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ANALYSIS AND CONVERGENCE OF A COVOLUME APPROXIMATION OF THE GINZBURG–LANDAU MODEL OF SUPERCONDUCTIVITY*

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Abstract. In this paper, we present the mathematical analysis of a covolume method for the approximations of the Ginzburg–Landau (GL) model for superconductivity. A nice feature of this approach is that the gauge invariance properties are retained in discrete approximations based on triangular grids. We also use properties of discrete vector fields to study issues such as the gauge choices and their enforcement.

Key words. Ginzburg–Landau model of superconductivity, covolume approximations, gauge invariance, convergence analysis

AMS subject classifications. 65N99, 82D55

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1. Introduction.

The macroscopic model of Ginzburg and Landau [12, 17] has been well accepted as a valid model for low-temperature superconductors. Even though for high-T_c superconductors, satisfactory microscopic models still await discovery, it is a reasonably simple matter to generalize the low-T_c GL model to account for high-T_c phenomena. Numerical simulations based on the GL models have been conducted extensively by physicists. The mathematical theories of numerical methods for the approximation of GL-type models have also been given by various authors. For example, finite element approximations were studied in [3, 4, 5, 6, 7, 8], which contain additional references on this subject. Here, we present a mathematical framework of a new approximation of the GL model on triangular grids based on the covolume techniques. This should be very useful for simulations of high-temperature superconductors for which triangular grids may be more convenient for modeling inhomogeneities in the spatial structure.

Covolume methods have been around for a long time [14]. Additional references are given in [15, 16] in which the covolume technique was developed to solve the div-curl systems and the Navier–Stokes equations. Covolume methods often preserve discrete versions of physical and mathematical laws, such as the divergence theorem, the Green's formula, and many forms of conservation laws. An application to the GL models of superconductivity which preserves the gauge invariance at the discrete level was presented in [9]. Similar ideas were used later in numerical simulations [10]. On a structured rectangular grid, the covolume method is also related to a nonstandard difference approximation given in [2] and also in [11, 13]. In this paper, we expand

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on the ideas of [9] and analyze the convergence of covolume methods for the steady state GL model.

The paper is organized as follows. In section 2, we describe the basic GL theory and the steady-state model in a bounded domain. In section 3, covolume approximations of div-curl systems and the Laplacian operator are given first along with some technical results. These are followed by the covolume approximation to the GL model. In section 4, we study properties concerning these approximations. The discrete gauge invariance of the approximation is established. In addition, we present the existence of the finite volume approximations as well as the pointwise bound for the magnitude of the discrete order parameter which mimics its continuous version. Then, we provide the convergence analysis of the covolume approximation. Some final comments are given in section 5.

2. The GL model for superconductivity. This section is concerned with some basic questions related to the steady-state GL model on a bounded domain and its approximations. Detailed discussion can be found in, for example, [5]. For an introduction to the theory of superconductivity, see [17].

2.1. The GL free energy. Let H be a constant applied magnetic field and $\Omega \subset R^2$ be the region occupied by the superconducting sample. For simplicity, we will assume that Ω is a bounded, connected polygonal domain. The basic thermodynamic postulate of the GL model is that the superconducting sample is in a state such that its Gibbs free energy is a minimum. Ginzburg and Landau postulated that the Gibbs free energy, in a nondimensionalized form, is given by

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left(\frac{1}{2}(1 - |\psi|^2)^2 + \left| \left(\frac{i}{\kappa}\nabla + \mathbf{A} \right) \psi \right|^2 + |\text{curl } \mathbf{A} - H|^2 \right) d\Omega \,, \tag{2.1}$$

where the first term is the potential energy, the second term is the kinetic energy, and the last term describes the magnetic energy and the work done by the applied field. The GL parameter κ is a material constant which, together with Ω and H, completely determines minimizers of the Gibbs free energy, or the solutions of the GL equations on a given domain Ω. The variables employed in GL models for superconductivity are the real, vector-valued *magnetic potential* $\mathbf{A}$ and the complex, scalar-valued *order parameter* ψ. They are related to (appropriately nondimensionalized) physical variables by

$$\begin{aligned} &\text{magnetic field:} && h = \text{curl}\,\mathbf{A}, \\ &\text{current:} && \mathbf{j} = \text{curl}\,h, \\ &\text{density of superconducting charge carriers:} && N_s = |\psi|^2 \,. \end{aligned} \tag{2.2}$$

The minimization of $\mathcal{G}$ with respect to variations in ψ and $\mathbf{A}$ yields the GL equations

$$\left(\frac{i}{\kappa}\nabla + \mathbf{A} \right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega \,, \tag{2.3}$$

$$\text{curl}\,\text{curl}\,\mathbf{A} = -\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2\mathbf{A} \quad \text{in } \Omega \,, \tag{2.4}$$

where $(\cdot)^*$ denotes the complex conjugate, along with natural boundary conditions on the boundary Γ:

$$\left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi \right) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \,, \tag{2.5}$$

$$\operatorname{curl} \mathbf{A} = H \quad \text{on } \Gamma\,. \tag{2.6}$$

Here, $\mathbf{n}$ is the unit outer normal vector to Γ. More general boundary conditions may also be considered; see [5].

2.2. Some properties of GL models. We use $H^m(\Omega)$ to denote the standard Sobolev space of real-valued functions, and $\mathcal{H}^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ to denote similar function spaces of complex-valued functions and vector-valued functions respectively. The norm in $H^m(\mathcal{D})$ is denoted by $\|\cdot\|_{m,\mathcal{D}}$, while $\|\cdot\|_{m,p,\mathcal{D}}$ denotes the norm in $W^{m,p}(\Omega)$. We also let

$$\mathbf{H}^1_n(\Omega) = \{\, \mathbf{Q} \in \mathbf{H}^1(\Omega) \;:\; \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\}\,,$$

$$\mathbf{H}^1_n(\operatorname{div};\Omega) = \{\, \mathbf{Q} \in \mathbf{H}^1(\Omega) \;:\; \operatorname{div}\mathbf{Q} = 0 \text{ in } \Omega \text{ and } \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\}\,.$$

The GL functional has an important property, namely, gauge invariance. To be specific, for any $\phi \in H^2(\Omega)$, let the linear transformation T_ϕ be defined by

$$T_\phi(\psi, \mathbf{A}) = (\psi e^{i\kappa\phi}\;,\; \mathbf{A} + \nabla\phi), \forall\, (\psi, \mathbf{A}) \in \mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega),$$

where κ is the GL parameter. Then, we have the following definition (see [5]).

DEFINITION 2.1. $(\psi, \mathbf{A})$ *and* $(\zeta, \mathbf{Q})$ *are said to be* gauge equivalent *if and only if there exists a smooth function* $\phi \in H^2(\Omega)$ *such that* $(\psi, \mathbf{A}) = T_\phi(\zeta, \mathbf{Q})$.

PROPOSITION 2.2. *For all* $\phi \in H^2(\Omega)$ *and* $(\psi, \mathbf{A}) \in \mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$, $\mathcal{G}(\psi, \mathbf{A}) = \mathcal{G}(T_\phi(\psi, \mathbf{A}))$*; i.e.,* $\mathcal{G}$ *is invariant under the gauge transformation* T_ϕ. □

LEMMA 2.3. *Any element* $(\psi, \mathbf{A}) \in \mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ *is gauge equivalent to an element* $(\zeta, \mathbf{Q})$ *of* $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\operatorname{div};\Omega)$. □

Consequently, one can prove the following theorem [5].

THEOREM 2.4. $\mathcal{G}$ *has at least one minimizer in* $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_{\mathrm{n}}(\operatorname{div};\Omega)$. *Moreover,*

$$\min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\operatorname{div};\Omega)} \mathcal{G} = \min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)} \mathcal{G}\,. \tag{2.7}$$

Thus, $\mathcal{G}$ *has a least one minimizer belonging to* $\mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$. □

2.3. Finite element approximations. Finite element approximations of the above models have been studied in detail in [3, 4, 5, 6, 7, 8]. They are based on the standard Ritz–Galerkin approach. The discrete approximation results in problems like

$$\min_{\mathcal{V}^h\times\mathbf{V}^h} \mathcal{G}(\psi^h, \mathbf{A}^h)\,, \tag{2.8}$$

where $\mathcal{V}^h$ and $\mathbf{V}^h$ are finite element subspaces of $\mathcal{H}^1(\Omega)$ and $\mathbf{H}^1(\Omega)$.

3. Covolume approximations. In general, discrete approximations given by (2.8) no longer enforce the gauge invariant property precisely. For problems on a rectangular region, the nonstandard difference-like approximation given in [2] and later in [11, 13] does preserve certain discrete gauge invariant properties. Covolume approximations have been constructed in [9] which serves as the generalization of the nonstandard difference methods to more complicated geometry partitioned by a triangular grid. (A brief summary was given in [4].) This new method is useful for high-temperature superconductors for which spatial inhomogeneities play a large role. In this section, we describe the construction previously given in the unpublished manuscript [9].

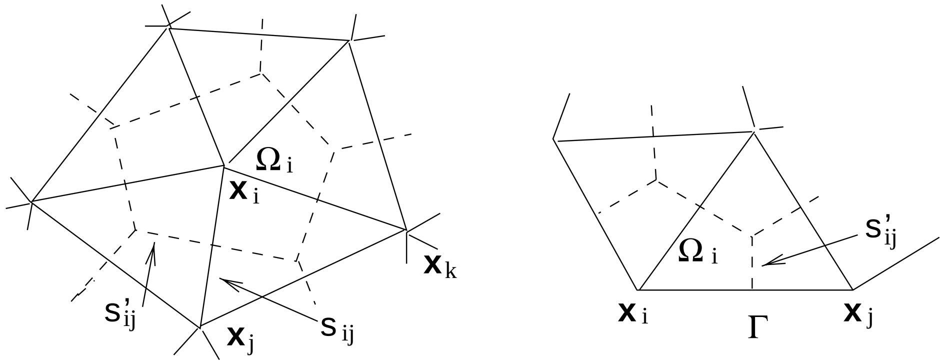

FIG. 1. *A triangular grid and its dual.*

3.1. Planar triangulation and covolume methods. We first define the following terminology for the given domain Ω with boundary Γ. Let Σ denote a regular triangulation of Ω with N_0 vertices $\mathbf{x}_j$, $j = 1, 2, ..., N_0$, N_1 edges that are denoted by s_{ij}, and N_2 triangles that are denoted by τ_{ijk}. A dual tessellation is formed by joining the circumcenters of adjacent triangles. Simple geometry implies that the line joining the circumcenters is normal to and bisects the common edge. The dual figures are polygons and are named as *covolumes* Σ'. The interior of the polygon associated with all the edges connecting the vertex $\mathbf{x}_j$ is denoted by Ω_j (see Figure 1).

