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INTEGRAL-ALGEBRAIC EQUATIONS: THEORY OF COLLOCATION METHODS II*

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Abstract. In a previous paper [*SIAM J. Numer. Anal.,* 51 (2013), pp. 2238–2259] we analyzed the optimal orders of convergence of piecewise polynomial collocation solutions for systems of integral-algebraic equations (IAEs) with tractability index $\mu = 1$. The present paper describes the decoupling of systems of IAEs of tractability index $\mu = 2$ and $(\nu + 1)$-smoothing $(\nu \geq 1)$. It is then shown that the application of the collocation method to the given system of IAEs is equivalent to the application to the decoupled system. While this in principle forms the basis for an elegant analysis of the optimal order of convergence of the method, we show by an example that collocation is not always feasible for general index-2 IAEs. Following the convergence analysis for semiexplicit index-2 IAEs we present two numerical examples: one to verify the predicted orders of convergence and one to show why the collocation method may break down for general IAEs with $\mu = 2$.

Key words. integral-algebraic equations of index 2, Volterra integral equations of the first kind, ν-smoothing, tractability index, collocation solutions, optimal order of convergence

AMS subject classifications. 65R20, 65L80, 45D05

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1. Introduction. The aim of this paper, the sequel to [6], is to analyze the optimal orders of (global and local) convergence of piecewise polynomial collocation solutions for a general system of linear integral-algebraic equations (IAEs) of the form

$$B(t)x(t) + (\mathcal{V}x)(t) = f(t), \quad t \in I := [0, T], \tag{1.1}$$

corresponding to the Volterra integral operator

$$(\mathcal{V}x)(t) := \int_0^t K(t, s)x(s)ds, \quad t \in I := [0, T], \tag{1.2}$$

and having index of tractability $\mu = 2$. Here, the matrices $B(\cdot),\ K(\cdot, \cdot) \in \mathbb{R}^{d\times d}$ $(d \geq 2)$ and the function $f(\cdot) \in \mathbb{R}^d$ are assumed to be continuous on their domains, and the matrix $B = B(t)$ is singular on I and has constant $\mathrm{rank(B(t)) = r_0 > 0\ (t \in I)}$. Moreover, $B(t)$ and $f(t)$ are such that the compatibility condition $B(0)x(0) = f(0)$ is satisfied.

In [6] we introduced the tractability index μ of a system of linear IAEs. It, together with the notion of ν-smoothing of a Volterra integral operator (a measure for

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the degree of its ill-posedness), provided the tools for the decoupling of the given IAE system into the inherent systems of second-kind and first-kind Volterra integral equations (VIEs). This decoupling formed the basis for establishing the optimal convergence properties of piecewise polynomial collocation solutions for (1.1) when $\mu = 1$.

In this sequel we extend the analysis to IAE systems with $\mu = 2$. It will be seen (cf. sections 3 and 4.2) that this extension is rather nontrivial, revealing also that the solution of such IAEs by piecewise polynomial collocation methods may no longer be feasible for arbitrary index-2 IAEs.

Collocation methods in piecewise polynomial spaces for IAEs (1.1) with special (semiexplicit or Hessenberg) structure have been studied in the recent papers [4, 3, 11, 10, 9]. In these papers the convergence analysis was based on index reduction techniques using the differential index or the rank-degree index. In particular, in [4] the Jacobi spectral method is applied to semiexplicit IAEs of index-2, but the regularity result (cf. Theorem 1) for the solution is not correct. In [3], the collocation method for the same IAEs as in [4] is analyzed, but we find there is a problem with the proof of the main convergence theorem (cf. Theorem 2), e.g., in (3.16), there is no h in D_2 actually, so the Gronwall inequality cannot be applied. The works [11] and [10] study the continuous and discontinuous collocation methods, respectively, for higher-index IAEs of Hessenberg type, which include semiexplicit IAEs of index-2, but again the regularity conditions are not correct. In [9], the collocation method for semiexplicit IAEs of index-3 is investigated.

However, as shown in [6] a projection-based approach using the tractability index appears to be a natural way of analyzing general systems of IAEs (in which the inherent first-kind VIEs do not occur explicitly). The resulting decoupling of the system of IAEs into the inherent second-kind and first-kind VIEs yields the basis both for obtaining regularity results for the exact solution and the analysis of the optimal order of convergence of collocation solutions.

The paper is organized as follows. In section 2 we recall, for the convenience of the reader, the definitions of the ν-smoothing property of the Volterra integral operator $\mathcal{V}$ and the tractability index μ of an IAE system (1.1). Section 3 describes the decoupling of an index-2 IAE system, for both 1-smoothing $\mathcal{V}$ and $(\nu+1)$-smoothing $\mathcal{V}$ with $\nu \geq 1$ (we will explain why we distinguish between these two cases in Remark 2.3). Moreover, in Theorem 3.3, we will see that the decouplings are different for these two cases. The collocation method for such IAE systems is introduced in section 4, and the approach in section 3 is then used to obtain the decoupled system of collocation equations. We also comment on some limitations in the applicability of the collocation method to certain IAEs with $\mu \geq 2$. This is followed by an analysis of the regularity of the solution, the existence and uniqueness of the collocation solution, and the optimal order of convergence for certain semiexplicit IAEs of index $\mu = 2$. In section 5 we display two numerical examples: they respectively illustrate the validity of the predicted orders of convergence and the accuracy of piecewise polynomial collocation solutions and why not all of the collocation methods are feasible for index-2 IAEs. The paper concludes with some remarks on open problems and future work.

2. A brief review of the tractability index. Before recalling the definition of the tractability index for systems of IAEs (1.1), we briefly review the definition of the notion of ν-smoothing of the Volterra integral operator $\mathcal{V}: C(I,\mathbb{R}^d) \to C(I,\mathbb{R}^d)$ in (1.1),

$$(\mathcal{V}x)(t) := \int_0^t K(t,s)x(s)ds, \ \ t \in I, \tag{2.1}$$

with continuous matrix kernel $K(\cdot,\cdot) \in \mathbb{R}^{d\times d}$.

Definition 2.1 (see [6]). *The Volterra integral operator* (2.1) *corresponding to the matrix kernel*

$$K(t,s) = \begin{pmatrix} K_{pq}(t,s) \\ (p,q=1,\dots,d) \end{pmatrix},$$

with $d \geq 2$, is said to be ν-smoothing *if there exist integers $\nu_{pq} \geq 1$ with*

$$\nu := \max_{1\leq p,q\leq d}\{\nu_{pq}\}$$

such that

(a) $\left.\frac{\partial^j K_{pq}(t,s)}{\partial t^j}\right|_{s=t} = 0, \quad t \in I, \quad j = 0,1,\dots,\nu_{pq}-2;$

(b) $\left.\frac{\partial^{\nu_{pq}-1} K_{pq}(t,s)}{\partial t^{\nu_{pq}-1}}\right|_{s=t} \neq 0, \quad t \in I;$ *and*

(c) $\frac{\partial^{\nu_{pq}} K_{pq}(t,s)}{\partial t^{\nu_{pq}}} \in C(D).$

We set $v_{pq} = 0$ when $K_{pq}(t,s) \equiv 0$. A first-kind VIE $\mathcal{V}x = f$ is called a ν-smoothing problem if $\mathcal{V}$ is a ν-smoothing operator and $f \in C^{\nu}(I)$.

We now recall the notion of the index-μ tractability of a system of linear IAEs (1.1) (see Definition 2.2 below) whose underlying Volterra integral operator $\mathcal{V}$ is $(\nu+1)$-smoothing with $\nu \geq 0$. As mentioned in [6], the use of the tractability index for systems of IAEs was motivated by the analogous index concept for systems of differential-algebraic equations (DAEs). As shown in [7, 8] and, especially, in the book [5], this projector-based approach allows us to indentify the structural properties of an IAE system (1.1). While the basic ideas underlying it are rather straightforward, the mathematical description of the projectors and the corresponding matrix chains is notationwise quite complex (see [5, pp. 106–117] for index-2 DAEs), and thus especially so for IAEs with $\mu = 2$, owing to the presence of the memory term $(\mathcal{V}x)(t)$ in (1.1).

For $i \in \mathbb{N}_0$ let $K^i, K_i, B_i \in \mathbb{R}^{d\times d}$, and denote by $(K^i)_{pq}$ and $(K_i)_{pq}$ the (p,q)-elements of the matrices K^i, K_i, respectively. We introduce the following chain of matrix functions:

$$K^0(t,s) := K(t,s),\ K_0 := K := K(t,t),\ B_0 := B,\ B_1 = B_1(t) := B_0 + K_0Q_0.$$

If $(K_i)_{pq}(t,t) \neq 0$ $(i \geq 0)$, we set $(K^{i+1})_{pq}(t,s) := 0$; otherwise

$$(K^{i+1})_{pq}(t,s) := \frac{\partial^{i+1}((K^i)_{pq}(t,s))}{\partial t^{i+1}}.$$

Moreover, let $K_{i+1} = K_{i+1}(t,t) := (K^{i+1}_{pq}(t,s)|_{s=t}\ (p,q=1,2,\dots,d))$ and

$$B_{i+2} = B_{i+2}(t) := B_{i+1} + \sum_{l=0}^{i+1} K_l \left(\prod_{j=0}^{i-l} P_j\right) Q_{i-l+1},\ 0 \leq i \leq \nu-1;$$

$$B_{i+2} = B_{i+2}(t) := B_{i+1} + \sum_{l=0}^{\nu} K_l \left(\prod_{j=0}^{i-l} P_j \right) Q_{i-l+1},\ i \geq \nu,$$

with $\prod_{j=0}^{-1} P_j = I_d$. Here, $Q_0 = Q_0(t)$ denotes a projector onto $\ker B_0$, while for $j \geq 1$, $Q_j = Q_j(t)$ is a projector onto $\ker B_j$ with $Q_j Q_k = 0$ for $k < j$, and $P_j = P_j(t) := I_d - Q_j$ (where I_d denotes the identity matrix in $\mathbb{R}^{d\times d}$).

Definition 2.2 (see [6]). *Assume that the Volterra integral operator describing the IAE system* (1.1) *is* $(\nu+1)$*-smoothing with* $\nu \geq 0$. *Then* (1.1) *is said to be index-*μ *tractable if all matrices* B_j $(j = 0, \ldots, \mu-1)$ *defined above are singular on* I *and have smooth null space, but* B_μ *is nonsingular at all points in* I.

Remark 2.3. From the definition of B_{i+2}, we can see that if $\mathcal{V}$ is 1-smoothing (i.e., $\nu = 0$), then the definition of B_{i+2} is taken as the second one, and only $K(t,s)$ is involved. However, if $\mathcal{V}$ is $\nu+1$-smoothing with $\nu \geq 1$, then not only $K(t,s)$ but also the derivatives of $K(t,s)$ are involved in B_{i+2}.

In the following, we give some simple examples of IAEs to illustrate that the tractability index μ may be greater than ν and also may be less than ν.

Example 2.4. Take $d = 2$ and $B(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has rank 1 for all $t \in I$.

(a) $K(t,s) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is ν-smoothing with $\nu = 1$. It can be easily checked that the tractability index is $\mu = 1 = \nu$.

(b) $K(t,s) = \begin{pmatrix} 0 & t-s \\ 0 & 1 \end{pmatrix}$ is ν-smoothing with $\nu = 2$. It is easily seen that the tractability index is $\mu = 1 < \nu$.

