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ANALYSIS OF COLLOCATION METHODS FOR GENERALIZED AUTO-CONVOLUTION VOLTERRA INTEGRAL EQUATIONS*

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Abstract. We first study the existence, uniqueness, and regularity properties of solutions to a generalized version of the auto-convolution Volterra integral equation of the second kind. These results are then used to establish the optimal global and local (super) convergence properties of piecewise polynomial collocation solutions for such integral equations. The theoretical results are illustrated by extensive numerical examples.

Key words. auto-convolution Volterra integral equations, existence and regularity of solutions, collocation methods, optimal order of convergence, superconvergence

AMS subject classifications. 65R20, 65Q20, 45D05

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1. Introduction.

The aim of this paper is to study the optimal convergence properties of piecewise polynomial collocation solutions for the generalized auto-convolution Volterra integral equation (AVIE)

$$u(t) = g(t) + \int_0^t K(t,s)u(t-s)u(s)ds, \quad t \in I := [0,T], \tag{1.1}$$

where $u(t)$ is the unknown solution and g, K are given functions. AVIEs of the second kind arise in many applications, for example, in the identification of memory kernels in the theory of viscoelasticity (see von Wolfersdorf [9, pp. 14–19] and its extensive list of references) and in the computation of certain special functions (von Wolfersdorf [10]).

There has been a great deal of work in the numerical analysis of the collocation methods for Volterra integral equations of Hammerstein form,

$$u(t) = g(t) + \int_0^t K(t,s)G(u(s))ds, \quad t \in I := [0,T] \tag{1.2}$$

(see, for example, Brunner [1] and its references).

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Recently, one of the authors [5] studied the existence, uniqueness, and regularity properties of solutions, as well as the convergence of collocation solutions, to the nonstandard Volterra integral equation

$$u(t) = g(t) + \int_0^t K(t, s, u(t), u(s))ds, \quad t \in I. \tag{1.3}$$

However, the convergence analysis of collocation solutions to a different kind of nonstandard Volterra integral equation, namely,

$$u(t) = g(t) + \int_0^t K(t, s)G(u(t-s))G(u(s))ds, \quad t \in I, \tag{1.4}$$

is still lacking. This is in particularly true for $G(u) = u^{\beta}$ $(\beta > 0)$. In this paper we take a first step toward filling this gap by studying collocation solutions to (1.1), whose nonlinearity corresponds to $\beta = 1$.

The paper is organized as follows. In section 2 we analyze the existence, uniqueness, and regularity of solutions to (1.1). Section 3 contains the detailed description of the collocation equations and their computational forms for the AVIE (1.1). The attainable global and local orders of superconvergence of collocation solutions (based on discontinuous piecewise polynomials) is established in sections 4, 5, and 6. It will be seen that the local and global superconvergence results for (1.1) differ from the ones for the nonstandard VIE (1.3) (see Remark 6.2). Section 7 contains a wide range of numerical examples that confirm our theoretical convergence results. The paper concludes with a brief description of open problems and future work.

2. Existence, uniqueness and regularity of solutions. The solvability of the basic AVIE

$$u(t) = g(t) + \lambda \int_0^t u(t-s)u(s)ds, \quad t \in I \ \ (\lambda \in R), \tag{2.1}$$

(that is, (1.1) with $K(t,s) \equiv \lambda \neq 0$) has been studied extensively (see, for example, Butzer [3], Bukhgeim [2], and, especially, von Wolfersdorf [9] and its comprehensive list of references). Since (2.1) contains the convolution of $u(s)$ with itself, it suggests to resort to Laplace transform techniques. They show that the Laplace transform $U(s) := (\mathcal{L}u)(t)$ of the solution $u(t)$ satisfies an algebraic quadratic equation and that only one of its two roots leads to a solution $u(t)$ of (2.1) that is continuous on $[0, \infty)$. (We note in passing that other classes of more general versions of (2.1) were studied by von Wolfersdorf and Janno [11] and von Wolfersdorf [9].)

The approach just described can of course no longer be employed for the generalized AVIE (1.1) whose its kernel $K(t,s)$ is not a constant. It is one aim of this section to establish the existence and uniqueness of a global continuous solution of (1.1), by (small) $\delta > 0$ the AVIE is linear on the interval $[\delta, T]$. Moreover, we show that the solution inherits the regularity properties of the given functions g and K.

THEOREM 2.1. *Assume that the given functions in the AVIE* (1.1) *satisfy* $g \in C(I)$, $K \in C(D)$, *where* $D := \{(t,s) : 0 \leq s \leq t \leq T\}$. *Then the AVIE* (1.1) *has a unique solution* $u \in C(I)$.

Proof. We first show that (1.1) has a local solution on some small interval $[0, \delta_0]$.

LEMMA 2.2. *Assume that $g \in C(I)$ and $K \in C(D)$. Then there exists a unique $u \in C[0,\delta_0]$ satisfying the AVIE (1.1) on $[0,\delta_0]$, where*

$$\delta_0 := \min\left\{\frac{1}{4\overline{K}(\overline{G}+1)^2}, T\right\}, \quad \overline{G} := \max_{t\in I}|g(t)|, \quad \overline{K} := \max_{(t,s)\in D}|K(t,s)|.$$

Proof. In order to establish the *existence* of such a solution we employ Picard iteration,

$$u_{n+1}(t) := g(t) + \int_0^t K(t,s)u_n(t-s)u_n(s)ds, \quad t\in[0,\delta_0]\ (n\geq 0), \tag{2.2}$$

with $u_0(t) := g(t)$, and show that

$$|u_n(t) - g(t)| \leq 1, \quad t\in[0,\delta_0]\ (n\geq 0). \tag{2.3}$$

Since for $n=0$ the estimate (2.3) is trivial, a straighforward induction step reveals that (2.3) holds for any $n\geq 1$:

$$|u_{n+1}(t) - g(t)| = \left|\int_0^t K(t,s)u_n(t-s)u_n(s)ds\right| \leq \overline{K}\left(\overline{G}+1\right)^2 t \leq 1$$

with each u_{n+1} defined on $[0,\delta_0]$. It follows from the continuity of g and K that $u_{n+1}\in C[0,\delta_0]$. We now prove that the sequence $\{u_n(t)\}$ is a Cauchy sequence on $[0,\delta_0]$. If we set $y_n(t) := u_{n+1}(t) - u_n(t)$, we obtain

$$|y_0(t)| = |u_1(t) - u_0(t)| = \left|\int_0^t K(t,s)g(t-s)g(s)ds\right| \leq \overline{K}\overline{G}^2 t \leq \overline{K}(\overline{G}+1)^2 t.$$

Hence, assuming that

$$|y_n(t)| \leq \frac{\overline{K}^{n+1}(\overline{G}+1)^{n+2}2^n t^{n+1}}{(n+1)!}, \tag{2.4}$$

a further induction argument shows that the above estimate holds for all n:

$$\begin{aligned}
|y_{n+1}(t)| &= \left|\int_0^t K(t,s)\left[u_{n+1}(t-s)u_{n+1}(s) - u_n(t-s)u_n(s)\right]ds\right| \\
&\leq \overline{K}\int_0^t \left[|u_{n+1}(t-s)||u_{n+1}(s)-u_n(s)| + |u_{n+1}(t-s)-u_n(t-s)||u_n(s)|\right]ds \\
&\leq 2\overline{K}\left(\overline{G}+1\right)\int_0^t |u_{n+1}(s)-u_n(s)|ds \\
&\leq 2\overline{K}\left(\overline{G}+1\right)\int_0^t \frac{\overline{K}^{n+1}(\overline{G}+1)^{n+2}2^n s^{n+1}}{(n+1)!}ds = \frac{\overline{K}^{n+2}(\overline{G}+1)^{n+3}2^{n+1}t^{n+2}}{(n+2)!}.
\end{aligned}$$

Therefore,

$$u_{n+m}(t) - u_n(t) = \sum_{j=0}^{m-1}\left[u_{n+j+1}(t) - u_{n+j}(t)\right]$$

implies that, for all $t \in [0, \delta_0]$,

$$|u_{n+m}(t) - u_n(t)| \leq \sum_{j=0}^{m-1} |y_{n+j}(t)| \leq \sum_{j=n}^{n+m-1} \frac{\overline{K}^{j+1}(\overline{G}+1)^{j+2}2^j t^{j+1}}{(j+1)!}.$$

This shows that $\lim_{n\to\infty} u_n(t) =: u(t)$ uniformly on $[0, \delta_0]$ with $u \in C([0, \delta_0])$. Since

$$\begin{aligned}
&\left|\int_0^t K(t,s)[u_n(t-s)u_n(s) - u(t-s)u(s)]ds\right| \\
&\quad \leq \int_0^t |K(t,s)|[|u_n(t-s)||u_n(s) - u(s)| + |u_n(t-s) - u(t-s)||u(s)|]ds \\
&\quad \leq 2\overline{K}(\overline{G}+1)\int_0^t |u_n(s) - u(s)|ds \to 0, \quad t \in [0, \delta_0],
\end{aligned}$$

as $n \to \infty$, we see that

$$\begin{aligned}
u(t) &= \lim_{n\to\infty} u_n(t) = g(t) + \lim_{n\to\infty}\int_0^t K(t,s)u_{n-1}(t-s)u_{n-1}(s)ds \\
&= g(t) + \int_0^t K(t,s)u(t-s)u(s)ds, \quad t \in [0, \delta_0],
\end{aligned}$$

and there exists a function $u \in C([0, \delta_0])$ which is a solution of the AVIE (1.1) on $[0, \delta_0]$.