To avoid the complexity associated with the self-intersecting covolumes, we require that τ is *locally equiangular* [15], i.e., for every pair of adjacent triangles that form a convex quadrilateral, the sum of the angles opposite the common side is at most 180 degrees. This ensures that

(1) each Ω_j is convex,

(2) $\Omega_j \cap \Omega_k = \emptyset$ if $j \neq k$.

Let $h_{ij} = |s_{ij}|$ denote the length of the edge s_{ij} and $h'_{ij} = |s'_{ij}|$ denote the length of the corresponding edge s'_{ij} of the covolume which bisects and is normal to s_{ij}. We use $\tau_{ijk} \in \Sigma$ to denote the triangle with adjacent vertices $\mathbf{x}_i, \mathbf{x}_j$, and $\mathbf{x}_k$ with area denoted by $|\tau_{ijk}|$. Note that for these adjacent vertices $\mathbf{x}_i, \mathbf{x}_j$, and $\mathbf{x}_k$, at most one of h'_{ij}, h'_{jk}, and h'_{ki} is zero. If a vertex $\mathbf{x}_i$ is on the boundary of Ω, the region Ω_i is modified to include only the portion which is inside Ω. Similarly, $h'_{ij} = |s'_{ij}|$ includes only the segment contained inside Ω for two adjacent vertices on the boundary Γ. The area of Ω_j is denoted by $|\Omega_j|$.

3.2. Covolume method for the div-curl operators. In the GL model, the magnetic field is computed from the curl of the magnetic potential $\mathbf{A}$. So, we will discuss first how this is done in the covolume setting. Meanwhile, the magnetic potential can often be taken in the so-called *Coulomb* gauge $\operatorname{div} \mathbf{A} = 0$. To discuss the discrete gauge choices, we also describe here how to approximate the div operator. Covolume methods for the div-curl operators have been discussed in [15]. Here, we present a discretization scheme which may be considered as the dual formulation of that in [15].

Our discrete representation of a given two-dimensional vector field $\mathbf{A}$ will proceed as follows. At the midpoint $\mathbf{x}_{ij}$ of each edge s_{ij}, we specify the tangential component of $\mathbf{A}$, $a_{ij}\mathbf{t}_{ij}$. Here,

$$\mathbf{t}_{ij} = \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|} \ . \tag{3.1}$$

Comparing with [15], where the component normal to the edge s_{ij} is specified, we can view the representation here as a dual formulation.

To approximate the curl operator, we simply apply Green's theorem to the triangle τ_{ijk}.

$$\int_{\tau_{ijk}} \operatorname{curl} \mathbf{A} dxdy = \int_{s_{ij}} \mathbf{A} \cdot \mathbf{t}_{ij} ds + \int_{s_{jk}} \mathbf{A} \cdot \mathbf{t}_{jk} ds + \int_{s_{ki}} \mathbf{A} \cdot \mathbf{t}_{ki} ds \ . \tag{3.2}$$

Letting g_{ijk} denote the approximation of $\operatorname{curl} \mathbf{A}$ in τ_{ijk}, we obtain the following approximation:

$$\frac{1}{|\tau_{ijk}|} \left(a_{ij} h_{ij} + a_{jk} h_{jk} + a_{ki} h_{ki} \right) = g_{ijk} \ . \tag{3.3}$$

For the div operator, we apply the divergence theorem in a covolume Ω_j.

$$\int_{\Omega_j} \operatorname{div} \mathbf{A} dxdy = \sum_k \int_{s'_{kj}} \mathbf{A} \cdot \mathbf{t}_{kj} ds \ ,$$

where the sum is taken for all $\mathbf{x}_k$'s that are connected to the vertex $\mathbf{x}_j$ through an edge of a triangle in τ with $h'_{jk} \neq 0$. If the vector potential $\mathbf{A}$ has normal component zero on the boundary Γ, then the above definition is extended to vertex $\mathbf{x}_j \in \Gamma$ and with Ω_j modified as we described earlier.

Denoting the approximation to $\operatorname{div} \mathbf{A}$ in Ω_j by f_j, we have

$$\frac{1}{|\Omega_j|} \sum_k a_{kj} h'_{kj} = f_j \ . \tag{3.4}$$

We use the convention that $a_{kj} = -a_{jk}$.

Let us state a property for a gradient field here.

LEMMA 3.1. *If $\vec{g} = \vec{0} \in R^{N_2}$ in* (3.3)*, then there exists a discrete potential ϕ, such that*

$$\frac{\phi_k - \phi_j}{h_{jk}} = a_{jk} \tag{3.5}$$

for all adjacent edges.

Proof. Take any vertex $\mathbf{x}_l$, set $\phi_l = 0$, then define $\phi_j = a_{lj} h_{lj}$ for any vertices $\mathbf{x}_j$ that are adjacent to $\mathbf{x}_l$. Similarly, defining ϕ_k at all other vertices using (3.5), it is easy to see that the definition is consistent due to equation (3.4) and the condition $\vec{g} = \vec{0}$. □

Remark. The above property is a discrete version of the fact that if $\operatorname{curl} \mathbf{A} = 0$ in a simply-connected domain, then $\mathbf{A} = \operatorname{grad} \phi$ for some function ϕ. Discrete versions of other identities may also be derived, e.g., if we let C, G be the coefficient matrices of the linear systems given by (3.3) and (3.5) respectively. Then, for any $\vec{\phi} \in R^{N_0}$,

$$CG\vec{\phi} = \vec{0} \ . \tag{3.6}$$

This corresponds to the identity $\operatorname{curl} \operatorname{grad} \phi = 0$ for any function ϕ.

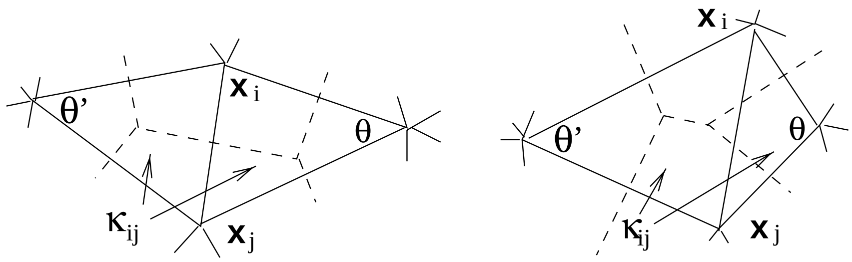

FIG. 2. *Two adjacent triangles (which form a kite area) with their common edge and the connection of their circumcenters.*

3.3. Covolume method for the Laplacian operator. The covolume formulation of the Laplacian operators has been studied by many. We may obtain it by integrating Δu in Ω_j to get

$$\int_{\Omega_j} \Delta u d\Omega = \sum_k \int_{s'_{kj}} \frac{\partial u}{\partial n} ds.$$

Approximating the boundary integral by $\nabla u \cdot \mathbf{t}_{jk} h'_{kj}$, and replacing the derivatives by differences, we get the approximation

$$d_j |\Omega_j| = \sum_k \frac{u_k - u_j}{h_{kj}} h'_{kj} \ . \tag{3.7}$$

The coefficient matrix of the covolume approximation is the same as the stiffness matrix resulting from the standard piecewise linear finite element approximations with the so-called hat-functions as basis. Using barycentric coordinates λ_j, $j = 1, 2, 3$, in a particular triangle τ_{123}, it is well known that

$$\int_{\tau_{123}} \nabla\lambda_2 \cdot \nabla\lambda_1 d\tau = -\cot\theta \ ,$$

where θ is the angle at the vertex $\mathbf{x}_3$.

Then, the ij entry of the stiffness matrix for piecewise linear elements is (see Figure 2)

$$M_{ij} = -\cot\theta - \cot\theta' = -h'_{ij}/h_{ij}. \tag{3.8}$$

One can also use the variational formulation

$$\min_{u^h \in S^h} \int_\Omega |\nabla u^h|^2 \, d\Omega = \min_{u^h \in S^h} \sum_{\tau \in \Sigma} \int_\tau |\nabla u^h|^2,$$

where S^h is the space of continuous piecewise linear elements. Similar to (3.8), the following elementary result can be easily established.

LEMMA 3.2. *Let $\mathbf{t}_{j_1 j_2}, \mathbf{t}_{j_2 j_3}, \mathbf{t}_{j_3 j_1}$ be three unit vectors for the edges of the triangle $\tau_{j_1 j_2 j_3}$, then for any vector $\mathbf{x}$ (real or complex), we have*

$$|\tau_{j_1 j_2 j_3}||\mathbf{x}|^2 = \cot\theta_{j_3} h^2_{j_1 j_2} |\mathbf{x}\cdot\mathbf{t}_{j_1 j_2}|^2 + \cot\theta_{j_2} h^2_{j_3 j_1} |\mathbf{x}\cdot\mathbf{t}_{j_3 j_1}|^2 + \cot\theta_{j_1} h^2_{j_2 j_3} |\mathbf{x}\cdot\mathbf{t}_{j_2 j_3}|^2,$$

where the θ_j's are the corresponding angles of the triangle. ∎

For a triangle $\tau_{j_1j_2j_3}$ with vertices $\mathbf{x}_{j_1}$, $\mathbf{x}_{j_2}$, and $\mathbf{x}_{j_3}$, let us define

$$\alpha_{j_1j_2} = \operatorname{grad} u^h \cdot \mathbf{t}_{j_1j_2} = \frac{u^h(\mathbf{x}_{j_2}) - u^h(\mathbf{x}_{j_1})}{h_{j_1j_2}} \,,$$

and define α_{lk} in a similar way for any pair $l, k \in \{j_1, j_2, j_3\}$. The previous lemma implies

$$|\operatorname{grad} u^h|^2 |\tau_{j_1j_2j_3}| = \cot\theta_{j_3} h^2_{j_1j_2}\alpha^2_{j_1j_2} + \cot\theta_{j_2} h^2_{j_3j_1}\alpha^2_{j_3j_1} + \cot\theta_{j_1} h^2_{j_2j_3}\alpha^2_{j_2j_3}.$$

Therefore, the covolume approximation is equivalent to

$$\min_{u^h} \sum_{\tau_j \in \tau} \left\{ \cot\theta_{j_3} h^2_{j_1j_2}\alpha^2_{j_1j_2} + \cot\theta_{j_2} h^2_{j_3j_1}\alpha^2_{j_3j_1} + \cot\theta_{j_1} h^2_{j_2j_3}\alpha^2_{j_2j_3} \right\}. \tag{3.9}$$

This formulation is more natural for the steady-state GL model where the problem also comes from a variational formulation.

