L^∞ estimates, we need a continuity from $L^2(\Omega_U)$ to $L^2(\Omega)$ uniformly with respect to t , while we have embedded U into X . For g we assume that its derivative is Lipschitz continuous. Thus we make the following assumptions:

$$(7) \quad |(Bv, w)_X| = |(v, B^*w)| \leq C \|v\|_{0, \Omega_U} \|w\|_{0, \Omega} \quad \forall v \in U, w \in Y,$$

$$(8) \quad |(g'(v) - g'(w), q)| \leq C \|v - w\|_{0, \Omega} \|q\|_{0, \Omega} \quad \forall v, w, q \in Y,$$

and there is a constant $c > 0$ such that

$$(9) \quad (h'(v) - h'(w), v - w) \geq c \|v - w\|_{0, \Omega_U}^2 \quad \forall v, w \in U,$$

$$(10) \quad (g'(v) - g'(w), v - w) \geq 0 \quad \forall v, w \in Y,$$

which are convex conditions on the functionals h and g . These conditions hold for the quadratic control problems where $\Omega = \Omega_U$ and $B = I$.

The following lemma is important in deriving residual type a posteriori error estimates.

LEMMA 3.1. *Let π_h be the average interpolation operator defined in [21]. For any $v \in W^{1,q}(\Omega^h)$ and $1 \leq q \leq \infty$,*

$$\|v - \pi_h v\|_{l,q,\tau} \leq C \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} h_{\tau'}^{m-l} |v|_{m,q,\tau'}, \quad v \in W^{m,q}(\tau'), \quad l = 0, 1, \quad l \leq m \leq 2.$$

REMARK 3.1. *One of the key steps in deriving a posteriori error estimates for the discontinuous Galerkin method is to construct a suitable L^2 stable approximation of the solution of the dual equation. In [12], this approximation is defined to be the space-time L^2 -projection of the solution. However, for this selection the spatial projection error cannot be bounded locally due to the global nature of the projecting onto continuous piecewise polynomial functions. This leads to the inconvenience restriction in [12] on the mesh used in the approximation: the change in the size of the elements in the mesh must be very smooth, which may be unrealistic in an adaptive finite element implementation. We shall define this approximation to be the L^2 -projection of the solution of the dual equation in time, but the quasi-interpolant of the solution in space as defined in [21]. It follows from Lemma 3.1 that this approximation is L^2 stable. Furthermore, optimal approximation results hold on local patches surrounding a particular element. It is then possible to derive a posteriori error estimates assuming only nondegeneracy of the mesh.*

LEMMA 3.2 (see [25]). *For all $v \in W^{1,q}(\Omega)$, $1 \leq q \leq \infty$,*

$$(11) \quad \|v\|_{0,q,\partial\tau} \leq C(h_{\tau}^{-1/q} \|v\|_{0,q,\tau} + h_{\tau}^{1-1/q} |v|_{1,q,\tau}).$$

3.1. $L^2(L^2)$ error estimates. First, let us present a lemma which is essential for our a posteriori error estimate analysis. Assuming that one can find an element v in K^δ to approximate the optimal control in an appropriate way, the approximation error in the control is then shown to be represented by an a posteriori error estimator, plus the approximation error in the costate. For constraints of obstacle type, this assumption can be verified for piecewise constant control approximation by taking v to be the integral average of the optimal control; see Examples 3.1 and 3.2.

LEMMA 3.3. *Let (y, p, u) and $(y_\delta, p_\delta, u_\delta)$ be the solutions of (2) and (6). Assume that (9), (10), and (7) hold; $K^\delta \subset K$; for all $1 \leq k \leq N$, $(h'(u_\delta) + B^* p_\delta)|_{\tau_U^k \times I_k} \in H^1(\tau_U^k \times I_k)$; and there is a $v \in K^\delta$ such that*

$$(12) \quad \left| \int_{I_k} (h'(u_\delta) + B^* p_\delta, v - u)_U dt \right| \leq C \int_{I_k} \sum_{\tau_U \in T_U^{h,k}} (h_{\tau_U} |h'(u_\delta) + B^* p_\delta|_{1,\tau_U} + \Delta t_k \|\partial_t(h'(u_\delta) + B^* p_\delta)\|_{0,\tau_U}) \|u - u_\delta\|_{0,\tau_U} dt.$$

Then we have

$$(13) \quad \|u - u_\delta\|_{L^2(0,T;L^2(\Omega_U))}^2 \leq C \left(\eta_1^2 + \|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 \right),$$

where

$$\eta_1^2 = \sum_{k=1}^N \sum_{\tau_U \in T_U^{h,k}} \int_{I_k} (h_{\tau_U}^2 |h'(u_\delta) + B^* p_\delta|_{1,\tau_U}^2 + \Delta t_k^2 \|\partial_t(h'(u_\delta) + B^* p_\delta)\|_{0,\tau_U}^2) dt$$

and $(y^{u_\delta}, p^{u_\delta}) \in W \times W$ is defined by the following system:

$$(14) \quad \begin{cases} (\partial_t y^{u_\delta}, w) + a(y^{u_\delta}, w) = (f + Bu_\delta, w) & \forall w \in Y, \quad t \in (0, T], \\ y^{u_\delta}(0) = y_0, \end{cases}$$

$$(15) \quad \begin{cases} -(\partial_t p^{u_\delta}, q) + a(q, p^{u_\delta}) = (g'(y^{u_\delta}), q) & \forall q \in Y, \quad t \in [0, T], \\ p^{u_\delta}(T) = 0. \end{cases}$$

Proof. It follows from (9), (2)₃, and (6)₃ that, for any $v \in K^\delta$,

$$(16) \quad \begin{aligned} c\|u - u_\delta\|_{L^2(0,T;L^2(\Omega_U))}^2 &\leq \int_0^T (h'(u), u - u_\delta)_U dt - \int_0^T (h'(u_\delta), u - u_\delta)_U dt \\ &\leq -\int_0^T (B^*p, u - u_\delta)_U dt - \int_0^T (h'(u_\delta), u - u_\delta)_U dt + \int_0^T (h'(u_\delta) + B^*p_\delta, v - u_\delta)_U dt \\ &\leq \int_0^T (h'(u_\delta) + B^*p_\delta, v - u)_U dt + \int_0^T (B^*(p_\delta - p^{u_\delta}), u - u_\delta)_U dt \\ &\quad + \int_0^T (B^*(p^{u_\delta} - p), u - u_\delta)_U dt, \end{aligned}$$

where p^{u_δ} is defined in (15). It is easy to see from (2), (14), and (15) that

$$(17) \quad (\partial_t(y^{u_\delta} - y), w) + a(y^{u_\delta} - y, w) = (B(u_\delta - u), w) \quad \forall w \in Y,$$

$$(18) \quad -(\partial_t(p^{u_\delta} - p), q) + a(q, p^{u_\delta} - p) = (g'(y^{u_\delta}) - g'(y), q) \quad \forall q \in Y.$$

Taking $w = p^{u_\delta} - p$ in (17) and $q = y^{u_\delta} - y$ in (18) and using $(y^{u_\delta} - y)|_{t=0} = (p^{u_\delta} - p)|_{t=T} = 0$ and (10) lead to

$$(19) \quad \begin{aligned} \int_0^T (B(u_\delta - u), p^{u_\delta} - p) dt &= (y^{u_\delta} - y, p^{u_\delta} - p) \Big|_0^T \\ &\quad + \int_0^T (g'(y^{u_\delta}) - g'(y), y^{u_\delta} - y) dt \geq 0. \end{aligned}$$

Let v be the function satisfying (12). Then by (12), (7), and (19),

$$(20) \quad c\|u - u_\delta\|_{L^2(0,T;L^2(\Omega_U))}^2 \leq C \left(\eta_1^2 + \|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 \right) + \frac{c}{2} \|u - u_\delta\|_{L^2(0,T;L^2(\Omega_U))}^2,$$

which completes the proof. \square

The assumption (12) is related to approximation properties of the convex set K . For instance, it always holds for unconstrained control, where $K = U$. For constraints of obstacle type, this assumption can also be verified.

We shall use the following dual equations: For given $f \in L^2(0, T; L^2(\Omega))$,

$$(21) \quad \begin{cases} \partial_t \varphi - \operatorname{div}(A \nabla \varphi) = f, & (x, t) \in \Omega \times (0, T], \\ \varphi|_{\partial\Omega} = 0, \quad t \in [0, T], & \varphi(x, 0) = 0, \quad x \in \Omega, \end{cases}$$

and

$$(22) \quad \begin{cases} -\partial_t \psi - \operatorname{div}(A^* \nabla \psi) = f, & (x, t) \in \Omega \times [0, T], \\ \psi|_{\partial\Omega} = 0, \quad t \in [0, T], & \psi(x, T) = 0, \quad x \in \Omega. \end{cases}$$

A similar idea is used in [21] for a Lagrange–Galerkin method.

LEMMA 3.4 (see [21]). *Assume that Ω is a convex domain. Let φ and ψ be the solutions of (21) and (22), respectively. Then, for $v = \varphi$ or $v = \psi$,*

$$\begin{aligned} \|v\|_{L^\infty(0,T;L^2(\Omega))} &\leq C\|f\|_{L^2(0,T;L^2(\Omega))}, \\ \|\nabla v\|_{L^2(0,T;L^2(\Omega))} &\leq C\|f\|_{L^2(0,T;L^2(\Omega))}, \\ \|D^2v\|_{L^2(0,T;L^2(\Omega))} &\leq C\|f\|_{L^2(0,T;L^2(\Omega))}, \\ \|\partial_t v\|_{L^2(0,T;L^2(\Omega))} &\leq C\|f\|_{L^2(0,T;L^2(\Omega))}, \end{aligned}$$

where $D^2v = \max_{1 \leq i, j \leq n} |\partial^2 v / \partial x_i \partial x_j|$.

In the following we deal with the error $\|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}$ to derive the final estimates. Let $\partial T^{h,k}$ be the set consisting of all the faces l of any $\tau^k \in T^{h,k}$ such that l is not on $\partial\Omega$. The A-normal derivative jump over the interior face l is defined by

$$[(A\nabla v) \cdot \mathbf{n}]_l = ((A\nabla v)|_{\partial\tau_l^1} - (A\nabla v)|_{\partial\tau_l^2}) \cdot \mathbf{n},$$

where \mathbf{n} is the unit outer normal vector of τ_l^1 on $l = \bar{\tau}_l^1 \cap \bar{\tau}_l^2$. Let h_l be the maximum diameter of the face l .