(c) $K(t,s) = \begin{pmatrix} 0 & 1 \\ t-s & 0 \end{pmatrix}$ is ν-smoothing with $\nu = 2$. In this case the tractability index is $\mu = 3 > \nu$.

Example 2.5. Let $d = 3$. The matrix $B(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has rank 2 for all $t \in I$.

(a) $K(t,s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is ν-smoothing with $\nu = 1$. It is readily seen that the tractability index is $\mu = 1 = \nu$.

(b) $K(t,s) = \begin{pmatrix} t-s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is ν-smoothing with $\nu = 2$. It can be easily checked that here the tractability index is $\mu = 1 < \nu$.

(c) $K(t,s) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ t-s & 0 & 0 \end{pmatrix}$ is ν-smoothing with $\nu = 2$. In this case it is easy to verify that the tractability index is $\mu = 3 > \nu$.

3. Decoupling of index-2 IAEs. Define

$$V_j = V_j(t) := \int_0^t K(t,s) P_0(s) P_1(s) \ldots P_j(s) x(s) ds \ (j \geq 0),$$

$$W_0 = W_0(t) := \int_0^t K(t,s) Q_0(s) x(s) ds,$$

$$W_j = W_j(t) := \int_0^t K(t,s) P_0(s) P_1(s) \ldots P_{j-1}(s) Q_j(s) x(s) ds \ (j \geq 1),$$

and set $\Pi_i := P_0 P_1 \cdots P_i$ $(i \geq 0)$.

Theorem 3.1 (Theorem 2.6 of [6]). *Assume that the IAE system* (1.1) *is index-*μ*-tractable and* 1*-smoothing. Then* (1.1) *can be decoupled into a regular part,*

$$u(t) + \Pi_{\mu-1} B_\mu^{-1} V_{\mu-1} + \Pi_{\mu-1} B_\mu^{-1} \sum_{j=0}^{\mu-1} W_j = \Pi_{\mu-1} B_\mu^{-1} f(t),$$

and a subsystem

$$\begin{bmatrix} 0 & -Q_0Q_1 & -Q_0P_1Q_2 & \dots & -Q_0P_1\dots P_{\mu-3}Q_{\mu-2} & -Q_0P_1\dots P_{\mu-2}Q_{\mu-1} \\ 0 & 0 & -\Pi_0Q_1Q_2 & \dots & -\Pi_0Q_1P_2\dots P_{\mu-3}Q_{\mu-2} & -\Pi_0Q_1P_2\dots P_{\mu-2}Q_{\mu-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\Pi_{\mu-4}Q_{\mu-3}Q_{\mu-2} & -\Pi_{\mu-4}Q_{\mu-3}P_{\mu-2}Q_{\mu-1} \\ 0 & 0 & 0 & \dots & 0 & -\Pi_{\mu-3}Q_{\mu-2}Q_{\mu-1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} v_0(t) \\ v_1(t) \\ \dots \\ v_{\mu-3}(t) \\ v_{\mu-2}(t) \\ v_{\mu-1}(t) \end{bmatrix}$$

$$+ \int_0^t \begin{bmatrix} Q_0P_1\dots P_{\mu-1}B_\mu^{-1}K(t,s) & \dots & Q_0P_1\dots P_{\mu-1}B_\mu^{-1}K(t,s) \\ \Pi_0Q_1P_2\dots P_{\mu-1}B_\mu^{-1}K(t,s) & \dots & \Pi_0Q_1P_2\dots P_{\mu-1}B_\mu^{-1}K(t,s) \\ \dots & \dots & \dots \\ \Pi_{\mu-4}Q_{\mu-3}P_{\mu-2}P_{\mu-1}B_\mu^{-1}K(t,s) & \dots & \Pi_{\mu-4}Q_{\mu-3}P_{\mu-2}P_{\mu-1}B_\mu^{-1}K(t,s) \\ \Pi_{\mu-3}Q_{\mu-2}P_{\mu-1}B_\mu^{-1}K(t,s) & \dots & \Pi_{\mu-3}Q_{\mu-2}P_{\mu-1}B_\mu^{-1}K(t,s) \\ Q_{\mu-1}B_\mu^{-1}K(t,s) & \dots & Q_{\mu-1}B_\mu^{-1}K(t,s) \end{bmatrix}$$

$$\cdot \begin{bmatrix} v_0(s) \\ v_1(s) \\ \dots \\ v_{\mu-3}(s) \\ v_{\mu-2}(s) \\ v_{\mu-1}(s) \end{bmatrix} ds$$

$$+ \begin{bmatrix} Q_0P_1\dots P_{\mu-1}B_\mu^{-1}\int_0^t K(t,s)u(s)ds \\ \Pi_0Q_1P_2\dots P_{\mu-1}B_\mu^{-1}\int_0^t K(t,s)u(s)ds \\ \dots \\ \Pi_{\mu-4}Q_{\mu-3}P_{\mu-2}P_{\mu-1}B_\mu^{-1}\int_0^t K(t,s)u(s)ds \\ \Pi_{\mu-3}Q_{\mu-2}P_{\mu-1}B_\mu^{-1}\int_0^t K(t,s)u(s)ds \\ Q_{\mu-1}B_\mu^{-1}\int_0^t K(t,s)u(s)ds \end{bmatrix} = \begin{bmatrix} Q_0P_1\dots P_{\mu-1}B_\mu^{-1}f(t) \\ \Pi_0Q_1P_2\dots P_{\mu-1}B_\mu^{-1}f(t) \\ \dots \\ \Pi_{\mu-4}Q_{\mu-3}P_{\mu-2}P_{\mu-1}B_\mu^{-1}f(t) \\ \Pi_{\mu-3}Q_{\mu-2}P_{\mu-1}B_\mu^{-1}f(t) \\ Q_{\mu-1}B_\mu^{-1}f(t) \end{bmatrix},$$

where $u(t) := \Pi_{\mu-1}x(t)$*,* $v_0(t) := Q_0x(t)$*, and* $v_i(t) := \Pi_{i-1}Q_ix(t)$ $(i = 1, \dots, \mu-1)$.

The proof of the results contained in the following lemma is straightforward.

LEMMA 3.2. *Let* (1.1) *be index-*2 *tractable and* $\nu+1$*-smoothing with* $\nu \geq 1$*. Then we have*

$$P_0P_1P_0P_1 = P_0P_1,\ P_0P_1P_0Q_1 = 0,\ P_0P_1Q_0 = 0,\ P_0P_1B_2^{-1}KQ_0 = -P_0P_1B_2^{-1}K_1Q_0, \tag{3.1}$$

$$\begin{aligned} Q_0P_1P_0P_1 = 0,\ Q_0P_1P_0Q_1 = -Q_0Q_1,\ Q_0P_1Q_0 = Q_0, \\ Q_0P_1B_2^{-1}KQ_0 = Q_0 - Q_0P_1B_2^{-1}K_1Q_0, \\ Q_0P_1B_2^{-1}KP_0Q_1 = -Q_0Q_1 + Q_0P_1B_2^{-1}KQ_0Q_1, \end{aligned} \tag{3.2}$$

$$Q_1B_2^{-1}KQ_0 = -Q_1B_2^{-1}K_1Q_0,\ Q_1B_2^{-1}KP_0Q_1 = Q_1 + Q_1B_2^{-1}KQ_0Q_1. \tag{3.3}$$

THEOREM 3.3. *Let* (1.1) *be index-*2 *tractable. Set* $u(t) := P_0P_1x(t)$*,* $v(t) := Q_1x(t)$*, and* $w(t) := Q_0x(t)$*.*

(a) *If* (1.1) *is* 1*-smoothing, then* (1.1) *can be decoupled into*

$$u + P_0P_1B_2^{-1}V_1 + P_0P_1B_2^{-1}W_1 + P_0P_1B_2^{-1}W_0 = P_0P_1B_2^{-1}f, \tag{3.4}$$

$$-Q_0v + Q_0P_1B_2^{-1}V_1 + Q_0P_1B_2^{-1}W_1 + Q_0P_1B_2^{-1}W_0 = Q_0P_1B_2^{-1}f, \tag{3.5}$$

$$Q_1B_2^{-1}V_1 + Q_1B_2^{-1}W_1 + Q_1B_2^{-1}W_0 = Q_1B_2^{-1}f. \tag{3.6}$$

(b) *If* (1.1) *is* $\nu+1$*-smoothing with* $\nu \geq 1$*, then* (1.1) *can be decoupled into*

$$u + P_0P_1B_2^{-1}V_1 + P_0P_1B_2^{-1}W_1 + P_0P_1B_2^{-1}W_0 - P_0P_1B_2^{-1}KP_0v = P_0P_1B_2^{-1}f, \tag{3.7}$$

$$Q_0P_1B_2^{-1}V_1 + Q_0P_1B_2^{-1}W_1 + Q_0P_1B_2^{-1}W_0 - Q_0P_1B_2^{-1}KQ_0v = Q_0P_1B_2^{-1}f, \tag{3.8}$$

$$Q_1B_2^{-1}V_1 + Q_1B_2^{-1}W_1 + Q_1B_2^{-1}W_0 - Q_1B_2^{-1}KQ_0v = Q_1B_2^{-1}f. \tag{3.9}$$

Furthermore, if (1.1) *is* 1*-smoothing and* $\partial_t(Q_1B_2^{-1}K(t,s))|_{s=t} = 0$*, assume that*

$$P_0(\cdot)P_1(\cdot)B_2^{-1}(\cdot)K(\cdot,\cdot), P_0(\cdot)Q_1(\cdot)B_2^{-1}(\cdot)K(\cdot,\cdot), Q_0(\cdot)P_1(\cdot)B_2^{-1}(\cdot)K(\cdot,\cdot) \in C^{l+1}(D),$$
$$Q_0(\cdot)Q_1(\cdot)B_2^{-1}(\cdot)K(\cdot,\cdot) \in C^{l+2}(D),$$
$$P_0(\cdot)P_1(\cdot)B_2^{-1}(\cdot)f(\cdot), P_0(\cdot)Q_1(\cdot)B_2^{-1}(\cdot)f(\cdot), Q_0(\cdot)P_1(\cdot)B_2^{-1}(\cdot)f(\cdot) \in C^{l+1}(I)$$

and

$$Q_0(\cdot)Q_1(\cdot)B_2^{-1}(\cdot)f(\cdot) \in C^{l+2}(I).$$

Then (1.1) *has a unique solution* $(u,v,w)^T \in C^l(I)$*, and there exist functions* E_i $(i = 1,2,\ldots,7), F_j$ $(j = 1,2,\ldots,7), G_k$ $(k = 1,2,\ldots,10) \in C^l(I); \kappa_{ij}$ $(i,j = 1,2,3) \in C^l(D)$ *such that the solution can be represented in the form*

$$\begin{aligned}
u(t) &= E_1(Q_1B_2^{-1}f)' + E_2\left((Q_1B_2^{-1}f)'|_{t=0}\right) + E_3P_0P_1B_2^{-1}f + E_4Q_0P_1B_2^{-1}f \\
&\quad + E_5Q_1B_2^{-1}f + E_6\left(P_0P_1B_2^{-1}f(0)\right) + E_7\left(Q_0P_1B_2^{-1}f(0)\right) \\
&\quad + \int_0^t \left[\kappa_{11}(t,s)P_0P_1B_2^{-1}f(s) + \kappa_{12}(t,s)Q_0P_1B_2^{-1}f(s) + \kappa_{13}(t,s)Q_1B_2^{-1}f(s)\right]ds, \\
v(t) &= F_1(Q_1B_2^{-1}f)' + F_2\left((Q_1B_2^{-1}f)'|_{t=0}\right) + F_3P_0P_1B_2^{-1}f + F_4Q_0P_1B_2^{-1}f \\
&\quad + F_5Q_1B_2^{-1}f + F_6\left(P_0P_1B_2^{-1}f(0)\right) + F_7\left(Q_0P_1B_2^{-1}f(0)\right) \\
&\quad + \int_0^t \left[\kappa_{21}(t,s)P_0P_1B_2^{-1}f(s) + \kappa_{22}(t,s)Q_0P_1B_2^{-1}f(s) + \kappa_{23}(t,s)Q_1B_2^{-1}f(s)\right]ds, \\
w(t) &= G_1(Q_1B_2^{-1}f)'' + G_2(P_0P_1B_2^{-1}f)' + G_3(Q_0P_1B_2^{-1}f)' + G_4(Q_1B_2^{-1}f)' \\
&\quad + G_5P_0P_1B_2^{-1}f + G_6Q_0P_1B_2^{-1}f + G_7Q_1B_2^{-1}f + G_8\left((Q_1B_2^{-1}f)'|_{t=0}\right) \\
&\quad + G_9\left(P_0P_1B_2^{-1}f(0)\right) + G_{10}\left(Q_0P_1B_2^{-1}f(0)\right) \\
&\quad + \int_0^t \left[\kappa_{31}(t,s)P_0P_1B_2^{-1}f(s) + \kappa_{32}(t,s)Q_0P_1B_2^{-1}f(s) + \kappa_{33}(t,s)Q_1B_2^{-1}f(s)\right]ds.
\end{aligned} \tag{3.10}$$

Proof. We distinguish between the following two cases.