Similarly to Lemma 3.2, we can prove a solvability condition regarding (3.7).

LEMMA 3.3. *For any $\vec{d}$, there exists $\vec{u}$ satisfying* (3.7) *if and only if*

$$\sum_j d_j |\Omega_j| = 0 \,.$$

The solution is unique up to a constant multiple of $\vec{e} = (1, 1, ..., 1)$.

In fact, Lemma 3.3 is a consequence of the following discrete maximum principle which can be easily verified.

LEMMA 3.4. *For any $\vec{d}$, if $d_j \geq 0$ for all j, then $\vec{u}$ is a constant multiple of $\vec{e} = (1, 1, ..., 1)$.*

3.4. Covolume method for the GL model. First of all, we use the above discussion to give the covolume approximation of the GL functional. The GL functional (2.1) is discretized as follows:

(3.10)

$$\begin{aligned}
\mathcal{G}^h(\vec{\psi}^h, \vec{A}^h) = &\sum_{\Omega_{j_1} \in \Sigma'} \left\{ \frac{|\Omega_{j_1}|}{2} (1 - |\psi_{j_1}|^2)^2 \right\} \\
&+ \sum_{\tau_{j_1j_2j_3} \in \Sigma} \left\{ \cot\theta_{j_3} h^2_{j_1j_2} |\alpha_{j_1j_2}|^2 + \cot\theta_{j_2} h^2_{j_3j_1} |\alpha_{j_3j_1}|^2 + \cot\theta_{j_1} h^2_{j_2j_3} |\alpha_{j_2j_3}|^2 \right\} \\
&+ \sum_{\tau_{j_1j_2j_3} \in \Sigma} \left\{ \frac{1}{|\tau_{j_1j_2j_3}|} \left(a_{j_1j_2} h_{j_1j_2} + a_{j_2j_3} h_{j_2j_3} + a_{j_3j_1} h_{j_3j_1} - H|\tau_{j_1j_2j_3}| \right)^2 \right\}.
\end{aligned}$$

Here, $\vec{\psi}^h$ is a vector whose components correspond to values of the discrete order parameter at the vertices of all the triangles, $\vec{A}^h$ is a vector whose components correspond to values of the tangential components of the discrete magnetic potential at the midpoint of the edges of all the triangles, while the term α_{jk} is a *gauge invariant* approximation to $(i/\kappa)\nabla\psi + \mathbf{A}\psi$ at the midpoint of the edge $\mathbf{x}_j\mathbf{x}_k$ and is defined by

$$\alpha_{jk} = \frac{i}{\kappa} \frac{\psi_k \exp(-i\kappa a_{jk} h_{jk}) - \psi_j}{h_{jk}}. \tag{3.11}$$

The covolume approximation of the steady-state GL model is given by the following minimization problem:

$$\min_{(\vec{\psi}^h, \vec{A}^h) \in R^{N_0} \times U} \mathcal{G}^h(\vec{\psi}^h, \vec{A}^h). \tag{3.12}$$

To obtain the discrete functional, the potential energy is discretized using the quadrature rule shown, the kinetic energy is discretized by generalizing (3.9) to the complex case, and in the magnetic energy, curl $\mathbf{A}$ is discretized using (3.3).

Examining the approximation of the kinetic energy term, we see that for edges whose corresponding angles in neighboring triangles sum up to π, the gauge invariant derivatives will disappear. For instance, consider the triangular grids obtained by dividing each square in a uniform square mesh into two triangles along the diagonal. If all diagonals are chosen in the same direction, one may not need to define the vector potential on the diagonal edges. As shown in [9], this leads to the widely used 5-point difference approximation (see, e.g., [2]).

4. Analysis of the covolume methods. To analyze the covolume approximation of the steady-state GL model, we first establish some theory for the approximation of the div-curl system which provides important estimates for discrete covolume solutions of the GL equations. Next, we study various properties of the covolume approximations of the GL model, such as the discrete gauge invariance and the implicit enforcement of the gauge choice. The convergence of the covolume methods is then obtained through the construction and comparison of minimizing sequences for both discrete and continuous functionals.

4.1. Some useful results for the div-curl problems. In order to analyze the errors, we first present the analysis of the related approximations of the div-curl systems given in section 3.2. The analysis of (3.3–3.4) is similar to that in [15] but simplifications are made whenever possible.

Let $\vec{u}$ be a vector whose components $\{u_{jk}\}$ are specified at the midpoint of the corresponding edge $\mathbf{x}_j\mathbf{x}_k$ in the triangulation with the convention that $u_{jk} = -u_{kj}$. Let U be the set of all such vectors. Note that U is isomorphic to a subspace of R^{N_1}.

First, we define the inner product:

$$[\vec{u}, \vec{v}] = \sum_{jk} u_{jk} v_{jk} h_{jk} h'_{jk}\ , \quad \forall\ \vec{u}, \vec{v} \in U,$$

where the sum is over all possible edges $\mathbf{x}_j\mathbf{x}_k$ of the given triangulation Σ. Define the discrete L^p norm $(2 \le p < \infty)$:

$$\|\vec{u}\|_{W,p} = \left(\sum_{jk} |u_{jk}|^p h_{jk} h'_{jk} \right)^{1/p}, \quad \forall\ \vec{u} \in U. \tag{4.1}$$

Strictly speaking, (4.1) does not define a norm on U since it is possible that $h'_{jk} = 0$ even for a locally equiangular grid. For convenience, we now make the following assumptions on the triangulation: we assume that the triangulation is *regular* and *quasi uniform*. In addition, it is *strictly locally equiangular* in the sense that there exist positive constants c_1, c_2 such that

$$c_1 h^2 \le h_{jk} h'_{jk} \le c_2 h^2 \ \ \forall\ j, k, \tag{4.2}$$

where h is the diameter of the triangulation. For a regular quasi-uniform triangulation, (4.2) is equivalent to the assumption that for every pair of adjacent triangles that form a convex quadrilateral, the sum of the angles opposite the common side is no bigger than a constant angle that is strictly less than 180 degrees. Possible generalization can be made to include the case where the sum of the angles opposite the common side is exactly 180 degrees. In this case, one may drop the unknown magnetic potential at the midpoint of the common side and also combine the approximations of the magnetic field on the two adjacent triangles into one equation specified on the convex quadrilateral. (See [9] for discussions on rectangular grids.) The discussion in such cases is similar but the notation is somewhat different and we do not intend to include it here.

With the above assumptions, we have the following inverse inequality.

LEMMA 4.1. *For $2 \le p < \infty$, there exists some generic constant c independent of h such that*

$$\|\vec{u}\|_{W,\infty} := \max_{jk} |u_{jk}| \le ch^{-1} \|\vec{u}\|_{W,2}$$

and

$$\|\vec{u}\|_{W,p} \le ch^{2/p-1} \|\vec{u}\|_{W,2}$$

for any $\vec{u} \in U$. ◻

Again, let C and D be the coefficient matrices of the linear systems given by (3.3) and (3.4) respectively.

LEMMA 4.2. *Assume that $\vec{u}$ and $\vec{v}$ satisfy the conditions*

$$D\vec{u} = \vec{0},$$
$$C\vec{v} = \vec{0}.$$

Then $[\vec{u}, \vec{v}] = 0$.

Proof. By Lemma 3.1, since $C\vec{v} = \vec{0}$, there exists a vector ϕ which is defined on the vertices of the triangles, subject to

$$v_{j_1 j_2} = \frac{\phi_{j_2} - \phi_{j_1}}{h_{j_1 j_2}}.$$

Then

$$\begin{aligned}[\vec{u}, \vec{v}] &= \sum_{j_1 j_2} u_{j_1 j_2} \frac{\phi_{j_2} - \phi_{j_1}}{h_{j_1 j_2}} h_{j_1 j_2} h'_{j_1 j_2} \\ &= \sum_{j_1 j_2} (u_{j_1 j_2} \phi_{j_2} h'_{j_1 j_2} - u_{j_1 j_2} \phi_{j_1} h'_{j_1 j_2}).\end{aligned}$$

Rearranging the summation according to the nodes we get

$$\begin{aligned}[\vec{u}, \vec{v}] &= \sum_{k} \phi_k (u_{k_1 k} h'_{k_1 k} + u_{k_2 k} h'_{k_2 k} + \cdots + u_{k_r k} h'_{k_r k}) \\ &= \sum_{k} \phi_k (Du)_k = 0.\end{aligned}$$

This proves the lemma. ◻

Consider the following problem.

$$\begin{aligned}\operatorname{div} \mathbf{u} &= f \quad \text{in } \Omega, \\ \operatorname{curl} \mathbf{u} &= g \quad \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma,\end{aligned} \tag{4.3}$$

with the corresponding covolume discretization

$$D\vec{u} = \vec{f}, \quad C\vec{u} = \vec{g}, \tag{4.4}$$

where the component of $\vec{f}$ corresponding to each Ω_j is defined as the average of the function f over that region, while the component of $\vec{g}$ corresponding to each triangle is defined as the average of the function g over that triangle. First, let us study the system (4.4).

