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CONSTRAINT PRECONDITIONERS FOR SYMMETRIC INDEFINITE MATRICES*

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Abstract. We study the eigenvalue bounds of block two-by-two nonsingular and symmetric indefinite matrices whose $(1,1)$ block is symmetric positive definite and Schur complement with respect to its $(2,2)$ block is symmetric indefinite. A constraint preconditioner for this matrix is constructed by simply replacing the $(1,1)$ block by a symmetric and positive definite approximation, and the spectral properties of the preconditioned matrix are discussed. Numerical results show that, for a suitably chosen $(1,1)$ block-matrix, this constraint preconditioner outperforms the block-diagonal and the block-tridiagonal ones in iteration step and computing time when they are used to accelerate the GMRES method for solving these block two-by-two symmetric positive indefinite linear systems. The new results extend the existing ones about block two-by-two matrices of symmetric negative semidefinite $(2,2)$ blocks to those of general symmetric $(2,2)$ blocks.

Key words. symmetric indefinite systems, constraint preconditioners

AMS subject classifications. 65F10, 65F50, CR, G1.3

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1. Introduction.

Consider the large, sparse, and symmetric system of linear equations

$$Az \equiv \begin{pmatrix} B & E \\ E^T & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \tag{1.1}$$

where $B \in \mathbb{R}^{p\times p}$ is symmetric positive definite, $C \in \mathbb{R}^{q\times q}$ is symmetric, and $S = C - E^T B^{-1} E \in \mathbb{R}^{q\times q}$, the Schur complement of A, is nonsingular, with $E \in \mathbb{R}^{p\times q}$. Then matrix $A \in \mathbb{R}^{n\times n}$, with $n = p + q$, is nonsingular, symmetric, and possibly indefinite. Let $z = (x^T, y^T)^T \in \mathbb{R}^n$ and $b = (f^T, g^T)^T \in \mathbb{R}^n$, with $x, f \in \mathbb{R}^p$ and $y, g \in \mathbb{R}^q$, be the unknown and the given right-hand side vectors, respectively. Note that we do not assume that C is negative semidefinite as is often done in the literature; see the references of this paper, e.g., [12].

To efficiently and accurately solve linear system (1.1) by a Krylov subspace iteration method, we often use preconditioner P to transform linear system (1.1) such that the resulting preconditioned matrix $P^{-1}A$ has the desired eigenvalue distribution, e.g., a tightly clustered spectrum or positive real spectrum; see, for example,

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[51, 4, 5, 11, 30, 40] and references therein. By considering the structure of matrix A, we can naturally choose such a preconditioner as

$$P = \begin{pmatrix} G & E \\ E^T & C \end{pmatrix}, \tag{1.2}$$

with $G \in \mathbb{R}^{p \times p}$ being a symmetric and positive-definite approximation to B; see [36, 32, 27]. Evidently, this choice of P is the symmetric specification of the one investigated in [30, 15, 8], and here, we call it the constraint preconditioner of matrix A; see also [22, 21, 20, 24]. The spectral analysis for several special cases of matrix A and corresponding preconditioner P are given in detail in some of the literature. For example, Keller, Gould, and Wathen [32] discussed the case where $C = 0$ and Silvester and Wathen [48] and Dollar [20] studied the case where C is symmetric negative semidefinite; see also [36, 21, 47]. We remark that, in both cases, the positive definiteness of matrices B and G is not necessarily demanded in the spectral analysis. For other types of preconditioning matrices for matrix A, see [33, 38, 43, 7, 4, 11, 50, 5, 6] and references therein. We should point out that all these discussions are focused on matrices A of the saddle-point or the generalized saddle-point form, for which the corresponding Schur complements are symmetric negative semidefinite, and are focused on preconditioning matrices P having a similar property to matrices A. Recently, a comprehensive survey of numerical methods for algebraic saddle-point problems has been written by Benzi, Golub, and Liesen [12].

In this paper, we will focus on the study of the eigenvalue bounds of the coefficient matrix A of linear system (1.1) and the spectral properties of preconditioned matrix $P^{-1}A$, with preconditioner P being given in (1.2). Our study extends the existing results to a more general setting in the sense that the eigenvalue bounds are now estimated for a symmetric indefinite matrix A of a symmetric *indefinite* Schur complement in section 2; the spectral properties are analyzed for such a matrix A and also the corresponding preconditioned matrix $P^{-1}A$ in section 3. Experimental results are presented in section 4 to demonstrate the performance of the constraint preconditioners. Finally, some concluding remarks are given in section 5.

The results in this paper are of theoretical importance and applicable value, since many practical problems arising from scientific computing and engineering applications may require the solutions of such systems of linear equations. For example, the Lagrange-type methods, possibly incorporated with certain regularization strategies, for constrained nonconvex optimization problems may lead to block two-by-two symmetric indefinite matrices with indefinite $(1,1)$ blocks and negative definite $(2,2)$ blocks, which can be reformulated to symmetric indefinite matrices of positive definite $(1,1)$ blocks and indefinite $(2,2)$ blocks by suitable block permutations; see [14, 44, 37]. Another set of examples is the generalized eigenvalue problems coming from the stability analysis of dynamical systems or from numerical simulations of circuit design, computational fluid dynamics, and structural mechanics [49, 3, 17, 23, 29, 28, 1, 13]. These problems are often solved by employing the shift-and-invert method, which requires the repeated solution of systems of linear equations of form (1.1) for several values of shift σ and different right-hand sides [41, 42]. The matrix subblocks may take the form $B = H - \sigma M$ and $C = K - \sigma N$, where $H, M \in \mathbb{R}^{p \times p}$, and $N \in \mathbb{R}^{q \times q}$ are symmetric positive definite matrices and $K \in \mathbb{R}^{q \times q}$ is a symmetric positive semidefinite matrix. When the generalized eigenvalue sets of matrix pencils (H, M) and (K, N) are well separated and shift σ is chosen to be an approximation to a generalized eigenvalue of matrix pencil (K, N), matrix B is symmetric positive definite, matrix C is symmetric indefinite, and matrix A as well as its Schur complement $S = C - E^T B^{-1} E$ are both symmetric indefinite [31, 10]. The last example is the nonlinear primal-dual methods for the Euler–Lagrange equations from total variation-based image restorations. When the continuous nonlinear primal-dual method is linearized by an approximated Newton iteration scheme, at each iteration step, we need to solve a system of linear equations with a coefficient matrix of block two-by-two form. This matrix can be reformulated to be symmetric indefinite with a positive definite $(1,1)$ block and an indefinite $(2,2)$ block by suitable block scaling and permutation [45, 16].

We end this section by introducing several necessary notations, concepts, and basic facts that will be used in the subsequent discussions. For symmetric matrix $M \in \mathbb{R}^{m\times m}$, we use $\mathrm{sp}(M)$ to represent the set of all of its eigenvalues and $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ to denote its smallest and largest eigenvalues. For two symmetric matrices $M, N \in \mathbb{R}^{m\times m}$, we say $M \preceq N$ if $z^T M z \le z^T N z$ holds for all $z \in \mathbb{R}^m$. Evidently, the following facts hold true:

(F_1) if $M \in \mathbb{R}^{m\times m}$ is a symmetric matrix, then $\mathrm{sp}(M) \subseteq [\alpha, \beta]$ if and only if $\alpha I \preceq M \preceq \beta I$;

(F_2) if $M, N \in \mathbb{R}^{m\times m}$ are symmetric matrices such that $M \preceq N$, then $\lambda_{\min}(M) \le \lambda_{\min}(N)$ and $\lambda_{\max}(M) \le \lambda_{\max}(N)$;

(F_3) if $M, N \in \mathbb{R}^{m\times m}$ are symmetric matrices such that $M \preceq N$, then $R^T M R \preceq R^T N R$ for any matrix $R \in \mathbb{R}^{m\times k}$. Conversely, if $M, N \in \mathbb{R}^{m\times m}$ are symmetric matrices such that $R^T M R \preceq R^T N R$ for nonsingular matrix $R \in \mathbb{R}^{m\times m}$, then $M \preceq N$;

(F_4) if $M, N \in \mathbb{R}^{m\times m}$ are symmetric positive definite matrices such that $M \preceq N$, then $N^{-1} \preceq M^{-1}$.

2. Eigenvalue bounds for *A*. The main aim of this section is to derive the eigenvalue bounds of matrix A. Because the coefficient matrix $A \in \mathbb{R}^{n\times n}$ of linear system (1.1) is symmetric, its eigenvalues are all real. Moreover, we can derive accurate bounds on its negative and positive eigenvalues.

THEOREM 2.1. *Let matrix $A \in \mathbb{R}^{n\times n}$ given in linear system* (1.1) *be nonsingular. Assume that $B \in \mathbb{R}^{p\times p}$ is symmetric positive definite, with* $\mathrm{sp}(B) \subseteq [\delta, \Delta]$, *and $S = C - E^T B^{-1} E \in \mathbb{R}^{q\times q}$ is symmetric indefinite, with* $\mathrm{sp}(S) \subseteq [-\Theta, -\theta] \cup [\gamma, \Gamma]$, *where δ, Δ, θ, Θ, and γ, Γ are positive reals. In addition, let* $\mathrm{sp}(E^T B^{-1} E) \subseteq [\omega, \Omega]$. *Then it holds that* $\mathrm{sp}(A) \subseteq \mathcal{I}_- \cup \mathcal{I}_+$, *where*

$$\mathcal{I}_- = \left[\frac{1}{2}\left(\omega + \Delta - \Theta - \sqrt{(\omega + \Delta - \Theta)^2 + 4\Theta\Delta}\right),\right.$$
$$\left.\frac{1}{2}\left(\Omega + \delta - \theta - \sqrt{(\Omega + \delta - \theta)^2 + 4\theta\delta}\right)\right]$$

and[1]

$$\mathcal{I}_+ = \left[\frac{1}{2}\left(\Omega + \delta + \gamma - \sqrt{(\Omega + \delta + \gamma)^2 - 4\delta\gamma}\right),\right.$$
$$\left.\frac{1}{2}\left(\Omega + \Delta + \Gamma + \sqrt{(\Omega + \Delta + \Gamma)^2 - 4\Delta\Gamma}\right)\right].$$

[1]Note that $(\Omega + \delta + \gamma)^2 - 4\delta\gamma = \Omega^2 + 2\Omega(\delta + \gamma) + (\delta - \gamma)^2 > 0$ and $(\Omega + \Delta + \Gamma)^2 - 4\Delta\Gamma = \Omega^2 + 2\Omega(\Delta + \Gamma) + (\Delta - \Gamma)^2 > 0$ hold true. Hence, the extremes of interval $\mathcal{I}_+$ are always well defined.