Case I. Assume that (1.1) is index-2 tractable and 1-smoothing. Then by Theorem 3.1, we can easily obtain (3.4), (3.5), and (3.6). By (3.4),

$$\begin{aligned}
u(t) &+ P_0P_1B_2^{-1}\int_0^t K(t,s)u(s)ds + P_0P_1B_2^{-1}\int_0^t K(t,s)P_0(s)v(s)ds \\
&+ P_0P_1B_2^{-1}\int_0^t K(t,s)w(s)ds = P_0P_1B_2^{-1}f(t),
\end{aligned}$$

which is a system of second-kind VIEs for u. It follows (see, for example, Theorem 2.1.2 of [1]) that there exists a resolvent kernel $R_1(t,s)$ corresponding to $-P_0P_1B_2^{-1}K(t,s)$ such that

(3.11)

$$\begin{aligned} u(t) &= -P_0P_1B_2^{-1}\int_0^t K(t,s)\left[P_0(s)v(s)+w(s)\right]ds + P_0P_1B_2^{-1}f(t) \\ &\quad + \int_0^t R_1(t,s)P_0(s)P_1(s)B_2^{-1}(s)\left[-\int_0^s K(s,x)\left(P_0(x)v(x)+w(x)\right)dx + f(s)\right]ds \\ &= \int_0^t N_1(t,s)\left[P_0(s)v(s)+w(s)\right]ds + g_1(t), \end{aligned}$$

where

$$N_1(t,s) := -P_0P_1B_2^{-1}K(t,s) - \int_s^t R_1(t,x)P_0(x)P_1(x)B_2^{-1}(x)K(x,s)dx$$

and

$$g_1(t) := P_0P_1B_2^{-1}f(t) + \int_0^t R_1(t,s)P_0(s)P_1(s)B_2^{-1}(s)f(s)ds.$$

If we substitute the above equation into (3.6), we obtain

$$\begin{aligned} &Q_1B_2^{-1}\int_0^t K(t,s)\left[\int_0^s N_1(s,x)\left(P_0(x)v(x)+w(x)\right)dx + g_1(s)\right]ds \\ &+ Q_1B_2^{-1}\int_0^t K(t,s)\left[P_0(s)v(s)+w(s)\right]ds = Q_1B_2^{-1}f(t) \end{aligned}$$

or

$$Q_1B_2^{-1}\int_0^t \left[K(t,s) + \int_s^t K(t,x)N_1(x,s)dx\right]\left[P_0(s)v(s)+w(s)\right]ds = g_2(t),$$

where

$$\begin{aligned} g_2(t) &:= Q_1B_2^{-1}f(t) - Q_1B_2^{-1}\int_0^t K(t,s)g_1(s)ds \\ &= Q_1B_2^{-1}f(t) - Q_1B_2^{-1}\int_0^t \left[K(t,s) + \int_s^t K(t,v)R_1(v,s)dv\right]P_0P_1B_2^{-1}f(s)ds \\ &=: Q_1B_2^{-1}f(t) + Q_1B_2^{-1}\int_0^t \iota_{11}(t,s)P_0P_1B_2^{-1}f(s)ds. \end{aligned}$$

It follows that

$$-\int_0^t N_2(t,s)P_0(s)v(s)ds = \int_0^t N_2(t,s)w(s)ds + g_2(t),$$

where $N_2(t,s) := -Q_1B_2^{-1}[K(t,s) + \int_s^t K(t,x)N_1(x,s)dx]$. Since

$$N_2(t,t)Q_0 = -Q_1B_2^{-1}KQ_0 = -Q_1B_2^{-1}(KP_0Q_1 + B_0 + KQ_0)Q_0 = -Q_1Q_0 = 0$$

and

$$-N_2(t,t)P_0Q_1 = Q_1B_2^{-1}KP_0Q_1 = Q_1B_2^{-1}(KP_0Q_1 + B_1)Q_1 = Q_1.$$

Therefore, by Theorem 2.1.7 of [1], there exists a resolvent kernel $R_2(t,s)$ corresponding to $\partial_t(N_2(t,s))P_0(s)$ such that

(3.12)
$$v(t) = g_2'(t)+\int_0^t \partial_t\left(N_2(t,s)\right)w(s)ds+\int_0^t R_2(t,s)\left[g_2'(s)+\int_0^s \partial_s(N_2(s,x))w(x)dx\right]ds$$
$$= \int_0^t N_3(t,s)w(s)ds + g_3(t),$$

where
$$N_3(t,s) := \partial_t\left(N_2(t,s)\right) + \int_s^t R_2(t,x)\partial_x(N_2(x,s))dx,$$
$$\begin{aligned} g_3(t) &:= g_2'(t) + \int_0^t R_2(t,s)g_2'(s)ds \\ &= (Q_1B_2^{-1}f)'(t) - Q_1B_2^{-1}KP_0P_1B_2^{-1}f(t) + R_2(t,t)Q_1B_2^{-1}f(t) \\ &\quad + \int_0^t \left[\iota_{21}(t,s)P_0P_1B_2^{-1}f(s) + \iota_{22}(t,s)Q_1B_2^{-1}f(s)\right]ds, \end{aligned}$$

with obvious meanings of ι_{21}, ι_{22}.

Substitution of (3.12) into (3.11) leads to

(3.13)
$$u(t) = \int_0^t N_1(t,s)P_0(s)\left[\int_0^s N_3(s,x)w(x)dx + g_3(s)\right]ds + \int_0^t N_1(t,s)w(s)ds + g_1(t)$$
$$= \int_0^t N_4(t,s)w(s)ds + g_4(t),$$

where
$$N_4(t,s) : = \int_s^t N_1(t,x)P_0(x)N_3(x,s)dx + N_1(t,s),$$
$$\begin{aligned} g_4(t) &:= g_1(t) + \int_0^t N_1(t,s)P_0(s)g_3(s)ds \\ &= P_0P_1B_2^{-1}f(t) + N_1(t,t)P_0Q_1B_2^{-1}f(t) + \int_0^t \left[\iota_{31}(t,s)P_0P_1B_2^{-1}f(s)\right. \\ &\quad \left. +\iota_{32}(t,s)Q_1B_2^{-1}f(s)\right]ds, \end{aligned}$$

with obvious meanings of ι_{31}, ι_{32}.

Substituting (3.12) and (3.13) into (3.5), we are led to

$$\begin{aligned} &- Q_0\left[\int_0^t N_3(t,s)w(s)ds+g_3(t)\right] + Q_0P_1B_2^{-1}\int_0^t K(t,s)\left[\int_0^s N_4(s,x)w(x)dx + g_4(s)\right]ds \\ &+ Q_0P_1B_2^{-1}\int_0^t K(t,s)P_0(s)\left[\int_0^s N_3(s,x)w(x)dx + g_3(s)\right]ds \\ &+ Q_0P_1B_2^{-1}\int_0^t K(t,s)w(s)ds \\ &= Q_0P_1B_2^{-1}f(t), \end{aligned}$$

i.e.,

$$\int_0^t N_5(t,s)w(s)ds = g_5(t), \tag{3.14}$$

where

$$N_5(t,s) := -Q_0N_3(t,s) + Q_0P_1B_2^{-1}\int_s^t K(t,x)N_4(x,s)dx$$
$$+ Q_0P_1B_2^{-1}\int_s^t K(t,x)P_0(x)N_3(x,s)dx + Q_0P_1B_2^{-1}K(t,s)$$

and

$$\begin{aligned}
g_5(t) :=\ & Q_0P_1B_2^{-1}f(t) + Q_0g_3(t)\\
& - Q_0P_1B_2^{-1}\int_0^t K(t,s)g_4(s)ds - Q_0P_1B_2^{-1}\int_0^t K(t,s)P_0(s)g_3(s)ds\\
=\ & Q_0P_1B_2^{-1}f(t) + Q_0(Q_1B_2^{-1}f)'(t) - Q_0Q_1B_2^{-1}KP_0P_1B_2^{-1}f(t)\\
& + Q_0R_2(t,t)Q_1B_2^{-1}f(t)\\
& + \int_0^t \left[\iota_{41}(t,s)P_0P_1B_2^{-1}f(s) + \iota_{42}(t,s)Q_1B_2^{-1}f(s)\right]ds,
\end{aligned}$$

with obvious meanings of ι_{41}, ι_{42}.

Observing that

$$\begin{aligned}
N_1(t,t)w(t) &= -P_0P_1B_2^{-1}KQ_0w(t) = -P_0P_1B_2^{-1}[B_0 + KP_0Q_1 + KQ_0]Q_0(t)w(t)\\
&= -P_0P_1Q_0w(t) = 0,
\end{aligned}$$

we obtain

$$\begin{aligned}
N_3(t,t)w(t) = \partial_t(N_2(t,s))|_{s=t}w(t) &= -\partial_t(Q_1B_2^{-1}K(t,s))|_{s=t}w(t)\\
&- Q_1B_2^{-1}KN_1(t,t)w(t) = 0,
\end{aligned}$$

and hence

$$\begin{aligned}
N_5(t,t)w(t) &= -Q_0N_3(t,t)w(t) + Q_0P_1B_2^{-1}Kw(t) = Q_0P_1B_2^{-1}KQ_0w(t)\\
&= Q_0P_1B_2^{-1}(KQ_0 + B + KP_0Q_1)Q_0w(t) = Q_0(I-Q_1)Q_0w(t)\\
&= Q_0w(t) = w(t).
\end{aligned}$$

This reveals that (3.14) is a first-kind VIE for w, and it has a unique continuous solution. Moreover, there exists a resolvent kernel $R_3(t,s)$ corresponding to $-\partial_t(N_5(t,s))$ such that

$$w(t) = g_5'(t) + \int_0^t R_3(t,s)g_5'(s)ds.$$

The solution representation (3.10) in Theorem 3.3 now follows.