THEOREM 4.3. *Given $\vec{f} \in R^{N_2}$ and $\vec{g} \in R^{N_0}$, the system*

$$D\vec{u} = \vec{f}\,, \quad C\vec{u} = \vec{g}$$

has a unique solution in U if and only if

$$\sum_{\Omega_j \in \Sigma'} f_j |\Omega_j| = 0\,.$$

Proof. We prove the uniqueness which, under the compatibility condition, implies the existence.

From Lemma 4.2, for any solution $\vec{u} \in U$ of

$$D\vec{u} = \vec{0}\,, \quad C\vec{u} = \vec{0}\,,$$

we have

$$\|\vec{u}\|^2_{W,2} = [\vec{u}, \vec{u}] = 0,$$

which implies $\vec{u} = \vec{0}$. □

Remark. One can see that the solvability condition is a discrete analogue of the solvability condition

$$\int_\Omega f d\Omega = 0$$

for the system (3.3). □

Assuming the compatibility condition holds, we now present an error estimate that is analogous to the one in [15].

THEOREM 4.4. *The following error estimate holds:*

$$\left\|\vec{u} - \vec{u}^{(1)}\right\|_{W,p} \le Ch^{2/p}\|\mathbf{u}\|_{1,\Omega}, \tag{4.5}$$

where c is a generic constant, $p \ge 2$, and $\vec{u}^{(1)}$ is given by

$$u^{(1)}_{j_1 j_2} = \frac{1}{h_{j_1 j_2}} \int_{s_{j_1 j_2}} \mathbf{u} \cdot \mathbf{t}_{j_1 j_2}\, ds,$$

and $\mathbf{t}_{j_1 j_2}$ is the unit vector in the direction $\mathbf{x}_{j_1}\mathbf{x}_{j_2}$.

Proof. Define $\vec{u}^{(2)}$ as

$$u^{(2)}_{j_1j_2} := \frac{1}{h'_{j_1j_2}} \int_{s'_{j_1j_2}} \mathbf{u}\cdot\mathbf{t}\, ds'$$

and let $\vec{e}^{(1)} = u - u^{(1)}$, $\vec{e}^{(2)} = u^{(2)} - u$. Then

$$D\vec{e}^{(2)} = \vec{0},$$
$$C\vec{e}^{(1)} = \vec{0},$$

which implies

$$[\vec{e}^{(1)}, \vec{e}^{(2)}] = 0.$$

It follows that

$$\left\|\vec{e}^{(1)}\right\|^2_{W,2} + \left\|\vec{e}^{(2)}\right\|^2_{W,2} = \left\|\vec{e}^{(1)} + \vec{e}^{(2)}\right\|^2_{W,2} = \left\|\vec{u}^{(2)} - \vec{u}^{(1)}\right\|^2_{W,2}.$$

So we have

$$\left\|\vec{u} - \vec{u}^{(1)}\right\|_{W,2} \le \left\|\vec{u}^{(2)} - \vec{u}^{(1)}\right\|_{W,2} \le ch\|\mathbf{u}\|_{1,\Omega},$$

where the last inequality follows from approximation theory by mapping into reference regions (see [15] and also similar arguments given later in the proof of Theorem 4.5). The estimate (4.5) then follows from the inverse inequality given in Lemma 4.1. ∎

THEOREM 4.5. *Let $2 \le p < \infty$, assume that $\vec{u} \in U$ satisfies*

$$D\vec{u} = \vec{f},$$
$$C\vec{u} = \vec{g}.$$

Then there exists a constant c independent of h such that

$$\|\vec{u}\|_{W,p} \le c\left(\|D\vec{u}\|_{\Sigma'} + \|C\vec{u}\|_{\Sigma}\right), \tag{4.6}$$

where

$$\left\|\vec{\phi}\right\|^2_{\Sigma'} = \sum_{\Omega_k\in\Sigma'} |\phi_k|^2|\Omega_k| \quad \forall\, \vec{\phi} \in R^{N_0},$$
$$\|\vec{\eta}\|^2_{\Sigma} = \sum_{\tau_j\in\Sigma} |\eta_j|^2|\tau_j| \quad \forall\, \vec{\eta} \in R^{N_2}.$$

Proof. Extend $\vec{f}$ to be a function f which has constant value over each $\Omega_j \in \Sigma'$ and extend $\vec{g}$ to be a piecewise constant function g with respect to the triangulation Σ. Consider

$$\operatorname{div}\mathbf{u} = f,$$
$$\operatorname{curl}\mathbf{u} = g,$$
$$\mathbf{u}\cdot\mathbf{n}\,|_{\Gamma} = 0.$$

Define $\vec{u}^*$ by

$$\vec{u}^*_{j_1j_2} = \frac{1}{h_{j_1j_2}} \int_{s_{j_1j_2}} \mathbf{u}\cdot\mathbf{t}_{j_1j_2} ds, \tag{4.7}$$

where $\mathbf{t}$ is the unit vector in the direction $\mathbf{x}_{j_1}\mathbf{x}_{j_2}$. Then by Theorem 4.4,

$$\|\vec{u} - \vec{u}^*\|_{W,p} \leq ch^{2/p}\|\mathbf{u}\|_{1,\Omega}, \tag{4.8}$$

where c is a generic constant. Define $\vec{u}_*$ by

$$\vec{u}_{*j_1j_2} := \frac{1}{|\kappa_{j_1j_2}|}\int_{\kappa_{j_1j_2}} \mathbf{u}\cdot\mathbf{t}_{j_1j_2}\, dxdy, \tag{4.9}$$

where $\kappa_{j_1j_2}$ is the convex quadrilateral formed by two triangles with $s_{j_1j_2}$ as the common edge (see Figure 2). By mapping $\kappa_{j_1j_2}$ to a standard reference quadrilateral $\hat{\kappa}$, letting $\hat{\mathbf{u}}$ be the image of $\mathbf{u}$ on the standard quadrilateral $\hat{\kappa}$, and letting $\hat{s}$ be the corresponding common edge, it follows that

$$\begin{aligned}\left|\frac{1}{|\kappa_{j_1j_2}|}\int_{\kappa_{j_1j_2}} \mathbf{u}\cdot\mathbf{t}_{j_1j_2}\, dxdy\right| &= \left|\frac{1}{|\hat{\kappa}|}\int_{\hat{\kappa}} \hat{\mathbf{u}}\cdot\hat{\mathbf{t}}\, d\hat{x}d\hat{y}\right| \\ &\leq c\|\hat{\mathbf{u}}\|_{0,p,\hat{\kappa}} \leq ch^{-2/p}\|\mathbf{u}\|_{0,p,\kappa_{j_1j_2}}\end{aligned}$$

and then

$$\begin{aligned}\|\vec{u}_*\|_{W,p} &= \left(\sum_{j_1j_2} h^{-2}\|\mathbf{u}\|^p_{0,p,\kappa_{j_1j_2}} h_{j_1j_2}h'_{j_1j_2}\right)^{1/p} \\ &\leq c\|\mathbf{u}\|_{0,p,\Omega} \leq c\|\mathbf{u}\|_{1,\Omega}\ .\end{aligned} \tag{4.10}$$

Similarly, using the same mapping techniques, the Bramble–Hilbert lemma, and the trace theorem, we have

$$\begin{aligned}|\vec{u}^*_{j_1j_2} - \vec{u}_{*j_1j_2}| &= \left|\frac{1}{h_{j_1j_2}}\int_{s_{j_1j_2}} \mathbf{u}\cdot\mathbf{t}_{j_1j_2} ds - \frac{1}{|\kappa_{j_1j_2}|}\int_{\kappa_{j_1j_2}} \mathbf{u}\cdot\mathbf{t}_{j_1j_2}\, dxdy\right| \\ &\leq \left|\frac{1}{|\hat{s}|}\int_{\hat{s}} \hat{\mathbf{u}}\cdot\hat{\mathbf{t}}\, d\hat{s} - \frac{1}{|\hat{\kappa}|}\int_{\hat{\kappa}} \hat{\mathbf{u}}\cdot\hat{\mathbf{t}}\, d\hat{x}d\hat{y}\right| \\ &\leq c \inf_{\mathbf{v}\in R^2} \|\hat{\mathbf{u}} - \mathbf{v}\|_{1,\hat{\kappa}} \\ &\leq c\|\nabla\hat{\mathbf{u}}\|_{0,\hat{\kappa}} \\ &\leq c\|\mathbf{u}\|_{1,\kappa_{ij}}\ .\end{aligned}$$

So

$$\|\vec{u}_* - \vec{u}^*\|_{W,2} \leq ch\|\mathbf{u}\|_{1,\Omega}$$

and therefore, by Lemma 4.1,

$$\|\vec{u}_* - \vec{u}^*\|_{W,p} \leq ch^{2/p}\|\mathbf{u}\|_{1,\Omega}. \tag{4.11}$$

Combining (4.8), (4.10), and (4.11) we get

$$\begin{aligned}\|\vec{u}\|_{W,p} &\leq \|\vec{u} - \vec{u}^*\|_{W,p} + \|\vec{u}^* - \vec{u}_*\|_{W,p} + \|\vec{u}_*\|_{W,p} \\ &\leq c\|\mathbf{u}\|_{1,\Omega}.\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{n}$ vanishes on the boundary, $\|\mathrm{curl}\,\mathbf{u}\|_{0,\Omega}+\|\mathrm{div}\,\mathbf{u}\|_{0,\Omega}$ is a norm for $\mathbf{H}^1(\Omega)$ and is equivalent to norm $\|\mathbf{u}\|_{1,\Omega}$. Therefore,

$$\begin{aligned}\|\vec{u}\|_{W,p} \le & c\left(\|\mathrm{curl}\,\mathbf{u}\|_{0,\Omega}+\|\mathrm{div}\,\mathbf{u}\|_{0,\Omega}\right)\\ \le & c\left(\|C\vec{u}\|_{\Sigma}+\|D\vec{u}\|_{\Sigma'}\right),\end{aligned}$$

where c is some generic constant independent of h. ∎

4.2. The discrete gauge invariance of the GL functional. We now focus on the covolume approximation (3.10). First, let us define the discrete gauge transformation.