Proof. Consider the case $\lambda \in \mathrm{sp}(A)$, with $\lambda > 0$. To prove that such a λ is located in interval $\mathcal{I}_+$, we need only to verify

$$\lambda_{\max}(A) \leq \frac{1}{2}\left(\Delta + \Gamma + \Omega + \sqrt{(\Delta+\Gamma+\Omega)^2 - 4\Delta\Gamma}\right)$$

and

$$\lambda_{\max}\left(A^{-1}\right) \leq \frac{1}{2}\left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{\Omega}{\delta\gamma} + \sqrt{\left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{\Omega}{\delta\gamma}\right)^2 - \frac{4}{\delta\gamma}}\right).$$

For the convenience of our statements, we define the matrices

$$D = \begin{pmatrix} B^{1/2} & 0 \\ 0 & I \end{pmatrix}, \quad L = \begin{pmatrix} I & 0 \\ E^T B^{-1/2} & I \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}.$$

After straightforward computations, we obtain

$$A = DLTL^TD \quad \text{and} \quad A^{-1} = D^{-1}L^{-T}T^{-1}L^{-1}D^{-1}.$$

Noticing that $S \preceq \Gamma I$ and $S^{-1} \preceq \frac{1}{\gamma}I$, we know $T \preceq T_R$ and $T^{-1} \preceq T_L^{-1}$, where

$$T_R = \begin{pmatrix} I & 0 \\ 0 & \Gamma I \end{pmatrix} \quad \text{and} \quad T_L = \begin{pmatrix} I & 0 \\ 0 & \gamma I \end{pmatrix}.$$

Making use of (F_3), we obtain

$$A \preceq DLT_RL^TD \quad \text{and} \quad A^{-1} \preceq D^{-1}L^{-T}T_L^{-1}L^{-1}D^{-1}.$$

Hence, from (F_2), we get

$$\lambda_{\max}(A) \leq \lambda_{\max}\left(DLT_RL^TD\right) \leq \lambda_{\max}\left(T_R^{1/2}L^TD^2LT_R^{1/2}\right) \tag{2.1}$$

and

$$\lambda_{\max}\left(A^{-1}\right) \leq \lambda_{\max}\left(D^{-1}L^{-T}T_L^{-1}L^{-1}D^{-1}\right) \leq \lambda_{\max}\left(T_L^{-1/2}L^{-1}D^{-2}L^{-T}T_L^{-1/2}\right). \tag{2.2}$$

Because of $B \preceq \Delta I$ and $B^{-1} \preceq \frac{1}{\delta}I$, based on (F_1), we see that $D^2 \preceq D_R^2$ and $D^{-2} \preceq D_L^{-2}$, where

$$D_R = \begin{pmatrix} \sqrt{\Delta}I & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad D_L = \begin{pmatrix} \sqrt{\delta}I & 0 \\ 0 & I \end{pmatrix}.$$

Making use of (F_3) again, we obtain

$$T_R^{1/2}L^TD^2LT_R^{1/2} \preceq T_R^{1/2}L^TD_R^2LT_R^{1/2}$$

and

$$T_L^{-1/2}L^{-1}D^{-2}L^{-T}T_L^{-1/2} \preceq T_L^{-1/2}L^{-1}D_L^{-2}L^{-T}T_L^{-1/2}.$$

It follows from (2.1) and (2.2) that

$$\lambda_{\max}(A) \le \lambda_{\max}\left(T_R^{1/2} L^T D_R^2 L T_R^{1/2}\right) \equiv \lambda_{\max}\left(\widetilde{L}^T \widetilde{L}\right) \tag{2.3}$$

and

$$\lambda_{\max}\left(A^{-1}\right) \le \lambda_{\max}\left(T_L^{-1/2} L^{-1} D_L^{-2} L^{-T} T_L^{-1/2}\right) \equiv \lambda_{\max}\left(\widehat{L}\widehat{L}^T\right) = \lambda_{\max}\left(\widehat{L}^T \widehat{L}\right), \tag{2.4}$$

where

$$\widetilde{L} = D_R L T_R^{1/2} = \begin{pmatrix} \sqrt{\Delta} I & 0 \\ E^T B^{-1/2} & \sqrt{\Gamma} I \end{pmatrix}$$

and

$$\widehat{L} = T_L^{-1/2} L^{-1} D_L^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\delta}} I & 0 \\ -\frac{1}{\sqrt{\delta\gamma}} E^T B^{-1/2} & \frac{1}{\sqrt{\gamma}} I \end{pmatrix}.$$

Let

$$B^{-1/2} E = U^T \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V$$

be the singular value decomposition of matrix $B^{-1/2}E$, where $U \in \mathbb{R}^{p\times p}$ and $V \in \mathbb{R}^{q\times q}$ are orthogonal matrices and $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) \in \mathbb{R}^{r\times r}$ is a diagonal matrix with $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$ being the nonzero singular values of $B^{-1/2}E$. It clearly holds that $\sigma_1^2 = \lambda_{\max}(E^T B^{-1} E) \le \Omega$. By direct calculation, we obtain that the eigenvalues of matrix $\widetilde{L}^T \widetilde{L}$ are Δ, Γ, and

$$\frac{1}{2}\left(\Delta + \Gamma + \sigma_j^2 \pm \sqrt{(\Delta + \Gamma + \sigma_j^2)^2 - 4\Delta\Gamma}\right), \quad j = 1, 2, \ldots, r,$$

and those of matrix $\widehat{L}^T \widehat{L}$ are $\frac{1}{\delta}$, $\frac{1}{\gamma}$, and

$$\frac{1}{2}\left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{\sigma_j^2}{\delta\gamma} \pm \sqrt{\left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{\sigma_j^2}{\delta\gamma}\right)^2 - \frac{4}{\delta\gamma}}\right), \quad j = 1, 2, \ldots, r.$$

Based on (2.3) and (2.4), we immediately obtain

$$\begin{aligned}
\lambda_{\max}(A) &\le \max\left\{\Delta, \quad \Gamma, \quad \frac{1}{2}\left(\Delta + \Gamma + \sigma_1^2 + \sqrt{(\Delta + \Gamma + \sigma_1^2)^2 - 4\Delta\Gamma}\right)\right\} \\
&\le \frac{1}{2}\left(\Delta + \Gamma + \Omega + \sqrt{(\Delta + \Gamma + \Omega)^2 - 4\Delta\Gamma}\right) \\
&\le \Delta + \Gamma + \Omega
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{\max}\left(A^{-1}\right) &\le \max\left\{\frac{1}{\delta}, \quad \frac{1}{\gamma}, \quad \frac{1}{2}\left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{\sigma_1^2}{\delta\gamma} + \sqrt{\left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{\sigma_1^2}{\delta\gamma}\right)^2 - \frac{4}{\delta\gamma}}\right)\right\} \\
&\le \frac{1}{2}\left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{\Omega}{\delta\gamma} + \sqrt{\left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{\Omega}{\delta\gamma}\right)^2 - \frac{4}{\delta\gamma}}\right) \\
&\le \frac{1}{\delta} + \frac{1}{\gamma} + \frac{\Omega}{\delta\gamma}.
\end{aligned}$$

Now, consider the case $\lambda \in \mathrm{sp}(A)$, with $\lambda < 0$. To prove that such a λ is located in interval $\mathcal{I}_-$, we let $z = (x^T, y^T)^T \in \mathbb{R}^n$, with $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$, be the corresponding eigenvector. Obviously, it holds that

$$\begin{cases} Bx + Ey = \lambda x, \\ E^T x + Cy = \lambda y. \end{cases}$$

We assert that $y \neq 0$. Otherwise, if $y = 0$, then $Bx = \lambda x$ and $E^T x = 0$. This shows $\lambda \in \mathrm{sp}(B)$ or $x = 0$, which cannot occur under the assumption as B is symmetric positive definite and $z = (x^T, y^T)^T \neq 0$. By using the variable transforms

$$\widetilde{x} = B^{1/2} x \quad \text{and} \quad \widetilde{E} = B^{-1/2} E,$$

we can rewrite the above equation as

$$\begin{cases} \widetilde{x} + \widetilde{E} y = \lambda B^{-1} \widetilde{x}, \\ \widetilde{E}^T \widetilde{x} + Cy = \lambda y, \end{cases}$$

or equivalently,

$$\begin{cases} \widetilde{x} + \widetilde{E} y = \lambda B^{-1} \widetilde{x}, \\ \widetilde{E}^T \widetilde{x} + \widetilde{E}^T \widetilde{E} y = (\lambda I - S) y. \end{cases} \tag{2.5}$$

Noting that $\widetilde{E}^T \widetilde{E} = E^T B^{-1} E$, we obtain

$$\mathrm{sp}\left(\widetilde{E}^T \widetilde{E}\right) \subseteq [\omega, \Omega]. \tag{2.6}$$