It remains to show that the above solution is *unique*. To do so, suppose that (1.1) possesses two continuous solutions z_1 and z_2 on $[0, \delta_0]$. Then

$$\begin{aligned}
|z_1(t) - z_2(t)| &= \left|\int_0^t K(t,s)[z_1(t-s)z_1(s) - z_2(t-s)z_2(s)]ds\right| \\
&\leq \overline{K}\int_0^t [|z_1(t-s)||z_1(s) - z_2(s)| + |z_1(t-s) - z_2(t-s)||z_2(s)|]ds \\
&\leq 2\overline{K}(\overline{G}+1)\int_0^t |z_1(s) - z_2(s)|ds, \quad t \in [0, \delta_0].
\end{aligned}$$

It follows from the classical Gronwall lemma (cf. [1, p. 79]) that

$$|z_1(t) - z_2(t)| \leq 0 \cdot \exp(2\overline{K}(\overline{G}+1)t) = 0, \quad t \in [0, \delta_0].$$

The continuity of z_1 and z_2 then implies that $z_1(t) = z_2(t)$ for all $t \in [0, \delta_0]$. ∎

We will denote this unique continuous solution of (1.1) on $[0, \delta_0]$ by $w_1(t)$. In order to extend the solution to the entire interval $[0, T]$, we first consider $t \in [\delta_0, 2\delta_0]$, assuming that $2\delta_0 \leq T$. It follows that

$$\begin{aligned}
u(t) &= g(t) + \int_0^{\delta_0} K(t,s)u(t-s)u(s)ds + \int_{\delta_0}^t K(t,s)u(t-s)u(s)ds \\
&= g(t) + \int_{t-\delta_0}^t K(t,t-s)u(t-s)u(s)ds + \int_{\delta_0}^t K(t,s)u(t-s)u(s)ds \\
&= g(t) + \int_{t-\delta_0}^{\delta_0} K(t,t-s)u(t-s)u(s)ds + \int_{\delta_0}^t [K(t,t-s) + K(t,s)]u(t-s)u(s)ds,
\end{aligned}$$

and hence

$$u(t) = g_1(t) + \int_{\delta_0}^{t} [K(t, t-s) + K(t,s)] w_1(t-s) u(s) ds, \tag{2.5}$$

where

$$g_1(t) := g(t) + \int_{t-\delta_0}^{\delta_0} K(t, t-s) w_1(t-s) w_1(s) ds, \quad t \in [\delta_0, 2\delta_0].$$

We observe that the VIE (2.5) is *linear*. Thus it follows from the classical Volterra theory (see, for example, Brunner [1, Chapter 2]) that there exists a unique continuous solution to (2.5) on $[\delta_0, 2\delta_0]$. This in turn implies that the AVIE (1.1) possesses a unique continuous solution on the interval $[0, 2\delta_0]$. We will denote this solution by $w_2(t)$.

Since $T < \infty$ we can continue this process. Suppose that $k\delta_0 < T$ and that, for $t \in [0, k\delta_0]$, the unique solution $w_k(t)$ of the AVIE (1.1) is in $C([0, k\delta_0])$. Then for $t \in [k\delta_0, (k+1)\delta_0]$ (1.1) can be written as

$$\begin{aligned} u(t) &= g(t) + \int_0^{k\delta_0} K(t,s) u(t-s) u(s) ds + \int_{k\delta_0}^{t} K(t,s) u(t-s) u(s) ds \\ &= g_k(t) + \int_{k\delta_0}^{t} [K(t, t-s) + K(t,s)] w_k(t-s) u(s) ds, \end{aligned} \tag{2.6}$$

where

$$g_k(t) := g(t) + \int_{t-k\delta_0}^{k\delta_0} K(t, t-s) w_k(t-s) w_k(s) ds.$$

Thus, (1.1) possesses a unique continuous solution for $t \in [0, (k+1)\delta_0]$. Since $\delta_0 > 0$ is fixed, the existence and uniqueness of a continuous solution can be extended to the entire interval $I = [0, T]$. ∎

Since the principal aim of this paper is to analyze the optimal order of convergence of a class of pieceewise polynomial collocation methods for (1.1), we need to know the regularity properties of the solution of this AVIE. We state these in Theorem 2.3.

THEOREM 2.3. *Assume that the given functions describing the AVIE* (1.1) *satisfy* $g \in C^m(I)$ *and* $K \in C^m(D)$ *for some integer* $m \geq 1$. *Then the solution* u *of* (1.1) *possesses the regularity* $u \in C^m(I)$.

Proof. As the proof of Theorem 2.1 revealed, the AVIE (1.1) is nonlinear only near $t = 0$, e.g., in $[0, \delta_0]$, and is linear when $t \in (\delta_0, T]$. Thus, since the solution of a linear Volterra integral equation inherits the regularity of the given functions (cf. Brunner [1, Theorem 2.1.3]), we only have to establish the regularity of u on an interval $[0, \delta_0]$. The basis for doing this is the Picard iteration equation (2.2): since $g \in C^m(I)$ and $K \in C^m(D)$, it follows that all the Picard iterates $u_n(t)$ are in $C^m(I)$. The uniform convergence of the series $\{u_n(t)\}$ on I then implies that its limit $u(t)$ has the same regularity $C^m(I)$. ∎

3. Collocation methods. Let

$$I_h := \{t_n = nh,\ n = 0, 1, \ldots, N\ \ (t_N = T)\}$$

be a given mesh on I. The solution u of (1.1) will be approximated by the element u_h in the piecewise polynomial space

$$S_{m-1}^{(-1)}(I_h) := \{v :\ v|_{e_n} \in \pi_{m-1}\ (0 \le n \le N-1)\}. \tag{3.1}$$

Here, $v|_{e_n}$ is the restriction of v to the subinterval $e_n := [t_n, t_{n+1}]$, and π_{m-1} $(m \ge 1)$ denotes the set of real polynomials of degree not exceeding $m-1$. Let X_h be given by

$$X_h := \{t = t_n + c_i h :\ 0 < c_1 < \cdots < c_m \le 1\ (0 \le n \le N-1)\}.$$

3.1. The collocation equations on uniform meshes. Setting $U_{n,j} := u_h(t_n + c_j h)(j = 1, \ldots, m)$, we express u_h on the subinterval e_n by the local Lagrange basis $l_j(v)$ with respect to the m (distinct) collocation parameters c_i, namely,

$$u_h(t_n + sh) = \sum_{j=1}^{m} l_j(s) U_{n,j},\ \ s \in (0, 1]$$

with

$$l_j(s) := \prod_{i=1, i\ne j}^{m} \frac{s - c_i}{c_j - c_i}.$$

Using the above notation, the restriction of u_h to the subinterval e_n by u_h^n given by the Lagrange interpolation

$$u_h|_{e_n} = u_h^n(t) = u_h^n(t_n + sh) = \sum_{i=1}^{m} l_i(s) U_{n,i}, \quad 0 < s \le 1. \tag{3.2}$$

Hence, our collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ is determined by the collocation equation

$$u_h(t) = g(t) + \int_0^t K(t,s) u_h(t-s) u_h(s) ds, \quad t \in X_h. \tag{3.3}$$

Using the notation $t_{n,i} = t_n + c_i h \in e_n$, we can rewrite (3.3) as

$$\begin{aligned} u_h^n(t_{n,i}) &= g(t_{n,i}) + \int_0^{t_{n,i}} K(t_{n,i}, s) u_h(t_{n,i} - s) u_h(s) ds \\ &= g(t_{n,i}) + \int_{t_n}^{t_{n,i}} K(t_{n,i}, s) u_h^0(t_{n,i} - s) u_h^n(s) ds \\ &\quad + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k,i}} K(t_{n,i}, s) u_h^{n-k}(t_{n,i} - s) u_h^k(s) ds \\ &\quad + \sum_{k=0}^{n-1} \int_{t_{k,i}}^{t_{k+1}} K(t_{n,i}, s) u_h^{n-k-1}(t_{n,i} - s) u_h^k(s) ds. \end{aligned} \tag{3.4}$$

It is easy to see that for each $u_h^n(t), n = 0, \ldots, N-1$, (3.4) is linear except for $u_h^0(t)$. In order to solve (3.4), we consider the following two cases.

Case I: $n = 0$.