LEMMA 3.5. *Let (y, p, u) , $(y_\delta, p_\delta, u_\delta)$, and p^{u_δ} be the solutions of (2), (6), and (15), respectively. Under the conditions of Lemma 3.4 and (8),*

$$\|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \sum_{i=0,2-7} \eta_i^2,$$

where

$$\begin{aligned} \eta_0^2 &= \|y_0^h - y_0\|_{0,\Omega}^2, \\ \eta_2^2 &= \sum_{k=1}^N \sum_{\tau \in T^{h,k}} \int_{I_k} h_\tau^4 \left\| \partial_t p_\delta + g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta) + \frac{[p_\delta]_k}{\Delta t_k} \right\|_{0,\tau}^2 dt, \\ \eta_3^2 &= \sum_{k=1}^N \sum_{\tau \in T^{h,k}} \int_{I_k} \Delta t_k^2 \|(\pi_k - I)(g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta))\|_{0,\tau}^2 dt, \\ \eta_4^2 &= \sum_{k=1}^N \sum_{\tau \in T^{h,k}} \int_{I_k} h_\tau^4 \left\| \partial_t y_\delta - f - B u_\delta - \operatorname{div}(A \nabla y_\delta) + \frac{[y_\delta]_{k-1}}{\Delta t_k} \right\|_{0,\tau}^2 dt, \\ \eta_5^2 &= \sum_{k=1}^N \sum_{\tau \in T^{h,k}} \int_{I_k} \Delta t_k^2 \|(\pi_k - I)(f + \operatorname{div}(A \nabla y_\delta))\|_{0,\tau}^2 dt, \\ \eta_6^2 &= \sum_{k=1}^N \sum_{l \in \partial T^{h,k}} \int_{I_k} h_l^3 (\|[(A \nabla y_\delta) \cdot \mathbf{n}]\|_{0,l}^2 + \|[A^* \nabla p_\delta] \cdot \mathbf{n}\|_{0,l}^2) dt, \\ \eta_7^2 &= \sum_{k=1}^N \sum_{\tau \in T^{h,k}} \Delta t_k (\|[y_\delta]_{k-1}\|_{0,\Omega}^2 + \|[p_\delta]_k\|_{0,\Omega}^2), \end{aligned}$$

where $\pi_k : L^2(I_k) \rightarrow \mathbb{P}_r(I_k)$ is the L^2 -projection operator on the variable t .

Proof. Let φ be the solution of (21) with $f = p_\delta - p^{u_\delta}$ and $\varphi_I \in X^\delta$ be the interpolation of φ such that

$$(23) \quad \varphi_I|_{\Omega \times I_k} = \pi_{h,k} \pi_k \varphi, \quad k = 1, 2, \dots, N,$$

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where $\pi_{h,k}$ is defined in Lemma 3.1 corresponding to the partitioning $T^{h,k}$ and $\pi_k : L^2(I_k) \rightarrow \mathbb{P}_r(I_k)$ is the L^2 -projection operator on the variable t . Then it follows from (21), (15), (6), and Green’s formula that

$$\begin{aligned} & \|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 = \int_0^T (p_\delta - p^{u_\delta}, f) dt \\ &= \int_0^T (p_\delta - p^{u_\delta}, \partial_t \varphi - \operatorname{div}(A \nabla \varphi)) dt \\ &= \int_0^T (-(\partial_t(p_\delta - p^{u_\delta}), \varphi) + a(\varphi, p_\delta - p^{u_\delta})) dt - \sum_{k=1}^N ([p_\delta]_k, \varphi_k^-) \\ &= \int_0^T (-(\partial_t p_\delta + g'(y_\delta), \varphi) + a(\varphi, p_\delta) - a(\varphi_I, p_\delta) + (\partial_t p_\delta + g'(y_\delta), \varphi_I)) dt \\ &\quad + \sum_{k=1}^N ([p_\delta]_k, (\varphi_I - \varphi)_k^-), \end{aligned}$$

which leads to

$$\begin{aligned} & \|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 \\ &= \sum_{k=1}^N \int_{I_k} - \left(\partial_t p_\delta + g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta) + \frac{[p_\delta]_k}{\Delta t_k}, \varphi - \varphi_I \right) dt \\ (24) \quad &+ \int_0^T (g'(y_\delta) - g'(y_\delta), \varphi) dt + \int_0^T \sum_{l \in \partial T^h} \int_l [(A^* \nabla p_\delta) \cdot \mathbf{n}] (\varphi - \varphi_I) dt \\ &+ \sum_{k=1}^N \int_{I_k} \left(\frac{[p_\delta]_k}{\Delta t_k}, (\varphi_I)_k^- - \varphi_I + \varphi - \varphi_k^- \right) dt \\ &:= \sum_{i=1}^4 I_i. \end{aligned}$$

For simplicity, let

$$r_p(x, t) \Big|_{\Omega \times I_k} := \partial_t p_\delta + g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta) + \frac{[p_\delta]_k}{\Delta t_k}.$$

By Lemmas 3.1 and 3.4,

$$\begin{aligned} (25) \quad & I_1 = \sum_{k=1}^N \int_{I_k} (r_p, (\pi_{h,k} - I) \pi_k \varphi + (\pi_k - I) \varphi) dt \\ &= \sum_{k=1}^N \int_{I_k} ((r_p, (\pi_{h,k} - I) \pi_k \varphi) - ((\pi_k - I)(g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta)), (\pi_k - I) \varphi)) dt \\ &\leq C \sum_{k=1}^N \sum_{\tau \in T^{h,k}} \int_{I_k} (h_\tau^4 \|r_p\|_{0,\tau}^2 + \Delta t_k^2 \|(\pi_k - I)(g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta))\|_{0,\tau}^2) dt \\ &\quad + \sigma (\|D^2(\pi_k \varphi)\|_{L^2(0,T;L^2(\Omega))}^2 + \|\partial_t \varphi\|_{L^2(0,T;L^2(\Omega))}^2) \\ &\leq C(\eta_2^2 + \eta_3^2) + C\sigma \|p^{u_\delta} - p_\delta\|_{L^2(0,T;L^2(\Omega))}^2. \end{aligned}$$

It is easy to see that from (8) and Lemma 3.4,

$$(26) \quad \begin{aligned} I_2 &= \int_0^T (g'(y_\delta) - g'(y^{u_\delta}), \varphi) dt \\ &\leq C \|y_\delta - y^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 + \sigma \|p^{u_\delta} - p_\delta\|_{L^2(0,T;L^2(\Omega))}^2. \end{aligned}$$

Similarly, by Lemmas 3.1, 3.2, and 3.4,

$$(27) \quad \begin{aligned} I_3 &= \int_0^T \sum_{l \in \partial T^h} \int_l [(A^* \nabla p_\delta) \cdot \mathbf{n}] (\varphi - \varphi_I) dt \\ &= \sum_{k=1}^N \sum_{l \in \partial T^{h,k}} \int_{I_k} \int_l [(A^* \nabla p_\delta) \cdot \mathbf{n}] (\varphi - \pi_{h,k} \varphi) dt \\ &\leq C \sum_{k=1}^N \sum_{l \in \partial T^{h,k}} \int_{I_k} h_l^3 \|[(A^* \nabla p_\delta) \cdot \mathbf{n}]\|_{0,l}^2 dt + \sigma \|D^2 \varphi\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\leq C \eta_6^2 + C \sigma \|p^{u_\delta} - p_\delta\|_{L^2(0,T;L^2(\Omega))}^2. \end{aligned}$$

It follows from Lemma 3.4 and the Schwarz inequality that

$$(28) \quad \begin{aligned} I_4 &= \sum_{k=1}^N \int_{I_k} \left(\frac{[p_\delta]_k}{\Delta t_k}, (\varphi_I)_k^- - \varphi_I + \varphi - \varphi_k^- \right) dt \\ &\leq \sum_{k=1}^N \Delta t_k \| [p_\delta]_k \|_{0,\Omega}^2 + \sigma \left(\|\partial_t \varphi_I\|_{L^2(0,T;L^2(\Omega))}^2 + \|\partial_t \varphi\|_{L^2(0,T;L^2(\Omega))}^2 \right) \\ &\leq C \eta_7^2 + C \sigma \|p^{u_\delta} - p_\delta\|_{L^2(0,T;L^2(\Omega))}^2. \end{aligned}$$

Thus, the above estimates give

$$(29) \quad \|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \sum_{i=2,3,6,7} \eta_i^2 + C \|y_\delta - y^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2.$$

Similarly, let ψ be the solution of (22) with $f = y_\delta - y^{u_\delta}$ and $\psi_I \in X^\delta$ be the interpolation of ψ such that

$$(30) \quad \psi_I|_{\Omega \times I_k} = \pi_{h,k} \pi_k \psi, \quad k = 1, 2, \dots, N.$$

Then, by Lemma 3.4, (14), (6), and Green’s formula,

$$\begin{aligned}
 (31) \quad & \|y_\delta - y^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 = \int_0^T (y_\delta - y^{u_\delta}, f) dt = \int_0^T (y_\delta - y^{u_\delta}, -\partial_t \psi - \operatorname{div}(A^* \psi)) dt \\
 & = \int_0^T ((\partial_t(y_\delta - y^{u_\delta}), \psi) + a(y_\delta - y^{u_\delta}, \psi)) dt + \sum_{k=1}^{N-1} ([y_\delta]_k, \psi_k^+) + ((y_\delta - y^{u_\delta})_0^+, \psi_0^+) \\
 & = \int_0^T ((\partial_t y_\delta - f - Bu_\delta, \psi) + a(y_\delta, \psi) - a(y_\delta, \psi_I) - (\partial_t y_\delta - f - Bu_\delta, \psi_I)) dt \\
 & \quad + \sum_{k=0}^{N-1} ([y_\delta]_k, (\psi - \psi_I)_k^+) + ((y_\delta - y^{u_\delta})_0^+, \psi_0^+) - ([y_\delta]_0, \psi_0^+) \\
 & = \sum_{k=1}^N \int_{I_k} \left(\partial_t y_\delta - f - Bu_\delta - \operatorname{div}(A \nabla y_\delta) + \frac{[y_\delta]_{k-1}}{\Delta t_k}, \psi - \psi_I \right) dt \\
 & \quad + \int_0^T \sum_{l \in \partial T^h} \int_l [(A \nabla y_\delta) \cdot \mathbf{n}] (\psi - \psi_I) dt \\
 & \quad + \sum_{k=1}^N \int_{I_k} \left(\frac{[y_\delta]_{k-1}}{\Delta t_k}, \psi_{k-1}^+ - \psi + \psi_I - (\psi_I)_{k-1}^+ \right) dt \\
 & \quad + ((y_\delta)_0^- - (y^{u_\delta})_0^+, \psi_0^+) := \sum_{i=1}^4 J_i.
 \end{aligned}$$