Case II. Assume that (1.1) is index-2 tractable and $\nu+1$-smoothing with $\nu \geq 1$. We then have

(3.15)
$$(B_1 + KP_0Q_1 + K_1Q_0)\left(P_1P_0P_1x + P_1P_0Q_1x + P_1Q_0x + Q_1x\right) + V_1 + W_1 + W_0$$
$$-KQ_0x - KP_0Q_1x - K_1Q_0x = f.$$

Multiplication of (3.15) by B_2^{-1} (where $B_2 = B_1 + KP_0Q_1 + K_1Q_0$ is nonsingular) yields

$$\begin{aligned}
& P_1P_0P_1x + P_1P_0Q_1x + P_1Q_0x + Q_1x + B_2^{-1}V_1 + B_2^{-1}W_1 + B_2^{-1}W_0\\
& -B_2^{-1}KQ_0x - B_2^{-1}KP_0Q_1x - B_2^{-1}K_1Q_0x = B_2^{-1}f.
\end{aligned} \tag{3.16}$$

Multiplying (3.16) by P_0P_1, Q_0P_1 and Q_1 respectively, we obtain

(3.17)
$$P_0P_1P_0P_1x + P_0P_1P_0Q_1x + P_0P_1Q_0x + P_0P_1B_2^{-1}V_1 + P_0P_1B_2^{-1}W_1 + P_0P_1B_2^{-1}W_0 \\ -P_0P_1B_2^{-1}KQ_0x - P_0P_1B_2^{-1}KP_0Q_1x - P_0P_1B_2^{-1}K_1Q_0x = P_0P_1B_2^{-1}f,$$

(3.18)
$$Q_0P_1P_0P_1x + Q_0P_1P_0Q_1x + Q_0P_1Q_0x + Q_0P_1B_2^{-1}V_1 + Q_0P_1B_2^{-1}W_1 + Q_0P_1B_2^{-1}W_0 \\ - Q_0P_1B_2^{-1}KQ_0x - Q_0P_1B_2^{-1}KP_0Q_1x - Q_0P_1B_2^{-1}K_1Q_0x = Q_0P_1B_2^{-1}f,$$

$$\begin{aligned} &Q_1x + Q_1B_2^{-1}V_1 + Q_1B_2^{-1}W_1 + Q_1B_2^{-1}W_0 \\ &\quad - Q_1B_2^{-1}KQ_0x - Q_1B_2^{-1}KP_0Q_1x - Q_1B_2^{-1}K_1Q_0x \\ &= Q_1B_2^{-1}f. \end{aligned} \tag{3.19}$$

By Lemma 3.2, we can rewrite (3.17) (3.18), and (3.19) as

(3.20)
$$P_0P_1x+P_0P_1B_2^{-1}V_1+P_0P_1B_2^{-1}W_1+P_0P_1B_2^{-1}W_0 - P_0P_1B_2^{-1}KP_0Q_1x=P_0P_1B_2^{-1}f,$$

(3.21)
$$Q_0P_1B_2^{-1}V_1 + Q_0P_1B_2^{-1}W_1 + Q_0P_1B_2^{-1}W_0 - Q_0P_1B_2^{-1}KQ_0Q_1x = Q_0P_1B_2^{-1}f,$$

$$Q_1B_2^{-1}V_1 + Q_1B_2^{-1}W_1 + Q_1B_2^{-1}W_0 - Q_1B_2^{-1}KQ_0Q_1x = Q_1B_2^{-1}f. \tag{3.22}$$

This yields (3.7), (3.8), and (3.9).

4. Collocation methods for index-2 IAEs

4.1. Collocation for a general IAE system. In this section, we consider the IAE system (1.1) under the assumption that B_2^{-1} is uniformly bounded on I. Let

$$I_h := \{t_n := nh :\ n = 0, 1, \dots, N\ \ (t_N = T)\}$$

be a given mesh on I. The solution x of (1.1) will be approximated by an element x_h of the piecewise polynomial space

$$S_{m-1}^{(-1)}(I_h) := \{v :\ v|_{e_n} \in \pi_{m-1}\ (0 \le n \le N-1)\}. \tag{4.1}$$

Here, $e_n := (t_n, t_{n+1}]$. If $w \in \mathbb{R}^d$, the notation $w \in \pi_k$ means that each component of w is a real polynomial of degree not exceeding k. Assume that the collocation points X_h are given by

$$X_h := \{t = t_n + c_ih :\ 0 < c_1 < \cdots < c_m \le 1\ (0 \le n \le N-1)\},$$

where the c_i are judiciously chosen collocation parameters. Consequently the collocation polynomial can be written as

$$x_h(t_n + sh) = \sum_{j=1}^{m} L_j(s)X_{n,j}, \tag{4.2}$$

where $X_{n,i} := x_h(t_n + c_ih)$, and $L_j(s) := \prod_{k=1,k\neq j}^{m} \frac{s-c_k}{c_j-c_k}$.

The collocation solution $x_h \in S_{m-1}^{(-1)}(I_h)$ is determined by the collocation equation

$$B(t)x_h(t) + \int_0^t K(t,s)x_h(s)ds = f(t),\ t \in X_h. \tag{4.3}$$

In the subsequent analysis of the decoupling of the collocation equation (4.3) we again distinguish, in analogy to section 5, between the following two cases.

Case I. The IAE (1.1) is index-2 tractable and 1-smoothing. We then have for $t \in X_h$,

$$\begin{aligned}(B_1 + KP_0Q_1)\left(P_1P_0P_1x_h + P_1P_0Q_1x_h + P_1Q_0x_h + Q_1x_h\right) + V_1^h + W_1^h + W_0^h \\ - KQ_0x_h - KP_0Q_1x_h = f,\end{aligned} \tag{4.4}$$

where

$$\begin{aligned} V_1^h = V_1^h(t) &:= \int_0^t K(t,s)P_0(s)P_1(s)x_h(s)ds, \\ W_0^h = W_0^h(t) &:= \int_0^t K(t,s)Q_0(s)x_h(s)ds, \quad \text{and} \quad W_1^h = W_1^h(t) \\ &:= \int_0^t K(t,s)P_0(s)Q_1(s)x_h(s)ds. \end{aligned}$$

Because now (1.1) is index-2 tractable, $B_2 = B_2(t_{n,i})$ is nonsingular, and multiplication of (4.4) by B_2^{-1} yields, for $t = t_{n,i}$,

$$\begin{aligned} P_1P_0P_1x_h + P_1P_0Q_1x_h + P_1Q_0x_h + Q_1x_h + B_2^{-1}V_1^h + B_2^{-1}W_1^h + B_2^{-1}W_0^h \\ -B_2^{-1}KQ_0x_h - B_2^{-1}KP_0Q_1x_h = B_2^{-1}f. \end{aligned} \tag{4.5}$$

Multiplying (4.5) by P_0P_1, Q_0P_1, and Q_1 respectively, we have at $t = t_{n,i}$,

$$\begin{aligned} P_0P_1P_0P_1x_h + P_0P_1P_0Q_1x_h + P_0P_1Q_0x_h + P_0P_1B_2^{-1}V_1^h + P_0P_1B_2^{-1}W_1^h \\ + P_0P_1B_2^{-1}W_0^h - P_0P_1B_2^{-1}KQ_0x_h - P_0P_1B_2^{-1}KP_0Q_1x_h = P_0P_1B_2^{-1}f, \end{aligned} \tag{4.6}$$

$$\begin{aligned} Q_0P_1P_0P_1x_h + Q_0P_1P_0Q_1x_h + Q_0P_1Q_0x_h + Q_0P_1B_2^{-1}V_1^h + Q_0P_1B_2^{-1}W_1^h \\ + Q_0P_1B_2^{-1}W_0^h - Q_0P_1B_2^{-1}KQ_0x_h - Q_0P_1B_2^{-1}KP_0Q_1x_h = Q_0P_1B_2^{-1}f, \end{aligned} \tag{4.7}$$

$$\begin{aligned} Q_1x_h + Q_1B_2^{-1}V_1^h + Q_1B_2^{-1}W_1^h + Q_1B_2^{-1}W_0^h - Q_1B_2^{-1}KQ_0x_h \\ - Q_1B_2^{-1}KP_0Q_1x_h = Q_1B_2^{-1}f. \end{aligned} \tag{4.8}$$

Similarly to Case I in the proof of Theorem 3.3 (see also [6]), (4.6), (4.7), and (4.8) can be rewritten as

$$P_0P_1x_h + P_0P_1B_2^{-1}V_1^h + P_0P_1B_2^{-1}W_1^h + P_0P_1B_2^{-1}W_0^h = P_0P_1B_2^{-1}f, \tag{4.9}$$

$$-Q_0Q_1x_h + Q_0P_1B_2^{-1}V_1^h + Q_0P_1B_2^{-1}W_1^h + Q_0P_1B_2^{-1}W_0^h = Q_0P_1B_2^{-1}f, \tag{4.10}$$

$$Q_1B_2^{-1}V_1^h + Q_1B_2^{-1}W_1^h + Q_1B_2^{-1}W_0^h = Q_1B_2^{-1}f. \tag{4.11}$$

Let $u_h(t) := P_0P_1x_h(t), v_h(t) := Q_1x_h(t), w_h(t) := Q_0x_h(t)$. Then

$$u_h + P_0P_1B_2^{-1}V_1^h + P_0P_1B_2^{-1}W_1^h + P_0P_1B_2^{-1}W_0^h = P_0P_1B_2^{-1}f, \tag{4.12}$$

$$-Q_0v_h + Q_0P_1B_2^{-1}V_1^h + Q_0P_1B_2^{-1}W_1^h + Q_0P_1B_2^{-1}W_0^h = Q_0P_1B_2^{-1}f, \tag{4.13}$$

$$Q_1B_2^{-1}V_1^h + Q_1B_2^{-1}W_1^h + Q_1B_2^{-1}W_0^h = Q_1B_2^{-1}f. \tag{4.14}$$

Therefore, (4.3) can be decomposed into (4.12), (4.13), and (4.14), which is equivalent to applying the collocation method to the decoupled equations (3.4), (3.5), and (3.6).