For $\vec{\phi}\in R^{N_0}$, let the linear transformation $T^h_{\vec{\phi}}$ be defined by

$$T^h_{\vec{\phi}}(\vec{\psi}^h,\vec{A}^h)=(\tilde{\psi}^h,\tilde{A}^h), \tag{4.12}$$

where

$$\tilde{\psi}_j=\psi_j e^{i\kappa\phi_j} \tag{4.13}$$

at all vertices and

$$\tilde{a}_{jk}=a_{jk}+\frac{\phi_k-\phi_j}{h_{jk}}\quad \forall\, j,k \tag{4.14}$$

for all edges. Then, we have the following definition.

DEFINITION 4.6. $(\vec{\psi}^h,\vec{A}^h)$ *and* $(\vec{\zeta}^h,\vec{Q}^h)$ *are said to be discrete gauge equivalent if and only if there exists a* $\vec{\phi}$ *such that* $(\vec{\psi}^h,\vec{A}^h)=T^h_{\vec{\phi}}(\vec{\zeta}^h,\vec{Q}^h)$.

PROPOSITION 4.7. *For all* $\vec{\phi}\in R^{N_0}$ *and* $(\vec{\psi}^h,\vec{A}^h)$,

$$\mathcal{G}^h(\vec{\psi}^h,\vec{A}^h)=\mathcal{G}^h(T^h_{\vec{\phi}}(\vec{\psi}^h,\vec{A}^h));$$

i.e., $\mathcal{G}$ *is invariant under the discrete gauge transformation* $T^h_{\vec{\phi}}$.

Proof. Let $(\tilde{\psi}^h,\tilde{A}^h)=T^h_{\vec{\phi}}(\vec{\psi}^h,\vec{A}^h)$. Obviously, $|\tilde{\psi}_j|=|\psi_j|$ for any j. On any triangle $\tau_{j_1j_2j_3}\in\Sigma$, using (3.3), we have

$$\begin{aligned}\tilde{a}_{j_1j_2}h_{j_1j_2}+\tilde{a}_{j_2j_3}h_{j_2j_3}+\tilde{a}_{j_3j_1}h_{j_3j_1} &= a_{j_1j_2}h_{j_1j_2}+\phi_{j_2}-\phi_{j_1}\\ &\quad +a_{j_2j_3}h_{j_2j_3}+\phi_{j_3}-\phi_{j_2}+a_{j_3j_1}h_{j_3j_1}+\phi_{j_1}-\phi_{j_3}\\ &= a_{j_1j_2}h_{j_1j_2}+a_{j_2j_3}h_{j_2j_3}+a_{j_3j_1}h_{j_3j_1}\ .\end{aligned}$$

Using (3.11), we have

$$\begin{aligned}\tilde{\alpha}_{jk} &= \frac{i}{\kappa}\frac{\tilde{\psi}_k\exp(-i\kappa\tilde{a}_{jk}h_{jk})-\tilde{\psi}_j}{h_{jk}}\\ &= \frac{i}{\kappa}\frac{\psi_k\exp(i\kappa\phi_k)\exp(-i\kappa a_{jk}h_{jk}-i\kappa\phi_k+i\kappa\phi_j)-\psi_j\exp(i\kappa\phi_j)}{h_{jk}}\\ &= \frac{i}{\kappa}\frac{\psi_k\exp(-i\kappa a_{jk}h_{jk})-\psi_j}{h_{jk}}\exp(i\kappa\phi_j)\\ &= \alpha_{jk}\exp(i\kappa\phi_j).\end{aligned}$$

So

$$\cot\theta_{j_3} h_{j_1j_2}^2 |\tilde{\alpha}_{j_1j_2}|^2 + \cot\theta_{j_2} h_{j_3j_1}^2 |\tilde{\alpha}_{j_3j_1}|^2 + \cot\theta_{j_1} h_{j_2j_3}^2 |\tilde{\alpha}_{j_2j_3}|^2$$
$$= \cot\theta_{j_3} h_{j_1j_2}^2 |\alpha_{j_1j_2}|^2 + \cot\theta_{j_2} h_{j_3j_1}^2 |\alpha_{j_3j_1}|^2 + \cot\theta_{j_1} h_{j_2j_3}^2 |\alpha_{j_2j_3}|^2.$$

Thus, $\mathcal{G}^h(\vec{\psi}^h, \vec{A}^h) = \mathcal{G}^h(T^h_{\phi^h}(\vec{\psi}^h, \vec{A}^h))$. □

The above proposition provides a discrete gauge invariant property of the covolume approximations of the GL model of superconductivity. It gives the freedom of choosing a different gauge for convenience in a particular numerical simulation.

4.3. Fixing the gauge. When doing a particular numerical simulation, a gauge should be specified to avoid the nonuniqueness produced by the gauge transformations and to make the computation efficient. In this section, we discuss how to fix the discrete gauge. A popular choice is the Coulomb gauge. In the continuous case, this simply requires that div $\mathbf{A} = 0$. For the discrete case, using (3.4), we shall require

$$\sum_k a_{kj} h'_{kj} = 0 \quad \forall\, j \tag{4.15}$$

or equivalently

$$D\vec{A} = \vec{0}\,.$$

Then we have the following lemma.

LEMMA 4.8. *Any pair* $(\vec{\psi}^h, \vec{A}^h)$ *is discrete gauge equivalent to a pair* $(\vec{\zeta}^h, \vec{Q}^h)$ *which satisfies*

$$\sum_k q_{kj} h'_{kj} = 0 \quad \forall\, j,$$

where q_{kj} *are the components of* $\vec{Q}^h$.

Proof. Let us define $\vec{\phi}$ by the following

$$\sum_k \frac{\phi_k - \phi_j}{h_{kj}} h'_{kj} = -\sum_k a_{kj} h'_{kj}, \tag{4.16}$$

where ϕ_k are the values of $\vec{\phi}$ at the vertices $\mathbf{x}_k$. From an earlier result, $\vec{\phi}$ is determined uniquely up to a constant multiple of $\mathbf{e}$. Now, if we define $(\vec{\zeta}^h, \vec{Q}^h) = T^h_{\vec{\phi}}(\vec{\psi}^h, \vec{A}^h)$, then

$$\sum_k q_{kj} h'_{kj} = \sum_k a_{kj} h'_{kj} + \sum_k \frac{\phi_k - \phi_j}{h_{kj}} \quad \forall\, j\ ,$$

and the lemma follows from (3.16). □

COROLLARY 4.9. *Every minimizer of* $\mathcal{G}^h$ *is gauge equivalent to a minimizer* $(\vec{\psi}^h, \vec{A}^h)$ *which has a discrete divergence-free vector component* $\vec{A}^h$.

This result implies that it is only necessary to minimize $\mathcal{G}^h$ over a discrete divergence-free vector field in order to find approximate solutions to the GL equations.

4.4. The implicit enforcement of the Coulomb gauge. When the gauge is chosen, the minimization problem becomes a constrained optimization problem. However, similarly to the technique introduced in [5], we can use a modified functional to enforce the gauge choice implicitly.

Let us define

$$\mathcal{F}^h(\vec{\psi}^h, \vec{A}^h) = \mathcal{G}^h(\vec{\psi}^h, \vec{A}^h) + \sum_{\Omega_j \in \Sigma'} \frac{1}{|\Omega_j|} \left(\sum_k a_{kj} h'_{kj} \right)^2 . \tag{4.17}$$

THEOREM 4.10. *Any minimizer* $(\vec{\psi}^h, \vec{A}^h)$ *of* $\mathcal{F}^h$ *satisfies*

$$\sum_k a_{kj} h'_{kj} = 0 \quad \forall \ j \ .$$

Moreover, any minimizer $(\vec{\psi}^h, \vec{A}^h)$ *of* $\mathcal{F}^h$ *is also a minimizer of* $\mathcal{G}^h$*.*

Proof. As in the proof of Lemma 4.8, we define $\vec{\phi} \in R^{N_0}$ by

$$\sum_k \frac{\phi_k - \phi_j}{h_{kj}} h'_{kj} = -\sum_k a_{kj} h'_{kj}$$

and $(\vec{\zeta}^h, \vec{Q}^h) = T^h_{\vec{\phi}}(\vec{\psi}^h, \vec{A}^h)$. Obviously $\mathcal{G}^h(\vec{\zeta}^h, \vec{Q}^h) = \mathcal{G}^h(\vec{\psi}^h, \vec{A}^h)$. Since

$$\sum_k q_{kj} h'_{kj} = 0 \quad \forall \ j \ ,$$

we get $\mathcal{F}^h(\vec{\zeta}^h, \vec{Q}^h) = \mathcal{G}^h(\vec{\zeta}^h, \vec{Q}^h)$. Thus,

$$\mathcal{F}^h(\vec{\psi}^h, \vec{A}^h) = \mathcal{F}^h(\vec{\zeta}^h, \vec{Q}^h) + \sum_{\Omega_j \in \Sigma'} \frac{1}{|\Omega_j|} \left(\sum_k a_{kj} h'_{kj} \right)^2 .$$

Hence, if $(\vec{\psi}^h, \vec{A}^h)$ is a minimizer of $\mathcal{F}^h$, then we must have

$$\sum_k a_{kj} h'_{kj} = 0 \quad \forall \ j \ .$$

Meanwhile, for any $(\vec{\zeta}^h, \vec{Q}^h)$ in $R^{N_0} \times U$, by Lemma 4.8, there exists a gauge equivalent pair $(\tilde{\zeta}^h, \tilde{Q}^h)$ in $R^{N_0} \times U$ such that

$$\mathcal{G}^h(\tilde{\zeta}^h, \tilde{Q}^h) = \mathcal{G}^h(\vec{\zeta}^h, \vec{Q}^h)$$

and $D\tilde{Q}^h = \vec{0}$. So

$$\begin{aligned} \mathcal{G}^h(\vec{\zeta}^h, \vec{Q}^h) &= \mathcal{G}^h(\tilde{\zeta}^h, \tilde{Q}^h) = \mathcal{F}^h(\tilde{\zeta}^h, \tilde{Q}^h) \\ &\geq \mathcal{F}^h(\vec{\psi}^h, \vec{A}^h) = \mathcal{G}^h(\vec{\psi}^h, \vec{A}^h) \ . \end{aligned}$$

This shows that $(\vec{\psi}^h, \vec{A}^h)$ is also a minimizer of $\mathcal{G}^h$. ∎

4.5. Existence of the approximate solutions. We only need to prove the existence of minimizers for (4.17) since by Theorem 4.10, they are also the minimizers of (3.10). Before we proceed, let us state a technical result.