From (3.4) with $n = 0$, we get

$$\begin{aligned} U_{0,i} &= u_h^0(c_ih) = g(c_ih) + \int_0^{c_ih} K(c_ih, s)u_h^0(c_ih - s)u_h^0(s)ds \\ &= g(c_ih) + h\int_0^{c_i} K(c_ih, sh)u_h^0((c_i - s)h)u_h^0(sh)ds \\ &= g(c_ih) + h\sum_{j,k=1}^{m} \int_0^{c_i} K(c_ih, sh)l_j(c_i - s)l_k(s)ds u_h^0(c_jh)u_h^0(c_kh) \\ &=: g(c_ih) + h\sum_{j,k=1}^{m} \alpha_{jk}^{(i)} u_h^0(c_jh)u_h^0(c_kh) \\ &= g(c_ih) + h\sum_{j,k=1}^{m} \alpha_{jk}^{(i)} U_{0,j}U_{0,k}, \end{aligned} \tag{3.5}$$

where $\alpha_{jk}^{(i)} := \int_0^{c_i} K(c_ih, sh)l_j(c_i - s)l_k(s)ds$. Since K and l_j are continuous, it is easy to see that there exists a constant $\widetilde{M}$, such that $\int_0^1 |l_j(c_i - s)l_k(s)|ds \leq \widetilde{M}$ for all $i, j, k = 1, \dots m$. Thus, we have $|\alpha_{jk}^{(i)}| \leq \overline{K}\widetilde{M}$ for all $i, j, k = 1, \dots m$.

Let $\psi = (\psi_1, \dots, \psi_m)^T$. Define the operators $\mathcal{T} : \mathbf{R}^m \to \mathbf{R}^m$ and $\overline{\mathcal{T}} : \mathbf{R}^m \to \mathbf{R}^m$ by

$$(\mathcal{T}\psi)_i := h\sum_{j,k=1}^{m} \alpha_{jk}^{(i)} \psi_j\psi_k, \quad i = 1, \dots, m, \tag{3.6}$$

and

$$(\overline{\mathcal{T}}\psi)_i := g(c_ih) + (\mathcal{T}\psi)_i, \quad i = 1, \dots, m. \tag{3.7}$$

We consider the set $B(0, M_0) = \{\psi \in \mathbf{R}^m : \ \|\psi\|_\infty \leq M_0\}$, where $M_0 := 2\overline{G} + 1$ and $\|\psi\|_\infty := \max_{1\leq i\leq m} |\psi_i|$. For all $\lambda, \psi \in B(0, M_0)$, we have

$$\begin{aligned} \|\overline{\mathcal{T}}\lambda - \overline{\mathcal{T}}\psi\|_\infty &= h \max_{1\leq i\leq m} \left| \sum_{j,k=1}^{m} \alpha_{jk}^{(i)}(\lambda_j\lambda_k - \psi_j\psi_k) \right| \\ &= h \max_{1\leq i\leq m} \left| \sum_{j,k=1}^{m} \alpha_{jk}^{(i)}(\lambda_j(\lambda_k - \psi_k) + \psi_k(\lambda_j - \psi_j)) \right| \\ &\leq h \max_{1\leq i\leq m} \left(\sum_{j,k=1}^{m} |\alpha_{jk}^{(i)}||\lambda_j||\lambda_k - \psi_k| + \sum_{j,k=1}^{m} |\alpha_{jk}^{(i)}||\psi_k||\lambda_j - \psi_j| \right) \\ &= h \max_{1\leq i\leq m} \sum_{j,k=1}^{m} \left(|\alpha_{jk}^{(i)}||\lambda_j| + |\alpha_{kj}^{(i)}||\psi_j| \right) |\lambda_k - \psi_k| \\ &\leq 2hm^2M_0\overline{K}\widetilde{M}\|\lambda - \psi\|_\infty =: L\|\lambda - \psi\|_\infty. \end{aligned}$$

For $h \in (0, \frac{1}{4m^2M_0\overline{K}\widetilde{M}}) =: (0, \bar{h})$, we have $L < \frac{1}{2}$. Moreover, for any $\lambda \in B(0, M_0)$,

$$\|\overline{\mathcal{T}}\lambda\|_\infty \leq L\|\lambda\|_\infty + \|g\|_\infty \leq LM_0 + \overline{G} \leq M_0.$$

Since $\overline{\mathcal{T}} : B(0, M_0) \to B(0, M_0)$ is a contraction mapping it has a unique fixed point

ψ^* in $B(0, M_0)$ when $h \in (0, \bar{h})$, that is,

$$\psi^* = \overline{\mathcal{T}}\psi^*. \tag{3.8}$$

Thus, if we compute the solution vector by means of the iteration

$$\begin{cases} \psi^{(k+1)} = \overline{\mathcal{T}}\psi^{(k)}, & k = 0, 1, \ldots, \\ \text{with } \psi^{(0)} \in B(0, M_0), \end{cases} \tag{3.9}$$

then for any $h \in (0, \bar{h})$, we obtain a sequence $\psi^{(k)} \in B(0, M_0)$ which converges to $\psi^* \in B(0, M_0)$, the unique solution of (3.8).

Thus, for arbitrary starting vector $\psi^{(0)} \in B(0, M_0)$, if we compute u_h^0 by (3.5) and (3.6), the iterates $\psi^{(k)}$ converge to the unique solution $U_0 := (U_{0,1}, \ldots, U_{0,m})^T$ of (3.8) in $B(0, M_0)$ uniformly for $h \in (0, \bar{h})$. Furthermore, u_h^0 is bounded.

Case II: $1 \le n \le N - 1$.

In order to prove that u_h^n is bounded and is defined uniquely, we first study the case $n = 1$. We know that for unknown function $u_h^1(t)$, (3.4) is linear and we have

$$\begin{aligned} U_{1,i} &= g(t_{1,i}) + \int_{t_1}^{t_{1,i}} K(t_{1,i}, s) u_h^0(t_{1,i} - s) u_h^1(s) ds \\ &\quad + \int_{t_0}^{t_{0,i}} K(t_{1,i}, s) u_h^1(t_{1,i} - s) u_h^0(s) ds + \int_{t_{0,i}}^{t_1} K(t_{1,i}, s) u_h^0(t_{1,i} - s) u_h^0(s) ds \\ &= g(t_{1,i}) + h \int_0^{c_i} K(t_{1,i}, t_1 + sh) u_h^0((c_i - s)h) u_h^1(t_1 + sh) ds \\ &\quad + h \int_0^{c_i} K(t_{1,i}, sh) u_h^1(t_1 + (c_i - s)h) u_h^0(sh) ds \\ &\quad + h \int_{c_i}^1 K(t_{1,i}, sh) u_h^0(t_1 + (c_i - s)h) u_h^0(sh) ds \\ &= g(t_{1,i}) + h \sum_{j,k=1}^m \int_0^{c_i} K(t_{1,i}, t_1 + sh) l_j(c_i - s) l_k(s) ds U_{0,j} U_{1,k} \\ &\quad + h \sum_{j,k=1}^m \int_0^{c_i} K(t_{1,i}, sh) l_j(s) l_k(c_i - s) ds U_{0,j} U_{1,k} \\ &\quad + h \sum_{j,k=1}^m \int_{c_i}^1 K(t_{1,i}, sh) l_j(1 + c_i - s) l_k(s) ds U_{0,j} U_{0,k}. \end{aligned} \tag{3.10}$$

Since $|U_{0,j}| \le \|U_0\|_\infty < M_0$, we find

$$\left| \int_0^{c_i} K(t_{1,i}, t_1 + sh) l_j(c_i - s) l_k(s) ds U_{0,j} \right| \le \overline{K}\widetilde{M}M_0$$

and

$$\left| \int_0^{c_i} K(t_{1,i}, sh) l_j(s) l_k(c_i - s) ds U_{0,j} \right| \le \overline{K}\widetilde{M}M_0.$$

Introducing the vector $\mathbf{g}_1 := (g(t_{1,1}), \ldots, g(t_{1,m}))^T$ and the matrices $A_1 = [\,A_1(i,k)\,]$

and $A_1^{(0)} = [\, A_1^{(0)}(i,k)\,]$ with elements

$$A_1(i,k) := \sum_{j=1}^{m} \int_0^{c_i} [K(t_{1,i}, t_1 + sh) l_j(c_i - s) l_k(s) + K(t_{1,i}, sh) l_j(s) l_k(c_i - s)] ds \cdot U_{0,j},$$

$$A_1^{(0)}(i,k) := \sum_{j=1}^{m} \int_{c_i}^{1} K(t_{1,i}, sh) l_j(1 + c_i - s) l_k(s) ds \cdot U_{0,j} \quad (i,k = 1,\ldots,m)\ ,$$

the previous equation (3.10) can be written as

$$[\mathcal{I}_m - hA_1]U_1 = \mathbf{g}_1 + hA_1^{(0)}U_0 \quad (0 \le n \le N-1).$$

It is easy to verify that $\|hA_1\|_\infty \le \frac{1}{2}$ for uniform meshes with diameter $h \in (0, \bar{h})$. Thus we obtain that $\|(\mathcal{I}_m - hA_1)^{-1}\|_\infty \le 2$ and $\|U_1\|_\infty \le 2(G + \frac{1}{2}M_0) =: M_1$. Here M_1 is a positive constant independent of h, that is, u_h^1 is also bounded and uniquely exists. Therefore, $u_h(t)$ is bounded and uniquely exists on $[0, 2h]$ uniformly for $h \in (0, \bar{h})$.