Case II. Let (1.1) be index-2 tractable and $\nu + 1$-smoothing with $\nu \geq 1$. So we have for $t \in X_h$,

$$\begin{aligned}&(B_1 + KP_0Q_1 + K_1Q_0)\,(P_1P_0P_1x_h + P_1P_0Q_1x_h + P_1Q_0x_h + Q_1x_h)\\ &+V_1^h + W_1^h + W_0^h - KQ_0x_h - KP_0Q_1x_h - K_1Q_0x_h = f.\end{aligned} \tag{4.15}$$

Because (1.1) is index-2 tractable, $B_2 = B_2(t_{n,i})$ is nonsingular, and multiplication of (4.15) by B_2^{-1} yields, for $t = t_{n,i}$,

$$\begin{aligned}&P_1P_0P_1x_h + P_1P_0Q_1x_h + P_1Q_0x_h + Q_1x_h + B_2^{-1}V_1^h + B_2^{-1}W_1^h + B_2^{-1}W_0^h\\ &-B_2^{-1}KQ_0x_h - B_2^{-1}KP_0Q_1x_h - B_2^{-1}K_1Q_0x_h = B_2^{-1}f.\end{aligned} \tag{4.16}$$

Multiplying (4.16) by P_0P_1, Q_0P_1, and Q_1 respectively, we have at $t = t_{n,i}$,

$$\begin{aligned}&P_0P_1P_0P_1x_h + P_0P_1P_0Q_1x_h + P_0P_1Q_0x_h + P_0P_1B_2^{-1}V_1^h + P_0P_1B_2^{-1}W_1^h\\ &+ P_0P_1B_2^{-1}W_0^h - P_0P_1B_2^{-1}KQ_0x_h - P_0P_1B_2^{-1}KP_0Q_1x_h\\ &- P_0P_1B_2^{-1}K_1Q_0x_h = P_0P_1B_2^{-1}f,\end{aligned} \tag{4.17}$$

$$\begin{aligned}&Q_0P_1P_0P_1x_h + Q_0P_1P_0Q_1x_h + Q_0P_1Q_0x_h + Q_0P_1B_2^{-1}V_1^h + Q_0P_1B_2^{-1}W_1^h\\ &+ Q_0P_1B_2^{-1}W_0^h - Q_0P_1B_2^{-1}KQ_0x_h - Q_0P_1B_2^{-1}KP_0Q_1x_h\\ &- Q_0P_1B_2^{-1}K_1Q_0x_h = Q_0P_1B_2^{-1}f,\end{aligned} \tag{4.18}$$

$$\begin{aligned}&Q_1x_h + Q_1B_2^{-1}V_1^h + Q_1B_2^{-1}W_1^h + Q_1B_2^{-1}W_0^h - Q_1B_2^{-1}KQ_0x_h\\ &- Q_1B_2^{-1}KP_0Q_1x_h - Q_1B_2^{-1}K_1Q_0x_h\\ &= Q_1B_2^{-1}f.\end{aligned} \tag{4.19}$$

In analogy to Case II in the proof of Theorem 3.3, (4.17), (4.18), and (4.19) can be rewritten as

$$u_h + P_0P_1B_2^{-1}V_1^h + P_0P_1B_2^{-1}W_1^h + P_0P_1B_2^{-1}W_0^h - P_0P_1B_2^{-1}KP_0v_h = P_0P_1B_2^{-1}f, \tag{4.20}$$

$$Q_0P_1B_2^{-1}V_1^h + Q_0P_1B_2^{-1}W_1^h + Q_0P_1B_2^{-1}W_0^h - Q_0P_1B_2^{-1}KQ_0v_h = Q_0P_1B_2^{-1}f, \tag{4.21}$$

$$Q_1B_2^{-1}V_1^h + Q_1B_2^{-1}W_1^h + Q_1B_2^{-1}W_0^h - Q_1B_2^{-1}KQ_0v_h = Q_1B_2^{-1}f. \tag{4.22}$$

Therefore, (4.3) can be decomposed into (4.20), (4.21), and (4.22), which is equivalent to applying the collocation method to the decoupled equations (3.7), (3.8), and (3.9).

4.2. The difficulties of collocation methods for general index-2 IAEs. Motivated by [2] and [7], we will consider the system of linear IAEs

$$\begin{pmatrix} 0 & 0 \\ 1 & \eta t \end{pmatrix} x(t) + \int_0^t \begin{pmatrix} 1 & \eta\left(\frac{\eta+1}{2}t - \frac{\eta-1}{2}s\right) \\ 0 & 1 \end{pmatrix} x(s)ds = f(t). \tag{4.23}$$

It is index-2 tractable for all parameter values $\eta \in \mathbb{R}$.

If we use the collocation method with $m = 1$ for solving (4.23), we obtain the collocation equations

$$\begin{cases} \displaystyle\int_0^{t_{n,1}} \left[x_{1,h}(s) + \eta\left(\frac{\eta+1}{2}t_{n,1} - \frac{\eta-1}{2}s\right) x_{2,h}(s)\right] ds = f_1(t_{n,1}), \\ \displaystyle x_{1,h}(t_{n,1}) + \eta t_{n,1} x_{2,h}(t_{n,1}) + \int_0^{t_{n,1}} x_{2,h}(s)ds = f_2(t_{n,1}). \end{cases}$$

They can be written in the form

$$\begin{aligned} &h\int_0^{c_1} \left[x_{1,h}(t_n+sh) + \eta\left(\frac{\eta+1}{2}t_{n,1} - \frac{\eta-1}{2}(t_n+sh)\right) x_{2,h}(t_n+sh)\right] ds \\ &+h\sum_{l=0}^{n-1}\int_0^1\left[x_{1,h}(t_l+sh)+\eta\left(\frac{\eta+1}{2}t_{n,1}-\frac{\eta-1}{2}(t_l+sh)\right)x_{2,h}(t_l+sh)\right]ds=f_1(t_{n,1}), \end{aligned}$$

and

$$x_{1,h}(t_{n,1})+\eta t_{n,1}x_{2,h}(t_{n,1})+h\int_0^{c_1} x_{2,h}(t_n+sh)ds+h\sum_{l=0}^{n-1}\int_0^1 x_{2,h}(t_l+sh)ds = f_2(t_{n,1});$$

that is,

$$\begin{aligned} &hc_1x_{1,h}(t_{n,1}) + h\eta\left(\frac{\eta+1}{2}t_{n,1} - \frac{\eta-1}{2}t_n - \frac{\eta-1}{4}hc_1\right)c_1x_{2,h}(t_{n,1}) \\ &\quad + h\sum_{l=0}^{n-1}\left[x_{1,h}(t_{l,1}) + \eta\left(\frac{\eta+1}{2}t_{n,1} - \frac{\eta-1}{2}\left(t_l+\frac{h}{2}\right)\right)x_{2,h}(t_{l,1})\right] ds = f_1(t_{n,1}), \end{aligned}$$

and

$$x_{1,h}(t_{n,1}) + \eta t_{n,1}x_{2,h}(t_{n,1}) + hc_1x_{2,h}(t_{n,1}) + h\sum_{l=0}^{n-1}x_{2,h}(t_{l,1}) = f_2(t_{n,1}).$$

Since the determinant of the matrix characterizing the linear algebraic system

$$\begin{aligned} &\begin{pmatrix} hc_1 & h\eta\left(\frac{\eta+1}{2}t_{n,1} - \frac{\eta-1}{2}t_n - \frac{\eta-1}{4}hc_1\right)c_1 \\ 1 & \eta t_{n,1} + hc_1 \end{pmatrix}\begin{pmatrix} x_{1,h}(t_{n,1}) \\ x_{2,h}(t_{n,2}) \end{pmatrix} \\ &= -h\sum_{l=0}^{n-1}\begin{pmatrix} 1 & \eta\left(\frac{\eta+1}{2}t_{n,1} - \frac{\eta-1}{2}\left(t_l+\frac{h}{2}\right)\right) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_{1,h}(t_{l,1}) \\ x_{2,h}(t_{l,2}) \end{pmatrix} + \begin{pmatrix} f_1(t_{n,1}) \\ f_2(t_{n,1}) \end{pmatrix} \end{aligned}$$

is given by

$$\begin{aligned} &hc_1(\eta t_{n,1} + hc_1) - h\eta\left(\frac{\eta+1}{2}t_{n,1} - \frac{\eta-1}{2}t_n - \frac{\eta-1}{4}hc_1\right)c_1 \\ &= h^2c_1\left[(\eta(n+c_1)+c_1) - \eta\left(\frac{\eta+1}{2}(n+c_1) - \frac{\eta-1}{2}n - \frac{\eta-1}{4}c_1\right)\right] \\ &= -h^2c_1^2\frac{\eta^2-\eta-4}{4}, \end{aligned}$$

we see that for $\eta = \frac{1\pm\sqrt{17}}{2}$ the above determinant has value zero for any choice of h and $c_1 > 0$.

We summarize this observation in the following remark.

Remark 4.1. Not all collocation methods are feasible for index-2 IAEs, or for higher-index IAEs.

4.3. Collocation methods for semiexplicit index-2 IAE systems. In this subsection, we consider, for ease of exposition, semiexplicit index-2 IAE systems with $d = 2$, namely,

(4.24)
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \int_0^t \begin{pmatrix} K_{11}(t,s) & K_{12}(t,s) \\ K_{21}(t,s) & 0 \end{pmatrix}\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} ds = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix},$$

where $x_i, K_{ij}(\cdot,\cdot) \in \mathbb{R}$, and $|K_{21}(t,t)K_{12}(t,t)| \geq k_0 > 0$.

4.3.1. The decoupling method. Now, $B_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $K_0 = \left(\begin{smallmatrix} K_{11} & K_{12} \\ K_{21} & 0 \end{smallmatrix}\right)$. We choose $Q_0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then $B_1 = \left(\begin{smallmatrix} 1 & K_{12} \\ 0 & 0 \end{smallmatrix}\right)$. By choosing $Q_1 = \left(\begin{smallmatrix} 1 & 0 \\ -\frac{1}{K_{12}} & 0 \end{smallmatrix}\right)$, we can get $B_2 = \left(\begin{smallmatrix} 1+K_{11} & K_{12} \\ K_{21} & 0 \end{smallmatrix}\right)$ and

$$B_2^{-1} = \begin{pmatrix} 0 & \frac{1}{K_{21}} \\ \frac{1}{K_{12}} & -\frac{1+K_{11}}{K_{12}K_{21}} \end{pmatrix}.$$

It can be easily checked that $P_0P_1 = 0$, $Q_0P_1B_2^{-1}KQ_0 = Q_0$, and $Q_1B_2^{-1}KQ_0 = 0$. Therefore, the next theorem is a consequence of Theorem 3.3.

THEOREM 4.2. *Any $\nu+1$-smoothing index-2 IAE (4.24) with $\nu \geq 0$ can be decoupled into $u = 0$ and*

$$-Q_0v + Q_0P_1B_2^{-1}W_1 + Q_0P_1B_2^{-1}W_0 = Q_0P_1B_2^{-1}f, \tag{4.25}$$

$$Q_1B_2^{-1}W_1 + Q_1B_2^{-1}W_0 = Q_1B_2^{-1}f. \tag{4.26}$$

At $t = t_{n,i}$, the collocation equations are $u_h = 0$ and

$$-Q_0v_h + Q_0P_1B_2^{-1}W_1^h + Q_0P_1B_2^{-1}W_0^h = Q_0P_1B_2^{-1}f, \tag{4.27}$$

$$Q_1B_2^{-1}W_1^h + Q_1B_2^{-1}W_0^h = Q_1B_2^{-1}f. \tag{4.28}$$

Consider the special case with $m = 1$, $K_{11}(t,s) \equiv 0$, $K_{12}(t,s) \equiv K_{21}(t,s) \equiv 1$. Let $v_{i,h}, w_{i,h}$ denote the ith component of v_h, w_h, respectively. Then (4.27) and (4.28) can be written as

$$-\begin{pmatrix} 0 \\ v_{2,h}(t_{n,1}) \end{pmatrix} + \int_0^{t_{n,1}} \begin{pmatrix} 0 \\ w_{2,h}(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ f_1 \end{pmatrix} \tag{4.29}$$

and

$$\int_0^{t_{n,1}} \begin{pmatrix} (v_{1,h}(s) + w_{1,h}(s)) \\ -(v_{1,h}(s) + w_{1,h}(s)) \end{pmatrix} ds = \begin{pmatrix} f_2 \\ -f_2 \end{pmatrix}. \tag{4.30}$$

We notice that the first equation of (4.29) is trivial, and the two equations in (4.30) are identical. Therefore (4.29) and (4.30) can be rewritten as

$$v_{2,h}(t_{n,1}) - \int_0^{t_{n,1}} w_{2,h}(s)ds = -f_1,$$

$$\int_0^{t_{n,1}} (v_{1,h}(s) + w_{1,h}(s))\, ds = f_2.$$

Therefore,

$$v_{2,h}(t_{n,1}) - hc_1 w_{2,h}(t_{n,1}) = h \sum_{l=0}^{n-1} w_{2,h}(t_{l,1}) - f_1,$$

$$hc_1 \left(v_{1,h}(t_{n,1}) + w_{1,h}(t_{n,1})\right) = -h \sum_{l=0}^{n-1} (v_{1,h}(t_{l,1}) + w_{1,h}(t_{l,1})) + f_2.$$

This system is overdetermined (it consists of two equations for four unknowns). This means that in general the collocation solution is not defined.