LEMMA 4.11. *Assume that $\mathcal{F}^h(\vec{\psi}^h, \vec{A}^h)$ is uniformly bounded, then there exists a generic constant $c > 0$, independent of h, such that*

$$\left\| D\vec{A}^h \right\|_{\Sigma'} + \left\| C\vec{A}^h \right\|_{\Sigma} \le c \,, \tag{4.18}$$

$$\left\| \vec{A}^h \right\|_{W,p} \le c \,, \tag{4.19}$$

and

$$\|\psi^h\|_{1,\Omega} \le c \,. \tag{4.20}$$

Here ψ^h is the piecewise linear function that interpolates the data $\vec{\psi}^h$ with respect to all vertices of the triangulation Σ.

Proof. From the definition of $\mathcal{F}^h$ and the uniform boundedness, we get immediately that

$$\left\| D\vec{A}^h \right\|_{\Sigma'} + \left\| C\vec{A}^h \right\|_{\Sigma} \le c \,,$$

for some constant c which is dependent on H, but is independent of h. By Theorem 4.5, we have for some generic constant $c > 0$,

$$\left\| \vec{A}^h \right\|_{W,p} \le c \,.$$

For the order parameter, the bound on the potential energy part gives

$$\sum_{\Omega_j \in \Sigma'} |\psi_j|^4 |\Omega_j| \le c \,. \tag{4.21}$$

Using equivalence of norms in polynomial spaces and scaling arguments, we get

$$\|\psi^h\|_{0,4,\Omega} \le c$$

for the piecewise linear function ψ^h. The energy bound implies

$$\sum_{\tau_{j_1 j_2 j_3} \in \Sigma} \left\{ \cot\theta_{j_3} h_{j_1 j_2}^2 |\alpha_{j_1 j_2}|^2 + \cot\theta_{j_2} h_{j_3 j_1}^2 |\alpha_{j_3 j_1}|^2 + \cot\theta_{j_1} h_{j_2 j_3}^2 |\alpha_{j_2 j_3}|^2 \right\} \le c \,.$$

Notice that

$$\begin{aligned}
\alpha_{jk} &= \frac{i}{\kappa} \frac{\psi_k \exp(-i\kappa a_{jk} h_{jk}) - \psi_j}{h_{jk}} \\
&= \frac{i}{\kappa} \frac{\psi_k - \psi_j}{h_{jk}} + \psi_k \frac{i}{\kappa} \frac{\exp(-i\kappa a_{jk} h_{jk}) - 1}{h_{jk}} \\
&= \tilde{\alpha}_{jk} + \psi_k \frac{i}{\kappa} \frac{\exp(-i\kappa a_{jk} h_{jk}) - 1}{h_{jk}}
\end{aligned}$$

and

$$\sum_{jk}\left|\psi_k \frac{i}{\kappa}\frac{\exp(-i\kappa a_{jk}h_{jk})-1}{h_{jk}}\right|^2 \le \sum_{jk}|\psi_k|^2|a_{jk}|^2$$
$$\le \left\{\sum_{jk}|\psi_k|^4\right\}^{1/2}\left\{\sum_{jk}|a_{jk}|^4\right\}^{1/2},$$

where the summation is over all adjacent vertices $\mathbf{x}_j, \mathbf{x}_k$. So, from the known bounds on $\vec{\psi}^h, \vec{A}^h$, we get

$$\sum_{\tau_{j_1j_2j_3}\in\Sigma}\left\{\cot\theta_{j_3}h_{j_1j_2}^2|\tilde{\alpha}_{j_1j_2}|^2+\cot\theta_{j_2}h_{j_3j_1}^2|\tilde{\alpha}_{j_3j_1}|^2+\cot\theta_{j_1}h_{j_2j_3}^2|\tilde{\alpha}_{j_2j_3}|^2\right\}\le c$$

for some constant c. From the discussion in section 3.3, the left-hand side is equivalent to $\left\|\nabla\psi^h\right\|_{0,\Omega}$, so that

$$\left\|\psi^h\right\|_{1,\Omega}\le c\,,$$

because of the estimate (4.21). ☐

THEOREM 4.12. *There exists a minimizer* $(\vec{\psi}^h, \vec{A}^h)$ *of* $\mathcal{F}^h$ *in* $R^{N_0}\times U$ *satisfying*

$$\sum_k a_{kj}h'_{kj}=0 \quad \forall\ j\,,$$

and under the assumption that the triangulation is locally equiangular, we also have

$$|\psi_j|\le 1 \qquad \forall\ j\,.$$

Moreover, it is also a minimizer of $\mathcal{G}^h$.

Proof. The functional $\mathcal{F}^h$ is obviously continuous and bounded below, and thus has a minimizing sequence $\{(\vec{\psi}^h_n, \vec{A}^h_n)\}_{n=1}^{\infty}$. Using the previous lemma, $\mathcal{F}^h(\vec{\psi}^h_n, \vec{A}^h_n)$ is uniformly bounded above and this implies that $(\vec{\psi}^h_n, \vec{A}^h_n)$ is also uniformly bounded, so that a minimizing sequence must have a convergent subsequence. By continuity, the limit of this subsequence is a global minimizer of $\mathcal{F}^h$, which is also a minimizer of $\mathcal{G}^h$ by Theorem 4.10. The pointwise bound follows from the Euler–Lagrange equations for the minimizer using a similar argument as in the continuous case. ☐

4.6. Convergence of the covolume method for the GL model. The main convergence result is given in Theorem 4.17, but first we relate the discrete and continuous energy functionals.

Given a piecewise constant function g^h and a piecewise linear function ψ^h with respect to the triangulation Σ, let ψ^h interpolate data $\vec{\psi}^h\in R^{N_0}$ at the vertices, i.e., $\psi^h(\mathbf{x}_j)=\psi^h_j, 1\le j\le N_0$. Define $\mathbf{A}^h$ by

$$\begin{aligned}&\operatorname{div}\mathbf{A}^h=0\,,\\ &\operatorname{curl}\mathbf{A}^h=g^h\,,\\ &\mathbf{A}^h\cdot\mathbf{n}\,|_\Gamma=0\,.\end{aligned}\tag{4.22}$$

Define $\vec{A}^h$ by

$$\begin{aligned}&D\vec{A}^h=\vec{0}\in R^{N_0}\,,\\ &C\vec{A}^h=\vec{g}\in R^{N_2}\,,\end{aligned}\tag{4.23}$$

where the component of $\vec{g}$ corresponding to each triangle is equal to the value of the piecewise constant function g over that triangle. Then, we have the following lemma.

LEMMA 4.13. *If* $(\psi^h, \mathbf{A}^h)$ *defined above is uniformly bounded independent of* h *in* $\mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$, *then*

$$\mathcal{F}(\psi^h, \mathbf{A}^h) = \mathcal{F}^h(\vec{\psi}^h, \vec{A}^h) + o(1) \quad as \;\; h \to 0 . \tag{4.24}$$

Proof. First, since div $\mathbf{A}^h$ and curl $\mathbf{A}^h$ are piecewise constant functions, we have
(4.25)

$$\int_\Omega \left(|\mathrm{div}\mathbf{A}^h|^2 + |\mathrm{curl}\mathbf{A}^h - H|^2\right) d\Omega = \sum_{\Omega_j \in \Sigma'} \frac{1}{|\Omega_j|} \left(\sum_k a_{kj} h'_{kj}\right)^2$$
$$+ \sum_{\tau_{j_1 j_2 j_3} \in \Sigma} \left\{ \frac{1}{|\tau_{j_1 j_2 j_3}|} \left(a_{j_1 j_2} h_{j_1 j_2} + a_{j_2 j_3} h_{j_2 j_3} + a_{j_3 j_1} h_{j_3 j_1} - H|\tau_{j_1 j_2 j_3}|\right)^2 \right\} .$$

Using standard approximation results and scaling arguments, we have for any continuous piecewise polynomial ζ (of degree ≤ 4),

$$\left| \int_\Omega \zeta \, d\Omega - \sum_{\Omega_{j_1} \in \Sigma'} \zeta(\mathbf{x}_{j_1}) |\Omega_{j_1}| \right| \le ch \|\zeta\|_{1,1,\Omega} ,$$

where c is a generic constant, independent of h. Let $\zeta = \frac{1}{2}(1 - |\psi^h|^2)^2$, since

$$\|\zeta\|_{1,1,\Omega} = \frac{1}{2} \|1 - |\psi^h|^2\|^2_{0,2,\Omega} + \|2(1 - |\psi^h|^2) \Re\{\psi^{h*} \nabla \psi^h\}\|_{0,1,\Omega} ,$$

by Sobolev imbedding theorems and the uniform bound on $\|\psi^h\|_{1,2,\Omega}$, we get

$$\left| \int_\Omega \frac{1}{2}(1 - |\psi^h|^2)^2 \, d\Omega - \sum_{\Omega_{j_1} \in \Sigma'} \frac{|\Omega_{j_1}|}{2} (1 - |\psi_{j_1}|^2)^2 \right| \le ch . \tag{4.26}$$