From the first equation in (2.5), we have

$$\widetilde{E} y = \lambda B^{-1} \widetilde{x} - \widetilde{x}.$$

After substituting this relationship into the second equation in (2.5), we get

$$(\lambda I - S) y = \widetilde{E}^T \widetilde{x} + \widetilde{E}^T \left(\lambda B^{-1} - I\right) \widetilde{x} = \lambda \widetilde{E}^T B^{-1} \widetilde{x}.$$

As the first equation in (2.5) also implies

$$\widetilde{x} = -\left(I - \lambda B^{-1}\right)^{-1} \widetilde{E} y,$$

we obtain

$$(\lambda I - S) y = -\lambda \widetilde{E}^T B^{-1} \left(I - \lambda B^{-1}\right)^{-1} \widetilde{E} y$$

or

$$-Sy = -\lambda \left[I + \widetilde{E}^T (B - \lambda I)^{-1} \widetilde{E}\right] y.$$

It then immediately follows that

$$-M_\lambda^{-1/2} S M_\lambda^{-1/2} \widetilde{y} = -\lambda \widetilde{y},$$

with

$$M_\lambda = I + \widetilde{E}^T (B - \lambda I)^{-1} \widetilde{E} \quad \text{and} \quad \widetilde{y} = M_\lambda^{1/2} y,$$

or

$$-\lambda \in \mathrm{sp}(\widetilde{S}_\lambda), \quad \text{with} \quad \widetilde{S}_\lambda = -M_\lambda^{-1/2} S M_\lambda^{-1/2}. \tag{2.7}$$

Here, we have used the fact that $\lambda < 0$ implies that $B - \lambda I$ is symmetric positive definite and that $y \neq 0$ implies $\widetilde{y} \neq 0$. Noticing that $-S \preceq \Theta I$ and $-S^{-1} \preceq \frac{1}{\theta} I$, we know

$$\widetilde{S}_\lambda \preceq \Theta M_\lambda^{-1} \quad \text{and} \quad \widetilde{S}_\lambda^{-1} \preceq \frac{1}{\theta} M_\lambda. \tag{2.8}$$

Since $\mathrm{sp}(B) \subseteq [\delta, \Delta]$, we obtain

$$\frac{1}{\Delta - \lambda} I \preceq (B - \lambda I)^{-1} \preceq \frac{1}{\delta - \lambda} I,$$

and therefore,

$$\frac{1}{\Delta - \lambda} \widetilde{E}^T \widetilde{E} \preceq \widetilde{E}^T (B - \lambda I)^{-1} \widetilde{E} \preceq \frac{1}{\delta - \lambda} \widetilde{E}^T \widetilde{E}.$$

Hence, it holds that

$$I + \frac{1}{\Delta - \lambda} \widetilde{E}^T \widetilde{E} \preceq M_\lambda \preceq I + \frac{1}{\delta - \lambda} \widetilde{E}^T \widetilde{E}.$$

By using (2.6), we obtain the estimate

$$\frac{\omega + \Delta - \lambda}{\Delta - \lambda} I \preceq M_\lambda \preceq \frac{\Omega + \delta - \lambda}{\delta - \lambda} I.$$

Hence, it follows from (2.8) that

$$\widetilde{S}_\lambda \preceq \frac{\Theta(\Delta - \lambda)}{\omega + \Delta - \lambda} I \quad \text{and} \quad \widetilde{S}_\lambda^{-1} \preceq \frac{\Omega + \delta - \lambda}{\theta(\delta - \lambda)} I.$$

So, from (2.7), we have

$$\frac{\theta(\delta - \lambda)}{\Omega + \delta - \lambda} \leq -\lambda \leq \frac{\Theta(\Delta - \lambda)}{\omega + \Delta - \lambda}.$$

This evidently shows that

$$\begin{aligned} \frac{1}{2}\left[\omega + \Delta - \Theta - \sqrt{(\omega + \Delta - \Theta)^2 + 4\Theta\Delta}\right] &\leq \lambda \\ &\leq \frac{1}{2}\left[\Omega + \delta - \theta - \sqrt{(\Omega + \delta - \theta)^2 + 4\theta\delta}\right]. \qquad \square \end{aligned}$$

Remark 2.1. The negative eigenvalues of coefficient matrix A depend primarily on the negative eigenvalues of Schur complement S, while the positive eigenvalues of coefficient matrix A essentially depend on the positive eigenvalues of Schur complement S. It is easily seen from Theorem 2.1 that $\mathcal{I}_-$ is dependent on θ and Θ, while $\mathcal{I}_+$ is dependent on γ and Γ.

Remark 2.2. From the proof process of Theorem 2.1, we can see that

$$\mathcal{I}_+ \subseteq \left[\frac{\delta\gamma}{\delta + \gamma + \Omega}, \quad \Delta + \Gamma + \Omega\right].$$

Moreover, when the Schur complement S of matrix A is symmetric negative definite, with $\mathrm{sp}(S) \subseteq [-\Theta, -\theta]$, it holds that $\mathrm{sp}(A) \subseteq \mathcal{I}_- \cup \mathcal{I}_+^{(o)}$, where $\mathcal{I}_+^{(o)} = [\delta, \Delta + \Omega]$. When the Schur complement S of matrix A is symmetric positive definite, with $\mathrm{sp}(S) \subseteq [\gamma, \Gamma]$, it holds that $\mathrm{sp}(A) \subseteq \mathcal{I}_+$. Note that the latter case corresponds to the requirement that matrix A is symmetric positive definite. Similar bounds for both of these two special cases were investigated by many authors, e.g., Axelsson and Neytcheva [2], Rusten and Winther [46], and Silvester and Wathen [48]. To our knowledge, however, there is no discussion about the eigenvalue bounds of matrix A with Schur complement S having, in general, both negative and positive eigenvalues.

Remark 2.3. The eigenvalue bounds for matrix $A \in \mathbb{R}^{n\times n}$ given in Theorem 2.1 require spectral knowledge on matrix $E^T B^{-1} E$ in addition to spectral information on Schur complement $S \in \mathbb{R}^{q\times q}$. This seems a restrictive requirement in actual applications. Moreover, it may be not easy to assess the indefiniteness of Schur complement S in practical situations so ensuring nonsingularity of S in this case may be quite problematic. Note that, in most cases, only spectral information on matrix blocks $B \in \mathbb{R}^{p\times p}$, $C \in \mathbb{R}^{q\times q}$, and $E \in \mathbb{R}^{p\times q}$ are available. These are the main limitations of our new results.

Example 2.1.[2] Consider linear system (1.1) with coefficient matrix A being chosen as follows:

$$B = 2I \in \mathbb{R}^{2\times 2}, \quad E = I \in \mathbb{R}^{2\times 2}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}. \tag{2.9}$$

Then we easily have

$$\mathrm{sp}(A) = \left\{1 \pm \sqrt{2}, \quad \frac{1}{2}\left(3 \pm \sqrt{5}\right)\right\} \approx \{-0.41, \quad 0.38, \quad 2.41, \quad 2.62\}$$

and

$$E^T B^{-1} E = \frac{1}{2} I, \quad S = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Therefore, it holds that

$$\delta = \Delta = 2, \quad \theta = \Theta = \frac{1}{2}, \quad \gamma = \Gamma = \frac{1}{2}, \quad \text{and} \quad \omega = \Omega = \frac{1}{2}.$$

By Theorem 2.1, we have

$$\mathcal{I}_- = \left[1 - \sqrt{2}, \quad 1 - \sqrt{2}\right] \approx [-0.41, \quad -0.41]$$

and

$$\mathcal{I}_+ = \left[\frac{1}{2}\left(3 - \sqrt{5}\right), \quad \frac{1}{2}\left(3 + \sqrt{5}\right)\right] \approx [0.38, \quad 2.62].$$

Clearly, the estimated eigenvalue bounds given in Theorem 2.1 are the same as the exact ones, and both negative and positive eigenvalue intervals are perfectly sharp. See also the estimates in Tables 1, 3, and 5 for larger and more practical matrices.

[2] This example was generously offered by Dr. Sue Dollar.

Example 2.2. Consider linear system (1.1) with coefficient matrix A being chosen as follows:

$$B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \frac{1}{\phi} \end{pmatrix} \in \mathbb{R}^{3\times 3}, \quad E = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\psi} \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 2},$$

$$\text{and} \quad C = \begin{pmatrix} 2\phi & 0 \\ 0 & -\epsilon \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

where $\phi > 0$, $\psi \geq 0$, and $\epsilon \geq 0$ are given constants. Then we easily have

$$E^T B^{-1} E = \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}, \qquad S = \begin{pmatrix} \phi & 0 \\ 0 & -\epsilon - \psi \end{pmatrix},$$

and

$$\mathrm{sp}(B) = \left\{ \frac{1}{2}\left(3 \pm \sqrt{5}\right), \quad \frac{1}{\phi} \right\}, \qquad \mathrm{sp}\left(E^T B^{-1} E\right) = \{\phi, \quad \psi\}.$$

(a) $\phi = \frac{1}{3}$, $\psi = 1$, and $\epsilon = 0$. For this case, E is a rectangular matrix having full column rank 2.
By direct calculations, we can obtain

$$\mathrm{sp}(A) = \{-0.5321, 0.2967, 0.6527, 2.8794, 3.3699\}.$$

Because of

$$\delta = \frac{1}{2}\left(3 - \sqrt{5}\right), \quad \Delta = 3, \quad \theta = \Theta = 1, \quad \gamma = \Gamma = \frac{1}{3},$$

and

$$\omega = \frac{1}{3}, \quad \Omega = 1,$$

by making use of Theorem 2.1, we have

$$\mathcal{I}_- = \left[\frac{7 - \sqrt{157}}{6}, \quad \frac{3 - \sqrt{5} - \sqrt{38 - 14\sqrt{5}}}{4} \right] \approx [-0.9217, \quad -0.4559]$$

and

$$\mathcal{I}_+ = \left[\frac{17 - 3\sqrt{5} - \sqrt{322 - 98\sqrt{5}}}{12}, \quad \frac{13 + \sqrt{133}}{6} \right] \approx [0.0125, \quad 4.0888].$$