4.3.2. The direct method. The proof of the following regularity theorem is straightforward and is thus left to the reader.

THEOREM 4.3. *Assume that* $f_i \in C^{l+i}(I)(i = 1, 2)$ *with* $f_2(0) = 0$, $K_{11} \in C^{l+1}(D), K_{12}, K_{21} \in C^{l+2}(D)$ *with* $D := \{(t, s) : 0 \le s \le t \le T\}$*; then* (4.24) *has a unique solution* $x = \binom{x_1}{x_2} \in C^l(I)$*, and there exist functions* κ_{1i} $(p = 1, 2), \kappa_{2j} \in C^l(I)$ $(i = 1, 2; j = 1, \ldots, 5), Q_{11}, Q_{2p} \in C^l(D)$ $(p = 1, 2)$*, such that the solution can be represented in the form*

$$x_1(t) = \kappa_{11}(t)f_2(t) + \kappa_{12}(t)f_2'(t) + \int_0^t Q_{11}(t, s)f_2(s)ds,$$
$$x_2(t) = \kappa_{21}(t)f_1(t) + \kappa_{22}(t)f_1'(t) + \int_0^t Q_{21}(t, s)f_1(s)ds$$
$$+ \kappa_{23}(t)f_2(t) + \kappa_{24}(t)f_2'(t) + \kappa_{25}(t)f_2''(t) + \int_0^t Q_{22}(t, s)f_2(s)ds.$$

The components x_p $(i = 1, 2)$ of the solution of (4.24) will be approximated by elements $x_{p,h}$ $(p = 1, 2)$ of the piecewise polynomial space $S_{m-1}^{(-1)}(I_h)$. According to (4.2) they can be written as

$$x_{p,h}(t_n + sh) = \sum_{j=1}^{m} L_j(s)x_{p,h}(t_{n,j}),\ \ s \in (0, 1]\ \ (p = 1, 2). \tag{4.31}$$

The collocation equations then assume the form

$$\begin{cases} x_{1,h}(t) + \displaystyle\int_0^t [K_{11}(t, s)x_{1,h}(s) + K_{12}(t, s)x_{2,h}(s)]ds = f_1(t), \\ \displaystyle\int_0^t [K_{21}(t, s)x_{1,h}(s)]ds = f_2(t), \end{cases} \tag{4.32}$$

where $t \in X_h$.

For the collocation points $t_{n,i}$ in the subinterval e_n this becomes

$$x_{1,h}(t_{n,i}) + h \int_0^{c_i} [K_{11}(t_{n,i}, t_n + sh)x_{1,h}(t_n + sh) + K_{12}(t_{n,i}, t_n + sh)x_{2,h}(t_n + sh)]ds$$
$$+ h \sum_{l=0}^{n-1} \int_0^1 [K_{11}(t_{n,i}, t_l + sh)x_{1,h}(t_l + sh) + K_{12}(t_{n,i}, t_l + sh)x_{2,h}(t_l + sh)]ds$$
$$= f_1(t_{n,i}),$$

$$h\int_0^{c_i}[K_{21}(t_{n,i},t_n+sh)x_{1,h}(t_n+sh)]ds \\ + h\sum_{l=0}^{n-1}\int_0^1[K_{21}(t_{n,i},t_l+sh)x_{1,h}(t_l+sh)]ds = f_2(t_{n,i}).$$

Therefore,

(4.33)

$$x_{1,h}(t_{n,i}) + h\sum_{j=0}^{m}\int_0^{c_i}K_{11}(t_{n,i},t_n+sh)L_j(s)dsx_{1,h}(t_{n,j}) \\ + h\sum_{j=0}^{m}\int_0^{c_i}K_{12}(t_{n,i},t_n+sh)L_j(s)dsx_{2,h}(t_{n,j}) = f_1(t_{n,i}) + F^1(t_{n,i}),$$

$$h\sum_{j=0}^{m}\int_0^{c_i}K_{21}(t_{n,i},t_n+sh)L_j(s)dsx_{1,h}(t_{n,j}) = f_2(t_{n,i}) + F^2(t_{n,i}), \tag{4.34}$$

where

$$F^1(t_{n,i}) := -h\sum_{l=0}^{n-1}\sum_{j=0}^{m}\int_0^1[K_{11}(t_{n,i},t_l+sh)L_j(s)x_{1,h}(t_{l,j}) \\ + K_{12}(t_{n,i},t_l+sh)L_j(s)x_{2,h}(t_{l,j})]ds,$$

$$F^2(t_{n,i}) := -\,h\sum_{l=0}^{n-1}\sum_{j=0}^{m}\int_0^1[K_{21}(t_{n,i},t_l+sh)L_j(s)x_{1,h}(t_{l,j})]ds.$$

Setting

$$K_{pq}^n := \begin{pmatrix} \int_0^{c_i}K_{pq}(t_{n,i},t_n+sh)L_j(s)ds \\ (i,j=1,\ldots,m)\end{pmatrix}(p,q=1,2),$$

$$K_{pq}^{n,l} := \begin{pmatrix} \int_0^{1}K_{pq}(t_{n,i},t_l+sh)L_j(s)ds \\ (i,j=1,\ldots,m)\end{pmatrix}, \quad F_n^p := \left(F^p(t_{n,i})\ (i=1,\ldots,m)\right)^T,$$

and

$$f_p^n = \left(f_p(t_{n,i})\ (i=1,\ldots,m)\right)^T, \quad X_n^p = \left(x_{p,h}(t_{n,i})(i=1,\ldots,m)\right)^T,$$

the above system of algebraic equations becomes

$$\begin{pmatrix} I+hK_{11}^n & hK_{12}^n \\ hK_{21}^n & 0\end{pmatrix}\begin{pmatrix} X_n^1 \\ X_n^2\end{pmatrix} = \begin{pmatrix} f_1^n+F_n^1 \\ f_2^n+F_n^2\end{pmatrix} \tag{4.35}$$

with

$$F_n^1 = -h\sum_{l=0}^{n-1}\left(K_{11}^{n,l}X_l^1 + K_{12}^{n,l}X_l^2\right), \quad F_n^2 = -h\sum_{l=0}^{n-1}K_{21}^{n,l}X_l^1.$$

Since the determinant of the coefficient matrix is

$$(-1)^m\det(hK_{21}^n hK_{12}^n) = (-1)^m h^{2m}(K_{21}(t_n,t_n)+O(h))^m(K_{12}(t_n,t_n) \\ + O(h))^m(\det(A))^2,$$

where

$$|K_{21}(t_n,t_n)K_{12}(t_n,t_n)| \geq k_0 > 0$$

for all sufficiently small $h > 0$, the system (4.35) has a unique solution. The following theorem makes this precise.

THEOREM 4.4. *There exists an $\overline{h} > 0$ so that for any mesh I_h with mesh diameter $h > 0$ satisfying $h < \overline{h}$, each of the linear algebraic systems (4.35) has a unique solution $\begin{pmatrix} X_n^1 \\ X_n^2 \end{pmatrix} \in \mathbb{R}^{2m}$. Hence the collocation equation (4.32) defines a unique collocation solution ($x_{p,h} \in S_{m-1}^{(-1)}(I_h)(p = 1, 2)$) for the IAE (4.24), and its representation on the subinterval $(t_n, t_{n+1}]$ is given by (4.31).*

We now turn to the analysis of the collocation errors $e_{p,h}(t_n+sh) := x_p(t_n+sh) - x_{p,h}(t_n+sh)(p = 1, 2)$. It will be based on familiar representation of the interpolation error (see section 1.8 of [1]) by which the exact solution $x_p(t)$ can be expressed in the form

$$x_p(t_n + sh) = \sum_{j=1}^{m} L_j(s)x_p(t_n + c_j h) + h^m S_{m,n,p}(s), \tag{4.36}$$

where the Peano remainder terms and Peano kernels are given by

$$S_{m,n,p}(s) := \int_0^1 K_m(s, \nu) x_p^{(m)}(t_n + \nu h) d\nu$$

and

$$K_m(s, \nu) := \frac{1}{(m-1)!} \left\{ (s - \nu)_+^{m-1} - \sum_{k=1}^{m} L_k(s)(c_k - \nu)_+^{m-1} \right\}, \; s \in (0, 1].$$

Thus, by (4.31) and (4.36) we have

$$e_{p,h}(t_n + sh) = \sum_{j=1}^{m} L_j(s) e_{p,h}(t_{n,j}) + h^m S_{m,n,p}(s). \tag{4.37}$$

Equations (4.24) and (4.32) imply that for $t_{n,i}$,

$$e_{1,h}(t_{n,i}) + h \int_0^{c_i} [K_{11}(t_{n,i}, t_n + sh)e_{1,h}(t_n + sh) + K_{12}(t_{n,i}, t_n + sh)e_{2,h}(t_n + sh)]\, ds$$
$$+ h \sum_{l=0}^{n-1} \int_0^1 [K_{11}(t_{n,i}, t_l + sh)e_{1,h}(t_l + sh) + K_{12}(t_{n,i}, t_l + sh)e_{2,h}(t_l + sh)]\, ds = 0,$$

$$h \int_0^{c_i} [K_{21}(t_{n,i}, t_n + sh)e_{1,h}(t_n + sh)]\, ds$$
$$+ h \sum_{l=0}^{n-1} \int_0^1 [K_{21}(t_{n,i}, t_l + sh)e_{1,h}(t_l + sh)]\, ds = 0.$$

Therefore,

$$\begin{aligned} e_{1,h}(t_{n,i}) &+ h \sum_{j=0}^{m} \int_0^{c_i} [K_{11}(t_{n,i}, t_n + sh)L_j(s)e_{1,h}(t_{n,j}) \\ &+ K_{12}(t_{n,i}, t_n + sh)L_j(s)e_{2,h}(t_{n,j})]\, ds \\ &+ h \sum_{l=0}^{n-1} \sum_{j=0}^{m} \int_0^1 [K_{11}(t_{n,i}, t_l + sh)L_j(s)e_{1,h}(t_{l,j}) \\ &+ K_{12}(t_{n,i}, t_l + sh)L_j(s)e_{2,h}(t_{l,j})]\, ds = h^m \rho_{n,i}^1, \end{aligned} \tag{4.38}$$

and

$$\begin{aligned}&h\sum_{j=0}^{m}\int_0^{c_i}[K_{21}(t_{n,i},t_n+sh)L_j(s)e_{1,h}(t_{n,j})]\,ds\\ &+h\sum_{l=0}^{n-1}\sum_{j=0}^{m}\int_0^1[K_{21}(t_{n,i},t_l+sh)L_j(s)e_{1,h}(t_{l,j})]\,ds=h^m\rho^2_{n,i},\end{aligned}\tag{4.39}$$

where

$$\begin{aligned}\rho^1_{n,i}:=&-h\sum_{j=0}^{m}\int_0^{c_i}[K_{11}(t_{n,i},t_n+sh)S_{m,n,1}(s)+K_{12}(t_{n,i},t_n+sh)S_{m,n,2}(s)]\,ds\\ &-h\sum_{l=0}^{n-1}\sum_{j=0}^{m}\int_0^1[K_{11}(t_{n,i},t_l+sh)S_{m,l,1}(s)+K_{12}(t_{n,i},t_l+sh)S_{m,l,2}(s)]\,ds,\end{aligned}$$

$$\begin{aligned}\rho^2_{n,i}:=&-h\sum_{j=0}^{m}\int_0^{c_i}[K_{21}(t_{n,i},t_n+sh)S_{m,n,1}(s)]\,ds-h\sum_{l=0}^{n-1}\sum_{j=0}^{m}\int_0^1[K_{21}(t_{n,i},t_l\\ &+sh)S_{m,l,1}(s)]\,ds.\end{aligned}$$