Now, assuming that for all $p < n$, u_h^p is bounded by M_{n-1} and is unique, we will prove that u_h^n has the same property for all $h \in (0, \bar{h})$. Since $n \neq 0$ we know that (3.4) is linear, and we have

$$\begin{aligned}
U_{n,i} &= g(t_{n,i}) + \int_{t_n}^{t_{n,i}} K(t_{n,i}, s) u_h^0(t_{n,i} - s) u_h^n(s) ds \\
&\quad + \sum_{l=0}^{n-1} \int_{t_l}^{t_{l,i}} K(t_{n,i}, s) u_h^{n-l}(t_{n,i} - s) u_h^l(s) ds \\
&\quad + \sum_{l=0}^{n-1} \int_{t_{l,i}}^{t_{l+1}} K(t_{n,i}, s) u_h^{n-l-1}(t_{n,i} - s) u_h^l(s) ds \\
&= g(t_{n,i}) + h \int_0^{c_i} K(t_{n,i}, t_n + sh) u_h^0((c_i - s)h) u_h^n(t_n + sh) ds \\
&\quad + h \int_0^{c_i} K(t_{n,i}, sh) u_h^n(t_n + (c_i - s)h) u_h^0(sh) ds \\
&\quad + h \sum_{l=1}^{n-1} \int_0^{c_i} K(t_{n,i}, t_l + sh) u_h^{n-l}(t_{n-l} + (c_i - s)h) u_h^l(t_l + sh) ds \\
&\quad + h \sum_{l=0}^{n-1} \int_{c_i}^{1} K(t_{n,i}, t_l + sh) u_h^{n-l-1}(t_{n-l} + (c_i - s)h) u_h^l(t_l + sh) ds \\
&= g(t_{n,i}) + h \sum_{j,k=1}^{m} \int_0^{c_i} K(t_{n,i}, t_n + sh) l_j(c_i - s) l_k(s) ds U_{0,j} U_{n,k} \\
&\quad + h \sum_{j,k=1}^{m} \int_0^{c_i} K(t_{n,i}, sh) l_j(s) l_k(c_i - s) ds U_{0,j} U_{n,k} \\
&\quad + h \sum_{l=1}^{n-1} \sum_{j,k=1}^{m} \int_0^{c_i} K(t_{n,i}, t_l + sh) l_j(c_i - s) l_k(s) ds U_{n-l,j} U_{l,k} \\
&\quad + h \sum_{l=0}^{n-1} \sum_{j,k=1}^{m} \int_{c_i}^{1} K(t_{n,i}, t_l + sh) l_j(1 + c_i - s) l_k(s) ds U_{n-l-1,j} U_{l,k}.
\end{aligned} \tag{3.11}$$

Similar to the case $n = 1$, we have

$$\|U_n\|_\infty \le 2(\overline{G} + 2T\overline{K}M_{n-1}^2 m^2 \widetilde{M}).$$

Let $M_n = \max\{M_{n-1}, 2(\overline{G} + 2T\overline{K}M_{n-1}^2 m^2 \widetilde{M})\}$. Then we have for $p \le n$ that u_h^p is bounded by M_n. Here, M_n is a positive constant independent of h, that is, u_h^n is also bounded and unique exists for all $h \in (0, \bar{h})$. Thus, we obtain $u_h(t)$ is bounded and exists uniquely on $[0, T]$ whenever $h \in (0, \bar{h})$.

We summarize these results in the following theorem.

THEOREM 3.1. *Assume that the given functions in AVIE* (1.1) *satisfy* $g \in C(I)$, $K \in C(D)$. *Then there exists* $\bar{h} > 0$ *such that for all* $h \in (0, \bar{h})$, *the system* (3.4) *possesses a uniformly bounded collocation solution* $u_h \in S_{m-1}^{(-1)}(I_h)$.

4. Optimal order of convergence of the collocation solution. In this subsection we shall analyze the attainable order of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$.

For given abscissas $a \le \xi_1 < \cdots < \xi_m \le b$, let

$$e_m(y;t) := y(t) - \sum_{j=1}^{m} l_j(t) y(\xi_j), \ t \in [a, b]$$

denote the error between y and its Lagrange interpolation polynomial of degree at most $m - 1$ with respect to the given points $\{\xi_j\}$.

LEMMA 4.1 (see [4]). *Assume* $f \in C^d[a, b]$ *with* $1 \le d \le m$. *Then* $e_m(y;t)$ *possesses the integral representation*

$$e_m(y;t) = \int_a^b H_d(t, s) y^{(d)}(s) ds, t \in [a, b],$$

where the Peano kernel H_d *is given by*

$$H_d(t, s) := \frac{1}{(d-1)!} \left\{ (t-s)_+^{d-1} - \sum_{k=1}^{m} l_k(t)(\xi_k - s)_+^{d-1} \right\}.$$

Here, $(t-s)_+^p := 0$ *for* $t < s$ *and* $(t-s)_+^p := (t-s)^p$ *for* $t \ge s$.

LEMMA 4.2 (see [1]). *Assume* $y \in C^d[a, b]$ *with* $1 \le d \le m$ *and* $[a, b] = [t_n, t_{n+1}], \xi_j = t_n + c_j h_n (j = 1, \ldots, m)$ *the interpolation error* $e_m(y;t)$ *can be expressed in the form*

$$e_m(y; t_n + v h_n) = h_n^d \int_0^1 H_d(v, z) y^{(d)}(t_n + z h_n) dz, \ v \in [0, 1].$$

THEOREM 4.3. *Assume that the given functions* g *and* K *in the AVIE* (1.1) *are at least* d *times continuously differentiable on their respective domains with* $1 \le d \le m$. *Then there exists a* $\tilde{h} > 0$, *such that for any choice of the collocation parameters* $\{c_i\}$ *with* $0 < c_1 < \cdots < c_m \le 1$, *we have*

$$\|u - u_h\|_\infty \le C h^d, \tag{4.1}$$

where C *depends on the collocation parameters* $\{c_i\}$ *and on* $u^{(d)}$ *but not on* h. *If* $d \ge m + 1$, *the collocation solution* $u_h \in S_{m-1}^{(-1)}(I_h)$ *satisfies, for* $h \in (0, \bar{h})$,

$$\|u - u_h\|_\infty \le C h^m.$$

In general, the exponent m *can not be replaced by* $m + 1$.

Proof. Applying Lemma 4.2, we obtain

$$u(t_n+vh)=\sum_{j=1}^{m} l_j(v)Y_{n,j}+h^d R_{d,n}(v),\ \ v\in(0,1] \tag{4.2}$$

with $Y_{n,j}:=u(t_{n,j})$. The Peano remainder term is given by

$$R_{d,n}(v)=\int_0^1 H_d(v,z)u^{(d)}(t_n+zh)dz,\ \ v\in[0,1]. \tag{4.3}$$

It follows from (3.2) that the collocation error $e_h(t):=u(t)-u_h(t)$ possesses the local representation

$$e_h(t_n+vh)=\sum_{i=1}^{m} l_i(v)\mathcal{E}_{n,i}+h^d R_{d,n}(v),\ \ v\in(0,1]. \tag{4.4}$$