Next, let us define $\vec{A}^{(1)}$ by

$$a^{(1)}_{j_1 j_2} = \frac{1}{h_{j_1 j_2}} \int_{s_{j_1 j_2}} \mathbf{A}^h \cdot \mathbf{t}_{j_1 j_2} \, ds,$$

where $\mathbf{t}_{j_1 j_2}$ is the unit vector in the direction $\mathbf{x}_{j_1} \mathbf{x}_{j_2}$. Then, applying standard techniques in approximation theory, we get

$$\int_{\tau_{j_1 j_2 j_3}} \left\{ \cot\theta_{j_3} h^2_{j_1 j_2} \left| \mathbf{A}^h(\mathbf{x}) \cdot \mathbf{t}_{j_1 j_2} - a^{(1)}_{j_1 j_2} \right|^2 + \cot\theta_{j_2} h^2_{j_3 j_1} \left| \mathbf{A}^h(\mathbf{x}) \cdot \mathbf{t}_{j_3 j_1} - a^{(1)}_{j_3 j_1} \right|^2 \right.$$
$$\left. + \cot\theta_{j_1} h^2_{j_2 j_3} \left| \mathbf{A}^h(\mathbf{x}) \cdot \mathbf{t}_{j_2 j_3} - a^{(1)}_{j_2 j_3} \right|^2 \right\} d\tau \le ch^2 \left\| \mathbf{A}^h \right\|^2_{1,\tau_{j_1 j_2 j_3}} |\tau_{j_1 j_2 j_3}| .$$

Considering the kinetic energy on a typical triangle $\tau_{j_1 j_2 j_3}$ with vertices $\mathbf{x}_{j_1}$, $\mathbf{x}_{j_2}$, and $\mathbf{x}_{j_3}$, we have

$$
\begin{aligned}
(\mathrm{I})^{j_1,j_2}_{j_1j_2j_3} :=& \int_{\tau_{j_1j_2j_3}} \left| \left(\frac{i}{\kappa}\nabla + \mathbf{A}^h\right)\psi^h(\mathbf{x})\cdot \mathbf{t}_{j_1j_2} \right.\\
&\left. - \left(\frac{i}{\kappa}\nabla\psi^h(\mathbf{x}_{j_2})\cdot\mathbf{t}_{j_1j_2} + a^{(1)}_{j_1j_2}\psi^h(\mathbf{x}_{j_2})\right)\right|^2 d\tau\\
\le& \int_{\tau_{j_1j_2j_3}} \left|\mathbf{A}^h(\mathbf{x})\cdot\mathbf{t}_{j_1j_2}\psi^h(\mathbf{x}) - a^{(1)}_{j_1j_2}\psi^h(\mathbf{x}_{j_2})\right|^2 d\tau\\
\le& 2\int_{\tau_{j_1j_2j_3}} \Big(|\mathbf{A}^h(\mathbf{x})\cdot\mathbf{t}_{j_1j_2}|^2|\psi^h(\mathbf{x}) - \psi^h(\mathbf{x}_{j_2})|^2\\
&+|\mathbf{A}^h(\mathbf{x})\cdot\mathbf{t}_{j_1j_2} - a^{(1)}_{j_1j_2}|^2|\psi^h(\mathbf{x}_{j_2})|^2\Big)\, d\tau\\
\le& 2\|\psi^h - \psi^h(\mathbf{x}_{j_2})\|^2_{0,\infty,\tau_{j_1j_2j_3}}\|\mathbf{A}^h(\mathbf{x})\cdot\mathbf{t}_{j_1j_2}\|^2_{0,\tau_{j_1j_2j_3}}\\
&+ 2\|\psi^h\|^2_{0,\infty,\tau_{j_1j_2j_3}}\int_{\tau_{j_1j_2j_3}}|\mathbf{A}^h(\mathbf{x})\cdot\mathbf{t}_{j_1j_2} - a^{(1)}_{j_1j_2}|^2\, d\tau\ .
\end{aligned}
$$

So

$$
\begin{aligned}
&\frac{1}{|\tau_{j_1j_2j_3}|}\left(\cot\theta_{j_3}h^2_{j_1j_2}(\mathrm{I})^{j_1j_2}_{j_1j_2j_3} + \cot\theta_{j_1}h^2_{j_2j_3}(\mathrm{I})^{j_2j_3}_{j_1j_2j_3} + \cot\theta_{j_2}h^2_{j_3j_1}(\mathrm{I})^{j_3j_1}_{j_1j_2j_3}\right)\\
&\quad\le 2\max_{l=1,2,3}\|\psi^h - \psi^h(\mathbf{x}_{j_l})\|^2_{0,\infty,\tau_{j_1j_2j_3}}\|\mathbf{A}^h\|^2_{0,\tau_{j_1j_2j_3}} + ch^2\left\|\mathbf{A}^h\right\|_{1,\tau_{j_1j_2j_3}}\|\psi^h\|^2_{0,\infty,\tau_{j_1j_2j_3}}\ .
\end{aligned}
$$

Since ψ^h is a linear polynomial on $\tau_{j_1j_2j_3}$, by the inverse estimate and Holder's inequality, we get

$$
\|\psi^h - \psi^h(\mathbf{x}_{j_l})\|^2_{0,\infty,\tau_{j_1j_2j_3}}\|\mathbf{A}^h(\mathbf{x})\|^2_{0,\tau_{j_1j_2j_3}} \le c\|\psi^h - \psi^h(\mathbf{x}_{j_l})\|^2_{0,q,\tau_{j_1j_2j_3}}\|\mathbf{A}^h\|^2_{0,p,\tau_{j_1j_2j_3}},
$$

where $p \ge 2$ and $2/p + 2/q = 1$. By Sobolev imbedding theorems and scaling arguments, we get

(4.27)

$$
\begin{aligned}
&\sum_{\tau_{j_1j_2j_3}\in\Sigma}\frac{1}{|\tau_{j_1j_2j_3}|}\left\{\cot\theta_{j_3}h^2_{j_1j_2}(\mathrm{I})^{j_1j_2}_{j_1j_2j_3} + \cot\theta_{j_1}h^2_{j_2j_3}(\mathrm{I})^{j_2j_3}_{j_1j_2j_3} + \cot\theta_{j_2}h^2_{j_3j_1}(\mathrm{I})^{j_3j_1}_{j_1j_2j_3}\right\}\\
&\quad\le ch^{2-4/p}\|\mathbf{A}^h\|^2_{1,\Omega}\|\psi^h\|^2_{1,\Omega} + ch^2\|\mathbf{A}^h\|^2_{1,\Omega}\|\psi^h\|^2_{1,\Omega}\ .
\end{aligned}
$$

On the other hand, recalling the definition (3.11) we have

$$
\begin{aligned}
(\mathrm{II})^{j_1j_2}_{j_1j_2j_3} :=& \left|\left(\frac{i}{\kappa}\nabla\psi^h(\mathbf{x}_{j_2})\cdot\mathbf{t}_{j_1j_2} + a^{(1)}_{j_1j_2}\psi^h(\mathbf{x}_{j_2})\right) - \alpha_{j_1j_2}\right|^2\\
\le&\frac{1}{\kappa^2}\left|\psi^h(\mathbf{x}_{j_2})\left(\frac{1-\exp(-i\kappa a_{j_1j_2}h_{j_1j_2})}{h_{j_1j_2}} - i\kappa a^{(1)}_{j_1j_2}\right)\right|^2\\
\le&\frac{2}{\kappa^2}\left|\psi^h(\mathbf{x}_{j_2})\left(\frac{1-\exp(-i\kappa a_{j_1j_2}h_{j_1j_2})}{h_{j_1j_2}} - i\kappa a_{j_1j_2}\right)\right|^2\\
&+ 2\left|\psi^h(\mathbf{x}_{j_2})(a_{j_1j_2} - a^{(1)}_{j_1j_2})\right|^2\\
\le& c\|\psi^h\|^2_{0,\infty,\tau_{j_1j_2j_3}}\left\{|a_{j_1j_2}|^4h^2_{j_1j_2} + |a_{j_1j_2} - a^{(1)}_{j_1j_2}|^2\right\}\ .
\end{aligned}
$$

So, by the estimate of ψ^h, the following error estimate derived from Theorem 4.4,

$$
\left\|\vec{A}^h - \vec{A}^{(1)}\right\|_{W,2} \le ch\left\|\mathbf{A}^h\right\|_{1,\Omega},
$$

and the inequality (4.6), we have

(4.28)
$$\sum_{\tau_{j_1j_2j_3}\in\Sigma} \left\{\cot\theta_{j_3} h_{j_1j_2}^2 (\mathrm{II})_{j_1j_2j_3}^{j_1j_2} + \cot\theta_{j_1} h_{j_2j_3}^2 (\mathrm{II})_{j_1j_2j_3}^{j_2j_3} + \cot\theta_{j_2} h_{j_3j_1}^2 (\mathrm{II})_{j_1j_2j_3}^{j_3j_1}\right\}$$
$$\le ch^2 \|\mathbf{A}^h\|_{1,\Omega}^2 (\|\mathbf{A}^h\|_{1,\Omega}^2 + 1) \|\psi^h\|_{1,\Omega}^2 .$$

Using the vector decomposition and inequalities (4.27) and (4.28), we get

$$\left| \int_\Omega \left| \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right|^2 d\Omega \right.$$
$$\left. - \sum_{\tau_{j_1j_2j_3}\in\Sigma} \{\cot\theta_{j_3} h_{j_1j_2}^2 |\alpha_{j_1j_2}|^2 + \cot\theta_{j_2} h_{j_3j_1}^2 |\alpha_{j_3j_1}|^2 + \cot\theta_{j_1} h_{j_2j_3}^2 |\alpha_{j_2j_3}|^2\}\right|$$
$$\le c \sum_{\tau_{j_1j_2j_3}\in\Sigma} \left\{(\mathrm{I})_{j_1j_2j_3}^{j_1j_2} + (\mathrm{I})_{j_1j_2j_3}^{j_2j_3} + (\mathrm{I})_{j_1j_2j_3}^{j_3j_1} + (\mathrm{II})_{j_1j_2j_3}^{j_1,j_2} + (\mathrm{II})_{j_1j_2j_3}^{j_3j_1} + (\mathrm{II})_{j_1j_2j_3}^{j_3j_1}\right\}$$
$$\le ch^{2-4/p} \|\mathbf{A}^h\|_{1,\Omega}^2 (1 + h^{4/p} + h^{4/p}\|\mathbf{A}^h\|_{1,\Omega}^2) \|\psi^h\|_{1,\Omega}^2$$

for any $p \ge 2$ and h small. Taking p sufficiently large, and combining (4.25), (4.26) with the above inequality, we obtain

$$|\mathcal{F}(\psi^h, \mathbf{A}^h) - \mathcal{F}^h(\vec{\psi}^h, \vec{A}^h)| \le ch$$

for some constant c, independent of h. □

Remark. From the proof, we see that actually

$$|\mathcal{F}(\psi^h, \mathbf{A}^h) - \mathcal{F}^h(\vec{\psi}^h, \vec{A}^h)| \le ch$$

for some constant c for small h. □

Now, let $(\psi_*, \mathbf{A}_*)$ be a global minimizer of the problem (2.7), i.e.,

$$\mathcal{F}(\psi_*, \mathbf{A}_*) = \min_{\mathcal{H}^1(\Omega)\times\mathbf{H}_n^1(\Omega)} \mathcal{F} . \tag{4.29}$$