By Theorem 2.4.2 of [1] and (4.39), there exists a constant C_1 such that

$$|e_{1,h}(t_{n,i})|\le C_1\begin{cases}h^m & \text{if } -1\le\rho_m:=(-1)^m\prod_{j=1}^{m}\dfrac{1-c_j}{c_j}<1,\\ h^{m-1} & \text{if } \rho_m=1,\end{cases}$$

and by (4.37), there exists a constant C_2 such that

$$|e_{1,h}(t_n+sh)|\le C_2\begin{cases}h^m & \text{if } -1\le\rho_m<1,\\ h^{m-1} & \text{if } \rho_m=1.\end{cases}$$

Moreover,

$$\begin{aligned}&h\sum_{j=0}^{m}\int_0^{c_i}[K_{12}(t_{n,i},t_n+sh)L_j(s)e_{2,h}(t_{n,j})]ds\\ &+h\sum_{l=0}^{n-1}\sum_{j=0}^{m}\int_0^1[K_{12}(t_{n,i},t_l+sh)L_j(s)e_{2,h}(t_{l,j})]ds=\begin{cases}O(h^m) & \text{if } -1\le\rho_m<1,\\ O(h^{m-1}) & \text{if } \rho_m=1.\end{cases}\end{aligned}$$

Hence, proceeding again as in the proof of Theorem 2.4.2 in [1], there exists a constant C_3 such that

$$|e_{2,h}(t_n+sh)|\le C_3\begin{cases}h^{m-1} & \text{if } -1<\rho_m<1,\\ h^{m-2} & \text{if } \rho_m=-1,\\ h^{m-3} & \text{if } \rho_m=1.\end{cases}$$

We summarize these results in the following theorem.

Theorem 4.5. *Assume that*

(a) *the given functions in* (4.24) *satisfy the conditions of Theorem* 4.3 *with* $l\ge m$;

(b) $x_{p,h}\in S^{(-1)}_{m-1}(p=1,2)$ *are the collocation solutions to* x_p *of* (4.24);

(c) $\overline{h} > 0$ *is such that, for any* $h \in (0, \overline{h})$, *each of the linear systems* (4.35) *has a unique solution.*

Then for all uniform meshes I_h *with* $h \in (0, \overline{h})$ *the collocation solution* x_h *converges uniformly on* I *to the solution* x *of* (4.24) *if, and only if, the collocation parameters satisfy the condition*

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1 - c_i}{c_i} \leq 1. \tag{4.40}$$

The attainable global order of convergence is then described by

$$\|x_1 - x_{1,h}\|_\infty := \max_{t \in I} \|x_1(t) - x_{1,h}(t)\| \leq \overline{C}_1 \begin{cases} h^m & \text{if } -1 \leq \rho_m < 1, \\ h^{m-1} & \text{if } \rho_m = 1; \end{cases}$$

$$\|x_2 - x_{2,h}\|_\infty := \max_{t \in I} \|x_2(t) - x_{2,h}(t)\| \leq \overline{C}_2 \begin{cases} h^{m-1} & \text{if } -1 < \rho_m < 1, \\ h^{m-2} & \text{if } \rho_m = -1, \\ h^{m-3} & \text{if } \rho_m = 1. \end{cases}$$

This holds for all $h \in (0, \overline{h})$ *and any set of collocation points* X_h *corresponding to collocation parameters* $0 < c_1 < \cdots < c_m \leq 1$ *that satisfy* (4.40). *The constants* $\overline{C}_1$ *and* $\overline{C}_2$ *depend on the collocation parameters* $\{c_i\}$ *but are independent of* h.

5. Numerical examples. We now present an example to illustrate the foregoing convergence results.

Example 5.1. Consider the IAE system

$$\begin{cases} x_1(t) + \displaystyle\int_0^t \left[(t-s)x_1(s) + e^{t-s}x_2(s)\right] ds = f_1(t), \\ \displaystyle\int_0^t \left[e^{2t-s}x_1(s)\right] ds = f_2(t) \end{cases} \tag{5.1}$$

on $I = [0, 1]$. The kernel of the above Volterra integral operator,

$$K(t, s) = \begin{pmatrix} t - s & \mathrm{e}^{t-s} \\ \mathrm{e}^{2t-s} & 0 \end{pmatrix},$$

is 2-smoothing, and $B(t) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, has rank 1 for all $t \in I$. The tractability index of this system is $\mu = 2$. We choose $f_1(t) := rte^{\beta t} + r\left(-2e^{\beta t} + e^{\beta t}\beta t + 2 + \beta t\right)/\beta^3 - 1/2\cos t + 1/2\sin t + 1/2e^t$ and $f_2(t) := r(-e^{t+\beta t}t + e^{t+\beta t}\beta t - e^{t+\beta t} + e^{2t})/(-1+\beta)^2$. It can be verified that the exact solution is $x_1(t) = rte^{\beta t}$, $x_2(t) = \cos t$.

For the numerical solution of (5.1) we choose $m = 2$ and $m = 3$. For $m = 2$ we use the Gauss collocation parameters, $c_1 = \frac{3-\sqrt{3}}{6}$, $c_2 = \frac{3+\sqrt{3}}{6}$; the Radau IIA collocation parameters, $c_1 = \frac{1}{3}$, $c_2 = 1$; and four sets of arbitrary collocation parameters, $c_1 = \frac{1}{2}$, $c_2 = 1$; $c_1 = \frac{1}{4}$, $c_2 = \frac{5}{6}$; $c_1 = \frac{1}{3}$, $c_2 = \frac{2}{3}$, and $c_1 = \frac{1}{6}$, $c_2 = \frac{1}{2}$. For $m = 3$ we use the Gauss collocation parameters, $c_1 = \frac{5-\sqrt{15}}{10}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{5+\sqrt{15}}{10}$; the Radau IIA collocation parameters, $c_1 = \frac{4-\sqrt{6}}{10}$, $c_2 = \frac{4+\sqrt{6}}{10}$, $c_3 = 1$; and four sets of arbitrary collocation parameters, $c_1 = \frac{1}{2}$, $c_2 = \frac{2}{3}$, $c_3 = 1$; $c_1 = \frac{1}{3}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{2}{3}$; $c_1 = \frac{1}{2}$, $c_2 = \frac{2}{3}$, $c_3 = \frac{8}{9}$; and $c_1 = \frac{1}{9}$, $c_2 = \frac{1}{3}$, $c_3 = \frac{1}{2}$.

In Tables 1–8 we list the absolute values of the error components $e_{1,h}$ and $e_{2,h}$ at $t = T = 1$ for the six collocation parameters and for $m = 2$ and $m = 3$, respectively, and the ratios of the absolute values of the errors for $N = 16$ over that for $N = 32$.

TABLE 1
$|e_{1,h}|$ *for Example* 5.1 *with* $m = 2$, $r = 1$, *and* $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, 1)$	$(\frac{1}{4}, \frac{5}{6})$	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{6}, \frac{1}{2})$
2^2	5.0718e-02	3.2564e-03	2.3851e-03	1.8748e-02	5.0471e-02	4.3544e+00
2^3	2.5605e-02	7.2116e-04	5.3443e-04	4.0512e-03	2.5541e-02	7.9778e+02
2^4	1.2859e-02	1.6967e-04	1.2649e-04	7.7886e-04	1.2842e-02	8.4721e+07
2^5	6.4426e-03	4.1149e-05	3.0768e-05	1.6674e-04	6.4384e-03	3.3742e+18
Ratio	1.9959e+00	4.1233e+00	4.1111e+00	4.6711e+00	1.9946e+00	-

TABLE 2
$|e_{2,h}|$ *for Example* 5.1 *with* $m = 2$, $r = 1$, *and* $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, 1)$	$(\frac{1}{4}, \frac{5}{6})$	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{6}, \frac{1}{2})$
2^2	5.4919e+00	1.0086e-01	6.7210e-02	1.5672e+00	4.2064e+00	5.9016e+02
2^3	1.1990e+01	3.9294e-02	2.7757e-02	9.6748e-01	9.0648e+00	4.9768e+05
2^4	2.4989e+01	1.7351e-02	1.2627e-02	3.5175e-01	1.8807e+01	2.2660e+11
2^5	5.0986e+01	8.1539e-03	6.0235e-03	1.2819e-01	3.8301e+01	3.7374e+22
Ratio	-	2.1279e+00	2.0963e+00	2.7440e+00	-	-

TABLE 3
$|e_{1,h}|$ *for Example* 5.1 *with* $m = 2$, $r = 10$, *and* $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, 1)$	$(\frac{1}{4}, \frac{5}{6})$	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{6}, \frac{1}{2})$
2^2	5.0718e-01	3.2564e-02	2.3851e-02	1.8748e-01	5.0471e-01	4.3544e+01
2^3	2.5605e-01	7.2116e-03	5.3443e-03	4.0512e-02	2.5541e-01	7.9778e+03
2^4	1.2859e-01	1.6967e-03	1.2649e-03	7.7886e-03	1.2842e-01	8.4721e+08
2^5	6.4426e-02	4.1149e-04	3.0768e-04	1.6674e-03	6.4384e-02	3.3742e+19
Ratio	1.9959e+00	4.1233e+00	4.1110e+00	4.6711e+00	1.9946e+00	-

TABLE 4
$|e_{2,h}|$ *for Example* 5.1 *with* $m = 2$, $r = 10$, *and* $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, 1)$	$(\frac{1}{4}, \frac{5}{6})$	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{6}, \frac{1}{2})$
2^2	5.4540e+01	9.6813e-01	6.4219e-01	1.5505e+01	4.1684e+01	5.8776e+03
2^3	1.1971e+02	3.8366e-01	2.7067e-01	9.6330e+00	9.0459e+01	4.9727e+06
2^4	2.4979e+02	1.7130e-01	1.2461e-01	3.5082e+00	1.8797e+02	2.2656e+12
2^5	5.0982e+02	8.0998e-02	5.9831e-02	1.2798e+00	3.8297e+02	3.7372e+23
Ratio	-	2.1148e+00	2.0828e+00	2.7412e+00	-	-

In order to show that not all collocation methods are feasible for index-2 IAEs, we return to the counterexample in section 4.2 to see what happens when it is solved by collocation.

Example 5.2. Consider (4.23) on $I = [0, 1]$, and choose f_1 and f_2 such that the exact solution is $x_1(t) = e^t, x_2(t) = \cos t$.

We choose $m = 1$, and use $N = 4, 8$ and $c_1 = 0.5, 1$ to compute the numerical solution of Example 5.2 with $\eta = 1, 2, 2.5, \frac{1+\sqrt{17}}{2} \approx 2.5616$, respectively.