Here, $\mathcal{E}_{n,i}:=Y_{n,i}-U_{n,i}=e_h^n(t_{n,i})$ satisfies the equation

$$\begin{aligned}
\mathcal{E}_{n,i}&=\int_0^{t_{n,i}} K(t_{n,i},s)[u(t_{n,i}-s)u(s)-u_h(t_{n,i}-s)u_h(s)]ds\\
&=\int_{t_n}^{t_{n,i}} K(t_{n,i},s)[u^0(t_{n,i}-s)u^n(s)-u_h^0(t_{n,i}-s)u_h^n(s)]ds\\
&\quad+\sum_{k=0}^{n-1}\int_{t_k}^{t_{k,i}} K(t_{n,i},s)[u^{n-k}(t_{n,i}-s)u^k(s)-u_h^{n-k}(t_{n,i}-s)u_h^k(s)]ds\\
&\quad+\sum_{k=0}^{n-1}\int_{t_{k,i}}^{t_{k+1}} K(t_{n,i},s)[u^{n-k-1}(t_{n,i}-s)u^k(s)-u_h^{n-k-1}(t_{n,i}-s)u_h^k(s)]ds\\
&=\int_{t_n}^{t_{n,i}} K(t_{n,i},s)[u^0(t_{n,i}-s)e_h^n(s)+e_h^0(t_{n,i}-s)u_h^n(s)]ds\\
&\quad+\sum_{k=0}^{n-1}\int_{t_k}^{t_{k,i}} K(t_{n,i},s)[u^{n-k}(t_{n,i}-s)e_h^k(s)+e_h^{n-k}(t_{n,i}-s)u_h^k(s)]ds\\
&\quad+\sum_{k=0}^{n-1}\int_{t_{k,i}}^{t_{k+1}} K(t_{n,i},s)[u^{n-k-1}(t_{n,i}-s)e_h^k(s)+e_h^{n-k-1}(t_{n,i}-s)u_h^k(s)]ds\\
&=\int_{t_n}^{t_{n,i}} K(t_{n,i},s)u^0(t_{n,i}-s)e_h^n(s)ds+\int_0^{c_ih} K(t_{n,i},t_{n,i}-s)u_h^n(t_{n,i}-s)e_h^0(s)ds\\
&\quad+\sum_{k=0}^{n-1}\int_{t_k}^{t_{k,i}} K(t_{n,i},s)u^{n-k}(t_{n,i}-s)e_h^k(s)ds\\
&\quad+\sum_{k=0}^{n-1}\int_{t_{n-k}}^{t_{n-k,i}} K(t_{n,i},t_{n,i}-s)u_h^k(t_{n,i}-s)e_h^{n-k}(s)ds\\
&\quad+\sum_{k=0}^{n-1}\int_{t_{k,i}}^{t_{k+1}} K(t_{n,i},s)u^{n-k-1}(t_{n,i}-s)e_h^k(s)ds\\
&\quad+\sum_{k=0}^{n-1}\int_{t_{n-k-1,i}}^{t_{n-k}} K(t_{n,i},t_{n,i}-s)u_h^k(t_{n,i}-s)e_h^{n-k-1}(s)ds.
\end{aligned}$$

Thus, we obtain

$$
\begin{aligned}
\mathcal{E}_{n,i} &= \int_{t_n}^{t_{n,i}} [K(t_{n,i},s)u^0(t_{n,i}-s) + K(t_{n,i},t_{n,i}-s)u_h^0(t_{n,i}-s)]e_h^n(s)ds \\
&\quad + \sum_{l=0}^{n-1} \int_{t_l}^{t_{l,i}} [K(t_{n,i},t_{n,i}-s)u_h^{n-l}(t_{n,i}-s) + K(t_{n,i},s)u^{n-l}(t_{n,i}-s)]e_h^l(s)ds \\
&\quad + \sum_{l=0}^{n-1} \int_{t_{l,i}}^{t_{l+1}} [K(t_{n,i},t_{n,i}-s)u_h^{n-l-1}(t_{n,i}-s) \\
&\qquad\qquad + K(t_{n,i},s)u^{n-l-1}(t_{n,i}-s)]e_h^l(s)ds \\
&\quad + \int_0^{c_i h} K(t_{n,i},t_{n,i}-s)u_h^n(t_{n,i}-s)e_h^0(s)ds \\
&= h\int_0^{c_i} [K(t_{n,i},t_n+vh)u((c_i-v)h) \\
&\qquad + K(t_{n,i},(c_i-v)h)u_h((c_i-v)h)]e_h^n(t_n+vh)dv \\
&\quad + h\sum_{l=0}^{n-1} \int_0^1 [K(t_{n,i},t_l+vh)u(t_{n-l}+(c_i-v)h) \\
&\quad + K(t_{n,i},t_{n-l}+(c_i-v)h)u_h(t_{n-l}+(c_i-v)h)]e_h^l(t_l+vh)dv \\
&= h\int_0^{c_i} \hat{K}_n(t_{n,i},t_n+sh)e_h^n(t_n+vh)dv \\
&\quad + h\sum_{l=0}^{n-1} \int_0^1 \hat{K}_l(t_{n,i},t_l+sh)e_h^l(t_l+vh)dv, \qquad (4.5)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{K}_l(t_{n,i},t_l+sh) &= K(t_{n,i},t_l+sh)u(t_{n-l}+(c_i-s)h) \\
&\quad + K(t_{n,i},t_{n-l}+(c_i-s)h)u(t_{n-l}+(c_i-s)h) \;\; (l=0,1,\ldots,n).
\end{aligned}
$$

Case I: The convergence order on e_0 $(n=0)$.

On e_0 (4.5) reduces to

$$
\begin{aligned}
\mathcal{E}_{0,i} &= h\int_0^{c_i} [K(c_ih,(c_i-s)h)u_h^0((c_i-s)h) + K(c_ih,sh)u((c_i-s)h)]e_h^0(s)ds \\
&= h\int_0^{c_i} \hat{K}_0(c_ih,sh)e_h^0(s)ds \qquad (4.6) \\
&= h\sum_{j=1}^{m} \int_0^{c_i} \hat{K}_0(c_ih,sh)l_j(s)ds\mathcal{E}_{0,j} + h^{d+1}\int_0^{c_i} \hat{K}_0(c_ih,sh)R_{d,0}(s)ds,
\end{aligned}
$$

where

$$\hat{K}_0(c_ih,sh) = K(c_ih,sh)u((c_i-s)h) + K(c_ih,c_ih-sh)u_h(c_ih-sh).$$

If we introduce the vector $\rho_0 \in \mathbb{R}^m$ with components

$$\rho_{0,i} := \int_0^{c_i} R_{d,0}(s)\hat{K}_0(c_ih,sh)ds, \quad (i=1,\ldots,m),$$

and define the matrix $B^0 \in \mathbb{R}^{m\times m}$ with elements

$$B^0_{i,j} := \int_0^{c_i} \hat{K}_0(c_i h, sh) l_j(s) ds \quad (i, j = 1, \ldots, m)\ ,$$

(4.6) can be written as

$$[\mathcal{I}_m - hB^0]\mathcal{E}_0 = h^{d+1}\rho_0\,. \tag{4.7}$$

It is easy to verify that there exists a constant $D_0 > 0$ such that

$$\|(\mathcal{I}_m - hB_0)^{-1}\|_1 \le D_0 \quad (n = 0, 1, \ldots, N-1),$$

whenever the mesh diameter h is in $(0, \bar{h})$.

Let

$$\overline{M} := \max\{\|u\|_\infty, \|u_h\|_\infty\} \text{ and } \quad M_d := \|u^{(d)}\|_\infty, \quad k_d := \sup_{v\in[0,1]} \int_0^1 |H_d(v, z)|dz.$$

It follows from the assumptions on the kernel K that $\|\rho_0\|_1 \le 2m\overline{M}k_d M_d \overline{K}$. Then, (4.7) leads to the estimate

$$\|\mathcal{E}_0\|_1 \le 2D_0 m\overline{M}k_d M_d \overline{K} h^{d+1} =: BM_d h^{d+1}.$$

Recalling the local error representation (4.4), this yields

$$|e_h(vh)| \le \Lambda_m \|\mathcal{E}_0\|_1 + h^d k_d M_d \le (\Lambda_m Bh + k_d) M_d h^d,$$

uniformly for $v \in [0, 1]$ and $0 \le n \le N-1$, with $\Lambda_m := \max_{1\le j\le m} \|l_j\|_\infty$. This in turn leads to

$$\|e^0_h\|_\infty \le Ch^d.$$

Case II: The convergence order on e_n $(1 \le n \le N-1)$.

For $1 \le n \le N-1$ the equations for $\mathcal{E}_n := (\mathcal{E}_{n,1}, \ldots, \mathcal{E}_{n,m})^T \in \mathbb{R}^m$ are

$$\begin{aligned}\mathcal{E}_{n,i} &= h\sum_{j=1}^{m} \int_0^{c_i} \hat{K}_n(t_{n,i}, t_n + sh) l_j(s) ds \mathcal{E}_{n,j} + h\sum_{l=0}^{n-1}\sum_{j=1}^{m} \int_0^1 \hat{K}_l(t_{n,i}, t_l + sh) l_j(s) ds \mathcal{E}_{l,j} \\ &\quad + h^{d+1} \int_0^{c_i} \hat{K}_n(t_{n,i}, t_n + sh) R_{d,n}(s) ds + h^{d+1} \sum_{l=0}^{n-1} \int_0^1 \hat{K}_l(t_{n,i}, t_l + sh) R_{d,l}(s) ds\end{aligned} \tag{4.8}$$

$(i = 1, \ldots, m)$. In order to write (4.8) in a more concise form we introduce the vectors $\rho^{(l)}_n$ and ρ_n in $\mathbf{R}^m$ with components

$$\rho^{(l)}_n(i) := \int_0^1 \hat{K}_l(t_{n,i}, t_l + sh) R_{d,l}(s) ds \quad (l = 0, 1, \ldots, n-1),$$
$$\rho_n(i) := \int_0^{c_i} \hat{K}_n(t_{n,i}, t_n + sh) R_{d,n}(s) ds, \quad (i = 1, \ldots, m)$$

and define the matrices B^n and $B^{n,l}$ in $L(\mathbf{R}^m)$ with elements

$$B^{n,l}_{i,j} := \int_0^1 \hat{K}_l(t_{n,i}, t_l + sh) l_j(s) ds \quad (l = 0, 1, \ldots, n-1),$$
$$B^n_{i,j} := \int_0^{c_i} \hat{K}_n(t_{n,i}, t_n + sh) l_j(s) ds \quad (i, j = 1, \ldots, m)\ .$$

Equation (4.8) can now be written as

$$[\mathcal{I}_m - hB^n]\mathcal{E}_n = h\sum_{l=0}^{n-1} B^{n,l}\mathcal{E}_l + h^{d+1}\sum_{l=0}^{n-1}\rho_n^{(l)} + h^{d+1}\rho_n \quad (1 \le n \le N-1). \tag{4.9}$$

If $h \in (0, \bar{h})$, there exists a constant $D_0 > 0$ such that

$$\|(\mathcal{I}_m - hB^n)^{-1}\|_1 \le D_0 \quad (n = 0, 1, \dots, N-1).$$

Similarly, there exists a $D_1 > 0$ such that $\|B^{n,l}\|_1 \le D_1$ for $0 \le l < n \le N-1$. It follows from the assumptions on the regularity of the kernel K that

$$\|\rho_n^{(l)}\|_1 \le 2m\overline{M}k_d M_d\overline{K} \quad (l < n), \quad \|\rho_n\|_1 \le 2m\overline{M}k_d M_d\overline{K}.$$

Recalling (4.9) we are led to

$$\|\mathcal{E}_n\|_1 \le D_0 D_1 h\sum_{l=0}^{n-1}\|\mathcal{E}_l\|_1 + D_0\left[2m\overline{M}k_d M_d\overline{K}h^d T + 2h^{d+1}m\overline{M}k_d M_d\overline{K}\right]$$

and hence to

$$\|\mathcal{E}_n\|_1 \le \gamma_0\sum_{l=0}^{n-1} h\|\mathcal{E}_l\|_1 + \gamma_1 M_d h^d, \quad n = 0, 1, \dots, N-1, \tag{4.10}$$

where $\gamma_0 = D_0 D_1, \gamma_1 = 4mD_0\overline{K}k_d\overline{M}T$.

LEMMA 4.4 (see [1]). *Assume that the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \le \rho_0$ and*

$$\varepsilon_n \le \rho_0 + \sum_{j=0}^{n-1} q_j + \sum_{j=0}^{n-1} k_j\varepsilon_j, \; n \ge 1$$

with $\rho_0 \ge 0, q_j \ge 0$ and $k_j \ge 0$ for $j \ge 0$. Then

$$\varepsilon_n \le \left(\rho_0 + \sum_{j=0}^{n-1} q_j\right)\exp\left(\sum_{j=0}^{n-1} k_j\right), \; n \ge 1.$$

Applying Lemma 4.4, we obtain the estimate

$$\|\mathcal{E}_n\|_1 \le \gamma_1\exp(\gamma_0 T)M_d h^d =: BM_d h^d, \quad n = 0, 1, \dots, N-1,$$

and it then follows from the local error representation (4.4) that

$$|e_h(t_n + vh_n)| \le \Lambda_m \|\mathcal{E}_n\|_1 + h^d k_d M_d \le (\Lambda_m B + k_d) M_d h^d,$$

uniformly for $v \in (0, 1]$ and $0 \le n \le N - 1$. This leads to the desired estimate $\|e_h\|_\infty \le Ch^d$. ∎

5. Global superconvergence results on *I*. Once the collocation solution u_h is known, we can use it to compute the iterated collocation solution,

$$u_h^{it}(t) := g(t) + \int_0^t K(t,s) u_h(t-s) u_h(s) ds, \quad t \in I. \tag{5.1}$$

It is well known (cf. [1, Chapter 2]) that for standar VIEs u_h^{it} often exhibits a higher order of global or local convergence. We will show that this is also true for the generalized AVIE (1.1).

THEOREM 5.1. *Assume that the given functions in the AVIE* (1.1) *satisfy* $g \in C^{m+1}(I)$ *and* $K \in C^{m+1}(D)$. *If the m collocation parameters* $\{c_i\}$ *are subject to the orthogonality condition*

$$J_0 = \int_0^1 \prod_{i=1}^m (s - c_i) ds = 0, \tag{5.2}$$

then the corresponding iterated collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ *is globally superconvergent on* I*:*

$$\|u - u_h^{it}\|_\infty \le Ch^{m+1}. \tag{5.3}$$

Proof. The residual

$$\delta_h(t) := -u_h(t) + g(t) + \int_0^t K(t,s) u_h(t-s) u_h(s) ds, \quad t \in I,$$

associated with the collocation solution u_h to the AVIE (1.1) has the property that $\delta_h(t) = 0$ for all $t \in X_h$. Under the regularity assumptions for g and K, we know that δ_h is continuous and piecewise C^{m+1} smooth on I, and Theorem 4.3 implies that it has uniformly bounded derivatives on each subinterval e_n. We also have

$$\delta_h(t) = e_h(t) + \int_0^t K(t,s)[u_h(t-s)u_h(s) - u(t-s)u(s)] ds, \quad t \in I.$$

Thus, it follows from (4.1) that

$$\|\delta_h\|_\infty \le \|e_h\|_\infty + \widehat{K}\|e_h\|_\infty \le C(1 + \widehat{K})h^m := D_2 h^m,$$

where

$$\widehat{K} := \max_{t \in I} \int_0^t |K(t,s)u_h(t-s) + K(t, t-s)u(t-s)| ds.$$

Recalling Theorem 4.3 we see that

$$\begin{aligned}
e_h(t) &= \delta_h(t) + \int_0^t K(t,s)[u(t-s)u(s) - u_h(t-s)u_h(s)]ds \\
&= \delta_h(t) + \int_0^t K(t,s)[e_h(t-s)e_h(s) + u(t-s)u_h(s) \\
&\quad - 2u_h(t-s)u_h(s) + u_h(t-s)u(s)]ds \\
&= \delta_h(t) + \int_0^t K(t,s)[e_h(t-s)e_h(s) + e_h(t-s)u_h(s) + u_h(t-s)e_h(s)]ds \\
&= \delta_h(t) + \int_0^t K(t,s)[e_h(t-s)e_h(s) + u_h(t-s)e_h(s)]ds \\
&\quad + \int_0^t K(t,s)e_h(t-s)u_h(s)ds \\
&= \delta_h(t) + \int_0^t K(t,s)[e_h(t-s)e_h(s) + u_h(t-s)e_h(s)]ds \\
&\quad + \int_0^t K(t,t-s)u_h(t-s)e_h(s)ds \\
&= \delta_h(t) + \int_0^t K(t,s)e_h(t-s)e_h(s)ds \\
&\quad + \int_0^t [K(t,s) + K(t,t-s)](u(t-s) - e_h(t-s))e_h(s)ds \\
&= \delta_h(t) + O(h^{2m}) + \int_0^t [K(t,s) + K(t,t-s)]u(t-s)e_h(s)ds \\
&= \delta_h(t) + O(h^{2m}) + \int_0^t G(t,s)e_h(s)ds
\end{aligned}$$

with $G(t,s) := [K(t,s) + K(t,t-s)]u(t-s)$.

By [1, section 2.2.4], we can express $e_h(t)$ as

$$e_h(t) = \delta_h(t) + \int_0^t G(t,s)\delta_h(s)ds + O(h^{2m}).$$

This implies that

$$e_h^{it}(t) := u(t) - u_h^{it}(t) = e_h(t) - \delta_h(t) = O(h^{2m}) + \int_0^t G(t,s)\delta_h(s)ds,$$

so at $t := t_n + vh, v \in (0,1)$, we have

$$\begin{aligned}
e_h^{it}(t_n + vh) &= O(h^{2m}) + \int_0^{t_n+vh} G(t_n + vh, s)\delta_h(s)ds \\
&= O(h^{2m}) + h\int_0^v G(t_n + vh, t_n + sh)\delta_h(t_n + sh)ds \\
&\quad + h\sum_{l=0}^{n-1}\int_0^1 G(t_n + vh, t_l + sh)\delta_h(t_l + sh)ds.
\end{aligned}$$

If we replace the integrals in the last two lines of the above equation by m-point interpolatory quadrature rules (using the collocation parameters $\{c_i\}$ as abscissas), the corresponding quadrature errors will be $\mathcal{O}(h^{m+1})$ if the $\{c_i\}$ satisfy the orthogonality condition (5.2). This leads to the completion of the proof of theorem of global superconvergence. ∎

6. Local superconvergence results on I_h.

THEOREM 6.1. *Suppose that $g \in C^{m+\kappa}(I)$ and $K \in C^{m+\kappa}(D)$ for some integer κ with $1 \le \kappa \le m$. If the m collocation parameters $\{c_i\}$ are subject to the orthogonality condition*

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s-c_i)ds = 0, \quad \nu = 0, \dots, \kappa-1$$

with $J_\kappa \neq 0$, then the corresponding iterated collocation solution u_h^{it} is superconvergent at the mesh points t_n:

$$\max_{t\in I_h}\{|u(t) - u_h^{it}(t)|\} \le Ch^{m+\kappa}. \tag{6.1}$$

Moreover, for $c_m = 1$ we have

$$\max_{t\in I_h}\{|u(t) - u_h(t)|\} \le Ch^{m+\kappa}, \tag{6.2}$$

where $\kappa \le m-1$.

Proof. At $t = t_n$, we have

$$\begin{aligned} e_h(t_n) &= \delta_h(t_n) + \int_0^{t_n} G(t_n, s)\delta_h(s)ds + O(h^{2m}) \\ &= \delta_h(t_n) + h\sum_{l=0}^{n-1}\int_0^1 G(t_n, t_l + sh)\delta_h(t_l + sh)ds + O(h^{2m}) \end{aligned}$$

and

$$\begin{aligned} e_h^{it}(t_n) &= \int_0^{t_n} G(t_n, s)\delta_h(s)ds + O(h^{2m}) \\ &= h\sum_{l=0}^{n-1}\int_0^1 G(t_n, t_l + sh)\delta_h(t_l + sh)ds + O(h^{2m}), \end{aligned}$$

which completes the proof. ∎

Remark 6.2. In [5], Guan, Zou, and Zhang studied the general nonstandard Volterra integral equation

$$u(t) = g(t) + \int_0^t K(t, s, u(t), u(s))ds, \ t \in I := [0, T].$$

They showed that under appropriate regularity assumptions on the given functions the attainable order of local superconvergence of the numerical solution at the mesh points is $O(h^{m+\kappa})$, provided the collocation points are such that $c_m = 1$. In this paper we have established a similar local superconvergence result for the generalized auto-convolution equation (1.1). More importantly, we have shown that if $c_m \neq 1$, the iterated solution u_h^{it} also exhibits local superconvergence of order of $O(h^{m+\kappa})$, which is not true for the VIE in [5].

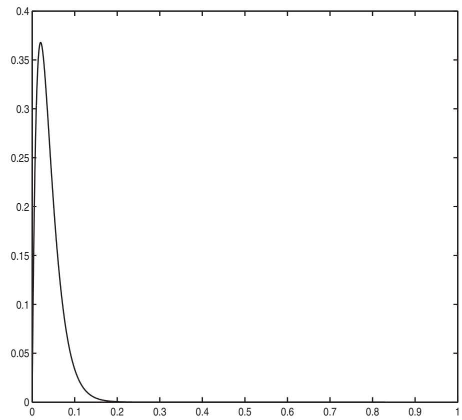

FIG. 1. *Example* 7.1: *The exact solution* $u(t) = \beta te^{-\gamma t}$ *with* $\beta = \gamma = 50$.

7. Numerical experiments. In this section, we apply the collocation method described in section 3 to (1.1) with different choices of the functions $g(t)$ and K. The numerical results illustrate the accuracy and the attainable order of convergence.

Example 7.1. Consider the equation

$$u(t) = \beta te^{-\gamma t} - \frac{\beta^2}{6} t^3 e^{-\gamma t} - \int_0^t K(t,s)u(t-s)u(s)ds, \ \ t \in [0,1]$$

with $K(t,s) = 1$. It has the analytic solution $u(t) = \beta te^{-\gamma t}$.

In our computations, we chose $\beta = \gamma = 50$. The graphs of the corresponding exact solutions is shown in Figure 1. The convergence orders of the piecewise quadratic collocation solution u_h $(m = 3)$ corresponding to equally spaced collocation parameters are shown in Figure 2: they confirm that the attainable order is indeed 3.

Figure 2 shows the numerical results for the piecewise quadratic approximation where the collocation parameters are either equally spaced or given by the Gauus and Radau II points. They show that in all cases the attainable order of global superconvergence is 4. The attainable order of local superconvergence (for $\beta = \gamma = 1, 10, 50, 100$) is 6 for the Gauss points, and 5 for the Radau II points.

Example 7.2. Consider the equation

$$u(t) = \beta te^{-\gamma t} - \beta^2 e^{-\gamma t}[2\sin t - t - t\cos t] - \int_0^t K(t,s)u(t-s)u(s)ds, \ \ t \in [0,1],$$

with $K(t,s) = \cos(t-s)$. It has the analytic solution $u(t) = \beta te^{-\gamma t}$.

The graph of the corresponding exact solution is shown in Figure 3, while the convergence behavior of the piecewise quadratic collocation solution with $\gamma = 1, \beta = 1$ based on the Gauss, Radau II and the uniform collocation (with $c_1 = 0.4, c_2 = 0.6, c_3 = 0.8$) points is shown in Tables 1–3.

Example 7.3. *The equation*

$$u(t) = \frac{1}{2}\sin t + \frac{1}{2}\int_0^t u(t-s)u(s)ds, \quad t \in [0,10],$$

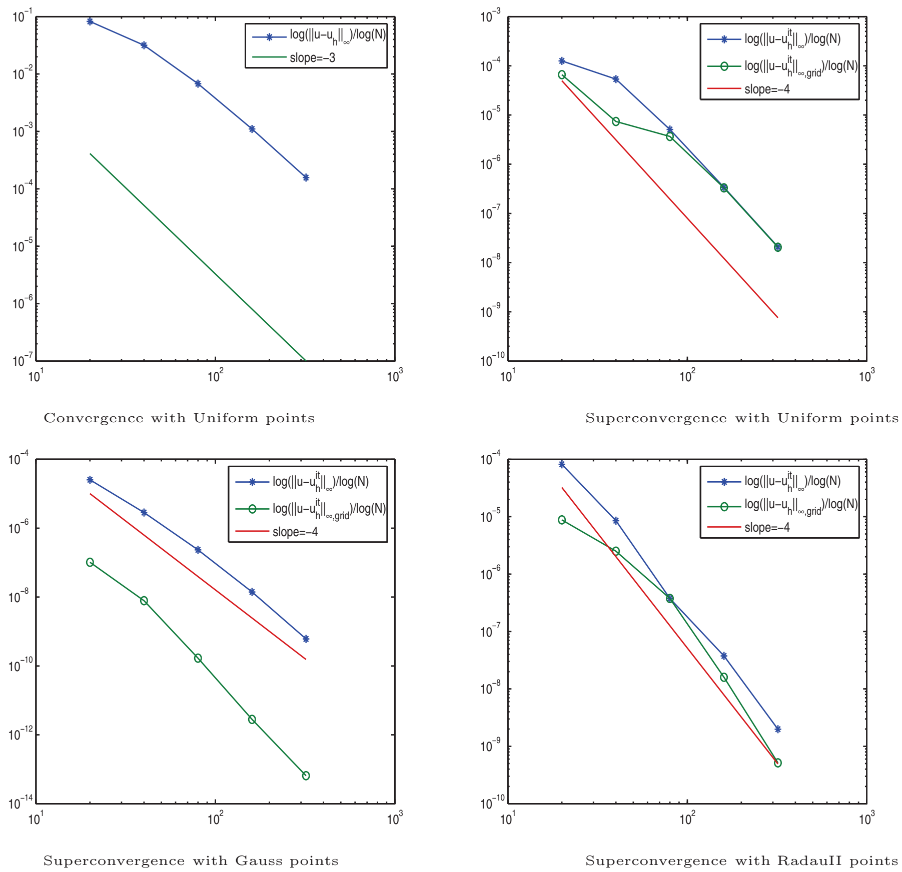

FIG. 2. *Example* 7.1: *The convergence and superconvergence for* $S_2^{-1}(I_h)$ *with different collocation points.*

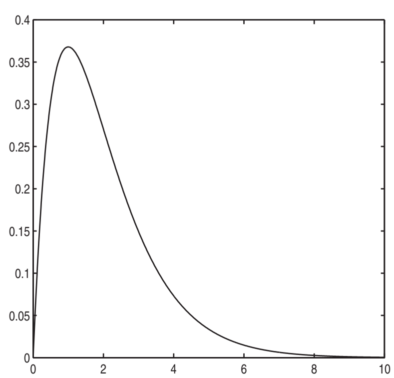

FIG. 3. *Example* 7.2: *The figures for exact solution* $u(t) = \beta t e^{-\gamma t}$ *with* $\gamma = 10$.

(cf. Butzer [3]) possesses the oscillatory solution $u(t) = J_1(t)$, where $J_1(t)$ is the Bessel function of order 1.

The convergence results for the piecewise quadratic approximation are shown in Figure 4. We see that the numerical convergence order is 3, confirming the result in Theorem 4.3.

Table 1
Example 7.2: Errors and orders of convergence for the Gauss points.

	$\|u - u_h\|_{grid}$		$\|u(t) - u_h^{it}(t)\|_{grid}$	
N	error	order	error	order
2^1	1.0193e-03	–	4.7902e-07	–
2^2	2.2541e-04	2.1770	8.7479e-09	5.7750
2^3	3.7172e-05	2.6003	1.3810e-10	5.9852
2^4	5.3280e-06	2.8025	2.1601e-12	5.9984
2^5	7.1291e-07	2.9018	3.4361e-14	5.9742

Table 2
Example 7.2: Errors and orders of convergence for the Radau II points.

	$\|u - u_h\|_{grid}$		$\|u(t) - u_h^{it}(t)\|_{grid}$	
N	error	order	error	order
2^1	6.2926e-06	–	7.1804e-06	–
2^2	2.2904e-07	4.7800	2.2904e-07	4.9704
2^3	7.3345e-09	4.9648	7.3345e-09	4.9648
2^4	2.3026e-10	4.9933	2.3026e-10	4.9933
2^5	7.2187e-12	4.9954	7.2188e-12	4.9954

Table 3
Example 7.2: Errors and orders of convergence for the uniform collocation points.

	$\|u - u_h\|_{grid}$		$\|u(t) - u_h^{it}(t)\|_{grid}$	
N	error	order	error	order
2^1	3.6202e-04	–	3.3151e-04	–
2^2	1.7664e-04	1.0352	5.0822e-05	2.7055
2^3	3.3746e-05	2.3881	6.7172e-06	2.9195
2^4	5.0279e-06	2.7467	8.5620e-07	2.9719
2^5	6.8038e-07	2.8855	1.0787e-07	2.9887

Figure 5 shows the numerical results for the piecewise quadratic approximation with the collocation parameters chosen as the Gauss points and the Radau II points. They confirm respectively order 5 of global superconvergence (Theorem 5.1) and the orders 5 and 6 of local superconvergence for the Radau II points and the Gauss points (Theorem 6.1).

8. Extensions and open problems. The convergence analysis of piecewise polynomial solutions for the more general AVIE (1.4),

$$u(t) = g(t) + \int_0^t G(u(t-s))G(u(s))ds, \quad t \in I, \tag{8.1}$$

remains to be carried out. An interesting example corresponds to $G(u) = u^\beta$ with $\beta > 0$.

The same holds for the adjoint version of the basic AVIE (2.1),

$$u(t) = 1 + \lambda \int_t^T u^\beta(t-s)u^\beta(s)\, ds, \;\; t \in [0,T]. \tag{8.2}$$

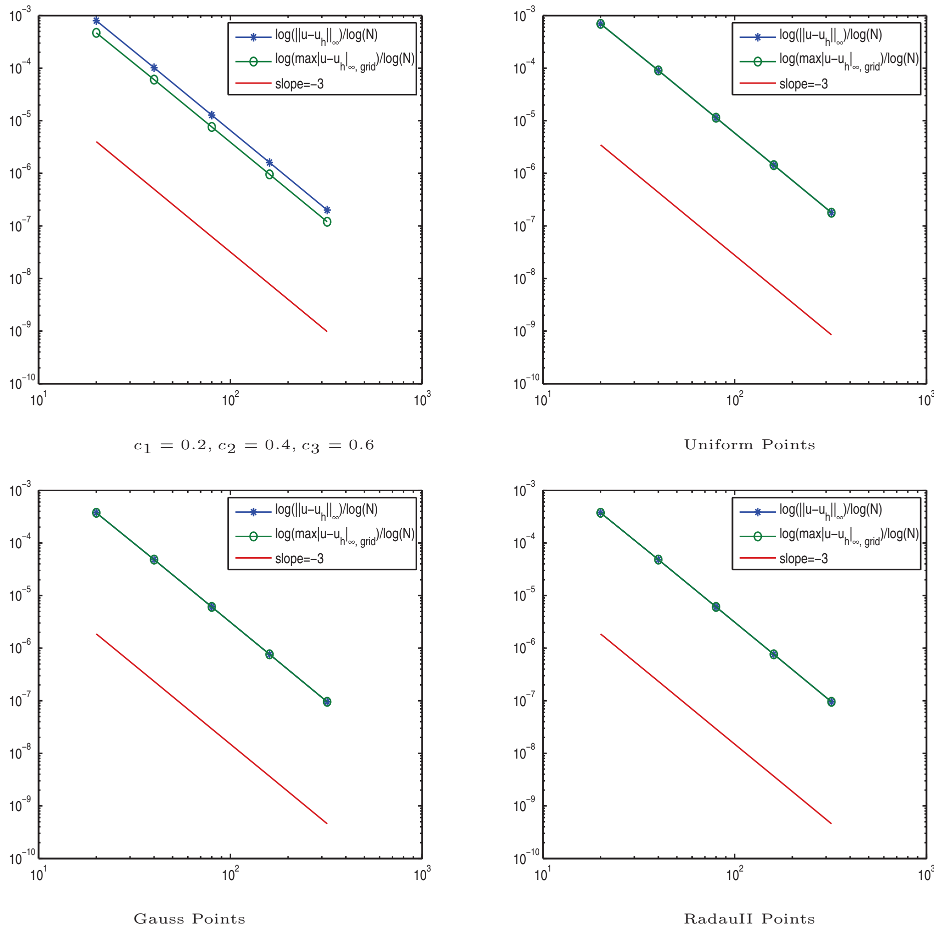

FIG. 4. *Example* 7.3: *The order of convergence for* $S_2^{(-1)}(I_h)$ *with different collocation points.*

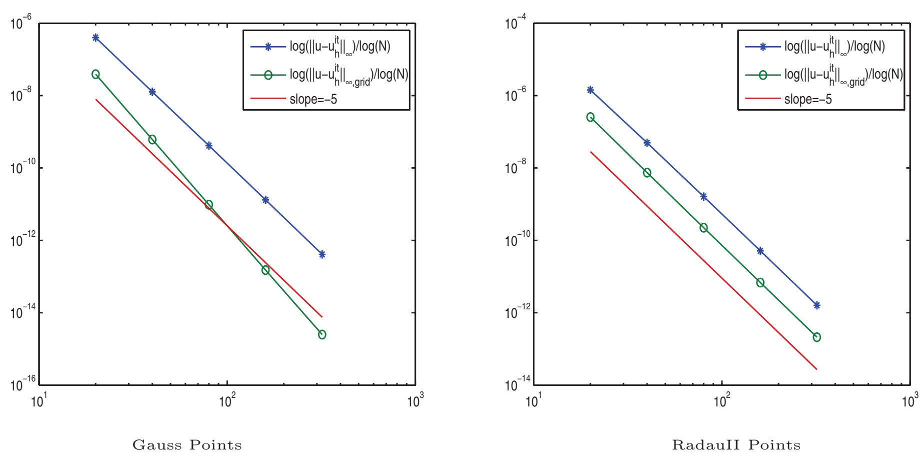

FIG. 5. *Example* 7.3: *Superconvergence for* $S_2^{(-1)}(I_h)$ *with different collocation points.*

Its theory for the case $\beta = 1$ was studied in Pimbley [7] and Ramalho [8], while Nussbaum and Baxter [6] deals with the general case $\beta > 0$. However, the numerical solution (e.g., by collocation methods) for this equation has to the best of our knowledge not yet been studied.

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REFERENCES

[1] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge University Press, Cambridge, 2004.

[2] A.L. Bukhgeim, *Inverse problems of memory reconstruction*, J. Inverse Ill-Posed Probl., 1 (1993), pp. 193–205.

[3] P.L. Butzer, *The application of Jan Mikusińsk's operator calculus to integral equations of convolution type*, Arch. Ration. Mech. Anal., 2 (1958), pp. 114–128.

[4] P.J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, 2nd ed., Academic Press, New York, 1984.

[5] Q.G. Guan, Y. K. Zou, and R. Zhang, *Analysis of collocation solutions for nonstandard Volterra integral equations*, IMA J. Numer. Anal., 32 (2012), pp. 1755–1785.

[6] R.D. Nussbaum and N. Baxter, *A nonlinear integral equation*, Nonlinear Anal., 5 (1981), pp. 1285–1307.

[7] G.H. Pimbley, Jr., *Positive solutions of a quadratic integral equation*, Arch. Ration. Mech. Anal., 24 (1967), pp. 107–127.

[8] R. Ramalho, *Existence and uniqueness theorems for a nonlinear integral equation*, Math. Ann., 221 (1976), pp. 35–44.

[9] L. von Wolfersdorf, *Einige Klassen quadratischer Integralgleichungen [Some classes of quadratic integral equations]*, Sitz.ber. Sächs. Akad. Wiss. Leipz. Math.-Nat.wiss. Kl., 2000.

[10] L. von Wolfersdorf, *Autoconvolution equations and special functions*, Integral Transforms Spec. Funct., 21 (2010), pp. 295–306.

[11] L. von Wolfersdorf and J. Janno, *On a class of nonlinear convolution equations*, Z. Anal. Anwend., 14 (1995), pp. 497–508.