(b) $\phi = \frac{1}{3}$, $\psi = 0$, and $\epsilon = 1$. For this case, E is a rectangular matrix having deficient column rank 1.
By direct calculations, we can obtain

$$\mathrm{sp}(A) = \{-1, 0.2967, 0.3820, 2.6180, 3.3699\}.$$

Because of

$$\delta = \frac{1}{2}\left(3 - \sqrt{5}\right), \quad \Delta = 3, \quad \theta = \Theta = 1, \quad \gamma = \Gamma = \frac{1}{3},$$

and

$$\omega = 0, \quad \Omega = \frac{1}{3},$$

by making use of Theorem 2.1, we have

$$\mathcal{I}_- = \left[-1, \quad \frac{5 - 3\sqrt{5} - \sqrt{286 - 102\sqrt{5}}}{12}\right] \approx [-1, \quad -0.7766]$$

and

$$\mathcal{I}_+ = \left[\frac{13 - 3\sqrt{5} - \sqrt{142 - 54\sqrt{5}}}{12}, \quad \frac{11 + \sqrt{85}}{6}\right] \approx [0.1401, \quad 3.3699].$$

Clearly, the estimated eigenvalue bounds given in Theorem 2.1 sharply include the exact ones. See also the estimates in Tables 1, 3, and 5 for larger and more practical matrices.

Theorem 2.1 also applies to the case that matrix block $E \in \mathbb{R}^{p\times q}$ is rank deficient. For this case, it clearly holds that $\omega = 0$; see Example 2.2(b).

Moreover, when $\mathbb{N}_o := \text{null}(E) \cap \text{null}(C) \neq \{0\}$, matrix $A \in \mathbb{R}^{n\times n}$ is symmetric and singular. For this case, we can construct nonsingular matrix $F \in \mathbb{R}^{n\times n}$ such that

$$F^T A F = \begin{pmatrix} \widetilde{A} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{with} \quad \widetilde{A} = \begin{pmatrix} B & \widetilde{E} \\ \widetilde{E}^T & \widetilde{C} \end{pmatrix} \in \mathbb{R}^{(n-n_o)\times(n-n_o)},$$

where n_o is the dimension of $\mathbb{N}_o$, $\widetilde{C}$ is a symmetric matrix, and the null spaces of $\widetilde{C}$ and $\widetilde{E}$ do not have overlapping, i.e., $\text{null}(\widetilde{E}) \cap \text{null}(\widetilde{C}) = \{0\}$. Hence, compressed matrix $\widetilde{A}$ is symmetric and nonsingular, and Theorem 2.1 can be applied to it instead.

More specifically, if we denote by

$$\overline{\mathbb{N}}_c = \text{null}(C) \setminus \mathbb{N}_o, \qquad \overline{\mathbb{N}}_e = \text{null}(E) \setminus \mathbb{N}_o,$$

and

$$\mathbb{N}_u = \text{null}(C) \cup \text{null}(E), \qquad \mathbb{R}_e = \mathbb{R}^q \setminus \mathbb{N}_u,$$

then it holds that

$$\text{range}(E) = \mathbb{R}_e \oplus \overline{\mathbb{N}}_c, \qquad \text{null}(E) = \overline{\mathbb{N}}_e \oplus \mathbb{N}_o,$$

and

$$\text{range}(C) = \mathbb{R}_e \oplus \overline{\mathbb{N}}_e, \qquad \text{null}(C) = \overline{\mathbb{N}}_c \oplus \mathbb{N}_o.$$

Assume $\text{rank}(E) = q - n_e$, and let $N_o \in \mathbb{R}^{q\times n_o}$, $\overline{N}_c \in \mathbb{R}^{q\times(n_c-n_o)}$, $\overline{N}_e \in \mathbb{R}^{q\times(n_e-n_o)}$, and $R_e \in \mathbb{R}^{q\times r_c}$ be the matrices whose columns form the bases of subspaces $\mathbb{N}_o$, $\overline{\mathbb{N}}_c$, $\overline{\mathbb{N}}_e$, and $\mathbb{R}_e$, respectively, where n_e, n_c, n_o, and r_c are nonnegative integers. Obviously, it holds that $r_c = q - n_e - n_c + n_o$ and

$$\mathbb{R}^q = \mathbb{R}_e \oplus \overline{\mathbb{N}}_e \oplus \overline{\mathbb{N}}_c \oplus \mathbb{N}_o.$$

So

$$q = r_c + (n_e - n_o) + (n_c - n_o) + n_o = r_c + n_e + n_c - n_o.$$

Now, by defining nonsingular matrix F to be

$$F = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & R_e & \overline{N}_e & \overline{N}_c & N_o \end{pmatrix},$$

after straightforward operations, we obtain

$$F^T A F = \begin{pmatrix} B & ER_e & 0 & E\overline{N}_c & 0 \\ R_e^T E^T & R_e^T C R_e & R_e^T C \overline{N}_e & 0 & 0 \\ 0 & \overline{N}_e^T C R_e & \overline{N}_e^T C \overline{N}_e & 0 & 0 \\ \overline{N}_c^T E^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Letting

$$\widetilde{C} = \begin{pmatrix} R_e^T C R_e & R_e^T C \overline{N}_e & 0 \\ \overline{N}_e^T C R_e & \overline{N}_e^T C \overline{N}_e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{E} = \begin{pmatrix} ER_e & 0 & E\overline{N}_c \end{pmatrix},$$

we immediately know that

$$F^T A F = \begin{pmatrix} B & \widetilde{E} & 0 \\ \widetilde{E}^T & \widetilde{C} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{A} & 0 \\ 0 & 0 \end{pmatrix}.$$

In addition, it follows from

$$\mathrm{range}(E) = \mathbb{R}_e \oplus \overline{\mathbb{N}}_c$$

that the matrix $\widetilde{E} \in \mathbb{R}^{p \times (q-n_o)}$ is of column rank $q - n_e$, i.e.,

$$\mathrm{rank}(\widetilde{E}) = \mathrm{rank}(E) = q - n_e,$$

and both matrices ER_e and $E\overline{N}_c$ are of full column rank. Note that matrix block $\overline{N}_e^T C \overline{N}_e$ is nonsingular. Hence, it holds that $\mathrm{null}(\widetilde{E}) \cap \mathrm{null}(\widetilde{C}) = \{0\}$ and, thereby, the compressed matrix $\widetilde{A}$ is nonsingular. Obviously, matrix block $\widetilde{C} \in \mathbb{R}^{(q-n_o)\times(q-n_o)}$ is symmetric; so is matrix $\widetilde{A}$.

In particular, as matrix block $C \in \mathbb{R}^{q\times q}$ is symmetric, we may choose orthogonal bases of subspaces $\mathrm{null}(C)$ and $\mathrm{range}(C)$ such that matrix $F \in \mathbb{R}^{n\times n}$ is orthogonal. Therefore, $F^T A F$ is orthogonally similar to A, and all eigenvalues of matrix A are equal to those of matrix $\widetilde{A}$ counting the multiplicity. In this fashion, Theorem 2.1 can be employed to estimate the bounds of the nonzero eigenvalues of matrix $A \in \mathbb{R}^{n\times n}$.

3. The spectral analysis of $P^{-1}A$. In this section, we will precisely describe the distribution of the eigenvalues and the characteristics of the eigenvectors of preconditioned matrix $P^{-1}A$. These results are essential for assessing the convergence properties of the Krylov subspace iteration methods, incorporated with preconditioner P for solving linear system (1.1).

THEOREM 3.1. *Let $A \in \mathbb{R}^{n\times n}$ given in (1.1) and $P \in \mathbb{R}^{n\times n}$ defined in (1.2) be nonsingular. Denote by $R = [E^T \quad C] \in \mathbb{R}^{q\times n}$ and $Z = [X^T \quad Y^T]^T \in \mathbb{R}^{n\times r}$, with $r \leq q$, where the columns of matrix Z form a basis of null space R (denoted as* $\operatorname{null}(R)$*). Represent by k the dimension of* $\operatorname{null}(B-G)$. *Assume that matrix $X \in \mathbb{R}^{p\times r}$ has rank $\ell \leq \min\{r,p\}$ and it is decomposed as $X = [X_f \quad X_o]$, with $X_f \in \mathbb{R}^{p\times \ell}$ having full column rank ℓ and $X_o \in \mathbb{R}^{p\times(r-\ell)}$. Then*

(i) *matrix $P^{-1}A$ has an eigenvalue 1 with algebraic multiplicity at least $q+k$ and at most $p+q-\ell$, and at least ℓ nonunit eigenvalues which are defined by the generalized eigenvalue problem*

$$X_f^T\left(B - EC^\dagger E^T\right)X_f u = \lambda X_f^T\left(G - EC^\dagger E^T\right)X_f u, \qquad \text{with} \quad u \in \mathbb{R}^\ell \setminus \{0\}, \tag{3.1}$$

where $C^\dagger$ denotes the Moore–Penrose generalized inverse of matrix C;[3]

(ii) *matrix $P^{-1}A$ has $q+k+l$ linearly independent eigenvectors. They are*
- (i_1) *q eigenvectors of form $z = \binom{0}{y}$ that correspond to case $\lambda = 1$;*
- (i_2) *k $(0 \leq k \leq p)$ eigenvectors of form $z = \binom{x}{y}$ that correspond to case $\lambda = 1$ for which all such vectors x form a basis of* $\operatorname{null}(B-G)$*;*
- (i_3) *l $(0 \leq l \leq \ell)$ eigenvectors of form $z = \binom{x}{y}$ that correspond to case $\lambda \neq 1$.*

Proof. Let λ be an eigenvalue of matrix $P^{-1}A$ and $z = \binom{x}{y}$ be the corresponding eigenvector. Clearly, it holds that

$$\begin{cases} Bx + Ey = \lambda(Gx + Ey), \\ E^T x + Cy = \lambda(E^T x + Cy). \end{cases} \tag{3.2}$$

From the second equation of (3.2), we obtain

$$(1-\lambda)\left(E^T x + Cy\right) = 0.$$

Hence, either $\lambda = 1$ or $E^T x + Cy = 0$ holds true.

For $\lambda = 1$, from the first equation of (3.2), we immediately have $Bx = Gx$, or $x \in \operatorname{null}(B-G)$. This shows that the corresponding eigenvectors are of form $z = \binom{0}{y}$ or $z = \binom{x}{y}$, with $x \in \mathbb{R}^p$ a basis vector of $\operatorname{null}(B-G)$. The number of the linearly independent vectors of such form is $q+k$. So, the multiplicity of eigenvalue $\lambda = 1$ is at least $q+k$, too.

For $E^T x + Cy = 0$, we know that $z := \binom{x}{y} \in \operatorname{null}(R)$. Let the columns of matrix $Z = \binom{X}{Y}$ form the basis of $\operatorname{null}(R)$. Then we have $z = Z\hat{u}$, $\forall \hat{u} \in \mathbb{R}^r \setminus \{0\}$, such that $RZ = 0$, or, in other words,

$$x = X\hat{u} \quad \text{and} \quad y = Y\hat{u}, \quad \text{with} \quad \forall \hat{u} \in \mathbb{R}^r \setminus \{0\}, \quad \text{such that} \quad E^T X + CY = 0,$$

where r is the dimension of $\operatorname{null}(R)$. It then follows from the first equation of (3.2) again that

$$(BX + EY)\hat{u} = \lambda(GX + EY)\hat{u}. \tag{3.3}$$

[3] $C^\dagger$ is defined as the Moore–Penrose generalized inverse of matrix C, if it satisfies the following conditions: (1) $CC^\dagger C = C$, (2) $C^\dagger CC^\dagger = C^\dagger$, (3) $(CC^\dagger)^T = CC^\dagger$, (4) $(C^\dagger C)^T = C^\dagger C$. In specific, if the singular value decomposition of matrix C is $C = U^T \Sigma U$, where U is an orthonormal matrix and $\Sigma = \operatorname{diag}(\Sigma_C, 0)$ is a diagonal matrix, with Σ_C being of positive diagonal entries, then $C^\dagger = U^T \operatorname{diag}(\Sigma_C^{-1}, 0)U$; see [26].

From [18] and [34, 35, 19], we easily know that matrix equation $CY = -E^T X$ is solvable if and only if $(I - CC^\dagger)E^T X = 0$. Moreover, in this case, its general solution has the expression

$$Y = -C^\dagger E^T X + \left(I - C^\dagger C\right) W, \qquad \forall W \in \mathbb{R}^{q\times q}. \tag{3.4}$$

Now, the substitution of (3.4) into (3.3), followed by straightforward operations, can lead to the generalized eigenvalue problem

$$\begin{aligned}\left[\left(B - EC^\dagger E^T\right) X + E\left(I - C^\dagger C\right) W\right] \hat{u} \\ = \lambda \left[\left(G - EC^\dagger E^T\right) X + E\left(I - C^\dagger C\right) W\right] \hat{u}.\end{aligned} \tag{3.5}$$

Because $X = [X_f \quad X_o] \in \mathbb{R}^{p\times r}$ and $X_f \in \mathbb{R}^{p\times \ell}$ have the same rank ℓ, the columns of X_f form a basis of the range space of matrix X. Let $T_f \in \mathbb{R}^{\ell\times(r-\ell)}$ be the transformation matrix such that $X_o = X_f T_f$. Conformably, we decompose vector $\hat{u} \in \mathbb{R}^r$ as

$$\hat{u} = \begin{pmatrix} u_f \\ u_o \end{pmatrix}, \quad \text{with} \quad u_f \in \mathbb{R}^\ell \quad \text{and} \quad u_o \in \mathbb{R}^{r-\ell}.$$

It follows from $X_o = X_f T_f$ that

$$\begin{cases} \left(B - EC^\dagger E^T\right) X\hat{u} = \left(B - EC^\dagger E^T\right) X_f u_f + \left(B - EC^\dagger E^T\right) X_o u_o \\ \qquad\qquad\qquad = \left(B - EC^\dagger E^T\right) X_f (u_f + T_f u_o), \\ \left(G - EC^\dagger E^T\right) X\hat{u} = \left(G - EC^\dagger E^T\right) X_f u_f + \left(G - EC^\dagger E^T\right) X_o u_o \\ \qquad\qquad\qquad = \left(G - EC^\dagger E^T\right) X_f (u_f + T_f u_o). \end{cases}$$

Therefore, generalized eigenvalue problem (3.5) can be equivalently rewritten as

$$\begin{aligned}\left(B - EC^\dagger E^T\right) X_f u + E\left(I - C^\dagger C\right) W\hat{u} \\ = \lambda \left[\left(G - EC^\dagger E^T\right) X_f u + E\left(I - C^\dagger C\right) W\hat{u}\right],\end{aligned} \tag{3.6}$$

with $u = u_f + T_f u_o \in \mathbb{R}^\ell$.

Evidently, if $x = X_f u \notin \operatorname{null}(B - G)$, then $\lambda = 1$ can only occur when $u = 0$, or, in other words, u_o is a solution of linear system $T_f u_o = -u_f$. For this case, we have

$$x = X\hat{u} = X_f u_f + X_o u_o = X_f (u_f + T_f u_o) = X_f u = 0,$$

and hence, $x \in \operatorname{null}(B - G)$. This causes a contradiction.

Suppose $\lambda \neq 1$. Premultiplying (3.6) by X_f^T and then substituting $X_f^T E = -Y_f^T C$ into the resulting equation, we obtain

$$X_f^T \left(B - EC^\dagger E^T\right) X_f u = \lambda X_f^T \left(G - EC^\dagger E^T\right) X_f u,$$

where $Y = [Y_f \quad Y_o] \in \mathbb{R}^{q\times r}$, with $Y_f \in \mathbb{R}^{q\times\ell}$ and $Y_o \in \mathbb{R}^{q\times(r-\ell)}$. Obviously, the number of all such kind of nonunit eigenvalues is at least ℓ.

This proves the validity of (i).

Conclusion (ii) follows straightforwardly from the basic fact in linear algebra that the eigenvectors corresponding to different eigenvalues of a matrix are linearly independent and from the actual choices of the three groups of linearly independent eigenvectors of matrix $P^{-1}A$. ∎

Theorem 3.1 generalizes the existing results about the spectral properties for symmetric indefinite matrix A and preconditioner P of symmetric negative semidefinite Schur complements obtained in [38, 32, 15, 30, 2, 20]. Of course, it can also be proved in an analogous fashion to the proofs of Theorems 4.1 and 4.2 in [20], with suitable generalization and technical modification. Here, we give a different proof by using the null spaces of the associated matrix blocks.

Similar to the remarks made in [20], if matrix block $C \in \mathbb{R}^{q\times q}$ has a small 2-norm, $\|B\|_2 = \mathcal{O}(1)$ and $\|G\|_2 = \mathcal{O}(1)$, then the $X_f^T EC^\dagger E^T X_f$ terms will dominate generalized eigenvalue problem (3.1) for $E^T X_f u \notin \text{null}(C)$, and hence, there will be at least $\ell - \dim(\text{null(C)})$ further eigenvalues clustered about 1 when $\|C\|_2 \ll 1$, where $\dim(\cdot)$ denotes the dimention of the corresponding subspace.

Of course, preconditioned matrix $P^{-1}A$ may be generally not diagonalizable due to the total count of its eigenvectors described in Theorem 3.1(ii). However, the eigenvectors corresponding to the unit eigenvalues of preconditioned matrix $P^{-1}A$ form an invariant subspace.

Remark 3.1. From Bai and Ng [8], we know that the degree of the minimal polynomial of matrix $P^{-1}A$ is at most $\tilde{p}+1$, where $\tilde{p}$ is the degree of the minimal polynomial of the matrix

$$W = \left(I + G^{-1}ES_G^{-1}E^T\right)G^{-1}(B-G), \quad \text{with} \quad S_G = C - E^T G^{-1}E.$$

Therefore, the dimension of the corresponding Krylov subspace

$$\mathcal{K}_n\left(P^{-1}A, b\right) \equiv \text{span}\left\{b, \left(P^{-1}A\right)b, \left(P^{-1}A\right)^2 b, \ldots, \left(P^{-1}A\right)^{n-1} b\right\}$$

is at most $\tilde{p}+1$ and, in exact arithmetic, a Krylov subspace iteration method with an optimality property, e.g., GMRES, will terminate and achieve the exact solution of the system of linear equations (1.1) within at most $\tilde{p}+1$ iteration steps. Note that it always holds $\tilde{p} \le p$; $\tilde{p} < p$ can occur when $B-G$ is a reduced low-rank matrix.

Remark 3.2. In the proof of Theorem 3.1, we do not use the assumption that matrix block B is symmetric positive definite, neither the assumption that B is nonsingular. The only condition used is that both matrices A and P are nonsingular.

Remark 3.3. Because $A \in \mathbb{R}^{n\times n}$ and $P \in \mathbb{R}^{n\times n}$ given in (1.1) and (1.2), respectively, are nonsingular, $B \in \mathbb{R}^{p\times p}$ is symmetric positive definite and $C \in \mathbb{R}^{q\times q}$ is symmetric, we know that generalized Schur complements $B - EC^\dagger E^T$ and $G - EC^\dagger E^T$ of matrices A and P are symmetric and nonsingular. As $X_f \in \mathbb{R}^{p\times \ell}$ has full column rank, both $X_f^T(B - EC^\dagger E^T)X_f$ and $X_f^T(G - EC^\dagger E^T)X_f$ are symmetric and nonsingular matrices. Hence, it follows that generalized eigenvalue problem (3.1) has only a finite number of nonzero eigenvalues. Moreover, when matrices $X_f^T(B - EC^\dagger E^T)X_f$ and $X_f^T(G - EC^\dagger E^T)X_f$ are indefinite, generalized eigenvalue problem (3.1) may have complex eigenvalues. If both of these matrices are positive/negative definite, then all eigenvalues of generalized eigenvalue problem (3.1) are real. In particular, for the case where both of these matrices are symmetric positive definite, a projected preconditioned conjugate gradient method can be used to find x and y [27, 20].

Remark 3.4. If matrix block C is symmetric and nonsingular, then $r = \ell$ and generalized eigenvalue problem (3.1) can be equivalently expressed as

$$X^T\left(B - E^T C^{-1} E\right)Xu = \lambda X^T\left(G - E^T C^{-1}E\right)Xu.$$

For this case, we can particularly let $X = [e_1\ e_2\ \cdots\ e_r]$, with $e_j,\ j = 1, 2, \ldots, r$, being the jth unit basis vector in $\mathbb{R}^p$, to further simplify this generalized eigenvalue problem.

Remark 3.5. The actual application of Theorem 3.1 requires a determination of the dimension of null space $\text{null}(B-G)$. The choice of matrix block G may often stem from algebraic considerations that do not take the nullity of $B-G$ into account, which, in turn, may make it difficult to assess the geometric multiplicities of the eigenvalues of the preconditioned matrix, and hence, the convergence rate of the corresponding preconditioned iteration method may be difficult to estimate; see Remark 3.1.

Example 3.1. Consider the coefficient matrix A of linear system (1.1) given by (2.9). Let preconditioner P defined in (1.2) have $(1,1)$ block $G = 3I$. Then we easily know that

$$\det(\lambda P - A) = -(\lambda - 1)^3(2\lambda - 1).$$

So, preconditioned matrix $P^{-1}A$ has an eigenvalue 1 with multiplicity 3 and an eigenvalue $\frac{1}{2}$ with multiplicity 1. It follows from straightforward calculations that the eigenvectors corresponding to eigenvalue $\lambda = 1$ are

$$(0,0,1,0)^T \quad \text{and} \quad (0,0,0,1)^T$$

and that corresponding to eigenvalue $\lambda = \frac{1}{2}$ is

$$(1,0,-1,0)^T.$$

By direct calculation, we have

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$X_f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad Y_f = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad Y_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad T_f = 0.$$

It is obvious that $\ell = 1$ and the dimension of $\text{null}(B-G)$ is 0, i.e., $k = 0$. In addition, generalized eigenvalue problem (3.1) becomes

$$u = 2\lambda u. \tag{3.7}$$

Hence, by Theorem 3.1, we know that matrix $P^{-1}A$ has an eigenvalue $\lambda = 1$ with multiplicity 3 and one eigenvalue $\lambda = \frac{1}{2}$, which is determined by generalized eigenvalue problem (3.7). Moreover, the eigenvectors corresponding to eigenvalue $\lambda = 1$ are

$$(0,0,1,0)^T \quad \text{and} \quad (0,0,0,1)^T$$

and that corresponding to eigenvalue $\lambda = \frac{1}{2}$ is

$$(1,0,-1,0)^T.$$

Clearly, the estimated result by Theorem 3.1 pertinently coincides with the exact one.

Analogously to Theorem 4.3 in [20], we can prove the following estimation about the dimension of Krylov subspace $\mathcal{K}_n(P^{-1}A, b)$.

THEOREM 3.2. *Let the conditions of Theorem* 3.1 *be satisfied. Then the dimension of Krylov subspace* $\mathcal{K}_n(P^{-1}A, b)$ *is at most* $\tilde{p}_b + 1$*, where*

$$\tilde{p}_b = \begin{cases} \min\{p-k+1, n-1\} & \textit{for} \quad p \le k+\ell, \\ \min\{\ell+1, n-1\} & \textit{for} \quad p > k+\ell. \end{cases}$$

This theorem shows that, in exact arithmetic, a Krylov subspace method with an optimality property, when incorporated with constraint preconditioner P, will achieve the exact solution of the system of linear equations (1.1) within at most the number $\tilde{p}_b + 1$ of iteration steps. Since $\tilde{p}_b \le p - k + 1$ and $\tilde{p} \le p$, we easily see that the number of iteration steps of the preconditioned Krylov subspace method estimated in Theorem 3.2 may be smaller than that in Remark 3.1.

4. Numerical results. In this section, we use three examples to show the numerical advantages of constraint preconditioner P defined in (1.2) over the existing preconditioners, such as the *block-diagonal* (BD) and the *block-tridiagonal* (BT) ones, by solving the corresponding system of linear equations (1.1) with the preconditioned GMRES method incorporated with these preconditioners. The numerical results about the GMRES method and its preconditioned variants with respect to the above-mentioned preconditioners are given, in order to compare their effectiveness in the sense of computing time (CPU), iteration step (IT), and relative residual error (RES).

We remark that the MINRES (minimum residual) method is also applicable to solving the system of linear equations (1.1) as its coefficient matrix is symmetric, indefinite, and nonsingular. When its preconditioned variant is employed, however, the preconditioning matrix is basically required to be symmetric positive definite. Note that constraint preconditioner P defined in (1.2) is not definite, neither the block-tridiagonal preconditioner. Hence, it is not suitable to examine the preconditioning effectiveness of this constraint preconditioner P by adopting MINRES as the linear solver.

In our computations, we take $(1,1)$-block matrix $G = (g_{ij}) \in \mathbb{R}^{p\times p}$ in constraint preconditioner P defined in (1.2) to be the tridiagonal matrix of $(1,1)$-block matrix $B = (b_{ij}) \in \mathbb{R}^{p\times p}$ in original matrix A, i.e.,

$$G = \begin{pmatrix} b_{11} & b_{12} & & & \\ b_{21} & b_{22} & b_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & b_{(p-1)(p-2)} & b_{(p-1)(p-1)} & b_{(p-1)p} \\ & & & b_{p(p-1)} & b_{pp} \end{pmatrix}.$$

All iterations are started from the zero vector, terminated when $\mathrm{ERR} := \|b - Az\|_2/\|b\|_2 \le 10^{-7}$ or when $\mathrm{IT} > 500$ and performed in MATLAB 6.5 on a personal computer with 512Mb of memory.

The actions of all preconditioners are implemented in exact fashion. For the BD and the BT preconditioners, the subsystems of linear equations associated with the diagonal block matrices are solved exactly by direct sparse Cholesky factorizations. The constraint preconditioner is first factorized into a product of block-lower and block-upper triangular factors by using the block LU decomposition technique and then, the induced block-triangular systems of linear equations are solved by forward elimination and backward substitution, in which the involved subsystems of linear equations associated with the diagonal block matrices are solved by using direct sparse Cholesky and LU factorizations, respectively.

TABLE 1
Estimated and exact eigenvalue bounds for the regularized driven-cavity flow problem.

Grid	$[-\Theta, -\theta] \cup [\gamma, \Gamma]$	$\mathcal{I}_- \cup \mathcal{I}_+$	$\mathrm{sp}(A)$
8×8	$[-.108, -1.0\mathrm{E}-16] \cup \emptyset$	$[-.108, -8.3\mathrm{E}-17] \cup [0.0\mathrm{E}+00, 3.87]$	$[-.106, -0.017] \cup [4.1\mathrm{E}-16, 3.86]$
16×16	$[-.027, -0.004] \cup \emptyset$	$[-.027, -0.003] \cup [0.0\mathrm{E}+00, 3.97]$	$[-.027, -0.004] \cup [1.3\mathrm{E}-16, 3.97]$
32×32	$[-.007, -1.7\mathrm{E}-18] \cup \emptyset$	$[-.007, -1.7\mathrm{E}-18] \cup [0.0\mathrm{E}+00, 3.99]$	$[-.007, -0.001] \cup [1.1\mathrm{E}-14, 3.99]$
64×64	$[-.002, -2.2\mathrm{E}-04] \cup \emptyset$	$[-.002, -2.2\mathrm{E}-04] \cup [4.8\mathrm{E}-18, 4.00]$	$[-.002, -2.2\mathrm{E}-04] \cup [1.6\mathrm{E}-14, 4.00]$

Our first example is the problem used in fluid dynamics, known as *the regularized driven-cavity flow*:

$$\left\{ \begin{array}{l} \left. \begin{array}{rcl} -\Delta \vec{u} + \nabla p & = & \vec{0} \\ \nabla \cdot \vec{u} & = & 0 \end{array} \right\}, \qquad (x, y) \in (-1, 1) \times (-1, 1), \\ \quad \frac{\partial u}{\partial x} \big|_{y=1} = 1 - x^4, \qquad\qquad x \in [-1, 1]. \end{array} \right.$$

It is a model of the flow in a regularized square cavity with the lid moving from left to right. Four types of test matrices on four uniform discretization grids, i.e., 8×8, 16×16, 32×32, and 64×64, for the cavity domain are generated with the stabilized Q1-P0 finite element approximation by using the IFISS software package; see [25].

We remark that these matrices are of singular or nearly singular $(2, 2)$ blocks. They are symmetric indefinite having positive definite $(1, 1)$ blocks and negative definite Schur complements; see Table 1. It is easily seen from Table 1 that, except for two instances, the new eigenvalue bounds given by Theorem 2.1 sharply estimate the exact spectral intervals.

In the block-diagonal and the block-tridiagonal preconditioners, the $(1, 1)$ blocks are taken to be matrices G, while the $(2, 2)$ blocks are taken as follows:

(a) The $(2, 2)$ blocks are set to be the matrices whose diagonal elements are the corresponding row sums and whose off-diagonal elements are those of matrices C. The obtained block-diagonal and block-tridiagonal preconditioners are denoted as BD(a) and BT(a), correspondingly;

(b) The $(2, 2)$ blocks are set to be the approximated Schur complement $C - E^T G^{-1} E$ of the coefficient matrix A of linear system (1.1). The obtained block-diagonal and block-tridiagonal preconditioners are denoted as BD(b) and BT(b), correspondingly.

For more details about choices of these matrix blocks, we refer to [51, 43, 4, 50, 5].

The numerical results for this example are listed in Table 2. From Table 2, we see that all iteration schemes can produce satisfactory approximate solutions for the system of linear equations (1.1), except the GMRES incorporated with the BD(a) preconditioner for the 64×64 grid. In addition, all preconditioners can reduce the number of iteration steps of the GMRES method, except the BD(a) preconditioner for the 32×32 and 64×64 grids. For the convergent cases, the number of iteration steps of the GMRES method preconditioned by constraint preconditioner P, say, CN-GMRES, is much less than the GMRES method as well as the BD- and the BT-preconditioned GMRES methods. According to the running time, we see that the CN-GMRES method costs much less than the GMRES method as well as the BD-GMRES and the BT-GMRES methods. For the convergent cases, BT-GMRES outperforms BD-GMRES in both iteration steps and computing times, and both of them cost more than CN-GMRES, in particular, when the grid size becomes large. Therefore, the constraint preconditioner is considerably superior to both BD and BT preconditioners, in the sense of IT and CPU, when they are employed to accelerate the GMRES method for solving the system of linear equations (1.1). In addition,

TABLE 2
Numerical results for the regularized driven-cavity flow problem.

Grid		8×8	16×16	32×32	64×64
n		226	834	3202	12546
GMRES	IT	55	114	226	433
	CPU	3.66E-02	8.94E-02	1.33E+00	1.75E+01
	RES	4.62E-08	9.95E-08	8.42E-08	9.76E-08
CN-GMRES	IT	14	27	54	95
	CPU	1.61E-02	3.89E-02	4.12E-01	5.73E+00
	RES	2.40E-08	5.82E-08	9.21E-08	9.78E-08
BD(a)-GMRES	IT	44	102	236	500
	CPU	2.06E-02	1.16E-01	1.76E+00	—
	RES	8.71E-08	8.89E-08	9.32E-08	1.46E-06
BT(a)-GMRES	IT	30	74	175	422
	CPU	1.46E-02	7.35E-02	1.05E+00	1.89E+01
	RES	3.66E-08	7.58E-08	9.37E-08	9.38E-08
BD(b)-GMRES	IT	41	71	124	241
	CPU	2.05E-02	8.49E-02	1.01E+00	1.66E+01
	RES	6.04E-08	9.14E-08	9.27E-08	9.61E-08
BT(b)-GMRES	IT	27	46	79	158
	CPU	1.43E-02	5.39E-02	5.71E-01	9.80E+00
	RES	5.26E-08	7.82E-08	9.17E-08	9.64E-08

we observe that both BD and BT preconditioners of type (a) show better numerical behavior than those of type (b) with respect to both ITs and CPUs.

Our second example is the system of linear equations (1.1), in which

$$\begin{cases} B = K \otimes I + I \otimes K + I \otimes D, \\ E = -(I \otimes v), \\ C = I - h\Xi, \end{cases}$$

with $K = \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$ being a tridiagonal matrix, $D = -4\pi^2 h^2 I$ being a diagonal matrix, $v = (0, 0, \ldots, 0, 1)^T \in \mathbb{R}^m$, and $\Xi \in \mathbb{R}^{m\times m}$ a Toeplitz matrix, resulting from the generating function $\xi(\theta) = 2|\theta|(\theta^2 - 1)$, of the jth diagonal element $\xi_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(\theta)e^{-\imath j\theta}d\theta$, $1 \le j \le m$. Hence, $p = m^2$, $q = m$, and $n = m^2 + m$. This problem is a finite difference discretization of the time-harmonic Maxwell equation

$$\begin{cases} \Delta u + (2\pi)^2 u = f(x, y), & (x, y) \in \Omega \cup \Re_2^+, \\ \qquad\qquad u = 0, & (x, y) \in \partial\left(\Omega \cup \Re_2^+\right), \end{cases}$$

on a uniform grid of domain $\Omega = [0, 1] \times [-1, 0]$, together with radiation boundary condition

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial \eta} - \imath 2\pi u\right) = 0,$$

where $h = \frac{1}{m+1}$, $\imath$ is the imaginary unit and $\Re_2^+$ denotes the upper half-space [9, 5].

We remark that these matrices are symmetric indefinite having positive definite $(1, 1)$ blocks and indefinite Schur complements; see Table 3. It is easily seen from Table 3 that the eigenvalue bounds estimated by Theorem 2.1 perfectly match the exact ones.

In the block-diagonal and the block-tridiagonal preconditioners, the $(1, 1)$ blocks are taken to be matrices G, while the $(2, 2)$ blocks are taken as follows:

Table 3
Estimated and exact eigenvalue bounds for the time-harmonic Maxwell equation.

h	$[-\Theta, -\theta] \cup [\gamma, \Gamma]$	$\mathcal{I}_- \cup \mathcal{I}_+$	$\mathrm{sp}(A)$
$\frac{1}{8}$	$[-31.5, -0.745] \cup [0.806, 1.02]$	$[-31.2, -0.648] \cup [0.597, 5.63]$	$[-31.2, -0.681] \cup [0.689, 5.24]$
$\frac{1}{16}$	$[-18.1, -0.461] \cup [0.193, 0.981]$	$[-17.9, -0.411] \cup [0.166, 6.14]$	$[-17.9 - 0.429] \cup [0.177, 5.81]$
$\frac{1}{32}$	$[-9.43, -0.008] \cup [0.195, 0.874]$	$[-9.26, -0.007] \cup [0.169, 6.27]$	$[-9.23, -0.008] \cup [0.180, 5.95]$
$\frac{1}{48}$	$[-6.21, -0.021] \cup [0.096, 0.830]$	$[-6.08, -0.019] \cup [0.085, 6.29]$	$[-6.04, -0.020] \cup [0.089, 5.98]$
$\frac{1}{64}$	$[-4.54, -0.027] \cup [0.052, 0.806]$	$[-4.43, -0.024] \cup [0.046, 6.30]$	$[-4.39, -0.025] \cup [0.048, 5.99]$
$\frac{1}{80}$	$[-3.52, -0.040] \cup [0.019, 0.792]$	$[-3.42, -0.036] \cup [0.016, 6.30]$	$[-3.39, -0.037] \cup [0.017, 5.99]$
$\frac{1}{96}$	$[-2.83, -0.014] \cup [0.031, 0.782]$	$[-2.74, -0.012] \cup [0.027, 6.30]$	$[-2.71, -0.013] \cup [0.028, 6.00]$

Table 4
Numerical results for the time-harmonic Maxwell equation.

h		$\frac{1}{32}$	$\frac{1}{48}$	$\frac{1}{64}$	$\frac{1}{80}$	$\frac{1}{96}$
n		992	2256	4032	6320	9120
GMRES	IT	162	248	310	385	480
	CPU	0.26	1.27	3.43	8.21	18.39
	RES	7.53E-08	7.14E-08	7.83E-08	8.17E-08	8.39E-08
CN-GMRES	IT	81	121	159	197	235
	CPU	0.12	0.49	1.41	3.37	6.89
	RES	8.72E-08	6.81E-08	7.86E-08	8.11E-08	8.88E-08
BD(a)-GMRES	IT	95	144	181	225	277
	CPU	0.14	0.63	1.63	3.97	8.08
	RES	8.47E-08	7.93E-08	8.87E-08	9.38E-08	8.70E-08
BT(a)-GMRES	IT	87	130	165	205	249
	CPU	0.13	0.54	1.45	3.49	7.18
	RES	8.00E-08	8.54E-08	8.79E-08	8.67E-08	9.05E-08
BD(b)-GMRES	IT	95	142	181	225	273
	CPU	0.14	0.6	1.63	3.93	7.88
	RES	7.54E-08	8.23E-08	8.37E-08	8.23E-08	8.16E-08
BT(b)-GMRES	IT	83	121	162	202	243
	CPU	0.12	0.48	1.41	3.46	6.91
	RES	7.73E-08	9.07E-08	7.57E-08	8.91E-08	9.41E-08

(a) The $(2, 2)$ blocks are set to be the diagonal matrices of matrices C. The obtained block-diagonal and block-tridiagonal preconditioners are denoted as BD(a) and BT(a), correspondingly;

(b) The $(2, 2)$ blocks are set to be the approximated Schur complement $C - E^T G^{-1} E$ of the coefficient matrix A of linear system (1.1). The obtained block-diagonal and block-tridiagonal preconditioners are denoted as BD(b) and BT(b), correspondingly.

For more details about choices of these matrix blocks, we refer to [51, 43, 4, 50, 5].

The numerical results for this example are listed in Table 4. From Table 4, we see that all iteration schemes can produce satisfactory approximate solutions for the system of linear equations (1.1), and all preconditioners can considerably reduce the number of iteration steps of the GMRES method. For each grid size, the running time and the number of iteration steps of the CN-GMRES method is much less than those of the GMRES as well as the BD-GMRES and the BT(a)-GMRES methods. However, the numerical properties of both CN-GMRES and BT(b)-GMRES methods are roughly comparable with respect to CPUs and ITs. Therefore, we conclude again that, for a suitably chosen $(1, 1)$-block matrix G, the constraint preconditioner is superior to both BD and BT(a) preconditioners and is comparable to the BT(b) preconditioner, in the sense of IT and CPU, when they are employed to accelerate the GMRES method for solving the system of linear equations (1.1).

Our third example is the system of linear equations (1.1) whose coefficient matrix has the following blocks:

$$B = 2\widehat{B}^T\widehat{B} + D_1 \in \mathbb{R}^{p\times p}, \quad E = \left(0, 0, \widehat{E}^T\right) \in \mathbb{R}^{p\times q},$$

and

$$C = \begin{pmatrix} D_2 & 0 & -I \\ 0 & D_3 & I \\ -I & I & 0 \end{pmatrix} \in \mathbb{R}^{q\times q}.$$

Here, for a given positive integer ne, $\widehat{\mathrm{ne}} = \mathrm{ne}(\mathrm{ne}+1)$; $\widetilde{\mathrm{ne}} = \mathrm{ne}^2$; $\widehat{B} = (\widehat{b}_{ij}) \in \mathbb{R}^{\widehat{\mathrm{ne}}\times\widehat{\mathrm{ne}}}$, with $\widehat{b}_{ij} = e^{-2((\frac{i}{3})^2+(\frac{j}{3})^2)}$; $D_1 = \mathrm{diag}(d_j^{(1)}) \in \mathbb{R}^{\widehat{\mathrm{ne}}\times\widehat{\mathrm{ne}}}$; $D_i = \mathrm{diag}(d_j^{(i)}) \in \mathbb{R}^{2\widetilde{\mathrm{ne}}\times 2\widetilde{\mathrm{ne}}}$; $i = 2, 3$, are diagonal matrices, with

$$d_j^{(1)} = \frac{1}{j^2} \quad \text{for} \quad 1 \le j \le \widehat{\mathrm{ne}}$$

and

$$d_j^{(2)} = d_j^{(3)} = (j - \widetilde{\mathrm{ne}})^2 \sin\left(\frac{j\pi}{2\,(2\widetilde{\mathrm{ne}}+1)}\right) \quad \text{for} \quad 1 \le j \le 2\widetilde{\mathrm{ne}};$$

$\widehat{E}$ is a blocked matrix given by

$$\widehat{E} = \begin{bmatrix} \widetilde{E} \otimes I_{\mathrm{ne}\times\mathrm{ne}} \\ I_{\mathrm{ne}\times\mathrm{ne}} \otimes \widetilde{E} \end{bmatrix},$$

with

$$\widetilde{E} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \in \mathbb{R}^{\mathrm{ne}\times(\mathrm{ne}+1)}.$$

This class of linear systems, with $p = \widehat{\mathrm{ne}}$ and $q = 6\widetilde{\mathrm{ne}}$, arises in computing the descent directions in the Newton steps involved in the modified primal-dual interior point method used to solve the nonsmooth and nonconvex minimization problems from restorations of piecewise constant images. The choice of matrix $\widetilde{E}$ being the first-order difference operator corresponds to the circles image; see [39] for details.

We remark that these matrices are symmetric indefinite having positive definite $(1,1)$ blocks and indefinite Schur complements; see Table 5. Again, it is easily seen from Table 5 that the eigenvalue bounds estimated by Theorem 2.1 include the exact ones.

From Table 5, we have also noticed that Theorem 2.1 results in much overestimated lower bounds for the negative eigenvalues of matrix $A \in \mathbb{R}^{n\times n}$. This is because for this example, the $(2,2)$-block matrix C is indefinite and matrix $E^TB^{-1}E$ is considerably ill-conditioned. As a result, Schur complement $S = C - E^TB^{-1}E$ has a

TABLE 5
Estimated and exact eigenvalue bounds for the piecewise-constant image restoration problem.

ne	$[-\Theta, -\theta] \cup [\gamma, \Gamma]$	$\mathcal{I}_- \cup \mathcal{I}_+$	$\mathrm{sp}(A)$
10	$[-6.6E+4, -7.3E-5] \cup [9.2E+1, 2.9E+4]$	$[-6.6E+4, 0] \cup [0, 9.5E+4]$	$[-2.8, -7.3E-5] \cup [3.8E-2, 2.9E+4]$
20	$[-1.1E+6, -4.5E-6] \cup [3.2E+2, 4.6E+5]$	$[-1.1E+6, 0] \cup [0, 1.6E+6]$	$[-2.8, -4.5E-6] \cup [1.1E-2, 4.6E+5]$
30	$[-5.8E+6, -8.8E-7] \cup [7.0E+2, 2.3E+6]$	$[-5.8E+6, 0] \cup [0, 8.1E+6]$	$[-2.8, -8.8E-7] \cup [5.0E-3, 2.3E+6]$

TABLE 6
Numerical results for the piecewise-constant image restoration problem.

ne		10	20	30	40	50
n		710	2820	6330	11240	17550
GMRES	IT	459	—	—	338	377
	CPU	1.54	6.05	13.57	12.83	27.01
	RES	6.51E-08	8.30E-07	1.63E-07	7.06E-08	8.51E-08
CN-GMRES	IT	7	7	5	5	5
	CPU	0.02	0.18	0.42	1.04	2.01
	RES	7.25E-08	1.81E-08	6.98E-08	4.00E-08	2.99E-08
BD(a)-GMRES	IT	220	—	—	—	—
	CPU	0.40	6.64	14.82	27.61	46.56
	RES	5.21E-14	1.00E+00	1.00E+00	1.00E+00	1.00E+00
BT(a)-GMRES	IT	111	421	—	—	—
	CPU	0.13	4.82	14.84	27.66	46.69
	RES	4.28E-12	2.92E-09	1.00E+00	1.00E+00	1.00E+00
BD(b)-GMRES	IT	12	12	12	12	12
	CPU	0.03	0.28	0.87	2.18	4.25
	RES	6.00E-09	9.00E-10	6.86E-09	6.71E-08	6.21E-08
BT(b)-GMRES	IT	9	9	8	8	8
	CPU	0.03	0.22	0.61	1.52	2.97
	RES	2.83E-08	3.10E-09	4.86E-08	2.88E-08	3.05E-08

very small negative eigenvalue. It then follows that, in Theorem 2.1, the quantities Ω and Θ are comparatively very large, but the quantity Δ is much less than both of them. So, in the computation of the left end-point of $\mathcal{I}_-$, we may get

$$\frac{1}{2}\left(\omega + \Delta - \Theta - \sqrt{(\omega + \Delta - \Theta)^2 + 4\Theta\Delta}\right)$$
$$\approx \frac{\Theta}{2}\left(\frac{\omega + \Delta}{\Theta} - 1 + \sqrt{\left(\frac{\omega + \Delta}{\Theta} - 1\right)^2 + \frac{4\Delta}{\Theta}}\right) \approx -\Theta.$$

In the block-diagonal and the block-tridiagonal preconditioners, the $(1,1)$ blocks are taken to be matrices G, while the $(2,2)$ blocks are taken as follows:

(a) The $(2,2)$ blocks are set to be matrices C. The obtained block-diagonal and block-tridiagonal preconditioners are denoted as BD(a) and BT(a), correspondingly;

(b) The $(2,2)$ blocks are set to be the approximated Schur complement $C - E^T G^{-1} E$ of the coefficient matrix A of linear system (1.1). The obtained block-diagonal and block-tridiagonal preconditioners are denoted as BD(b) and BT(b), correspondingly.

The numerical results for this example are listed in Table 6. From Table 6, we see that, among all iteration schemes, only CN-GMRES, BD(b)-GMRES, and BT(b)-GMRES can produce satisfactory approximate solutions for the system of linear equations (1.1). For each grid size, the CPU and the IT of the CN-GMRES method is less than those of the GMRES as well as the BD-GMRES and the BT-GMRES methods when they are convergent. Therefore, we conclude again that, for a suitably chosen $(1,1)$-block matrix G, the constraint preconditioner is superior to the BD preconditioner when it is employed to accelerate the GMRES method for solving the system of linear equations (1.1).

5. Concluding remarks. We have studied the eigenvalue bounds of the block two-by-two nonsingular and symmetric indefinite matrix whose $(1,1)$ block is symmetric positive definite and Schur complement with respect to its $(2,2)$ block is symmetric indefinite. We also investigated the spectral properties of preconditioned matrices using constraint preconditioners. Numerical results have shown the effectiveness of the constraint preconditioners for these symmetric indefinite block two-by-two matrices. In particular, for a suitably chosen $(1,1)$-block matrix G, the constraint preconditioner is often superior to both BD and BT(a) preconditioners and is comparable to the BT(b) preconditioner, in the sense of IT and CPU, when they are employed to accelerate the GMRES method for solving the system of linear equations (1.1).

Of course, we have realized that, in our experiments, the actions of all preconditioners are implemented in exact fashion, and the results are obtained by ignoring the act of inner iterations. In fact, inner solves often need to be done inexactly and, perhaps, iteratively, and many computational issues may arise here. In this sense, further study of inner solves is required before being able to determine whether the constraint preconditioning methodology is practically superior to other approaches.

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