TABLE 5
$|e_{1,h}|$ *for Example* 5.1 *with* $m = 3$, $r = 1$, *and* $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, \frac{2}{3}, 1)$	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, \frac{2}{3}, \frac{8}{9})$	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$
2^2	5.6237e-04	1.7437e-04	9.3809e-05	6.3937e-04	1.4228e-04	1.1148e+01
2^3	7.1990e-05	1.9782e-05	1.0815e-05	8.2789e-05	1.6739e-05	9.9634e+04
2^4	9.0769e-06	2.3562e-06	1.2985e-06	1.0494e-05	2.0281e-06	5.5840e+13
2^5	1.1386e-06	2.8751e-07	1.5909e-07	1.3197e-06	2.4955e-07	1.3153e+32
Ratio	7.9720e+00	8.1952e+00	8.1624e+00	7.9516e+00	8.1272e+00	-

TABLE 6
$|e_{2,h}|$ *for Example* 5.1 *with* $m = 3$, $r = 1$, *and* $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, \frac{2}{3}, 1)$	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, \frac{2}{3}, \frac{8}{9})$	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$
2^2	2.5676e-01	3.2322e-03	3.9226e-03	1.6860e-01	3.7754e-03	2.6693e+03
2^3	1.3755e-01	8.5492e-04	9.0866e-04	8.9976e-02	9.0001e-04	1.0713e+08
2^4	7.0652e-02	2.1568e-04	2.1831e-04	4.6137e-02	2.1867e-04	2.5323e+17
2^5	3.5739e-02	5.3963e-05	5.3480e-05	2.3320e-02	5.3836e-05	2.4474e+36
Ratio	1.9769e+00	3.9968e+00	4.0820e+00	1.9785e+00	4.0617e+00	-

TABLE 7
$|e_{1,h}|$ *for Example* 5.1 *with* $m = 3$, $r = 10$, *and* $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, \frac{2}{3}, 1)$	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, \frac{2}{3}, \frac{8}{9})$	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$
2^2	5.6237e-03	1.7437e-03	9.3809e-04	6.3937e-03	1.4228e-03	1.1148e+02
2^3	7.1990e-04	1.9782e-04	1.0815e-04	8.2789e-04	1.6739e-04	9.9634e+05
2^4	9.0769e-05	2.3562e-05	1.2985e-05	1.0494e-04	2.0281e-05	5.5840e+14
2^5	1.1386e-05	2.8751e-06	1.5909e-06	1.3197e-05	2.4955e-06	1.3153e+33
Ratio	7.9719e+00	8.1952e+00	8.1624e+00	7.9516e+00	8.1272e+00	-

TABLE 8
$|e_{2,h}|$ *for Example* 5.1 *with* $m = 3$, $r = 10$, *and* $\beta = -1$.

N	Gauss	Radau IIA	$(\frac{1}{2}, \frac{2}{3}, 1)$	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, \frac{2}{3}, \frac{8}{9})$	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$
2^2	2.5693e+00	3.0920e-02	3.8439e-02	1.6883e+00	3.6519e-02	2.6690e+04
2^3	1.3758e+00	8.3687e-03	8.9858e-03	9.0005e-01	8.8419e-03	1.0713e+09
2^4	7.0655e-01	2.1339e-03	2.1703e-03	4.6140e-01	2.1667e-03	2.5323e+18
2^5	3.5739e-01	5.3676e-04	5.3321e-04	2.3320e-01	5.3585e-04	2.4474e+37
Ratio	1.9770e+00	3.9756e+00	4.0704e+00	1.9786e+00	4.0435e+00	-

Tables 9–16 reveal that as $\eta \to \frac{1+\sqrt{17}}{2}$, the numerical solutions fail to approximate the exact solution at $t = T = 1$ regardless of $c_1 = 1$ or $c_1 = 0.5$, and $N = 4$ or $N = 8$ (cf. $x_1(1) = 2.7183$, $x_2(1) = 0.5403$).

6. Concluding remarks. The analyses of the IAE systems and their numerical solution by piecewise polynomial collocation methods described in the paper [6] and the present paper are based on the assumption that the underlying Volterra integral operator $\mathcal{V}$ (cf. (2.1)) has a smooth kernel. In order to complete this analysis we are currently studying systems of IAEs (1.1) with underlying Volterra integral operators

TABLE 9
$x_{1,h}$ *for Example* 5.2 *with* $N = 4, c_1 = 1$.

t_n	$\eta = 1$	$\eta = 2$	$\eta = 2.5$	$\eta = \frac{1+\sqrt{17}}{2}$
t_1	9.9335e-01	1.6633e+00	1.3764e+01	−4.7236e+16
t_2	9.8666e-01	8.5948e+00	1.2461e+03	1.5665e+34
t_3	1.0076e+00	5.5395e+01	9.6969e+04	7.4492e+50
t_4	1.0552e+00	3.4734e+02	6.8984e+06	−1.8607e+68

TABLE 10
$x_{2,h}$ *for Example* 5.2 *with* $N = 4, c_1 = 1$.

t_n	$\eta = 1$	$\eta = 2$	$\eta = 2.5$	$\eta = \frac{1+\sqrt{17}}{2}$
t_1	1.5606e+00	1.4718e-01	−1.3703e+01	5.3051e+16
t_2	1.5868e+00	−4.8517e+00	−8.2672e+02	−1.0233e+34
t_3	1.5529e+00	−2.9070e+01	−4.5532e+04	−3.4310e+50
t_4	1.4958e+00	−1.4880e+02	−2.5043e+06	6.6181e+67

TABLE 11
$x_{1,h}$ *for Example* 5.2 *with* $N = 8, c_1 = 1$.

t_n	$\eta = 1$	$\eta = 2$	$\eta = 2.5$	$\eta = \frac{1+\sqrt{17}}{2}$
t_1	9.9788e-01	1.3525e+00	7.6573e+00	−2.4601e+16
t_2	9.9215e-01	5.1673e+00	6.5246e+02	8.1583e+33
t_3	9.8784e-01	3.1605e+01	5.0750e+04	3.8796e+50
t_4	9.8890e-01	1.9822e+02	3.6104e+06	−9.6908e+67
t_5	9.9774e-01	1.2020e+03	2.4363e+08	−1.9876e+83
t_6	1.0148e+00	7.0745e+03	1.5877e+10	−3.6344e+99
t_7	1.0383e+00	4.0706e+04	1.0095e+12	−1.4092e+116
t_8	1.0638e+00	2.3022e+05	6.3021e+13	3.0861e+133

TABLE 12
$x_{2,h}$ *for Example* 5.2 *with* $N = 8, c_1 = 1$.

t_n	$\eta = 1$	$\eta = 2$	$\eta = 2.5$	$\eta = \frac{1+\sqrt{17}}{2}$
t_1	1.5359e+00	7.8274e-02	−1.4344e+01	5.5259e+16
t_2	1.5721e+00	−5.4454e+00	−8.6519e+02	−1.0659e+34
t_3	1.5877e+00	−3.2873e+01	−4.7659e+04	−3.5738e+50
t_4	1.5857e+00	−1.6967e+02	−2.6213e+06	6.8936e+67
t_5	1.5697e+00	−8.5314e+02	−1.4417e+08	1.1516e+83
t_6	1.5442e+00	−4.2699e+03	−7.9295e+09	1.7762e+99
t_7	1.5145e+00	−2.1353e+04	−4.3612e+11	5.9552e+115
t_8	1.4867e+00	−1.0677e+05	−2.3987e+13	−1.1487e+133

of the form

$$(\mathcal{V}_\alpha x)(t) := \int_0^t K_\alpha(t,s)x(s)\,ds, \;\; t \in [0,T],$$

Table 13
$x_{1,h}$ *for Example* 5.2 *with* $N = 4, c_1 = 0.5$.

t_n	$\eta = 1$	$\eta = 2$	$\eta = 2.5$	$\eta = \frac{1+\sqrt{17}}{2}$
t_1	9.9788e-01	1.3525e+00	7.6573e+00	−2.4601e+16
t_2	1.0056e+00	2.1489e+01	3.6371e+03	−8.0452e+33
t_3	1.0288e+00	5.8464e+02	1.2586e+06	−8.5473e+49
t_4	1.0635e+00	1.4246e+04	3.7591e+08	−1.5716e+67

Table 14
$x_{2,h}$ *for Example* 5.2 *with* $N = 4, c_1 = 0.5$.

t_n	$\eta = 1$	$\eta = 2$	$\eta = 2.5$	$\eta = \frac{1+\sqrt{17}}{2}$
t_1	1.5359e+00	7.8274e-02	−1.4344e+01	5.5259e+16
t_2	1.5612e+00	−2.2101e+01	−3.4177e+03	7.4110e+33
t_3	1.5428e+00	−4.1903e+02	−7.4532e+05	4.9522e+49
t_4	1.5038e+00	−7.5370e+03	−1.6248e+08	6.6415e+66

Table 15
$x_{1,h}$ *for Example* 5.2 *with* $N = 8, c_1 = 0.5$.

t_n	$\eta = 1$	$\eta = 2$	$\eta = 2.5$	$\eta = \frac{1+\sqrt{17}}{2}$
t_1	9.9941e-01	1.1818e+00	4.3983e+00	−1.2548e+16
t_2	1.0005e+00	1.1691e+01	1.8592e+03	−4.1035e+33
t_3	1.0041e+00	3.0713e+02	6.4322e+05	−4.3596e+49
t_4	1.0115e+00	7.4786e+03	1.9212e+08	−8.0161e+66
t_5	1.0231e+00	1.6969e+05	5.3194e+10	1.0136e+84
t_6	1.0385e+00	3.6820e+06	1.4062e+13	−4.4623e+100
t_7	1.0561e+00	7.7504e+07	3.6031e+15	1.1642e+117
t_8	1.0733e+00	1.5959e+09	9.0267e+17	−2.1388e+133

Table 16
$x_{2,h}$ *for Example* 5.2 *with* $N = 8, c_1 = 0.5$.

t_n	$\eta = 1$	$\eta = 2$	$\eta = 2.5$	$\eta = \frac{1+\sqrt{17}}{2}$
t_1	1.5194e+00	4.0382e-02	−1.4669e+01	5.6371e+16
t_2	1.5459e+00	−2.3131e+01	−3.4932e+03	7.5601e+33
t_3	1.5584e+00	−4.3967e+02	−7.6181e+05	5.0518e+49
t_4	1.5588e+00	−7.9129e+03	−1.6607e+08	6.7751e+66
t_5	1.5493e+00	−1.4201e+05	−3.6203e+10	−6.7424e+83
t_6	1.5330e+00	−2.5483e+06	−7.8920e+12	2.4470e+100
t_7	1.5132e+00	−4.5728e+07	−1.7204e+15	−5.4307e+116
t_8	1.4940e+00	−8.2056e+08	−3.7504e+17	8.6804e+132

where the elements of the matrix kernel $K_\alpha(t,s)$ may be weakly singular; that is, they have the form

$$(t-s)^{\alpha_{pq}-1}K_{pq}(t,s) \;\; (p,q = 1,\dots,d)$$

with smooth K_{pq} and $0 < \alpha_{pq} \leq 1$. Owing to the resulting low regularity of the solution of such an IAE, the optimal orders of convergence of the collocation solution $x(t)$ described in Theorem 4.5 will no longer be valid for uniform meshes.

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REFERENCES

[1] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge University Press, Cambridge, UK, 2004.

[2] C. W. Gear, *Differential algebraic equations, indices, and integral algebraic equations*, SIAM J. Numer. Anal., 27 (1990), pp. 1527–1534.

[3] F. Ghoreishi, M. Hadizadeh, and S. Pishbin, *On the convergence analysis of the spline collocation method for system of integral algebraic equations of index*-2, Int. J. Comput. