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ON PRECONDITIONED ITERATIVE METHODS FOR CERTAIN TIME-DEPENDENT PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. When the Newton method or the fixed-point method is employed to solve the systems of nonlinear equations arising in the sinc-Galerkin discretization of certain time-dependent partial differential equations, in each iteration step we need to solve a structured subsystem of linear equations iteratively by, for example, a Krylov subspace method such as the preconditioned GMRES. In this paper, based on the tensor and the Toeplitz structures of the linear subsystems we construct structured preconditioners for their coefficient matrices and estimate the eigenvalue bounds of the preconditioned matrices under certain assumptions. Numerical examples are given to illustrate the effectiveness of the proposed preconditioning methods. It has been shown that a combination of the Newton/fixed-point iteration with the preconditioned GMRES method is efficient and robust for solving the systems of nonlinear equations arising from the sinc-Galerkin discretization of the time-dependent partial differential equations.

Key words. time-dependent partial differential equation, sinc-Galerkin discretization, Toeplitz-like matrix, preconditioning, eigenvalue bound, GMRES method

AMS subject classifications. 65F10, 65F15, 65T10; CR: G1.3

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1. Introduction.

We consider the numerical solution of time-dependent partial differential equations of the form

$$(1.1)\quad \begin{cases} p_t(t)\dfrac{\partial u}{\partial t}(x,t) + p_x(x)u(x,t)\dfrac{\partial u}{\partial x}(x,t) - \varepsilon\dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), & a < x < b,\ t \geq 0, \\ u(a,t) = \gamma(t) \quad \text{and} \quad u(b,t) = \delta(t), & t \geq 0, \\ u(x,0) = g(x), & a \leq x \leq b, \end{cases}$$

where $p_z(z)$, $z \in \{x,t\}$, are given continuously differentiable functions, $f(x,t)$, $\gamma(t)$, $\delta(t)$, and $g(x)$ are given bounded functions, and ε is a prescribed small positive parameter. Note that when $p_z(z) \equiv 1$, $z \in \{x,t\}$, the partial differential equation (1.1) reduces to the Burgers equation; see [16] for more details.

When the time-dependent partial differential equation (1.1) is discretized by the sinc-Galerkin method, in an analogous approach to [5] we can obtain systems of nonlinear equations of the form

$$(1.2)\quad \mathbf{F}(\mathbf{u}) := B\mathbf{u} + C\Psi(\mathbf{u}) - \mathbf{b} = 0,$$

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where B and C are known n-by-n matrices, $\mathbf{b}$ is a given n-vector, and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, with

$$\Psi(\mathbf{u}) = (\psi_1(u_1), \psi_2(u_2), \ldots, \psi_n(u_n))^T \quad \text{and} \quad \mathbf{u} = (u_1, u_2, \ldots, u_n)^T,$$

is a continuous diagonal mapping defined on the open ball

$$\mathcal{U}_\delta := \{u \in \mathbb{R}^n \mid \|\mathbf{u}\| < \delta\}.$$

Here, δ is a positive constant. The matrices B and C are given by

$$\begin{aligned} B = {} & \varepsilon \left(T_x^{(2)} + D_x^{(1)} T_x^{(1)} + T_x^{(1)} D_x^{(1)} + D_x^{(2)}\right) \otimes Q_t \\ & + Q_x \otimes \left(D_t^{(3)} T_t^{(1)} + T_t^{(1)} D_t^{(3)} + D_t^{(4)}\right) \end{aligned} \tag{1.3}$$

and

$$C = \left(D_x^{(3)} T_x^{(1)} + T_x^{(1)} D_x^{(3)} + D_x^{(4)}\right) \otimes Q_t, \tag{1.4}$$

and the mapping Ψ is given by

$$\Psi(\mathbf{u}) = \left(u_1^2, u_2^2, \ldots, u_n^2\right)^T, \tag{1.5}$$

where $T_z^{(i)}$ ($i = 1, 2$ and $z \in \{x, t\}$) are $(m_z + n_z + 1)$-by-$(m_z + n_z + 1)$ Toeplitz matrices, with

$$T_z^{(1)} = \begin{bmatrix} 0 & -1 & \frac{1}{2} & \cdots & \frac{(-1)^{m_z+n_z}}{m_z+n_z} \\ 1 & & & & \vdots \\ -\frac{1}{2} & & \ddots & & \frac{1}{2} \\ \vdots & & & & -1 \\ -\frac{(-1)^{m_z+n_z}}{m_z+n_z} & \cdots & -\frac{1}{2} & 1 & 0 \end{bmatrix}, \tag{1.6}$$

$$T_z^{(2)} = \begin{bmatrix} \frac{\pi^2}{3} & -2 & \frac{2}{2^2} & \cdots & \frac{(-1)^{m_z+n_z}2}{(m_z+n_z)^2} \\ -2 & & & & \vdots \\ \frac{2}{2^2} & & \ddots & & \frac{2}{2^2} \\ \vdots & & & & -2 \\ \frac{(-1)^{m_z+n_z}2}{(m_z+n_z)^2} & \cdots & \frac{2}{2^2} & -2 & \frac{\pi^2}{3} \end{bmatrix}, \tag{1.7}$$

and $D_z^{(i)}$ and Q_z ($i = 1, 2, 3, 4$ and $z \in \{x, t\}$) are $(m_z + n_z + 1)$-by-$(m_z + n_z + 1)$ diagonal matrices, with

$$D_z^{(1)} = \frac{h_z}{2} \cdot \operatorname{diag}\left[\left\{-\frac{\phi_z''(z)}{(\phi_z'(z))^2} - \frac{2\omega_z'(z)}{\phi_z'(z)\omega_z(z)}\right\}_{z=-m_z}^{n_z}\right], \tag{1.8}$$

$$D_z^{(2)} = \frac{h_z^2}{2} \cdot \operatorname{diag}\left[\left\{-\frac{\omega_z''(z)}{(\phi_z'(z))^2\omega_z(z)}\right\}_{z=-m_z}^{n_z}\right], \tag{1.9}$$

$$D_z^{(3)} = \frac{h_z}{2} \cdot \operatorname{diag}\left[\{-p_z(z)\omega_z(z)\}_{z=-m_z}^{n_z}\right], \tag{1.10}$$

$$D_z^{(4)} = \frac{h_z^2}{2} \cdot \operatorname{diag}\left[\left\{-\frac{(p_z(z)\omega_z(z))'}{\phi_z'(z)}\right\}_{z=-m_z}^{n_z}\right], \tag{1.11}$$

and

$$Q_z = \operatorname{diag}\left[\left\{\frac{\omega_z(z)}{\phi_z'(z)}\right\}_{z=-m_z}^{n_z}\right]. \tag{1.12}$$

Here, m_x, n_x and m_t, n_t are positive integers representing the numbers of the bases used in the spatial and the temporal spaces, respectively, $\phi_x(x)$ and $\phi_t(t)$ are the restrictions of the conformal mapping $\phi_z(z)$ onto the real intervals (a,b) and $(0,+\infty)$, respectively, with $\phi_z(z)$ a mapping from a simply connected domain $\mathcal{D}$ onto

$$\mathcal{D}_d := \{z \mid z = x + \imath y, |y| < d, d > 0\},$$

with $\imath$ the imaginary unit; and $\omega_x(x)$ and $\omega_t(t)$ are two weighting functions with respect to the spatial and the temporal variables, respectively. See [16, 5] for a detailed description about the sinc-Galerkin discretization. We remark that the first and the second derivatives of $\phi_z(z)$ and $\omega_z(z)$ with respect to the variable z will be denoted as $\phi_z^{'}(z)$, $\omega_z^{'}(z)$ and $\phi_z^{''}(z)$, $\omega_z^{''}(z)$, respectively, and the matrices $T_z^{(1)}$, $z \in \{x,t\}$, defined in (1.6) are skew-symmetric, while the matrices $T_z^{(2)}$, $z \in \{x,t\}$, defined in (1.7) are symmetric positive definite; see Lemmas 2.1 and 2.2.

The system of nonlinear equations (1.2) is usually termed as a mildly nonlinear system in literature; see [19, 21] for general backgrounds and applications, [2, 5] for the basic existence and uniqueness theory about the solution, and [1, 2, 7, 8, 21, 22] for several splitting iteration methods in the sequential and parallel computing senses. When the system of mildly nonlinear equations (1.2) is solved by the Newton or the fixed-point iteration method, at each step we need to solve a subsystem of linear equations of the form

$$(B + CD)\mathbf{z} = \mathbf{r}, \tag{1.13}$$

where D is a diagonal matrix approximating the Jacobian matrix of the mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{r}$ is the current residual vector. Unfortunately, direct methods such as the Gaussian elimination or the fast Toeplitz algorithms [15, 14] are not applicable to effectively solve this class of diagonally scaled Toeplitz-plus-diagonal linear systems due to the considerably high computational complexity; see [9, 10, 11, 12, 13]. However, noticing that the matrix-vector product $(B{+}CD)\mathbf{q}$ can be computed in $\mathcal{O}(n \log n)$ operations for any vector $\mathbf{q} \in \mathbb{R}^n$, we can employ Krylov subspace iteration methods such as GMRES [20] to iteratively solve the linear subsystem (1.13) in an economical cost. Usually, in order to accelerate the convergence speeds of the Krylov subspace iteration methods, we need to precondition the linear subsystem (1.13) by a good approximating matrix with respect to the coefficient matrix $A := B + CD$. Therefore, in order to solve the original linear subsystem, we turn to solving the corresponding preconditioned linear subsystem instead; see [6, 5] and the references therein.