Let

$$r_y(x, t) \Big|_{I_k} := \partial_t y_\delta - f - Bu_\delta - \operatorname{div}(A \nabla y_\delta) + \frac{[y_\delta]_{k-1}}{\Delta t_k}.$$

Then, as in (25), (27), and (28),

$$\begin{aligned}
 (32) \quad J_1 & = \sum_{k=1}^N \int_{I_k} (r_y, (\pi_{h,k} - I) \pi_k \psi + (\pi_k - I) \psi) dt \\
 & = \sum_{k=1}^N \int_{I_k} ((r_y, (\pi_{h,k} - I) \pi_k \psi) + ((\pi_k - I)(f + \operatorname{div}(A \nabla y_\delta)), (\pi_k - I) \psi)) dt \\
 & \leq C \sum_{k=1}^N \sum_{\tau \in T^{h,k}} \int_{I_k} (h_\tau^4 \|r_y\|_{0,\tau}^2 + \Delta t_k^2 \|(\pi_k - I)(f + \operatorname{div}(A \nabla y_\delta))\|_{0,\tau}^2) dt \\
 & \quad + \sigma (\|D^2(\pi_k \psi)\|_{L^2(0,T;L^2(\Omega))}^2 + \|\partial_t \psi\|_{L^2(0,T;L^2(\Omega))}^2) \\
 & \leq C(\eta_4^2 + \eta_5^2) + C\sigma \|y^{u_\delta} - y_\delta\|_{L^2(0,T;L^2(\Omega))}^2,
 \end{aligned}$$

$$(33) \quad J_2 = \int_0^T \sum_{l \in \partial T^h} \int_l [(A \nabla y_\delta) \cdot \mathbf{n}] (\psi - \psi_I) dt \leq C\eta_6^2 + \sigma \|y^{u_\delta} - y_\delta\|_{L^2(0,T;L^2(\Omega))}^2,$$

$$\begin{aligned}
 (34) \quad J_3 & = \sum_{k=1}^N \int_{I_k} \left(\frac{[y_\delta]_{k-1}}{\Delta t_k}, \psi_{k-1}^+ - \psi + \psi_I - (\psi_I)_{k-1}^+ \right) dt \\
 & \leq C\eta_7^2 + \sigma \|y^{u_\delta} - y_\delta\|_{L^2(0,T;L^2(\Omega))}^2,
 \end{aligned}$$

and

$$J_4 = ((y_\delta)_0^- - (y^{u_\delta})_0^+, \psi_0^+) \leq C\eta_0^2 + \sigma \|y^{u_\delta} - y_\delta\|_{L^2(0,T;L^2(\Omega))}^2.$$

Hence

$$(35) \quad \|y_\delta - y^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \sum_{i=0,4-7} \eta_i^2.$$

We complete the proof by combining the estimates (29) and (35). \square

From Lemmas 3.3 and 3.5, we have the following a posteriori error estimates.

THEOREM 3.1. *Let (y, p, u) and $(y_\delta, p_\delta, u_\delta)$ be the solutions of (2) and (6). Assume that the conditions in Lemmas 3.3–3.5 are valid; then*

$$\|u - u_\delta\|_{L^2(0,T;L^2(\Omega))}^2 + \|y - y_\delta\|_{L^2(0,T;L^2(\Omega))}^2 + \|p - p_\delta\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \sum_{i=0}^7 \eta_i^2,$$

where η_i are defined in Lemmas 3.3 and 3.5.

Proof. We obtain from (13), (35), and (29) that

$$\|u - u_\delta\|_{L^2(0,T;L^2(\Omega))}^2 + \|y^{u_\delta} - y_\delta\|_{L^2(0,T;L^2(\Omega))}^2 + \|p^{u_\delta} - p_\delta\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \sum_{i=0}^7 \eta_i^2.$$

Then the desired results follows from the triangle inequality and

$$(36) \quad \|p - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))} \leq C \|y - y^{u_\delta}\|_{L^2(0,T;L^2(\Omega))} \leq C \|u - u_\delta\|_{L^2(0,T;L^2(\Omega))},$$

which can be derived from (17) and (18). \square

It seems to be difficult to derive any lower error bounds for the control problem. As matter of fact, there seem to be no good lower a posteriori error bounds in the literature even for the full backward-Euler finite element approximation of linear parabolic equations. The main difficulty seems to be that the properties of the time variable and its discretization are quite different from those of the space variables. Novel techniques are yet to be developed to derive lower bounds for such mixed approximations.

REMARK 3.2. *It is clear that the above a posteriori error estimator consists of two parts. The η_1^2 part results from the approximation error of the inequality in the optimality condition (2). The other (more familiar) part (η_i^2 ($i = 0, 2, \dots, 7$)) is contributed from the approximation error of the state and costate equations and in this sense is more or less standard. Among them, η_1^2 mainly indicates the approximation error for the control, and the other part mainly reflects the approximation error for the state and costate.*

The part (η_i^2 ($i = 0, 2, \dots, 7$)) can be further divided into two parts: one from the approximation error of the state equation and the other from that of the costate equation. Clearly, a posteriori error estimators obtained solely from the state equation, which only present the part contributed from the state equation, may fail to reflect the main approximation error of the optimal control problem and thus fail to yield efficient mesh refinements.

The above error estimates are applicable to a wide range of control problems. It may be possible to further improve them in some individual cases, as will be seen in the next section. To this end, it is clear that one needs to derive improved error

estimates for the approximation of the inequality in (2), and thus one requires explicit information on the structure of K .

REMARK 3.3. *It is generally difficult to know the exact bounding constant C in Theorem 3.1, as is true for most a posteriori error estimates of residual type. The constant is contributed from those in the interpolation results (e.g., Lemmas 3.1–3.2), the stability results (e.g., Lemmas 3.3–3.4), and the Sobolev embedding theorems. For simpler situations, it may be possible to trace down all those constants and to give the bounding constant good upper bounds; see [9] for some of the latest advances on this aspect. Generally this is a complex procedure. On the other hand, a posteriori estimators of residual type can be (actually have widely been) used to guide mesh refinements without having exact knowledge on the bounding constants, provided they are not too large. It seems that the magnitude of the bounding constants does not cause any serious problems in guiding mesh refinements for elliptic and parabolic equations, although it does bring up serious concerns in CFD (see [23]), since it can indeed be extremely large there.*

In our case, it seems that the bounding constant in Theorems 3.1–3.2 will have a similar magnitude as those for the standard parabolic equation case, as the only new contribution here is from the constant C in Lemma 3.3. This constant can be traced down in Examples 3.1–3.2, which in turns depends on the bounding constant for the integral averaging interpolator $\pi_{\delta,k}^a$. It is known that the bounding constant associated with $\pi_{\delta,k}^a$ will not be very large; see [9] for the details.

REMARK 3.4. *It is not straightforward to develop suitable implementation techniques for $(x-t)$ mesh adaptivity of parabolic control problems. To the best of our knowledge, there seems to be no existing work in the literature, even using the same meshes for the state and the control. For instance, it seems impossible to simply extend the mesh adaptivity techniques developed for evolutionary equations (e.g., parabolic or Navier–Stokes equations) to the control problem that we have just studied. Although the state equation is evolutionary, the optimal control problem itself is clearly not. It is impossible to solve the control problem step by step in time, although this is possible for the state equation. This calls for new implementation techniques on mesh adaptivity for the optimal control governed by evolutionary state equations. From the above analysis of η_1^2 (η_i^2), it is also clear that the most suitable implementation, and thus the optimal mesh refinements will greatly depend on what is the most important quantity to be computed in a particular control problem. It also depends on the structure of the meshes used in the computations. Furthermore, as some large discretized optimization problems may need to be repeatedly solved, one may have to use a suitable multigrids method together with mesh adaptivity. Issues like which items in the estimator are more important and how to pick up the constant C are also important. It is clear that a systematic study of this is much needed. These issues will be investigated in our future research.*

3.2. $L^\infty(L^2)$ error estimates. In some adaptive schemes, it is more desirable to have $L^\infty(L^2)$ estimates. In this subsection, we give error estimates in $L^\infty(L^2)$ -norm. Concretely, we shall use the norm of the following form:

$$\|v\|_{I_k, Q} = \left\{ \frac{1}{\Delta t_k} \int_{I_k} \|v(t)\|_{0, Q}^2 dt \right\}^{1/2}, \quad Q = \Omega_U, \tau_U, \Omega, \tau, l.$$

We now need to consider the following dual equations for any $1 \leq k \leq N - 1$:

$$(37) \quad \begin{cases} \partial_t \varphi - \operatorname{div}(A \nabla \varphi) = 0, & (x, t) \in \Omega \times (t_k, T], \\ \varphi|_{\partial\Omega} = 0, \quad t \in [t_k, T], & \varphi(x, t_k) = \varphi_*(x), \quad x \in \Omega, \end{cases}$$

and

$$(38) \quad \begin{cases} -\partial_t \psi - \operatorname{div}(A^* \nabla \psi) = 0, & (x, t) \in \Omega \times [0, t_k), \\ \psi|_{\partial\Omega} = 0, \quad t \in [0, t_k], & \psi(x, t_k) = \psi_*(x), \quad x \in \Omega. \end{cases}$$

We have the following stability results [12].

LEMMA 3.6. *Assume that Ω is a convex domain. Let φ and ψ be the solutions of (37) and (38), respectively. Then*

$$\begin{aligned} \|\varphi\|_{L^\infty(t_k, T; L^2(\Omega))} &\leq C \|\varphi_*\|_{L^2(\Omega)}, \\ \|\varphi\|_{L^2(t_k, t_k + \varepsilon; L^2(\Omega))} &\leq C \sqrt{\varepsilon} \|\varphi_*\|_{L^2(\Omega)}, \quad 0 < \varepsilon < T - t_k, \\ \|\nabla \varphi\|_{L^2(t_k, T; L^2(\Omega))} &\leq C \|\varphi_*\|_{L^2(\Omega)}, \\ \|\sqrt{t - t_k} |D^2 \varphi|\|_{L^2(t_k, T; L^2(\Omega))} &\leq C \|\varphi_*\|_{L^2(\Omega)}, \\ \|\sqrt{t - t_k} \partial_t \varphi\|_{L^2(t_k, T; L^2(\Omega))} &\leq C \|\varphi_*\|_{L^2(\Omega)} \end{aligned}$$

and

$$\begin{aligned} \|\psi\|_{L^\infty(0, t_k; L^2(\Omega))} &\leq C \|\psi_*\|_{L^2(\Omega)}, \\ \|\psi\|_{L^2(t_k - \varepsilon, t_k; L^2(\Omega))} &\leq C \sqrt{\varepsilon} \|\psi_*\|_{L^2(\Omega)}, \quad 0 < \varepsilon < t_k, \\ \|\nabla \psi\|_{L^2(0, t_k; L^2(\Omega))} &\leq C \|\psi_*\|_{L^2(\Omega)}, \\ \|\sqrt{t_k - t} |D^2 \psi|\|_{L^2(0, t_k; L^2(\Omega))} &\leq C \|\psi_*\|_{L^2(\Omega)}, \\ \|\sqrt{t_k - t} \partial_t \psi\|_{L^2(0, t_k; L^2(\Omega))} &\leq C \|\psi_*\|_{L^2(\Omega)}, \end{aligned}$$

where $D^2 v = \max_{1 \leq i, j \leq n} |\partial^2 v / \partial x_i \partial x_j|$.