Naturally,

$$\mathrm{div}\mathbf{A}_* = 0 \quad \text{in} \quad \Omega$$

and

$$\mathbf{A}_* \cdot \mathbf{n}\,|_\Gamma = 0 .$$

Let $I^h\psi_*$ be a piecewise linear approximant of ψ_* using either the pointwise values of ψ_* at the vertices (if they are well defined) or the averages of ψ_* over the Ω_j's. Let $\bar{\mathbf{A}}$ be defined by

$$\begin{aligned} &\mathrm{div}\,\bar{\mathbf{A}} = 0, \\ &\mathrm{curl}\,\bar{\mathbf{A}} = \bar{g}, \\ &\bar{\mathbf{A}}\cdot\mathbf{n}\,|_\Gamma = 0, \end{aligned} \tag{4.30}$$

where, on each triangle, $\bar{g}$ is defined to be a constant which equals the average of $g = \mathrm{curl}\ \mathbf{A}_*$ over that triangle. It is straightforward to see that the bounds on the energy functional imply the following corollary.

COROLLARY 4.14. *There exists a constant $c > 0$ such that*

$$\|I^h\psi_*\|_{1,\Omega} \le c\|\psi_*\|_{1,\Omega}$$

and

$$\|\bar{\mathbf{A}}\|_{1,\Omega} \le c\|\mathbf{A}_*\|_{1,\Omega} \ .$$

Moreover,

$$\mathcal{F}(I^h\psi_*, \bar{\mathbf{A}}) = \mathcal{F}(\psi_*, \mathbf{A}_*) + o(1) \tag{4.31}$$

as $h \to 0$. □

Remark. If we assume more regularity on the minimizer $(\psi_*, \mathbf{A}_*)$, then we can show

$$|\mathcal{F}(I^h\psi_*, \bar{\mathbf{A}}) - \mathcal{F}(\psi_*, \mathbf{A}_*)| \le ch$$

for some constant c and for sufficiently small h. □

Let us define $\vec{\psi}_*^h$ to be the vector whose components correspond to values of $I^h\psi_*$ at the vertices of all triangles. We also define $\vec{A}_*^h$ by

$$D\vec{A}_*^h = \vec{0}\ ,$$
$$C\vec{A}_*^h = \vec{g}\ ,$$

where $\vec{g}$ is the vector whose component is the value of $\operatorname{curl}\bar{\mathbf{A}} = \bar{g}$ for any triangle in Σ. Then, using the uniform bounds on

$$\|\bar{\mathbf{A}}\|_{1,\Omega} \le c$$

and

$$\|I^h\psi_*\|_{1,\Omega} \le c\ ,$$

Lemma 4.15 follows as in Lemma 4.13.

LEMMA 4.15. *Using the definitions given above, we have*

$$\mathcal{F}(I^h\psi_*, \bar{\mathbf{A}}) = \mathcal{F}^h(\vec{\psi}_*^h, \vec{A}_*^h) + o(1) \tag{4.32}$$

as $h \to 0$. □

For any h, let $(\vec{\psi}^h, \vec{A}^h)$ be a minimizer of $\mathcal{F}^h$, i.e.,

$$\mathcal{F}^h(\vec{\psi}^h, \vec{A}^h) = \min_{R^{N_0}\times U} \mathcal{F}^h\ .$$

It is easy to see that $\mathcal{F}^h(\vec{\psi}^h, \vec{A}^h)$ is uniformly bounded.

Let us define ψ^h as the piecewise linear interpolant of the data $\vec{\psi}^h$ in Ω, g^h as a piecewise constant function in Ω that equals the component of $D\vec{A}^h$ on each triangle in Σ, and $\mathbf{A}^h$ as

$$\begin{aligned} \operatorname{div}\mathbf{A}^h &= 0\ ,\\ \operatorname{curl}\mathbf{A}^h &= g^h\ ,\\ \mathbf{A}^h\cdot\mathbf{n}\,|_\Gamma &= 0\ . \end{aligned} \tag{4.33}$$

By Lemma 4.11, we see that

$$\|\mathbf{A}^h\|_{1,\Omega} \le C$$

and

$$\|\psi^h\|_{1,\Omega} \le C$$

for some generic constant $C > 0$. Moreover Lemma 4.13 implies the following lemma.

LEMMA 4.16. *Using definitions given above, we have*

$$\mathcal{F}(\psi^h, \mathbf{A}^h) = \mathcal{F}^h(\vec{\psi}^h, \vec{A}^h) + o(1) \tag{4.34}$$

as $h \to 0$. ∎

We now prove the following convergence theorem.

THEOREM 4.17. *For any* h*, let* $(\vec{\psi}^h, \vec{A}^h)$ *and* $(\psi^h, \mathbf{A}^h)$ *be defined as above. Then as* $h \to 0$*, there is a subsequence of* $\{(\psi^h, \mathbf{A}^h)\}$ *that converges to a global minimizer of* $\mathcal{F}$ *in* $\mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$*, and*

$$\lim_{h\to 0} \mathcal{F}^h(\vec{\psi}^h, \vec{A}^h) = \min_{\mathcal{H}^1(\Omega)\times \mathbf{H}^1_n(\Omega)} \mathcal{F} \ . \tag{4.35}$$

Proof. Let $(\psi_*, \mathbf{A}_*)$ be a global minimizer of $\mathcal{F}$. By definition,

$$\mathcal{F}(\psi_*, \mathbf{A}_*) \le \mathcal{F}(\psi^h, \mathbf{A}^h) \ .$$

For any $\epsilon > 0$, if h is small enough, we have by Lemma 4.16

$$\mathcal{F}(\psi^h, \mathbf{A}^h) \le \mathcal{F}^h(\vec{\psi}^h, \vec{A}^h) + \epsilon \ .$$

Thus,

$$\mathcal{F}(\psi_*, \mathbf{A}_*) \le \mathcal{F}^h(\vec{\psi}^h_*, \vec{A}^h_*) + \epsilon \ .$$

It follows from (4.31) and (4.32) that, for small h,

$$\mathcal{F}^h(\vec{\psi}^h_*, \vec{A}^h_*) \le \mathcal{F}(\psi_*, \mathbf{A}_*) + \epsilon \ .$$

So,

$$\mathcal{F}(\psi_*, \mathbf{A}_*) \le \mathcal{F}(\psi^h, \mathbf{A}^h) \le \mathcal{F}(\psi_*, \mathbf{A}_*) + 2\epsilon,$$

and

$$\mathcal{F}(\psi_*, \mathbf{A}_*) \le \mathcal{F}^h(\vec{\psi}^h, \vec{A}^h) + \epsilon \le \mathcal{F}(\psi_*, \mathbf{A}_*) + 2\epsilon. \tag{4.36}$$

If we let $\epsilon \to 0$, we see that $\{(\psi^h, \mathbf{A}^h)\}$ forms a minimizing sequence of $\mathcal{F}$, thus it has a weakly convergent subsequence $\{(\psi^h_n, \mathbf{A}^h_n)\}$ that converges to a global minimizer of $\mathcal{F}$. Using results in functional analysis (see [5]), we see that the subsequence also converges strongly in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$. Moreover, equation (4.35) follows from (4.36). ∎

Remark. If we assume more regularity on the minimizer $(\psi_*, \mathbf{A}_*)$, then from earlier remarks, we have the following result, which is stronger than (4.35):

$$|\mathcal{F}^h(\vec{\psi}^h, \vec{A}^h) - \mathcal{F}(\psi_*, \mathbf{A}_*)| \le ch$$

for some constant c and sufficiently small h. The convergence of the whole sequence $\{(\vec{\psi}^h, \vec{A}^h)\}$ does not follow immediately since the minimizers of $\mathcal{F}$ are not unique in general. However, it is reasonable to expect convergence to a regular solution branch when viewing the GL equations as a system of parameterized equations with κ as the parameter. For finite element approximations, such results have been obtained in [5]. □

Using the properties we have proved before, the convergence of $\mathbf{A}^h$ to $\mathbf{A}_*$ in $\mathbf{H}^1(\Omega)$ implies the following corollary.

COROLLARY 4.18. *If the components $a_{j_1j_2}$ of $\vec{A}_*$ are defined by*

$$a^{(1)}_{j_1j_2} := \frac{1}{h_{j_1j_2}} \int_{s_{j_1j_2}} \mathbf{A}_* \cdot \mathbf{t}_{j_1j_2}\, ds,$$

where $\mathbf{A}_*$ is part of a minimizer of $\mathcal{F}$ which is the limit of the convergent subsequence $\mathbf{A}^{h_n}$, and $\mathbf{t}_{j_1j_2}$ is the unit vector in the direction $\mathbf{x}_{j_1}\mathbf{x}_{j_2}$, then for any $2 \le p < \infty$,

$$\|\vec{A}^h_n - \vec{A}_*\|_{W,p} \to 0\ ,$$

as $h_n \to 0$. □

5. Conclusion. The covolume methods for the GL model of superconductivity presented here enjoy discrete gauge invariance. In some sense, having discrete gauge invariant properties is analogous to having discrete conservation laws for approximations of other physical problems [1]. The physically meaningful pointwise bound on the order parameter is also preserved at the discrete level. These features are viewed by many physicists as important for a successful numerical method. Our convergence analysis is restricted to triangulations that satisfy certain angle conditions. These conditions are not essential when other alternative formulations [9] are used. Rigorous theory for these and other generalizations will be discussed in a future work.

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