In this paper, we construct a structured preconditioner M for the matrix A by making use of the tensor-product structure of the original matrix A and the diagonally scaled Toeplitz-plus-diagonal structure of the matrix blocks involved. The positive definiteness of both matrices A and M are discussed in detail, and the eigenvalue bounds about the preconditioned matrix $M^{-1}A$ are estimated precisely by utilizing the generalized Bendixson theorem [6]. Theoretical analysis shows that the eigenvalues of the matrix $M^{-1}A$ are tightly and uniformly bounded in a rectangle on the complex plane independent of the size of the matrix. Numerical implementations show that the Newton-GMRES and the fixed-point-GMRES iteration methods, when incorporated with the structured preconditioner M, are effective and robust nonlinear solvers for the systems of mildly nonlinear equations arising from the sinc-Galerkin discretization of the referred time-dependent partial differential equations.

The organization of the paper is as follows. In section 2, we construct a structured preconditioner for the coefficient matrix of the linear subsystem (1.13) and analyze basic properties of the original and the preconditioning matrices. In section 3, we demonstrate several preliminary results associated with the spectral analysis of the preconditioned matrix. The eigenvalue bounds of the preconditioned matrix are estimated in section 4, and numerical examples are given in section 5 to show the effectiveness of the proposed preconditioning and the corresponding preconditioned iteration methods. Finally, in section 6, we end this paper with some concluding remarks.

2. The structured preconditioners. Consider the system of mildly nonlinear equations (1.2), with the function $\Psi(\mathbf{u})$ being given in (1.5) and the matrices B and C being given in (1.3) and (1.4), respectively, where $T_z^{(i)}$ ($i = 1, 2,\ z \in \{x, t\}$), $D_z^{(i)}$ ($i = 1, 2, 3, 4$ and $z \in \{x, t\}$) and Q_z ($z \in \{x, t\}$) are defined in (1.6)–(1.12). Denote by I the identity matrix. Let Ω be a positive definite diagonal matrix such that $D := I \otimes \Omega$ is an approximation to the Jacobian matrix of $\Psi(\mathbf{u})$. Then the target matrix under consideration is

$$\begin{aligned} A &= B + CD \\ &= \varepsilon \left(T_x^{(2)} + D_x^{(1)} T_x^{(1)} + T_x^{(1)} D_x^{(1)} + D_x^{(2)} \right) \otimes Q_t \\ &\quad + Q_x \otimes \left(D_t^{(3)} T_t^{(1)} + T_t^{(1)} D_t^{(3)} + D_t^{(4)} \right) \\ &\quad + \left(D_x^{(3)} T_x^{(1)} + T_x^{(1)} D_x^{(3)} + D_x^{(4)} \right) \otimes (Q_t \Omega). \end{aligned} \tag{2.1}$$

By utilizing the special structure of the matrix A, we can construct its preconditioner M as

$$\begin{aligned} M &= \widehat{B} + \widehat{C} D \\ &= \varepsilon \left(B_x^{(2)} + D_x^{(1)} B_x^{(1)} + B_x^{(1)} D_x^{(1)} + D_x^{(2)} \right) \otimes Q_t \\ &\quad + Q_x \otimes \left(D_t^{(3)} B_t^{(1)} + B_t^{(1)} D_t^{(3)} + D_t^{(4)} \right) \\ &\quad + \left(D_x^{(3)} B_x^{(1)} + B_x^{(1)} D_x^{(3)} + D_x^{(4)} \right) \otimes (Q_t \Omega), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \widehat{B} &= \varepsilon \left(B_x^{(2)} + D_x^{(1)} B_x^{(1)} + B_x^{(1)} D_x^{(1)} + D_x^{(2)} \right) \otimes Q_t \\ &\quad + Q_x \otimes \left(D_t^{(3)} B_t^{(1)} + B_t^{(1)} D_t^{(3)} + D_t^{(4)} \right) \end{aligned}$$

and

$$\widehat{C} = \left(D_x^{(3)} B_x^{(1)} + B_x^{(1)} D_x^{(3)} + D_x^{(4)} \right) \otimes Q_t,$$

and, for $z \in \{x, t\}$,

$$B_z^{(1)} = \text{tridiag}\,[1, 0, -1] \quad \text{and} \quad B_z^{(2)} = \text{tridiag}\,[-1, 2, -1] \tag{2.3}$$

are tridiagonal approximations to $T_z^{(1)}$ and $T_z^{(2)}$, respectively. Note that the preconditioning matrix M is obtained by replacing only $T_z^{(i)}$ ($i = 1, 2,\ z \in \{x, t\}$) in the matrix A by $B_z^{(i)}$ ($i = 1, 2,\ z \in \{x, t\}$), correspondingly.

We remark that the preconditioner M is a block tridiagonal matrix and is usually of mild size as, compared with the finite-difference system, the sinc-Galerkin system needs not be very large and is of mild size in order to achieve the same discretization accuracy [17, 18, 5]. Therefore, for any given vector $\mathbf{r}$, the generalized residual equation $M\mathbf{w} = \mathbf{r}$ involved in the preconditioned GMRES iteration method can be solved in $\mathcal{O}(N_x N_t^2)$ or $\mathcal{O}(N_x^2 N_t)$ operations by using a variety of linear solvers such as the sparse direct methods, where $N_z = m_z + n_z + 1$, with $z \in \{x, t\}$.

It was proved in [16] that the Toeplitz matrix $T_x^{(2)}$ is symmetric positive definite and its eigenvalues are located in a positive interval. This result, together with some eigenproperties of the Toeplitz matrices $T_z^{(1)}$ ($z \in \{x, t\}$), is precisely described in the following lemma.

Lemma 2.1 (see [16, Theorems 4.18 and 4.19]). *Let the matrices* $T_z^{(1)}$ ($z \in \{x, t\}$) *and* $T_x^{(2)}$ *be defined as in* (1.6) *and* (1.7), *respectively. Then*

(i) *for* $z \in \{x, t\}$, $T_z^{(1)}$ *is a skew-symmetric matrix and its eigenvalues* $\{\imath\lambda_j^{(1)}\}_{j=-m_z}^{n_z}$ *satisfy* $\lambda_j^{(1)} \in [-\pi, \pi]$, $-m_z \le j \le n_z$;

(ii) $T_x^{(2)}$ *is a symmetric positive definite matrix and its eigenvalues* $\{\lambda_j^{(2)}\}_{j=-m_x}^{n_x}$ *satisfy* $\lambda_j^{(2)} \in [4\sin^2(\frac{\pi}{2(N_x+1)}), \pi^2]$, *where* $N_x = m_x + n_x + 1$.

Analogously, the structural properties and the eigenvalue locations about the matrices $B_z^{(1)}$ ($z \in \{x, t\}$) and $B_x^{(2)}$ are precisely described in the following lemma; see [4].

Lemma 2.2 (see [4, Lemma A.1]). *Let the matrices* $B_z^{(1)}$ ($z \in \{x, t\}$) *and* $B_x^{(2)}$ *be defined as in* (2.3). *Then*

(i) *for* $z \in \{x, t\}$, $B_z^{(1)}$ *is a skew-symmetric matrix and its eigenvalues* $\{\imath\lambda_j^{(1)}\}_{j=-m_z}^{n_z}$ *satisfy* $\lambda_j^{(1)} \in [-\cos(\frac{\pi}{N_z+1}), \cos(\frac{\pi}{N_z+1})]$, $-m_z \le j \le n_z$, *where* $N_z = m_z + n_z + 1$;

(ii) $B_x^{(2)}$ *is a symmetric positive definite matrix and its eigenvalues* $\{\lambda_j^{(2)}\}_{j=-m_x}^{n_x}$ *satisfy* $\lambda_j^{(2)} \in [4\sin^2(\frac{\pi}{2(N_x+1)}), 4\cos^2(\frac{\pi}{2(N_x+1)})]$, *where* $N_x = m_x + n_x + 1$.

Based on these two lemmas, we now demonstrate the positive definiteness of the matrix A defined in (2.1) and its preconditioning matrix M defined in (2.2).

To this end, in what follows we use $(\cdot)^*$ to denote the conjugate transpose of either a vector or a square matrix. For a given square matrix X, we use $\mathcal{H}(X)$ and $\mathcal{S}(X)$ to denote, respectively, its Hermitian and skew-Hermitian parts [4] and $\lambda(X)$ its spectral set.

Theorem 2.1. *Assume that* $D_x^{(2)}$, $D_x^{(4)}$, *and* $D_t^{(4)}$ *are positive semidefinite diagonal matrices and* Q_z ($z \in \{x, t\}$) *and* Ω *are positive definite diagonal matrices. Then both* $\mathcal{H}(A)$ *and* $\mathcal{H}(M)$ *are symmetric positive definite matrices. Hence,* A *and* M *are positive definite* [1] *and, thus, are nonsingular.*

[1] A matrix is positive definite if its Hermitian part is positive definite. Note that a positive definite matrix is not necessarily Hermitian; see [4, 3].

Proof. The Hermitian and the skew-Hermitian parts of A and M are

$$\begin{aligned}\mathcal{H}(A) &= \frac{1}{2}(A + A^*) \\ &= \varepsilon\left(T_x^{(2)} + D_x^{(2)}\right) \otimes Q_t + Q_x \otimes D_t^{(4)} + D_x^{(4)} \otimes (Q_t\Omega),\end{aligned}$$

$$\begin{aligned}\mathcal{S}(A) &= \frac{1}{2}(A - A^*) \\ &= \varepsilon\left(D_x^{(1)}T_x^{(1)} + T_x^{(1)}D_x^{(1)}\right) \otimes Q_t + Q_x \otimes \left(D_t^{(3)}T_t^{(1)} + T_t^{(1)}D_t^{(3)}\right) \\ &\quad + \left(D_x^{(3)}T_x^{(1)} + T_x^{(1)}D_x^{(3)}\right) \otimes (Q_t\Omega)\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}(M) &= \frac{1}{2}(M + M^*) \\ &= \varepsilon\left(B_x^{(2)} + D_x^{(2)}\right) \otimes Q_t + Q_x \otimes D_t^{(4)} + D_x^{(4)} \otimes (Q_t\Omega),\end{aligned}$$

$$\begin{aligned}\mathcal{S}(M) &= \frac{1}{2}(M - M^*) \\ &= \varepsilon\left(D_x^{(1)}B_x^{(1)} + B_x^{(1)}D_x^{(1)}\right) \otimes Q_t + Q_x \otimes \left(D_t^{(3)}B_t^{(1)} + B_t^{(1)}D_t^{(3)}\right) \\ &\quad + \left(D_x^{(3)}B_x^{(1)} + B_x^{(1)}D_x^{(3)}\right) \otimes (Q_t\Omega).\end{aligned}$$

Because the diagonal matrices $D_x^{(2)}$, $D_x^{(4)}$, and $D_t^{(4)}$ are positive semidefinite, the diagonal matrices Q_z $(z \in \{x, t\})$ and Ω are positive definite, and from Lemma 2.1 the Toeplitz matrices $T_x^{(2)}$ are symmetric positive definite, so we know that $\mathcal{H}(A)$ is symmetric positive definite. Therefore, A is a positive definite matrix and, thus, is nonsingular.

From Lemma 2.2 the matrix $B_x^{(2)}$ is symmetric positive definite. By applying the same arguments to the preconditioning matrix M, we can immediately show that M is positive definite and nonsingular, too. ▯

3. Several preliminary lemmas.

3. Several preliminary lemmas. In this section, we are going to demonstrate several lemmas that are indispensable for estimating the eigenvalue bounds of the preconditioned matrix $M^{-1}A$.

LEMMA 3.1. *Let $\Delta = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ be an n-by-n positive diagonal matrix and $H \in \mathbb{C}^{n\times n}$ be a Hermitian positive definite matrix. Then it holds that*

$$\frac{v^*(\Delta \otimes H)v}{v^*(H \otimes \Delta)v} \le \kappa(\Delta)\kappa(H) \qquad \forall v \in \mathbb{C}^n\backslash\{0\},$$

where $\kappa(\cdot)$ denotes the Euclidean condition number of the corresponding matrix.

Proof. Because $H \in \mathbb{C}^{n\times n}$ is a Hermitian positive definite matrix, there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a positive diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in$

$\mathbb{R}^{n\times n}$ such that $H = U^*\Lambda U$. Therefore, for all $v \in \mathbb{C}^n \setminus \{0\}$ we have

$$\begin{aligned}\frac{v^*(\Delta\otimes H)v}{v^*(H\otimes\Delta)v} &= \frac{v^*[\Delta\otimes(U^*\Lambda U)]v}{v^*[(U^*\Lambda U)\otimes\Delta]v}\\ &= \frac{v^*[(I\otimes U)^*(\Delta\otimes\Lambda)(I\otimes U)]v}{v^*[(U\otimes I)^*(\Lambda\otimes\Delta)(U\otimes I)]v}\\ &\le \frac{\max_{1\le\ell,j\le n}\{\delta_\ell\lambda_j\}}{\min_{1\le\ell,j\le n}\{\delta_j\lambda_\ell\}}\\ &= \frac{\max_{1\le\ell\le n}\delta_\ell}{\min_{1\le\ell\le n}\delta_\ell}\cdot\frac{\max_{1\le\ell\le n}\lambda_\ell}{\min_{1\le\ell\le n}\lambda_\ell}\\ &= \kappa(\Delta)\kappa(H). \qquad \square\end{aligned}$$

While Lemma 3.1 gives an upper bound about the generalized Rayleigh quotient with respect to the Hermitian positive definite matrix H, the following lemma presents an estimate about the generalized Rayleigh quotient with respect to the Hermitian and the skew-Hermitian matrices H and S.

LEMMA 3.2. *Let $\Gamma = \operatorname{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$ and $\Delta = \operatorname{diag}(\delta_1, \delta_2, \dots, \delta_n)$ be n-by-n positive diagonal matrices, $H \in \mathbb{C}^{n\times n}$ be a Hermitian positive definite matrix, and $S \in \mathbb{C}^{n\times n}$ be a skew-Hermitian matrix. Then it holds that*

$$\left|\frac{v^*(S\otimes\Gamma)v}{v^*(H\otimes\Delta)v}\right| \le \tau\left|\frac{v^*(S\otimes\Gamma)v}{v^*(H\otimes\Gamma)v}\right| \qquad \forall v\in\mathbb{C}^n\backslash\{0\},$$

where $\tau = \max_{1\le\ell\le n}\{\frac{\gamma_\ell}{\delta_\ell}\}$.

Proof. Because $H \in \mathbb{C}^{n\times n}$ is Hermitian positive definite, there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a positive diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ such that $H = U^*\Lambda U$. Therefore, for all $v \in \mathbb{C}^n \setminus \{0\}$ we have

$$\begin{aligned}v^*(H\otimes\Delta)v = v^*(U^*\Lambda U\otimes\Delta)v &= v^*((U^*\otimes I)(\Lambda\otimes\Delta)(U\otimes I))v\\ &\ge \frac{1}{\tau}(v^*((U^*\otimes I)(\Lambda\otimes\Gamma)(U\otimes I))v) = \frac{1}{\tau}(v^*(H\otimes\Gamma)v).\end{aligned}$$

It then follows that

$$\left|\frac{v^*(S\otimes\Gamma)v}{v^*(H\otimes\Delta)v}\right| \le \tau\left|\frac{v^*(S\otimes\Gamma)v}{v^*(H\otimes\Gamma)v}\right|. \qquad \square$$

The following *generalized Bendixson theorem*, established in [6], is essential for us to derive a rectangular domain for bounding the eigenvalues of the preconditioned matrix $M^{-1}A$.

THEOREM 3.1 (see [6, Theorem 2.4]). *Let $A, M \in C^{n\times n}$ be n-by-n complex matrices, and, for $\forall\ v \in C^n \setminus \{0\}$, it holds that $v^*\mathcal{H}(A)v \ne 0$ and $v^*\mathcal{H}(M)v \ne 0$. Let the functions $h(v)$, $f_A(v)$, and $f_M(v)$ be defined as*

$$h(v) = \frac{v^*\mathcal{H}(A)v}{v^*\mathcal{H}(M)v}, \quad f_A(v) = \frac{1}{\imath}\cdot\frac{v^*\mathcal{S}(A)v}{v^*\mathcal{H}(A)v}, \quad \text{and} \quad f_M(v) = \frac{1}{\imath}\cdot\frac{v^*\mathcal{S}(M)v}{v^*\mathcal{H}(M)v},$$

respectively. Assume that there exist positive constants γ_1 and γ_2 such that

$$\gamma_1 \le h(v) \le \gamma_2 \quad \forall\ v \in C^n \setminus \{0\}$$

and nonnegative constants η and μ such that

$$-\mu \le f_A(v) \le \mu \quad \text{and} \quad -\eta \le f_M(v) \le \eta \ \ \forall\ v \in C^n \setminus \{0\}.$$

Then, when $\eta\mu \le 1$, *we have*

$$\begin{cases} \dfrac{(1-\eta\mu)\gamma_1}{1+\eta^2} \le \operatorname{Re}\left(\lambda\left(M^{-1}A\right)\right) \le (1+\eta\mu)\gamma_2, \\ -(\eta+\mu)\gamma_2 \le \operatorname{Im}\left(\lambda\left(M^{-1}A\right)\right) \le (\eta+\mu)\gamma_2. \end{cases}$$

Here, $\operatorname{Re}(\cdot)$ *and* $\operatorname{Im}(\cdot)$ *represent the real and the imaginary parts of the corresponding complex, respectively.*

In order to derive the bounded domain about the eigenvalues of the matrix $M^{-1}A$ by making use of the generalized Bendixson theorem, we essentially need the bounds of several generalized Rayleigh quotients with respect to certain parts of the matrices A and M defined in (2.1) and (2.2). These bounds are precisely stated in the following two lemmas.

LEMMA 3.3 (see [6, Lemma 4.2]). *Assume that* $D_x^{(2)}$ *defined in* (1.9) *is a positive semidefinite diagonal matrix. Let* $T_x^{(2)}$ *be the Toeplitz matrix defined in* (1.7) *and* $B_x^{(2)}$ *the tridiagonal matrix defined in* (2.3), *respectively. Then it holds that*

$$1 \le \frac{v^*\left(T_x^{(2)}+D_x^{(2)}\right)v}{v^*\left(B_x^{(2)}+D_x^{(2)}\right)v} \le \frac{\pi^2}{4} \qquad \forall v \in \mathbb{C}^n\backslash\{0\}.$$

LEMMA 3.4. *Assume that* $D_x^{(2)}$ *defined in* (1.9) *is a positive semidefinite diagonal matrix,* Q_t *defined in* (1.12) *is a positive definite diagonal matrix, and* $D_z^{(j)}$ ($j=1,3$, $z\in\{x,t\}$) *are the diagonal matrices defined in* (1.8) *and* (1.10). *Let* $T_z^{(1)}$ ($z\in\{x,t\}$) *and* $T_x^{(2)}$ *be the Toeplitz matrices defined in* (1.6) *and* (1.7) *and* $B_z^{(1)}$ ($z\in\{x,t\}$) *and* $B_x^{(2)}$ *be the tridiagonal matrices defined in* (2.3), *respectively. Denote* $c_x^{(2)} = 4\sin^2(\frac{\pi}{2(N_x+1)})$. *For* $z\in\{x,t\}$, *let* $N_z = m_z+n_z+1$ *and assume* $N := N_x = N_t$. *Define*

$$\bar{d}_z^{(j)} = \max_{1\le\ell\le N}\left\{\left[D_z^{(j)}\right]_{\ell\ell}\right\} \quad (j=1,3, \quad z\in\{x,t\}), \quad d_x^{(2)} = \min_{1\le\ell\le N}\left\{\left[D_x^{(2)}\right]_{\ell\ell}\right\}$$

and

$$\mu_z^{(j)} = \frac{2\pi\bar{d}_z^{(j)}}{\sqrt{\left(c_x^{(2)}+d_x^{(2)}\right)\left(\pi^2+d_x^{(2)}\right)}},$$

$$\eta_z^{(j)} = \frac{\bar{d}_z^{(j)}\left(\sqrt{d_x^{(2)}+4}-\sqrt{d_x^{(2)}}\right)}{\sqrt{c_x^{(2)}+d_x^{(2)}}}, \qquad j=1,3, \quad z\in\{x,t\}.$$

Then, for $j=1,3$, $z\in\{x,t\}$, *and all* $v\in\mathbb{C}^n\setminus\{0\}$, *it holds that*

$$\max\left\{\left|\frac{v^*\left[\left(D_z^{(j)}T_z^{(1)}+T_z^{(1)}D_z^{(j)}\right)\otimes Q_t\right]v}{v^*\left[\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right|, \left|\frac{v^*\left[Q_t\otimes\left(D_z^{(j)}T_z^{(1)}+T_z^{(1)}D_z^{(j)}\right)\right]v}{v^*\left[Q_t\otimes\left(T_x^{(2)}+D_x^{(2)}\right)\right]v}\right|\right\} \le \mu_z^{(j)}$$

and

$$\max\left\{\left|\frac{v^*\left[\left(D_z^{(j)}B_z^{(1)}+B_z^{(1)}D_z^{(j)}\right)\otimes Q_t\right]v}{v^*\left[\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right|,\right.$$
$$\left.\left|\frac{v^*\left[Q_t\otimes\left(D_z^{(j)}B_z^{(1)}+B_z^{(1)}D_z^{(j)}\right)\right]v}{v^*\left[Q_t\otimes\left(B_x^{(2)}+D_x^{(2)}\right)\right]v}\right|\right\}\leq\eta_z^{(j)}.$$

Proof. By making use of Lemma 2.1, following the same arguments as in the proof of [6, Lemma 4.3] we can obtain these estimates. ∎

4. The spectral analysis. In this section, we will derive precise bounds for the eigenvalues of the preconditioned matrix $M^{-1}A$, where the matrices A and M are defined in (2.1) and (2.2), respectively. To this end, we first estimate the bounds of the function $h(v)$ defined in Theorem 3.1.

LEMMA 4.1. *Assume that $D_x^{(2)}$ and $D_z^{(4)}$ $(z\in\{x,t\})$ defined in* (1.9) *and* (1.11) *are positive semidefinite diagonal matrices and Q_z $(z\in\{x,t\})$ defined in* (1.12) *and Ω are positive definite diagonal matrices. Let $T_x^{(2)}$ be the Toeplitz matrix defined in* (1.7) *and $B_x^{(2)}$ be the tridiagonal matrix defined in* (2.3). *Then*

$$1\leq\frac{v^*\mathcal{H}(A)v}{v^*\mathcal{H}(M)v}\leq\frac{\pi^2}{4}\qquad\forall v\in\mathbb{C}^n\setminus\{0\}. \tag{4.1}$$

Proof. For notational simplicity we denote

$$D_\delta=Q_x\otimes D_t^{(4)}+D_x^{(4)}\otimes(Q_t\Omega)+\delta I,$$

where $\delta>0$ is arbitrary. Evidently, D_δ is a positive definite diagonal matrix. Therefore, for any $v\in\mathbb{C}^n\setminus\{0\}$, according to the proof of Theorem 2.1 we have

$$\begin{aligned}\frac{v^*[\mathcal{H}(A)+\delta I]v}{v^*[\mathcal{H}(M)+\delta I]v}&=\frac{v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t+D_\delta\right]v}{v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t+D_\delta\right]v}\\&\leq\max\left\{\frac{v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}{v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v},\quad\frac{v^*D_\delta v}{v^*D_\delta v}\right\}\\&=\max\left\{\frac{v^*\left[\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}{v^*\left[\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v},\quad 1\right\}.\end{aligned}$$

The above inequality follows from the basic inequality:

$$\frac{\beta_1+\beta_2}{\alpha_1+\alpha_2}\leq\max\left\{\frac{\beta_1}{\alpha_1},\quad\frac{\beta_2}{\alpha_2}\right\}\qquad\forall\alpha_j,\beta_j>0,\quad j=1,2.$$

Based on Lemma 3.3, we can demonstrate the validity of the estimate

$$\frac{v^*[\mathcal{H}(A)+\delta I]v}{v^*[\mathcal{H}(M)+\delta I]v}\leq\frac{\pi^2}{4}$$

in an analogous fashion to [6, Lemma 4.2]. Moreover, as $\delta > 0$ is arbitrary, it then follows that

$$\frac{v^*\mathcal{H}(A)v}{v^*\mathcal{H}(M)v} \le \frac{\pi^2}{4}.$$

Similarly, the left-hand side of the inequality (4.1) can be verified. ◻

For the bounds of the functions $f_A(v)$ and $f_M(v)$ defined in Theorem 3.1, we can give the following estimates.

LEMMA 4.2. *Assume that* $D_x^{(2)}$ *and* $D_z^{(4)}$ $(z \in \{x,t\})$ *defined in* (1.9) *and* (1.11) *are positive semidefinite diagonal matrices,* Q_z $(z \in \{x,t\})$ *defined in* (1.12) *and* Ω *are positive definite diagonal matrices, and* $D_z^{(j)} (j = 1, 3, z \in \{x,t\})$ *are the diagonal matrices defined in* (1.8) *and* (1.10). *Let* $T_z^{(1)} (z \in \{x,t\})$ *and* $T_x^{(2)}$ *be the Toeplitz matrices defined in* (1.6) *and* (1.7) *and* $B_z^{(1)}$ $(z \in \{x,t\})$ *and* $B_x^{(2)}$ *be the tridiagonal matrices defined in* (2.3), *respectively. Denote* $c_x^{(2)} = 4\sin^2(\frac{\pi}{2(N_x+1)})$. *For* $z \in \{x,t\}$, *let* $N_z = m_z + n_z + 1$ *and assume* $N := N_x = N_t$. *Define*

$$\bar{d}_z^{(j)} = \max_{1\le \ell \le N}\left\{\left[D_z^{(j)}\right]_{\ell\ell}\right\} \quad (j = 1,2,3), \quad d_x^{(2)} = \min_{1\le \ell \le N}\left\{\left[D_x^{(2)}\right]_{\ell\ell}\right\}$$

and

$$\mu_z^{(j)} = \frac{2\pi \bar{d}_z^{(j)}}{\sqrt{\left(c_x^{(2)} + d_x^{(2)}\right)\left(\pi^2 + d_x^{(2)}\right)}},$$

$$\eta_z^{(j)} = \frac{\bar{d}_z^{(j)}\left(\sqrt{d_x^{(2)} + 4} - \sqrt{d_x^{(2)}}\right)}{\sqrt{c_x^{(2)} + d_x^{(2)}}}, \qquad j = 1,3, \quad z \in \{x,t\}.$$

Let

$$\begin{cases} \mu = \mu_x^{(1)} + \dfrac{\varepsilon\left(\pi^2 + \bar{d}_x^{(2)}\right)\kappa(Q_t)\max\limits_{1\le\ell\le N}\left\{\left[Q_t^{-1}Q_x\right]_{\ell\ell}\right\}}{c_x^{(2)} + d_x^{(2)}}\mu_t^{(3)} + \max\limits_{1\le\ell\le N}\{[\Omega]_{\ell\ell}\}\mu_x^{(3)}, \\ \eta = \eta_x^{(1)} + \dfrac{\varepsilon\left(4 - c_x^{(2)} + \bar{d}_x^{(2)}\right)\kappa(Q_t)\max\limits_{1\le\ell\le N}\left\{\left[Q_t^{-1}Q_x\right]_{\ell\ell}\right\}}{c_x^{(2)} + d_x^{(2)}}\eta_t^{(3)} + \max\limits_{1\le\ell\le N}\{[\Omega]_{\ell\ell}\}\eta_x^{(3)}. \end{cases}$$

Then it holds that

$$\left|\frac{v^*\mathcal{S}(A)v}{v^*\mathcal{H}(A)v}\right| \le \mu \quad \textit{and} \quad \left|\frac{v^*\mathcal{S}(M)v}{v^*\mathcal{H}(M)v}\right| \le \eta \qquad \forall v \in \mathbb{C}^n \setminus \{0\}.$$

Proof. For notational simplicity we denote

$$D^{(4)} = Q_x \otimes D_t^{(4)} + D_x^{(4)} \otimes (Q_t\Omega).$$

Because $D_z^{(4)} (z \in \{x,t\})$ are positive semidefinite diagonal matrices and $Q_z (z \in \{x,t\})$ and Ω are positive definite diagonal matrices, we see that $D^{(4)}$ is a positive semidefinite diagonal matrix.

For any $v \in \mathbb{C}^n \setminus \{0\}$, according to the proof of Theorem 2.1 we have

$$\begin{aligned}\left|\frac{v^*\mathcal{S}(A)v}{v^*\mathcal{H}(A)v}\right| &\leq \left|\frac{v^*\left[\varepsilon\left(D_x^{(1)}T_x^{(1)}+T_x^{(1)}D_x^{(1)}\right)\otimes Q_t\right]v}{v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t+D^{(4)}\right]v}\right| \\ &\quad+\left|\frac{v^*\left[Q_x\otimes\left(D_t^{(3)}T_t^{(1)}+T_t^{(1)}D_t^{(3)}\right)\right]v}{v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t+D^{(4)}\right]v}\right| \\ &\quad+\left|\frac{v^*\left[\left(D_x^{(3)}T_x^{(1)}+T_x^{(1)}D_x^{(3)}\right)\otimes(Q_t\Omega)\right]v}{v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t+D^{(4)}\right]v}\right| \\ &\leq \left|\frac{v^*\left[\left(D_x^{(1)}T_x^{(1)}+T_x^{(1)}D_x^{(1)}\right)\otimes Q_t\right]v}{v^*\left[\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right| \\ &\quad+\left|\frac{v^*\left[Q_x\otimes\left(D_t^{(3)}T_t^{(1)}+T_t^{(1)}D_t^{(3)}\right)\right]v}{v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right| \\ &\quad+\left|\frac{v^*\left[\left(D_x^{(3)}T_x^{(1)}+T_x^{(1)}D_x^{(3)}\right)\otimes(Q_t\Omega)\right]v}{v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right|\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\left|\frac{v^*\mathcal{S}(M)v}{v^*\mathcal{H}(M)v}\right| &\leq \left|\frac{v^*\left[\varepsilon\left(D_x^{(1)}B_x^{(1)}+B_x^{(1)}D_x^{(1)}\right)\otimes Q_t\right]v}{v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t+D^{(4)}\right]v}\right| \\ &\quad+\left|\frac{v^*\left[Q_x\otimes\left(D_t^{(3)}B_t^{(1)}+B_t^{(1)}D_t^{(3)}\right)\right]v}{v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t+D^{(4)}\right]v}\right| \\ &\quad+\left|\frac{v^*\left[\left(D_x^{(3)}B_x^{(1)}+B_x^{(1)}D_x^{(3)}\right)\otimes(Q_t\Omega)\right]v}{v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t+D^{(4)}\right]v}\right| \\ &\leq \left|\frac{v^*\left[\left(D_x^{(1)}B_x^{(1)}+B_x^{(1)}D_x^{(1)}\right)\otimes Q_t\right]v}{v^*\left[\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right| \\ &\quad+\left|\frac{v^*\left[Q_x\otimes\left(D_t^{(3)}B_t^{(1)}+B_t^{(1)}D_t^{(3)}\right)\right]v}{v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right| \\ &\quad+\left|\frac{v^*\left[\left(D_x^{(3)}B_x^{(1)}+B_x^{(1)}D_x^{(3)}\right)\otimes(Q_t\Omega)\right]v}{v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right|.\end{aligned} \tag{4.3}$$

Here in both estimates we have technically split the nominators into three parts and then used the triangular inequality to obtain the first inequalities. The second inequalities are directly obtained by using the positive semidefiniteness of the diagonal matrix $D^{(4)}$. In addition, we have used the facts that $D_x^{(2)}$ is a positive semidefinite diagonal matrix and both $T_x^{(2)}$ and $B_x^{(2)}$ are positive definite Toeplitz matrices; see Lemma 2.1.

From Lemma 3.4 we easily see that

$$\left|\frac{v^*\left[\left(D_x^{(1)}T_x^{(1)}+T_x^{(1)}D_x^{(1)}\right)\otimes Q_t\right]v}{v^*\left[\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right|\le \mu_x^{(1)} \tag{4.4}$$

and

$$\left|\frac{v^*\left[\left(D_x^{(1)}B_x^{(1)}+B_x^{(1)}D_x^{(1)}\right)\otimes Q_t\right]v}{v^*\left[\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right|\le \eta_x^{(1)} \tag{4.5}$$

hold true. It follows from Lemmas 2.1 and 2.2 that

$$\kappa\left(T_x^{(2)}+D_x^{(2)}\right)\le\frac{\pi^2+\bar{d}_x^{(2)}}{c_x^{(2)}+d_x^{(2)}}\quad\text{and}\quad\kappa\left(B_x^{(2)}+D_x^{(2)}\right)\le\frac{4-c_x^{(2)}+\bar{d}_x^{(2)}}{c_x^{(2)}+d_x^{(2)}}. \tag{4.6}$$

By making use of Lemma 3.1 and (4.6), we have

$$\begin{aligned} v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v &\ge \frac{v^*\left[Q_t\otimes\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\right]v}{\kappa\left(\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\right)\kappa(Q_t)} \\ &\ge \frac{c_x^{(2)}+d_x^{(2)}}{\varepsilon\left(\pi^2+\bar{d}_x^{(2)}\right)\kappa(Q_t)}\cdot v^*\left[Q_t\otimes\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\right]v \\ &= \frac{1}{\sigma_T}\cdot v^*\left[Q_t\otimes\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\right]v \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v &\ge \frac{v^*\left[Q_t\otimes\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\right]v}{\kappa\left(\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\right)\kappa(Q_t)} \\ &\ge \frac{c_x^{(2)}+d_x^{(2)}}{\varepsilon\left(4-c_x^{(2)}+\bar{d}_x^{(2)}\right)\kappa(Q_t)}\cdot v^*\left[Q_t\otimes\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\right]v \\ &= \frac{1}{\sigma_B}\cdot v^*\left[Q_t\otimes\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\right]v, \end{aligned} \tag{4.8}$$

where

$$\sigma_T=\frac{\varepsilon\left(\pi^2+\bar{d}_x^{(2)}\right)\kappa(Q_t)}{c_x^{(2)}+d_x^{(2)}}\quad\text{and}\quad\sigma_B=\frac{\varepsilon\left(4-c_x^{(2)}+\bar{d}_x^{(2)}\right)\kappa(Q_t)}{c_x^{(2)}+d_x^{(2)}}.$$

Therefore, according to Lemmas 3.2 and 3.4, as well as (4.7)–(4.8), it holds that

$$\begin{aligned}\left|\frac{v^*\left[Q_x\otimes\left(D_t^{(3)}T_t^{(1)}+T_t^{(1)}D_t^{(3)}\right)\right]v}{v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right| &\leq \sigma_T\left|\frac{v^*\left[Q_x\otimes\left(D_t^{(3)}T_t^{(1)}+T_t^{(1)}D_t^{(3)}\right)\right]v}{v^*\left[Q_t\otimes\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\right]v}\right|\\ &\leq \sigma_T\tau_Q\left|\frac{v^*\left[Q_x\otimes\left(D_t^{(3)}T_t^{(1)}+T_t^{(1)}D_t^{(3)}\right)\right]v}{v^*\left[Q_x\otimes\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\right]v}\right|\\ &\leq \sigma_T\tau_Q\mu_t^{(3)} \qquad (4.9)\end{aligned}$$

and

$$\begin{aligned}\left|\frac{v^*\left[Q_x\otimes\left(D_t^{(3)}B_t^{(1)}+B_t^{(1)}D_t^{(3)}\right)\right]v}{v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right| &\leq \sigma_B\left|\frac{v^*\left[Q_x\otimes\left(D_t^{(3)}B_t^{(1)}+B_t^{(1)}D_t^{(3)}\right)\right]v}{v^*\left[Q_t\otimes\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\right]v}\right|\\ &\leq \sigma_B\tau_Q\left|\frac{v^*\left[Q_x\otimes\left(D_t^{(3)}B_t^{(1)}+B_t^{(1)}D_t^{(3)}\right)\right]v}{v^*\left[Q_x\otimes\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\right]v}\right|\\ &\leq \sigma_B\tau_Q\eta_t^{(3)}, \qquad (4.10)\end{aligned}$$

where $\tau_Q=\max_{1\leq\ell\leq N}\{[Q_t^{-1}Q_x]_{\ell\ell}\}$. In addition, according to Lemmas 3.2 and 3.4 it holds that

$$\begin{aligned}\left|\frac{v^*\left[\left(D_x^{(3)}T_x^{(1)}+T_x^{(1)}D_x^{(3)}\right)\otimes(Q_t\Omega)\right]v}{v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right| &\leq \tau_\Omega\left|\frac{v^*\left[\left(D_x^{(3)}T_x^{(1)}+T_x^{(1)}D_x^{(3)}\right)\otimes(Q_t\Omega)\right]v}{v^*\left[\varepsilon\left(T_x^{(2)}+D_x^{(2)}\right)\otimes(Q_t\Omega)\right]v}\right|\\ &\leq \tau_\Omega\mu_x^{(3)} \qquad (4.11)\end{aligned}$$

and

$$\begin{aligned}\left|\frac{v^*\left[\left(D_x^{(3)}B_x^{(1)}+B_x^{(1)}D_x^{(3)}\right)\otimes(Q_t\Omega)\right]v}{v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes Q_t\right]v}\right| &\leq \tau_\Omega\left|\frac{v^*\left[\left(D_x^{(3)}B_x^{(1)}+B_x^{(1)}D_x^{(3)}\right)\otimes(Q_t\Omega)\right]v}{v^*\left[\varepsilon\left(B_x^{(2)}+D_x^{(2)}\right)\otimes(Q_t\Omega)\right]v}\right|\\ &\leq \tau_\Omega\eta_x^{(3)}, \qquad (4.12)\end{aligned}$$

where $\tau_Q=\max_{1\leq\ell\leq N}\{[\Omega]_{\ell\ell}\}$.

Now, by substituting the inequalities (4.4), (4.5), (4.9), (4.10), (4.11), and (4.12) into (4.2) and (4.3), we immediately obtain the estimates that we are deriving. ∎

By using Theorem 3.1 and Lemmas 4.1 and 4.2, we can straightforwardly obtain the main theorem of this paper.

THEOREM 4.1. *Let the conditions of Lemma 4.2 be satisfied. Without loss of generality, we make use of scaling on the original system of linear equations such that* $\mu\eta<1$. *Then it holds that*

$$\frac{1-\mu\eta}{1+\eta^2}\leq \operatorname{Re}\left(\lambda\left(M^{-1}A\right)\right)\leq\frac{\pi^2(1+\mu\eta)}{4}$$

and

$$-\frac{\pi^2(\mu+\eta)}{4}\leq \operatorname{Im}\left(\lambda\left(M^{-1}A\right)\right)\leq\frac{\pi^2(\mu+\eta)}{4}.$$

Based on Theorem 4.1, we can immediately obtain a theoretical estimate about the asymptotic convergence rate of the preconditioned GMRES method with the preconditioner M in (2.2) for solving the system of linear equations (1.13). Here, we should suitably scale the partial differential equation (1.1) and appropriately choose the weighting functions $\omega_x(x)$ and $\omega_t(t)$ and the conformal mappings $\phi_x(x)$ and $\phi_t(t)$ such that $\mu\eta < 1$. For details, we refer to [20, 6].

We remark that, when Theorem 4.1 is specialized to the matrices A and M, arising from the sinc-Galerkin discretization of the Burgers equation, much sharper bounds than those given in [5] about the eigenvalues of the preconditioned matrix $M^{-1}A$ can be straightforwardly obtained under weaker restrictions. This is one of the theoretical advantages of our new result.

5. Numerical experiments. In this section, we use two examples of the time-dependent partial differential equation (1.1) to demonstrate the effectiveness of the preconditioning and the corresponding preconditioned GMRES iteration method. Here, both Newton and fixed-point methods are applied to solve the discretized system of nonlinear equations (1.2).

In our computations, the initial guess is set to be the zero vector and the outer nonlinear iteration is stopped once the current residual satisfies the criteria

$$\frac{\left\|r^{(k)}\right\|_2}{\left\|r^{(0)}\right\|_2} \le 10^{-6}.$$

In each outer iteration step, a preconditioned linear system

$$M^{-1}A\mathbf{z} = M^{-1}\mathbf{r}, \quad \text{with} \quad A = B + CD \quad \text{and} \quad M = \widehat{B} + \widehat{C}D, \tag{5.1}$$

is solved, which forms the inner iteration process for solving the linear subsystems involved in each step of the Newton or the fixed-point method; see (1.13) and (2.2). Here, the stopping criteria for the inner iteration, i.e., the preconditioned GMRES method, is that the relative reduction on the norm of the residual is less than 10^{-6}. Besides, all codes are written in MATLAB 7.01 and all experiments are implemented on a personal computer with 2.66GHz central processing unit and 0.99G memory.

For the positive diagonal matrix $\Omega = \text{diag}([\Omega]_{11}, [\Omega]_{22}, \ldots, [\Omega]_{N_tN_t})$, we can construct it according to a certain approximating rule. With respect to the Newton iteration method, we may minimize $\|I \otimes \Omega - \Psi'(\mathbf{u}^{(c)})\|_2$ to obtain the Ω, where $\mathbf{u}^{(c)} = (u_1^{(c)}, u_2^{(c)}, \ldots, u_n^{(c)})^T$ is the current Newton iterate. As now $\Psi'(\mathbf{u}) = 2 \cdot \text{diag}(u_1, u_2, \ldots, u_n)$, with $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$ and $n = N_xN_t$, by direct computations we can obtain the formulas for the diagonal elements of Ω as follows:

$$[\Omega]_{jj} = \frac{2}{N_x}\sum_{k=0}^{N_x-1} u_{kN_t+j}^{(c)}, \qquad j = 1, 2, \ldots, N_t.$$

Analogously, with respect to the fixed-point iteration method, we can choose

$$[\Omega]_{jj} = \frac{1}{N_x}\sum_{k=0}^{N_x-1} u_{kN_t+j}^{(c)}, \qquad j = 1, 2, \ldots, N_t,$$

where $\mathbf{u}^{(c)} = (u_1^{(c)}, u_2^{(c)}, \ldots, u_n^{(c)})^T$ denotes the current fixed-point iterate. Note that the difference between these two Ω's is just a factor of 2.

The following two equations in the form of (1.1) are used to examine the numerical performance of the new preconditioner M defined in (2.2) and to show the accuracy of the computed solution.

Example 5.1. The time-dependent partial differential equation

$$\begin{cases} -\dfrac{\partial u}{\partial t}(x,t) + \dfrac{u(x,t)}{x}\dfrac{\partial u}{\partial x}(x,t) - \varepsilon\dfrac{\partial^2 u}{\partial x^2}(x,t) \\ \quad = e^{-\pi^2 t}\sin(\pi x) \\ \qquad \cdot\left(\pi^2 t - 1 + \dfrac{\pi t e^{-\pi^2 t}\cos(\pi x)}{x} + \varepsilon\pi^2 t\right), & 0 < x < 1 \quad \text{and} \quad t \geq 0, \\ u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0, & t \geq 0, \\ u(x,0) = 0, & 0 \leq x \leq 1, \end{cases}$$

with the exact solution being $u(x,t) = te^{-\pi^2 t}\sin(\pi x)$.

Example 5.2. The time-dependent partial differential equation

$$\begin{cases} -\dfrac{\partial u}{\partial t}(x,t) + \dfrac{u(x,t)}{x}\dfrac{\partial u}{\partial x}(x,t) - \varepsilon\dfrac{\partial^2 u}{\partial x^2}(x,t) \\ \quad = -xe^{-t}(1-x)(1-t) \\ \qquad + t^2 e^{-2t}(1-x)(1-2x) - 2\varepsilon t e^{-t}, & 0 < x < 1 \quad \text{and} \quad t \geq 0, \\ u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0, & t \geq 0, \\ u(x,0) = 0, & 0 \leq x \leq 1, \end{cases}$$

with the exact solution being $u(x,t) = x(1-x)te^{-t}$.

The conformal mappings are chosen as $\phi(z) = \ln(\frac{z}{1-z})$ and $\psi(z) = \ln(\sinh(z))$ so that their restrictions onto the real intervals $(0,1)$ and $(0,+\infty)$ are $\phi_x(x) := \phi(x) = \ln(\frac{x}{1-x})$ and $\phi_t(t) := \psi(t) = \ln(\sinh(t))$, which are used for the discretizations of x and t variables, respectively. And the weighting functions are chosen to be $\omega_x(x) = 1/\phi_x'(x)$ and $\omega_t(t) = 1/\phi_t'(t)$.

In the numerical tables, the symbol I means that no preconditioner is used when solving the linear subsystems involved in the nonlinear iterations, while M represents that the preconditioner M defined in (2.2) is used. We use N_{IT} to denote the number of the Newton iteration steps, F_{IT} that of the fixed-point iteration steps, G_{IT} the average number of GMRES iteration steps in each Newton or fixed-point iteration, CPU the total computing timings, and Se the maximum absolute discretization error at the sinc grid points and Ue that on the corresponding uniform grid points, while we use "average Se" and "average Ue" to represent the average absolute errors at all of the sinc grid points and at all of the uniform grid points, respectively. In addition, the symbol $*$ is used to denote that the iteration does not satisfy the terminating criterion within 50 steps of the Newton or the fixed-point iteration while $+$ that the inner iteration does not satisfy the GMRES terminating criterion within 1000 iteration steps.

We solve Example 5.1 when $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$. Tables 5.1–5.2 list the numbers of iteration steps and the CPU timings required for the convergence of the Newton iteration, and Tables 5.3–5.4 list those required for the convergence of the fixed-point iteration, respectively, when they are applied to solve the system of nonlinear equations (1.2) resulting from the sinc-Galerkin discretization of Example 5.1. Tables 5.5 and 5.6 list iteration numbers and CPU timings when the Newton and the fixed-point methods are applied, respectively, to Example 5.2, with $\varepsilon = 10^{-3}$. In all tables, some errors for reflecting the accuracy of the computed solutions are also shown.

Table 5.1
Results for Example 5.1. $\varepsilon = 10^{-3}$, and the Newton method is applied.

	I			M						
n	N_{IT}	G_{IT}	CPU	N_{IT}	G_{IT}	Se	average Se	Ue	average Ue	CPU
81	4	80	0.33	4	32	2.22×10^{-3}	7.73×10^{-4}	2.14×10^{-3}	6.54×10^{-4}	0.33
289	4	282	3.00	4	58	1.21×10^{-3}	1.72×10^{-4}	1.08×10^{-3}	8.70×10^{-5}	0.98
1089	4	977	62.36	4	111	1.55×10^{-3}	1.60×10^{-4}	1.30×10^{-3}	1.41×10^{-5}	6.48
4225	*	+	—	4	246	1.69×10^{-3}	1.86×10^{-4}	1.48×10^{-3}	1.05×10^{-5}	78.59

Table 5.2
Results for Example 5.1. $\varepsilon = 10^{-4}$, and the Newton method is applied.

	I			M						
n	N_{IT}	G_{IT}	CPU	N_{IT}	G_{IT}	Se	average Se	Ue	average Ue	CPU
81	4	80	0.33	4	35	2.36×10^{-3}	7.31×10^{-4}	2.23×10^{-3}	6.22×10^{-4}	0.25
289	4	283	3.02	4	68	5.11×10^{-4}	8.08×10^{-5}	2.61×10^{-4}	7.05×10^{-5}	1.13
1089	4	963	61.25	4	148	3.70×10^{-4}	3.03×10^{-5}	1.71×10^{-4}	4.31×10^{-6}	8.72
4225	*	+	—	5	359	4.62×10^{-4}	2.91×10^{-5}	1.85×10^{-4}	1.43×10^{-6}	170.20

Table 5.3
Results for Example 5.1. $\varepsilon = 10^{-3}$, and the fixed-point method is applied.

	I			M						
n	F_{IT}	G_{IT}	CPU	F_{IT}	G_{IT}	Se	average Se	Ue	average Ue	CPU
81	5	65	0.33	5	25	2.22×10^{-3}	7.73×10^{-4}	2.14×10^{-3}	6.54×10^{-4}	0.23
289	4	210	2.25	4	40	1.21×10^{-3}	1.72×10^{-4}	1.08×10^{-3}	8.70×10^{-5}	0.64
1089	6	824	83.67	6	76	1.54×10^{-3}	1.62×10^{-4}	1.30×10^{-3}	1.41×10^{-5}	6.32
4225	*	+	—	12	140	1.69×10^{-3}	1.90×10^{-4}	1.47×10^{-3}	1.05×10^{-5}	111.63

Table 5.4
Results for Example 5.1. $\varepsilon = 10^{-4}$, and the fixed-point method is applied.

	I			M						
n	F_{IT}	G_{IT}	CPU	F_{IT}	G_{IT}	Se	average Se	Ue	average Ue	CPU
81	6	68	0.34	6	27	2.36×10^{-3}	7.31×10^{-4}	2.23×10^{-3}	6.22×10^{-4}	0.30
289	4	211	2.27	4	46	5.11×10^{-4}	8.08×10^{-5}	2.60×10^{-4}	7.06×10^{-5}	0.75
1089	4	711	47.06	3	72	2.04×10^{-4}	2.49×10^{-5}	1.74×10^{-4}	4.30×10^{-6}	3.23
4225	*	+	—	3	123	2.44×10^{-4}	2.77×10^{-5}	1.99×10^{-4}	1.47×10^{-6}	25.67

Table 5.5
Results for Example 5.2. $\varepsilon = 10^{-3}$, and the Newton method is applied.

	I			M						
n	N_{IT}	G_{IT}	CPU	N_{IT}	G_{IT}	Se	average Se	Ue	average Ue	CPU
289	9	285	7.08	9	87	4.23×10^{-3}	1.30×10^{-3}	4.27×10^{-3}	1.41×10^{-3}	3.31
1089	9	996	149.11	9	179	1.82×10^{-3}	6.59×10^{-4}	1.80×10^{-3}	4.87×10^{-4}	25.27
4225	*	+	—	10	508	2.35×10^{-3}	5.77×10^{-4}	2.17×10^{-3}	3.17×10^{-4}	653.31

Table 5.6
Results for Example 5.2. $\varepsilon = 10^{-3}$, and the fixed-point method is applied.

	I			M						
n	F_{IT}	G_{IT}	CPU	F_{IT}	G_{IT}	Se	average Se	Ue	average Ue	CPU
289	12	246	7.64	11	53	4.23×10^{-3}	1.30×10^{-3}	4.27×10^{-3}	1.41×10^{-3}	2.38
1089	16	808	204.30	14	93	1.82×10^{-3}	6.60×10^{-4}	1.80×10^{-3}	4.87×10^{-4}	17.98
4225	*	+	—	33	168	2.35×10^{-3}	5.77×10^{-4}	2.17×10^{-3}	3.17×10^{-4}	377.16

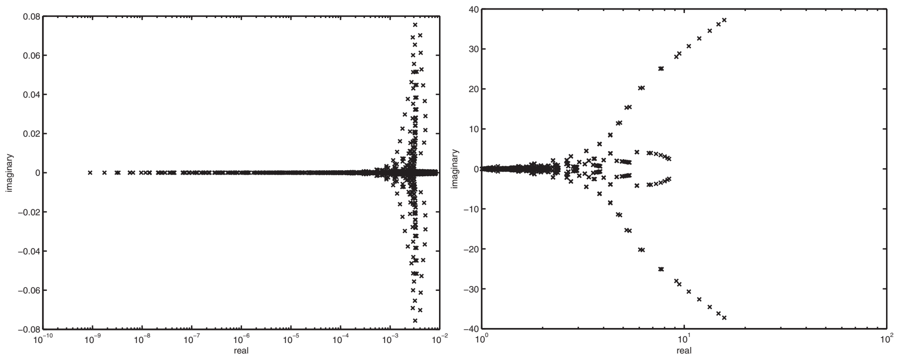

FIG. 5.1. *Spectral distribution of Example* 5.1. $\varepsilon = 10^{-3}$ *and* $n = 1089$*; without preconditioning (left), with the preconditioner* M *(right); and the Newton method is applied.*

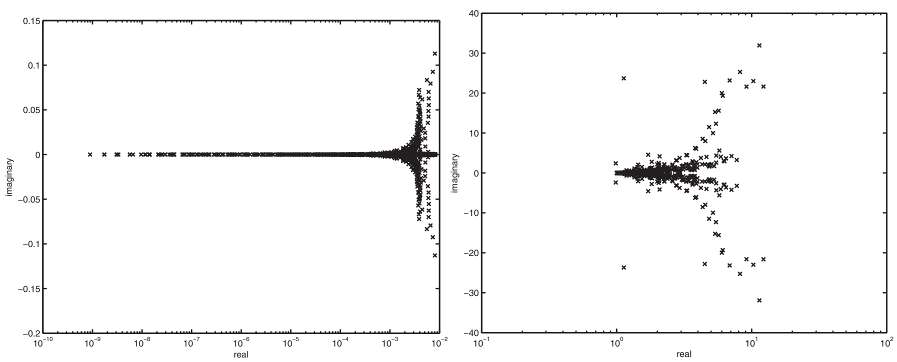

FIG. 5.2. *Spectral distribution of Example* 5.2. $\varepsilon = 10^{-3}$ *and* $n = 1089$*; without preconditioning (left), with the preconditioner* M *(right); and the fixed-point method is applied.*

From these tables, we see that the new preconditioner can considerably improve the convergence properties of both Newton and fixed-point iteration methods and greatly reduce the running times. Moreover, with increasing of the problem size n, the number of the Newton or the fixed-point iteration steps keeps almost the same or increases slowly if the inner iteration solver, i.e., GMRES, is preconditioned by the new preconditioner while GMRES cannot achieve the prescribed tolerance within 1000 iteration steps and, therefore, the Newton or the fixed-point iteration cannot achieve the prescribed tolerance within 50 iteration steps if GMRES without using a preconditioner is employed as the inner iteration solver. Therefore, the new preconditioning method can substantially improve the convergence behaviors of both Newton and fixed-point iterations and, consequently, lead to fast convergent nonlinear solvers for the systems of nonlinear equations (1.2) arising in the sinc-Galerkin discretization of the time-dependent partial differential equation (1.1).

Figures 5.1 and 5.2 depict the spectral distributions of the original coefficient matrix A and the preconditioned matrix $M^{-1}A$ when the Newton method is applied to Example 5.1 and the fixed-point method is applied to Example 5.2, respectively. The figures clearly show that the matrices without preconditioning are very ill-conditioned and, therefore, the corresponding GMRES method may be convergent very slowly or even divergent, while the matrices with preconditioning are well-conditioned as they

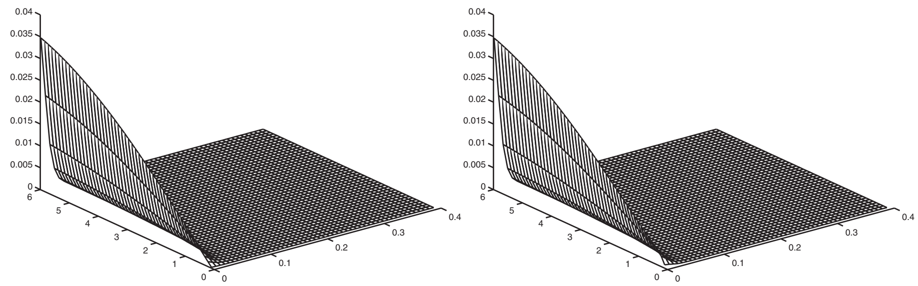

FIG. 5.3. *Solutions of Example* 5.1. $\varepsilon = 10^{-3}$ *and* $n = 1089$*; exact solution (left), computed solution (right); and the Newton method is applied.*

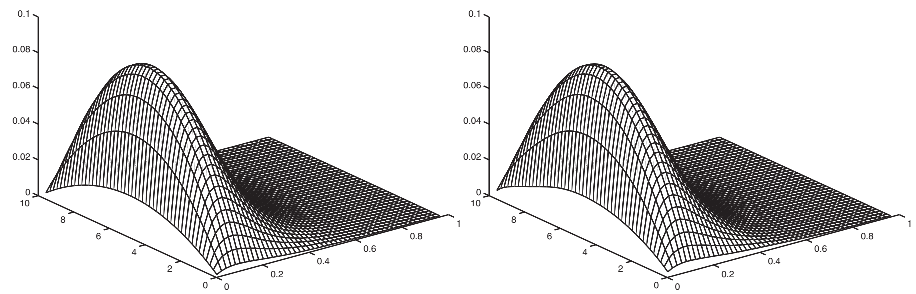

FIG. 5.4. *Solutions of Example* 5.2. $\varepsilon = 10^{-3}$ *and* $n = 1089$*; exact solution (left), computed solution (right); and the fixed-point method is applied.*

have tightly clustered eigenvalues and, thus, the corresponding preconditioned GMRES method may be convergent very quickly to the exact solutions of the subsystems of linear equations. As a result, the preconditioned GMRES method used as the inner linear solver may lead to a fast convergent Newton or fixed-point method for solving the sinc-Galerkin nonlinear systems of the form (1.2).

In Figures 5.3 and 5.4, we plot the exact and the computed solutions of Examples 5.1 and 5.2 corresponding to the cases shown in Figures 5.1 and 5.2, respectively, where the computed solution is obtained by using either the Newton or the fixed-point method. It is clear from Figures 5.3 and 5.4 that the new preconditioned iteration methods can compute reasonably accurate results.

6. Concluding remarks. We have constructed a structured preconditioner that can efficiently improve the convergence property of the GMRES iteration employed to inexactly solve the subsystem of linear equations involved in each Newton or fixed-point iteration for solving the system of nonlinear equations resulting from the sinc-Galerkin discretization of the time-dependent partial differential equation (1.1). The bounds of the eigenvalues of the preconditioned matrix were precisely estimated by making use of the generalized Bendixson theorem, which, in particular, can lead to sharper eigenvalue bounds than those derived in [5] for the preconditioned matrix arising from the sinc-Galerkin discretization of the Burgers equation. Numerical experiments have shown the effectiveness of this new preconditioning method.

REFERENCES

[1] Z.-Z. Bai, *Parallel multisplitting AOR method for solving a class of system of nonlinear algebraic equations,* Appl. Math. Mech., 16 (1995), pp. 675–682.

[2] Z.-Z. Bai, *Parallel nonlinear AOR method and its convergence,* Comput. Math. Appl., 31 (1996), pp. 21–31.

[3] Z.-Z. Bai, G.H. Golub, L.-Z. Lu, and J.-F. Yin, *Block triangular and skew-Hermitian splitting methods for positive-definite linear systems,* SIAM J. Sci. Comput., 26 (2005), pp. 844–863.

[4] Z.-Z. Bai, G.H. Golub, and M.K. Ng, *Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems,* SIAM J. Matrix Anal. Appl., 24 (2003), pp. 603–626.

[5] Z.-Z. Bai, Y-M. Huang, and M.K. Ng, *On preconditioned iterative methods for Burgers equations,* SIAM J. Sci. Comput., 29 (2007), pp. 415–439.

[6] Z.-Z. Bai and M.K. Ng, *Preconditioners for nonsymmetric block Toeplitz-like-plus-diagonal linear systems,* Numer. Math., 96 (2003), pp. 197–220.

[7] Z.-Z. Bai and D.-R. Wang, *Asynchronous multisplitting nonlinear Gauss-Seidel type method,* Appl. Math. J. Chinese Univ. Ser. B, 9 (1994), pp. 189–194.

[8] Z.-Z. Bai and D.-R. Wang, *Asynchronous parallel multisplitting nonlinear Gauss-Seidel iteration,* Appl. Math. Chinese Univ. Ser. B, 12 (1997), pp. 179–194.

[9] R.H. Chan and X.-Q. Jin, *A family of block preconditioners for block systems,* SIAM J. Sci. Statist. Comput., 13 (1992), pp. 1218–1235.

[10] R.H. Chan, W.-F. Ng, and H.-W. Sun, *Fast construction of optimal circulant preconditioners for matrices from the fast dense matrix method,* BIT, 40 (2000), pp. 24–40.

[11] X.-Q. Jin, *A note on preconditioned block Toeplitz matrices,* SIAM J. Sci. Comput., 16 (1995), pp. 951–955.

[12] X.-Q. Jin, *Band Toeplitz preconditioners for block Toeplitz systems,* J. Comput. Appl. Math., 70 (1996), pp. 225–230.

[13] X.-Q. Jin, *Developments and Applications of Block Toeplitz Iterative Solvers*, Kluwer Academic Publishers, Dordrecht Science Press, Beijing, 2002.

[14] T. Kailath and A.H. Sayed, *Displacement structure: Theory and applications,* SIAM Rev., 37 (1995), pp. 297–386.

[15] N. Levinson, *The Wiener RMS (root mean square) error criterion in filter design and prediction,* J. Math. Phys. Mass. Inst. Tech., 25 (1947), pp. 261–278.

[16] J. Lund and K.L. Bowers, *Sinc Methods for Quadrature and Differential Equations*, SIAM, Philadelphia, 1992.

[17] M.K. Ng, *Fast iterative methods for symmetric sinc-Galerkin systems,* IMA J. Numer. Anal., 19 (1999), pp. 357–373.

[18] M.K. Ng and D. Potts, *Fast iterative methods for sinc systems,* SIAM J. Matrix Anal. Appl., 24 (2002), pp. 581–598.

[19] J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, London, 1970.

[20] Y. Saad, *Iterative Methods for Sparse Linear Systems*, PWS, Boston, 1996.

[21] D.-R. Wang, Z.-Z. Bai, and D.J. Evans, *Asynchronous multisplitting relaxed iterations for weakly nonlinear systems,* Int. J. Comput. Math., 54 (1994), pp. 57–76.

[22] D.-R. Wang, Z.-Z. Bai, and D.J. Evans, *On the monotone convergence of multisplitting method for a class of system of weakly nonlinear equations,* Int. J. Comput Math., 60 (1996), pp. 229–242.