THEOREM 3.2. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (2) and (6), respectively. Assume that the conditions in Theorem 3.1 and Lemma 3.6 are valid; then*

$$\max_{1 \leq k \leq N} (\|u - u_h\|_{I_k, \Omega_U}^2 + \|y - y_h\|_{I_k, \Omega}^2 + \|p - p_h\|_{I_k, \Omega}^2) \leq C \sum_{i=0}^8 \mathfrak{N}_i^2,$$

where

$$\begin{aligned} \mathfrak{N}_0^2 &= \|y_0^h - y_0\|_{0, \Omega}^2, \\ \mathfrak{N}_1^2 &= \max_{1 \leq k \leq N} \sum_{\tau_U \in T_U^{h, k}} (h_{\tau_U}^2 \|\nabla(h'(u_\delta) + B^* p_\delta)\|_{I_k, \tau_U}^2 + \Delta t_k^2 \|\partial_t(h'(u_\delta) + B^* p_\delta)\|_{I_k, \tau_U}^2), \\ \mathfrak{N}_2^2 &= \max_{1 \leq k \leq N} \sum_{\tau \in T^{h, k}} h_\tau^2 (\Delta t_k + L_N h_\tau^2) \left\| \partial_t p_\delta + g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta) + \frac{[p_\delta]_k}{\Delta t_k} \right\|_{I_k, \tau}^2, \\ \mathfrak{N}_3^2 &= \max_{1 \leq k \leq N} \sum_{\tau \in T^{h, k}} \Delta t_k^2 \|(\pi_k - I)(g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta))\|_{I_k, \tau}^2, \\ \mathfrak{N}_4^2 &= \max_{1 \leq k \leq N} \sum_{l \in \partial T^{h, k}} h_l (\Delta t_k + L_N h_l^2) \|(A^* \nabla p_\delta) \cdot \mathbf{n}\|_{I_k, l}^2, \end{aligned}$$

$$\begin{aligned} \mathfrak{N}_5^2 &= \max_{1 \leq k \leq N} \sum_{\tau \in T^{h,k}} h_\tau^2 (\Delta t_k + L_N h_\tau^2) \left\| \partial_t y_\delta - f - B u_\delta - \operatorname{div}(A \nabla y_\delta) + \frac{[y_\delta]_{k-1}}{\Delta t_k} \right\|_{I_k, \tau}^2, \\ \mathfrak{N}_6^2 &= \max_{1 \leq k \leq N} \sum_{\tau \in T^{h,k}} \Delta t_k^2 \|(\pi_k - I)(f + \operatorname{div}(A \nabla y_\delta))\|_{I_k, \tau}^2, \\ \mathfrak{N}_7^2 &= \max_{1 \leq k \leq N} \sum_{l \in \partial T^{h,k}} h_l (\Delta t_k + L_N h_l^2) \|[(A \nabla y_\delta) \cdot \mathbf{n}]\|_{I_k, l}^2, \\ \mathfrak{N}_8^2 &= \max_{1 \leq k \leq N} (\| [y_\delta]_{k-1} \|_{0, \Omega}^2 + \| [p_\delta]_k \|_{0, \Omega}^2), \end{aligned}$$

where

$$L_N = \max \left\{ \max_{1 \leq k \leq N-2} \sum_{k'=k+2}^N \frac{\Delta t_{k'}}{t_{k'-1} - t_k}, \max_{2 \leq k \leq N} \sum_{k'=1}^{k-1} \frac{\Delta t_{k'}}{t_k - t_{k'}} \right\}.$$

Proof. We first consider $\|u - u_h\|_{L^2(I_k; L^2(\Omega_U))}$. As in (16) and (20), for any $v \in K^\delta$, we have

$$\begin{aligned} c \|u - u_\delta\|_{L^2(I_k; L^2(\Omega_U))}^2 &\leq \int_{I_k} (h'(u), u - u_\delta)_U dt - \int_{I_k} (h'(u_\delta), u - u_\delta)_U dt \\ &\leq \int_{I_k} (h'(u_\delta) + B^* p_\delta, v - u)_U dt + \int_{I_k} (B^*(p_\delta - p), u - u_\delta)_U dt \\ &\leq C \int_{I_k} (h_{\tau_U}^2 |h'(u_\delta) + B^* p_\delta|_{1, \tau_U}^2 + \Delta t_k^2 \|\partial_t (h'(u_\delta) + B^* p_\delta)\|_{0, \tau_U}^2) dt \\ &\quad + C \left(\|p_\delta - p^{u_\delta}\|_{L^2(I_k; L^2(\Omega))}^2 + \|p^{u_\delta} - p\|_{L^2(I_k; L^2(\Omega))}^2 \right) + \frac{c}{2} \|u - u_\delta\|_{L^2(I_k; L^2(\Omega_U))}^2. \end{aligned}$$

It is easy to see from (18) and (8) that

$$\begin{aligned} \|p^{u_\delta} - p\|_{I_k, \Omega} &\leq \|p^{u_\delta} - p\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq C \|y^{u_\delta} - y\|_{L^2(0, T; L^2(\Omega))}^2 \\ &\leq C \|u - u_\delta\|_{L^2(0, T; L^2(\Omega_U))}^2. \end{aligned}$$

We thus obtain

$$(39) \quad \|u - u_\delta\|_{I_k, \Omega_U}^2 \leq C (\mathfrak{N}_1^2 + \|p_\delta - p^{u_\delta}\|_{I_k, \Omega}^2) + C \|u - u_\delta\|_{L^2(0, T; L^2(\Omega_U))}^2.$$

The last term above has been estimated in Theorem 3.1.

We consider $\|p_\delta - p^{u_\delta}\|_{I_k, \Omega}^2$ for any $1 \leq k \leq N$. Let φ be the solution of the dual problem

$$\begin{cases} \partial_t \varphi - \operatorname{div}(A \nabla \varphi) = p_\delta - p^{u_\delta}, & (x, t) \in \Omega \times I_k, \\ \varphi|_{\partial \Omega} = 0, \quad t \in I_k, & \varphi(x, t_{k-1}) = 0, \quad x \in \Omega, \end{cases}$$

and let φ_I be defined as in (23). Then, similarly to (24),

$$\begin{aligned} & \|p_\delta - p^{u_\delta}\|_{L^2(I_k;L^2(\Omega))}^2 = \int_{I_k} (p_\delta - p^{u_\delta}, \partial_t \varphi - \operatorname{div}(A \nabla \varphi)) dt \\ &= \int_{I_k} (-\partial_t(p_\delta - p^{u_\delta}), \varphi) + a(\varphi, p_\delta - p^{u_\delta}) dt + (p_\delta - p^{u_\delta}, \varphi)_k^- \\ &= \int_{I_k} (-\partial_t p_\delta + g'(y^{u_\delta}), \varphi) + a(\varphi, p_\delta) - a(\varphi_I, p_\delta) + (\partial_t p_\delta + g'(y_\delta), \varphi_I) dt \\ &\quad + ([p_\delta]_k, (\varphi_I)_k^-) + (p_\delta - p^{u_\delta}, \varphi)_k^- \\ &= \int_{I_k} \left(\partial_t p_\delta + g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta) + \frac{[p_\delta]_k}{\Delta t_k}, \varphi_I - \varphi \right) dt + \int_{I_k} (g'(y_\delta) - g'(y^{u_\delta}), \varphi) dt \\ &\quad + \int_{I_k} \sum_{l \in \partial T^h} \int_l [(A^* \nabla p_\delta) \cdot \mathbf{n}](\varphi - \varphi_I) dt + \int_{I_k} \left(\frac{[p_\delta]_k}{\Delta t_k}, (\varphi_I)_k^- - \varphi_I + \varphi - \varphi_{k-1}^+ \right) dt \\ &\quad + (p_\delta - p^{u_\delta}, \varphi)_k^- := \sum_{i=1}^5 \mathcal{I}_i. \end{aligned}$$

It is easy to see that \mathcal{I}_i ($i = 1-4$) can be estimated in the same way as in (25)–(28) such that

$$\begin{aligned} (40) \quad \mathcal{I}_1 &\leq C \sum_{\tau \in T^h} \int_{I_k} (h_\tau^4 \|r_p\|_{0,\tau}^2 + \Delta t_k^2 \|(\pi_k - I)(g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta))\|_{0,\tau}^2) dt \\ &\quad + \sigma \|p_\delta - p^{u_\delta}\|_{L^2(I_k;L^2(\Omega))}^2, \\ \mathcal{I}_2 &\leq C \|y_\delta - y^{u_\delta}\|_{L^2(I_k;L^2(\Omega))}^2 + \sigma \|p_\delta - p^{u_\delta}\|_{L^2(I_k;L^2(\Omega))}^2, \\ (41) \quad \mathcal{I}_3 &\leq C \sum_{l \in \partial T^h} \int_{I_k} h_l^3 \|[(A^* \nabla p_\delta) \cdot \mathbf{n}]\|_{0,l}^2 dt + \sigma \|p_\delta - p^{u_\delta}\|_{L^2(I_k;L^2(\Omega))}^2, \\ (42) \quad \mathcal{I}_4 &\leq \Delta t_k \| [p_\delta]_k \|_{0,\Omega}^2 + \sigma \|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2. \end{aligned}$$

We bound \mathcal{I}_5 by

$$(43) \quad \begin{aligned} \mathcal{I}_5 &\leq \|(p_\delta - p^{u_\delta})_k^-\|_{0,\Omega} \sqrt{\Delta t_k} \|\partial_t \varphi\|_{L^2(I_k;L^2(\Omega))} \\ &\leq C \Delta t_k \|(p_\delta - p^{u_\delta})_k^-\|_{0,\Omega}^2 + \sigma \|p_\delta - p^{u_\delta}\|_{L^2(I_k;L^2(\Omega))}^2. \end{aligned}$$

Thus, the above estimates give

$$(44) \quad \|p_\delta - p^{u_\delta}\|_{I_k,\Omega}^2 \leq C \left(\sum_{i=2-4,8} \mathfrak{N}_i^2 + \|y_\delta - y^{u_\delta}\|_{I_k,\Omega}^2 + \|(p_\delta - p^{u_\delta})_k^-\|_{0,\Omega}^2 \right).$$

We then consider $\|y_\delta - y^{u_\delta}\|_{I_k,\Omega}^2$. Let ψ be the solution of the dual problem

$$\begin{cases} -\partial_t \psi - \operatorname{div}(A^* \nabla \psi) = y_\delta - y^{u_\delta}, & (x, t) \in \Omega \times I_k, \\ \psi|_{\partial\Omega} = 0, \quad t \in I_k, & \psi(x, t_k) = 0, \quad x \in \Omega, \end{cases}$$

and let ψ_I be defined as in (30). Then, similarly to (31), for $1 \leq k \leq N$,

$$\begin{aligned} & \|y_\delta - y^{u_\delta}\|_{L^2(I_k; L^2(\Omega))}^2 = \int_{I_k} (y_\delta - y^{u_\delta}, -\partial_t \psi - \operatorname{div}(A^* \psi)) dt \\ &= \int_{I_k} ((\partial_t(y_\delta - y^{u_\delta}), \psi) + a(y_\delta - y^{u_\delta}, \psi)) dt + (y_\delta - y^{u_\delta}, \psi)_{k-1}^+ \\ &= \int_{I_k} ((\partial_t y_\delta - f - Bu_\delta, \psi) + a(y_\delta, \psi) - a(y_\delta, \psi_I) - (\partial_t y_\delta - f - Bu_\delta, \psi_I)) dt \\ &\quad - ([y_\delta]_{k-1}, (\psi_I)_{k-1}^+) + (y_\delta - y^{u_\delta}, \psi)_{k-1}^+ \\ &= \int_{I_k} \left(\partial_t y_\delta - f - Bu_\delta - \operatorname{div}(A \nabla y_\delta) + \frac{[y_\delta]_{k-1}}{\Delta t_k}, \psi - \psi_I \right) dt \\ &\quad + \int_{I_k} \sum_{l \in \partial T^h} \int_l [(A \nabla y_\delta) \cdot \mathbf{n}] (\psi - \psi_I) dt + \int_{I_k} \left(\frac{[y_\delta]_{k-1}}{\Delta t_k}, \psi_k^- - \psi + \psi_I - (\psi_I)_{k-1}^+ \right) dt \\ &\quad + (y_\delta - y^{u_\delta}, \psi)_{k-1}^+ := \sum_{i=1}^4 \mathcal{J}_i, \end{aligned}$$

where \mathcal{J}_i ($i = 1-3$) can be estimated as in (40)–(43) so that

$$\begin{aligned} \mathcal{J}_1 &\leq C \int_{I_k} (h_\tau^4 \|r_y\|_{0,\tau}^2 + \Delta t_k^2 \|(\pi_k - I)(f + \operatorname{div}(A \nabla y_\delta))\|_{0,\tau}^2) dt \\ &\quad + \sigma \|y_\delta - y^{u_\delta}\|_{L^2(I_k; L^2(\Omega))}^2, \\ \mathcal{J}_2 &\leq C \sum_{l \in \partial T^h} \int_{I_k} h_l^3 \|[(A \nabla y_\delta) \cdot \mathbf{n}]\|_{0,l}^2 dt + \sigma \|y_\delta - y^{u_\delta}\|_{L^2(I_k; L^2(\Omega))}^2, \\ \mathcal{J}_3 &\leq \Delta t_k \| [y_\delta]_{k-1} \|_{0,\Omega}^2 + \sigma \|y_\delta - y^{u_\delta}\|_{L^2(I_k; L^2(\Omega))}^2, \\ \mathcal{J}_4 &\leq C \Delta t_k \| (y_\delta - y^{u_\delta})_{k-1}^+ \|_{0,\Omega}^2 + \sigma \|y_\delta - y^{u_\delta}\|_{L^2(I_k; L^2(\Omega))}^2. \end{aligned}$$

Therefore,

$$(45) \quad \|y_\delta - y^{u_\delta}\|_{I_k, \Omega}^2 \leq C \left(\sum_{i=5-8} \mathfrak{N}_i^2 + \| (y_\delta - y^{u_\delta})_{k-1}^+ \|_{0,\Omega}^2 \right).$$

We need to further consider $\| (p_\delta - p^{u_\delta})_k^- \|_{0,\Omega}^2$ and $\| (y_\delta - y^{u_\delta})_{k-1}^+ \|_{0,\Omega}^2$ ($1 \leq k \leq N$). We note that $\| (p_\delta - p^{u_\delta})_N^- \|^2 = \| [p_\delta]_N \|_{0,\Omega}^2 \leq \mathfrak{N}_8^2$. For any $1 \leq k \leq N - 1$, let φ be the solution of (37) with $\varphi_* = (p_\delta - p^{u_\delta})_k^-$ and φ_I be defined as in (23). Then, by (37), (15), and (6),

$$\begin{aligned} & \| (p_\delta - p^{u_\delta})_k^- \|_{0,\Omega}^2 = ((p_\delta - p^{u_\delta})_k^-, \varphi_*) - ((p_\delta - p^{u_\delta})_k^+, \varphi_*) + (p_\delta - p^{u_\delta}, \varphi)_k^+ \\ &= \int_{t_k}^T (-\partial_t(p_\delta - p^{u_\delta}), \varphi) + a(\varphi, p_\delta - p^{u_\delta}) dt - \sum_{k'=k+1}^N ([p_\delta]_{k'}, \varphi_{k'}^-) - ([p_\delta]_k, \varphi_*) \\ &= \int_{t_k}^T (-\partial_t p_\delta + g'(y^{u_\delta}), \varphi) + a(\varphi, p_\delta) - a(\varphi_I, p_\delta) + (\partial_t p_\delta + g'(y_\delta), \varphi_I) dt \\ &\quad + \sum_{k'=k+1}^N ([p_\delta]_{k'}, (\varphi_I - \varphi)_{k'}^-) - ([p_\delta]_k, \varphi_*) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k'=k+1}^N \int_{I_{k'}} \left(\partial_t p_\delta + g'(y_\delta) + \operatorname{div}(A^* \nabla p_\delta) + \frac{[p_\delta]_{k'}}{\Delta t_{k'}}, \varphi_I - \varphi \right) \\
 &\quad + \int_{t_k}^T (g'(y_\delta) - g'(y^{u_\delta}), \varphi) dt \\
 &\quad + \int_{t_k}^T \sum_{l \in \partial T^h} \int_l [(A^* \nabla p_\delta) \cdot \mathbf{n}] (\varphi - \varphi_I) dt \\
 &\quad + \sum_{k'=k+1}^N \int_{I_{k'}} \left(\frac{[p_\delta]_{k'}}{\Delta t_{k'}}, (\varphi_I)_{k'}^- - \varphi_I + \varphi - \varphi_{k'}^- \right) \\
 &\quad - ([p_\delta]_k, \varphi_*) := \sum_{i=1}^5 \mathcal{I}_i.
 \end{aligned}$$

We have to treat the cases in which t_k is near T and away from T differently. For simplicity, let $c_k = 1$ for $1 \leq k \leq N - 2$ and $c_{N-1} = 0$. We decompose \mathcal{I}_1 as follows:

$$\begin{aligned}
 \mathcal{I}_1 &= \left(\sum_{k'=k+1} + c_k \sum_{k'=k+2}^N \right) \int_{I_{k'}} ((r_p, (\pi_{h,k} - I)\pi_k \varphi) + ((\pi_k - I)r_p, (\pi_k - I)\varphi)) dt \\
 &:= \mathcal{I}_{11} + c_k \mathcal{I}_{12}.
 \end{aligned}$$

By Lemmas 3.1 and 3.6, we have

$$\begin{aligned}
 \mathcal{I}_{11} &\leq C \int_{I_{k+1}} \sum_{\tau \in T^h} h_\tau \|r_p\|_{0,\tau} |\pi_k \varphi|_{1,\tau} dt + \int_{I_{k+1}} \|(\pi_k - I)r_p\|_{0,\Omega} \|\varphi\|_{0,\Omega} dt \\
 &\leq C \int_{I_{k+1}} \sum_{\tau \in T^h} h_\tau^2 \|r_p\|_{0,\tau}^2 dt + \sigma \int_{I_{k+1}} |\varphi|_{1,\Omega}^2 dt \\
 &\quad + C \Delta t_{k+1} \int_{I_{k+1}} \|(\pi_k - I)r_p\|_{0,\Omega}^2 dt + \sigma \|\varphi\|_{L^\infty(I_{k+1}; L^2(\Omega))}^2 \\
 &\leq C(\mathfrak{N}_2^2 + \mathfrak{N}_3^2) + C\sigma \|(p_\delta - p^{u_\delta})_k^-\|_{0,\Omega}^2,
 \end{aligned} \tag{46}$$

and

$$\begin{aligned}
 \mathcal{I}_{12} &\leq C \sum_{k'=k+2}^N \left(\int_{I_{k'}} \sum_{\tau \in T^h} h_\tau^2 \|r_p\|_{0,\tau} |\pi_k \varphi|_{2,\tau} dt \right. \\
 &\quad \left. + \Delta t_{k'} \|(\pi_k - I)r_p\|_{L^2(I_{k'}; L^2(\Omega))} \|\partial_t \varphi\|_{L^2(I_{k'}; L^2(\Omega))} \right) \\
 &\leq C \sum_{k'=k+2}^N \int_{I_{k'}} (t_{k'-1} - t_k)^{-1} \sum_{\tau \in T^h} h_\tau^4 \|r_p\|_{0,\tau}^2 dt + \sigma \int_{t_{k+1}}^T (t - t_k) \|D^2 \varphi\|_{0,\Omega}^2 dt \\
 &\quad + C \sum_{k'=k+2}^N \Delta t_{k'} \|(\pi_k - I)r_p\|_{L^2(I_{k'}; L^2(\Omega))} \frac{1}{\sqrt{t_{k'-1} - t_k}} \|\sqrt{t - t_k} \partial_t \varphi\|_{L^2(I_{k'}; L^2(\Omega))} \\
 &\leq CL_N \max_{k+2 \leq k' \leq N} \sum_{\tau \in T^h} \left(h_\tau^4 \|r_p\|_{I_{k},\tau}^2 + \Delta t_{k'} \|(\pi_k - I)r_p\|_{I_{k},\tau}^2 \right)
 \end{aligned} \tag{47}$$

$$\begin{aligned}
 & + \sigma \int_{t_{k+1}}^T |t - t_k| (\|D^2\varphi\|_{0,\Omega}^2 + \|\partial_t\varphi\|_{0,\Omega}^2) dt \\
 & \leq C(\mathfrak{N}_2^2 + \mathfrak{N}_3^2) + C\sigma\|(p_\delta - p^{u_\delta})_k^-\|_{0,\Omega}^2.
 \end{aligned}$$

It follows from (8) and Lemma 3.6 that

$$\mathcal{I}_2 \leq C\|y_\delta - y^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 + \sigma\|(p_\delta - p^{u_\delta})_k^-\|_{0,\Omega}^2.$$

By using (11) and Lemma 3.1, we can estimate \mathcal{I}_3 in the same way as for \mathcal{I}_1 such that

(48)

$$\begin{aligned}
 \mathcal{I}_3 & = \left(\int_{t_k}^{t_{k+1}} + c_k \int_{t_{k+1}}^T \right) \sum_{l \in \partial T^h} \int_l [(A^* \nabla p_\delta) \cdot \mathbf{n}] (\varphi - \pi_{h,k} \varphi) dt \\
 & \leq C \int_{t_k}^{t_{k+1}} \sum_{l \in \partial T^h} h_l \|[(A^* \nabla p_\delta) \cdot \mathbf{n}]\|_{0,l}^2 dt + \sigma \int_{t_k}^{t_{k+1}} |\varphi|_{1,\Omega}^2 dt \\
 & \quad + C c_k \int_{t_{k+1}}^T |t - t_k|^{-1} \sum_{l \in \partial T^h} h_l^3 \|[(A^* \nabla p_\delta) \cdot \mathbf{n}]\|_{0,l}^2 dt + \sigma \int_{t_{k+1}}^T |t - t_k| \|D^2\varphi\|_{0,\Omega}^2 dt \\
 & \leq C\mathfrak{N}_4^2 + C\sigma\|(p_\delta - p^{u_\delta})_k^-\|_{0,\Omega}^2.
 \end{aligned}$$

We rewrite \mathcal{I}_4 as

$$\mathcal{I}_4 = \left(\sum_{k'=k+1} + c_k \sum_{k'=k+2}^N \right) \int_{I_{k'}} \left(\frac{[p_\delta]_{k'}}{\Delta t_{k'}}, (\varphi_I)_{k'}^- - \varphi_I + \varphi - \varphi_{k'}^- \right) dt := \mathcal{I}_{41} + c_k \mathcal{I}_{42}.$$

We then use Lemma 3.6 again to obtain

(49)

$$\begin{aligned}
 \mathcal{I}_{41} & = ([p_\delta]_{k+1}, (\varphi_I - \varphi)_{k+1}^-) + \int_{t_k}^{t_{k+1}} \left(\frac{[p_\delta]_{k+1}}{\Delta t_{k+1}}, \varphi - \pi_{h,k} \varphi \right) dt \\
 & \leq C\|[p_\delta]_{k+1}\|_{0,\Omega} (\Delta t_{k+1}^{-1/2} \|\varphi\|_{L^2(I_{k+1};L^2(\Omega))} + \|\varphi\|_{L^\infty(I_{k+1};L^2(\Omega))}) \\
 & \leq C\|[p_\delta]_{k+1}\|_{0,\Omega}^2 + \sigma\|(p^{u_\delta} - p_\delta)_k^-\|_{0,\Omega}^2,
 \end{aligned}$$

(50)

$$\begin{aligned}
 \mathcal{I}_{42} & = \sum_{k'=k+2}^N \int_{k'} \left(\frac{[p_\delta]_{k'}}{\Delta t_{k'}}, (\varphi_I)_{k'}^- - \varphi_I + \varphi - \varphi_{k'}^- \right) dt \\
 & \leq C \sum_{k'=k+2}^N \|[p_\delta]_{k'}\|_{0,\Omega} \sqrt{\Delta t_{k'}} \|\partial_t \varphi\|_{L^2(I_{k'};L^2(\Omega))} \\
 & \leq C \sum_{k'=k+2}^N \frac{\Delta t_{k'}}{t_{k'-1} - t_k} \|[p_\delta]_{k'}\|_{0,\Omega}^2 + \sigma \|\sqrt{t - t_k} \partial_t \varphi\|_{L^2(t_{k+1},T;L^2(\Omega))}^2 \\
 & \leq C L_N \max_{k+2 \leq k' \leq N} \|[p_\delta]_{k'}\|_{0,\Omega}^2 + C\sigma\|(p^{u_\delta} - p_\delta)_k^-\|_{0,\Omega}^2,
 \end{aligned}$$

and

(51)

$$\mathcal{I}_5 \leq C\|[p_\delta]_k\|_{0,\Omega}^2 + \sigma\|(p^{u_\delta} - p_\delta)_k^-\|_{0,\Omega}^2.$$

We thus have shown that

(52)

$$\|(p^{u_\delta} - p_\delta)_k^-\|_{0,\Omega}^2 \leq C \sum_{i=2-4,8} \mathfrak{N}_i^2 + C\|y_\delta - y^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2.$$

The last term above has been estimated in Theorem 3.1.

It remains to estimate $\|(y_\delta - y^{u_\delta})_k^+\|_{0,\Omega}^2$ ($0 \leq k \leq N - 1$). Since

$$\|(y_\delta - y^{u_\delta})_0^+\|_{0,\Omega}^2 \leq \|[y_\delta]_0\|_{0,\Omega}^2 + \|(y_\delta)_0^- - (y^{u_\delta})_0^+\|_{0,\Omega}^2 \leq \mathfrak{N}_8^2 + \mathfrak{N}_0^2,$$

we need only to consider the cases of $1 \leq k \leq N - 1$. Let ψ be the solution of (38) with $\psi_* = (y_\delta - y^{u_\delta})_k^+$ and ψ_I be defined as in (30). Then, by (38) and (14),

$$\begin{aligned} & \|(y_\delta - y^{u_\delta})_k^+\|_{0,\Omega}^2 = ((y_\delta - y^{u_\delta})_k^+, \psi_*) - (y_\delta - y^{u_\delta}, \psi)_k^- + (y_\delta - y^{u_\delta}, \psi)_k^- \\ &= \int_0^{t_k} ((\partial_t(y_\delta - y^{u_\delta}), \psi) + a(y_\delta - y^{u_\delta}, \psi)) dt + \sum_{k'=0}^{k-1} ([y_\delta]_{k'}, \psi_{k'}^+) \\ & \quad + (y_0^h - y_0, \psi_0^+) + ([y_\delta]_k, \psi_*) \\ &= \int_0^{t_k} ((\partial_t y_\delta - f - Bu_\delta, \psi) + a(y_\delta, \psi) - a(y_\delta, \psi_I) - (\partial_t y_\delta - f - Bu_\delta, \psi_I)) dt \\ & \quad + \sum_{k'=1}^k ([y_\delta]_{k'-1}, (\psi - \psi_I)_{k'-1}^+) + (y_0^h - y_0, \psi_0^+) + ([y_\delta]_k, \psi_*) \\ &= \sum_{k'=1}^k \int_{I_{k'}} \left(\partial_t y_\delta - f - Bu_\delta - \operatorname{div}(A\nabla y_\delta + \frac{[y_\delta]_{k'-1}}{\Delta t_{k'}}), \psi - \psi_I \right) \\ & \quad + \int_0^{t_k} \left(\sum_{l \in \partial T^h} \int_l [(A\nabla y_\delta) \cdot \mathbf{n}] (\psi - \psi_I) \right) dt \\ & \quad + \sum_{k'=1}^k \int_{I_{k'}} \left(\frac{[y_\delta]_{k'-1}}{\Delta t_{k'}}, \psi_{k'-1}^+ - \psi + \psi_I - (\psi_I)_{k'-1}^+ \right) \\ & \quad + (y_0^h - y_0, \psi_0^+) + ([y_\delta]_k, \psi_*) := \sum_{i=1}^5 \mathcal{J}_i. \end{aligned}$$

Let $c_1 = 0$ and $c_k = 1$ for $2 \leq k \leq N - 1$. Then, as in (46)–(51),

$$\begin{aligned} \mathcal{J}_1 &= \left(c_k \sum_{k'=1}^{k-1} + \sum_{k'=k} \right) \int_{I_{k'}} \{ (r_y, (\pi_{h,k} - I)\pi_k \psi) + ((\pi_k - I)r_y, (\pi_k - I)\psi) \} dt \\ &\leq C(\mathfrak{N}_5^2 + \mathfrak{N}_6^2) + \sigma \|(y^{u_\delta} - y_\delta)_k^+\|_{0,\Omega}^2, \\ \mathcal{J}_2 &= \left(c_k \sum_{k'=1}^{k-1} + \sum_{k'=k} \right) \int_{I_{k'}} \sum_{l \in \partial T^h} \int_l [(A\nabla y_\delta) \cdot \mathbf{n}] (\psi - \pi_{h,k} \psi) dt \\ &\leq C\mathfrak{N}_7^2 + \sigma \|(y^{u_\delta} - y_\delta)_k^+\|_{0,\Omega}^2, \\ \mathcal{J}_3 &= \left(c_k \sum_{k'=1}^{k-1} + \sum_{k'=k} \right) \int_{I_{k'}} \left(\frac{[y_\delta]_{k'-1}}{\Delta t_{k'}}, (\psi_I)_{k'}^- - \psi_I + \psi - \psi_{k'}^- \right) dt \\ &\leq C\mathfrak{N}_8^2 + \sigma \|(y^{u_\delta} - y_\delta)_k^+\|_{0,\Omega}^2, \\ \mathcal{J}_4 &\leq C\|y_0^h - y_0\|_{0,\Omega}^2 + \sigma \|(y^{u_\delta} - y_\delta)_k^+\|_{0,\Omega}^2, \\ \mathcal{J}_5 &\leq C\|[y_\delta]_k\|_{0,\Omega}^2 + \sigma \|(y^{u_\delta} - y_\delta)_k^+\|_{0,\Omega}^2. \end{aligned}$$

Hence

$$(53) \quad \|(y_\delta - y^{u_\delta})_k^+\|_{0,\Omega}^2 \leq C \sum_{i=0,5-8} \mathfrak{N}_i^2.$$

We complete the proof by combining the estimates (39), (44), (45), (52), and (53) and the result of Theorem 3.1. \square

In the rest of the section, we apply the results obtained to some model control problems. We only consider the piecewise constant finite element space for the approximation of the control.

Example 3.1. Consider the case $K = \{v \in X : v \geq \phi_0\}$, where ϕ_0 is a constant. Let $K^\delta = \{v \in X^\delta : v \geq \phi_0\}$. Then it is easy to see that $K^\delta \subset K$. Let v in Lemma 3.3 be such that $v|_{\tau_U^k \times I_k} = \pi_{\delta,k}^a u$, where $\pi_{\delta,k}^a u$ is the integral average of u on $\tau_U^k \times I_k$. Then $v = \pi_{\delta,k}^a u \in K^\delta$, and for $1 \leq k \leq N$,

$$\begin{aligned} & \left| \int_{I_k} (h'(u_\delta) + B^* p_\delta, v - u)_U dt \right| = \left| \int_{I_k} (h'(u_\delta) + B^* p_\delta, \pi_{\delta,k}^a u - u)_U dt \right| \\ &= \left| \int_{I_k} ((\pi_{\delta,k}^a - I)(h'(u_\delta) + B^* p_\delta), (\pi_{\delta,k}^a - I)(u - u_\delta))_U dt \right| \\ &\leq C \int_{I_k} \sum_{\tau_U \in T_U^{h,k}} (h_{\tau_U} |h'(u_\delta) + B^* p_\delta|_{1,\tau_U} + \Delta t_k \|\partial_t (h'(u_\delta) + B^* p_\delta)\|_{0,\tau_U}) \|u - u_\delta\|_{0,\tau_U} dt. \end{aligned}$$

Hence, the condition (12) in Lemma 3.3 is satisfied. Consequently the estimates obtained in Theorems 3.1–3.2 are applicable.

Example 3.2. Consider the case $K = \{v \in X : \int_{\Omega_U} v \geq 0\}$. Let $K^\delta = \{v \in X^\delta : \int_{\Omega_U} v \geq 0\}$. Then it is easy to see that $K^\delta \subset K$. Let v in Lemma 3.3 be defined as in Example 3.1. Then, the condition (12) in Lemma 3.3 is also satisfied.

4. Improved error estimates for the constraint of obstacle type. It seems to be difficult to further improve the estimates obtained in Theorems 3.1 and 3.2 without having structure information on the constraint set K . In this section, we consider a case where the constraint set is of obstacle type, which is met very frequently in real applications. We are then able to derive improved error estimates for the DG scheme of the finite element approximation to the parabolic optimal control problem (6). As mentioned in section 3, the essential step is to derive improved estimates for the approximation of the inequality in (2), via utilizing the structure information of K . Such improved estimates are found to be useful in computing elliptic control problems; see [27]. We shall only examine piecewise constant or piecewise linear control approximation.

We assume that the constraint on the control is an obstacle such that

$$K = \{v \in X : v \geq \phi \text{ a.e. in } \Omega_U \times (0, T]\},$$

where $\phi \in X$. We define the coincidence set (contact set) $\Omega_U^-(t)$ and the noncoincidence set (noncontact set) $\Omega_U^+(t)$ as follows:

$$\Omega_U^-(t) := \{x \in \Omega_U : u(x, t) = \phi(x, t)\}, \quad \Omega_U^+(t) := \{x \in \Omega_U : u(x, t) > \phi(x, t)\}.$$

Let

$$(54) \quad K^\delta = \{v \in X^\delta : v \geq \phi^\delta \text{ in } \Omega_U \times (0, T]\},$$

where $\phi^\delta \in X^\delta$ is an approximation to ϕ satisfying $\phi^\delta \geq \phi$. Hence, we have that $K^\delta \subset K$. In this section, we assume that

$$h(u) = \int_{\Omega_U} j(u),$$

where $j(\cdot)$ is a convex continuously differentiable function on \mathbb{R} . Then, it is easy to see that

$$\int_0^T (h'(u), v)_U = \int_0^T (j'(u), v)_U = \int_0^T \int_{\Omega_U} j'(u)v.$$

We shall assume the following uniform convexity condition:

$$(j'(t) - j'(s))(t - s) \geq c(t - s)^2 \quad \forall s, t \in \mathbb{R}.$$

It can be seen that the inequality in (2) is now equivalent to the following:

$$(55) \quad j'(u) + B^*p \geq 0, \quad u \geq \phi, \quad (j'(u) + B^*p)(u - \phi) = 0, \quad \text{a.e. in } \Omega_U \times (0, T].$$

In order to have the improved a posteriori error estimate, we divide $\Omega_U \times (0, T]$ into the following three subsets:

$$\begin{aligned} \Omega_\phi &= \{(x, t) \in \Omega_U \times (0, T] : (B^*p_\delta)(x, t) \leq -j'(\phi^\delta)\}, \\ \Omega_\phi^0 &= \{(x, t) \in \Omega_U \times (0, T] : (B^*p_\delta)(x, t) > -j'(\phi^\delta), u_\delta = \phi^\delta\}, \\ \Omega_\phi^+ &= \{(x, t) \in \Omega_U \times (0, T] : (B^*p_\delta)(x, t) > -j'(\phi^\delta), u_\delta > \phi^\delta\}. \end{aligned}$$

Then, it is easy to see that the above three subsets do not overlap each other, and

$$\bar{\Omega}_U \times (0, T] = \bar{\Omega}_\phi \cup \bar{\Omega}_\phi^0 \cup \bar{\Omega}_\phi^+.$$

We shall show that $h'(u_\delta) + B^*p_\delta$ can be replaced by $(j'(u_\delta) + B^*p_\delta)|_{\Omega_\phi}$ in the error estimates. Note that $j'(u) + B^*p = 0$ when $u > \phi$. Thus in a sense, the set Ω_ϕ is an approximation of the noncoincidence set $\{(x, t) : x \in \Omega_U^+(t), t \in (0, T]\}$.

THEOREM 4.1. *Let (y, p, u) and $(y_\delta, p_\delta, u_\delta)$ be the solutions of (2) and (6), respectively. Assume that all the conditions of Lemma 3.5 hold, and K^δ is defined in (54) with $\phi \in L^2(0, T; L^2(\Omega_U))$. Moreover, assume that $j'(\cdot)$ and $g'(\cdot)$ are locally Lipschitz continuous. Then*

$$\|u_\delta - u\|_{L^2(0, T; L^2(\Omega_U))}^2 + \|y_\delta - y\|_{L^2(0, T; L^2(\Omega))}^2 + \|p_\delta - p\|_{L^2(0, T; L^2(\Omega))}^2 \leq C \sum_{i=0}^8 \hat{\eta}_i^2,$$

where $\hat{\eta}_i^2 = \eta_i^2$ ($i = 0, 2-7$) are given in Lemma 3.5 and

$$\begin{aligned} \hat{\eta}_1^2 &= \int_{\Omega_\phi} |j'(u_\delta) + B^*p_\delta|^2, \\ \hat{\eta}_8^2 &= \|\phi - \phi^\delta\|_{0, \Omega_\phi^0}^2. \end{aligned}$$

Proof. We consider $\|u_\delta - u\|_{L^2(0, T; L^2(\Omega_U))}^2$. From the uniform convexity of j , we

have that

$$\begin{aligned}
 c\|u - u_\delta\|_{L^2(0,T;L^2(\Omega_U))}^2 &\leq \int_0^T (j'(u) - j'(u_\delta), u - u_\delta)_U \\
 &= \int_0^T (j'(u) + B^*p, u - u_\delta)_U + \int_0^T (j'(u_\delta) + B^*p_\delta, u_\delta - u)_U \\
 (56) \quad &+ \int_0^T (B^*(p_\delta - p^{u_\delta}), u - u_\delta)_U + \int_0^T (B^*(p^{u_\delta} - p), u - u_\delta)_U \\
 &= \int_0^T (j'(u) + B^*p, u - u_\delta)_U + \int_0^T (j'(u_\delta) + B^*p_\delta, u_\delta - u)_U \\
 &+ \int_0^T (B^*(p_\delta - p^{u_\delta}), u - u_\delta)_U + \int_0^T (y^{u_\delta} - y, y - y^{u_\delta}) \\
 &\leq \int_0^T (j'(u) + B^*p, u - u_\delta)_U + \int_0^T (j'(u_\delta) + B^*p_\delta, u_\delta - u)_U \\
 &+ \int_0^T (B^*(p_\delta - p^{u_\delta}), u - u_\delta)_U := \sum_1^3 I_i.
 \end{aligned}$$

We first estimate I_1 . Note that

$$\begin{aligned}
 (57) \quad &\int_0^T (j'(u) + B^*p, u - u_\delta)_U \\
 &= \int_{\Omega_\phi \cup \Omega_\phi^+} (j'(u) + B^*p)(u - u_\delta) + \int_{\Omega_\phi^0} (j'(u) + B^*p)(u - \phi^\delta).
 \end{aligned}$$

Let

$$w = \begin{cases} u_\delta, & (x, t) \in \Omega_\phi \cup \Omega_\phi^+, \\ u, & (x, t) \in \Omega_\phi^0. \end{cases}$$

Then, $w \in K$, and hence

$$(58) \quad \int_{\Omega_\phi \cup \Omega_\phi^+} (j'(u) + B^*p)(u - u_\delta) = \int_0^T \int_{\Omega_U} (j'(u) + B^*p)(u - w) \leq 0.$$

Note that $(j'(u) + B^*p)(u - \phi) = 0$. We have that

$$\begin{aligned}
 (59) \quad &\int_{\Omega_\phi^0} (j'(u) + B^*p)(u - \phi^\delta) = \int_{\Omega_\phi^0} (j'(u) + B^*p)((u - \phi) + (\phi - \phi^\delta)) \\
 &= \int_{\Omega_\phi^0} (j'(u) + B^*p)(\phi - \phi^\delta).
 \end{aligned}$$

It follows from (57)–(59) that

$$(60) \quad I_1 = \int_0^T (j'(u) + B^*p, u - u_\delta)_U \leq \int_{\Omega_\phi^0} (j'(u) + B^*p)(\phi - \phi^\delta).$$

Next we estimate I_2 . It is clear that

$$\begin{aligned}
 & \int_0^T (j'(u_\delta) + B^*p_\delta, u_\delta - u)_U \\
 &= \int_{\Omega_\phi} (j'(u_\delta) + B^*p_\delta)(u_\delta - u) + \int_{\Omega_\phi^+} (j'(u_\delta) + B^*p_\delta)(u_\delta - u) \\
 (61) \quad &+ \int_{\Omega_\phi^0} (j'(\phi_h) + B^*p_\delta)(\phi^\delta - u).
 \end{aligned}$$

First it is easy to see that

$$\begin{aligned}
 \int_{\Omega_\phi} (j'(u_\delta) + B^*p_\delta)(u_\delta - u) &\leq C \int_{\Omega_\phi} (j'(u_\delta) + B^*p_\delta)^2 + C\sigma \|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2 \\
 (62) \quad &= C\hat{\eta}_1^2 + C\sigma \|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2.
 \end{aligned}$$

Second, let $\tau_U \times (t_i, t_{i+1}]$ be such that $u_\delta|_{\tau_U \times (t_i, t_{i+1}]} > \phi^\delta$; it follows from (6) that there exist $\epsilon > 0$ and $\psi \in X^\delta$, such that $\psi \geq 0$, $\|\psi\|_{L^\infty(t_i, t_{i+1}; L^\infty(\tau_U))} = 1$, and

$$\int_{t_i}^{t_{i+1}} \int_{\tau_U} (j'(u_\delta) + B^*p_\delta)(u_\delta - (u_\delta - \epsilon\psi)) = \epsilon \int_{t_i}^{t_{i+1}} \int_{\tau_U} (j'(u_\delta) + B^*p_\delta)\psi \leq 0.$$

Note that on Ω_ϕ^+ , $(j'(u_\delta) + B^*p_\delta) > (j'(\phi^\delta) + B^*p_\delta) > 0$. We have that

$$\begin{aligned}
 & \int_{(\tau_U \times (t_i, t_{i+1})) \cap \Omega_\phi^+} |j'(u_\delta) + B^*p_\delta| \psi = \int_{(\tau_U \times (t_i, t_{i+1})) \cap \Omega_\phi^+} (j'(u_\delta) + B^*p_\delta) \psi \\
 & \leq - \int_{(\tau_U \times (t_i, t_{i+1})) \cap \Omega_\phi} (j'(u_\delta) + B^*p_\delta) \psi \leq \int_{(\tau_U \times (t_i, t_{i+1})) \cap \Omega_\phi} |j'(u_\delta) + B^*p_\delta|.
 \end{aligned}$$

Let $\hat{\tau}_{U t_i}$ be the reference element of $\tau_U \times (t_i, t_{i+1}]$, $\tau_{U t_i}^0 = (\tau_U \times (t_i, t_{i+1})) \cap \Omega_\phi^+$, and $\hat{\tau}_{U t_i}^0 \subset \hat{\tau}_{U t_i}$ be the image of $\tau_{U t_i}^0$. Let n be the dimension of Ω_U and $k_i = t_{i+1} - t_i$. Note that $j'(\cdot)$ is locally Lipschitz continuous. It follows from the equivalence of the norm in a finite dimensional space that

$$\begin{aligned}
 & \int_{\tau_{U t_i}^0} |j'(u_\delta) + B^*p_\delta|^2 \leq Ch_{\tau_U}^n k_i \int_{\hat{\tau}_{U t_i}^0} |j'(u_\delta) + B^*p_\delta|^2 \\
 & \leq Ch_{\tau_U}^n k_i \left(\int_{\hat{\tau}_{U t_i}^0} |j'(u_\delta) + B^*p_\delta| \psi \right)^2 \leq Ch_{\tau_U}^{-n} k_i^{-1} \left(\int_{\tau_{U t_i}^0} |j'(u_\delta) + B^*p_\delta| \psi \right)^2 \\
 & \leq Ch_{\tau_U}^{-n} k_i^{-1} \left(\int_{\tau_{U t_i} \cap \Omega_\phi} |j'(u_\delta) + B^*p_\delta| \right)^2 \leq C \int_{\tau_{U t_i} \cap \Omega_\phi} |j'(u_\delta) + B^*p_\delta|^2.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \int_{\Omega_\phi^+} (j'(u_\delta) + B^*p_\delta)(u_\delta - u) \\
 & \leq C \int_{\Omega_\phi^+} (j'(u_\delta) + B^*p_\delta)^2 + C\sigma \|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2 \\
 (63) \quad & \leq C \int_{\Omega_\phi} (j'(u_\delta) + B^*p_\delta)^2 + C\sigma \|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2 \\
 & = C\hat{\eta}_1^2 + C\sigma \|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2.
 \end{aligned}$$

It follows from the definition of Ω_ϕ^0 that $(j'(\phi^\delta) + B^*p_\delta) > 0$ on Ω_ϕ^0 . Then we have

$$(64) \quad \begin{aligned} \int_{\Omega_\phi^0} (j'(\phi_h) + B^*p_\delta)(\phi^\delta - u) &= \int_{\Omega_\phi^0} (j'(\phi^\delta) + B^*p_\delta)((\phi^\delta - \phi) + (\phi - u)) \\ &\leq \int_{\Omega_\phi^0} (j'(u_\delta) + B^*p_\delta)(\phi^\delta - \phi). \end{aligned}$$

Thus it follows from (61)–(64) that

$$(65) \quad \begin{aligned} I_2 = \int_0^T (j'(u_\delta) + B^*p_\delta, u_\delta - u)_U &\leq C\hat{\eta}_1^2 + \int_{\Omega_\phi^0} (j'(u_\delta) + B^*p_\delta)(\phi^\delta - \phi) \\ &\quad + C\delta \|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2. \end{aligned}$$

Then it follows from (60) and (65) that

$$(66) \quad \begin{aligned} I_1 + I_2 &= \int_0^T (j'(u) + B^*p, u - u_\delta)_U + \int_0^T (j'(u_\delta) + B^*p_\delta, u_\delta - u)_U \\ &\leq C\hat{\eta}_1^2 + \int_{\Omega_\phi^0} (j'(u) + B^*p - j'(u_\delta) - B^*p_\delta)(\phi - \phi^\delta) \\ &\quad + C\sigma \|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2 \\ &\leq C(\hat{\eta}_1^2 + \|\phi - \phi^\delta\|_{0,\Omega_\phi^0}^2) + C\sigma (\|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2 \\ &\quad + \|j'(u_\delta) - j'(u)\|_{L^2(0,T;L^2(\Omega_U))}^2 + \|B^*(p_\delta - p^{u_\delta})\|_{L^2(0,T;L^2(\Omega_U))}^2 \\ &\quad + \|B^*(p^{u_\delta} - p)\|_{L^2(0,T;L^2(\Omega_U))}^2) \\ &\leq C(\hat{\eta}_1^2 + \hat{\eta}_8^2) + C\sigma \|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2 + C\|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega_U))}^2. \end{aligned}$$

Here we used the inequalities

$$\|j'(u_\delta) - j'(u)\|_{L^2(0,T;L^2(\Omega_U))}^2 \leq C\|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2,$$

$$\|B^*(p_\delta - p^{u_\delta})\|_{L^2(0,T;L^2(\Omega_U))}^2 \leq C\|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega_U))}^2,$$

and

$$\|B^*(p^{u_\delta} - p)\|_{L^2(0,T;L^2(\Omega_U))}^2 \leq C\|p^{u_\delta} - p\|_{L^2(0,T;L^2(\Omega))}^2 \leq C\|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2.$$

Finally for I_3 , it is easy to show that

$$(67) \quad \begin{aligned} I_3 &= \int_0^T (B^*(p_\delta - p^{u_\delta}), u - u_\delta)_U \\ &\leq C\|B^*(p_\delta - p^{u_\delta})\|_{L^2(0,T;L^2(\Omega))}^2 + C\sigma \|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2 \\ &\leq C\|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2 + C\sigma \|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2. \end{aligned}$$

Thus, we obtain from (56), (66), and (67) that

$$\|u_\delta - u\|_{L^2(0,T;L^2(\Omega_U))}^2 \leq C(\hat{\eta}_1 + \hat{\eta}_8 + \|p_\delta - p^{u_\delta}\|_{L^2(0,T;L^2(\Omega))}^2).$$

The remainder of the proof is the same as for Lemma 3.5 and Theorem 3.1. \square

REMARK 4.1. *By the same argument, we can obtain a similar estimate in the $L^\infty(L^2)$ norm considered in Theorem 3.2. It is worth noting that there may be different approaches to derive sharp a posteriori error bounds for the obstacle constraints. Noticeably, it may be possible to design some penalty schemes to solve the optimality system, and then apply the techniques used in [8, 17, 22] to derive sharp bounds.*

REMARK 4.2. *Here the key idea is to remove some inactive data in the coincidence set and to thus obtain sharper error estimates for the approximation of the inequality in (2). In fact, as seen in the above proof, only the part where $j'(u_\delta) + B^*p_\delta \leq 0$ needs to be left in the estimator $\hat{\eta}_1^2$. Let us define*

$$\hat{\Omega}_\phi = \{(x, t) \in \Omega_U \times (0, T] : (B^*p_\delta)(x, t) \leq -j'(u_\delta)\}.$$

*In a sense, the set $\hat{\Omega}_\phi$ is an approximation of the noncoincidence set. It follows that $(j'(u_\delta) + B^*p_\delta)|_{\hat{\Omega}_\phi} \leq 0$, while $j'(u) + B^*p \geq 0$. Thus on $\hat{\Omega}_\phi$, $j'(u_\delta) + B^*p_\delta$ truly indicates the error. In fact, we have*

$$\begin{aligned} \int_{\hat{\Omega}_\phi} |j'(u_\delta) + B^*p_\delta|^2 &\leq \int_{\hat{\Omega}_\phi} |j'(u_\delta) + B^*p_\delta - (j'(u) + B^*p)|^2 \\ &\leq C(\|u - u_\delta\|_{L^2(0,T;L^2(\Omega_U))}^2 + \|p - p_\delta\|_{L^2(0,T;L^2(\Omega))}^2). \end{aligned}$$

For ease of computation, we have used the set Ω_ϕ , which is a little larger than $\hat{\Omega}_\phi$. However, we still have

$$\hat{\eta}_1^2 \leq C(\|u - u_\delta\|_{L^2(0,T;L^2(\Omega_U))}^2 + \|p - p_\delta\|_{L^2(0,T;L^2(\Omega))}^2 + \hat{\eta}_8^2).$$

*On the coincidence set, $u = \phi$. Therefore the error should be indicated by $\hat{\eta}_8$, and the term $j'(u_\delta) + B^*p_\delta$ should not appear there.*

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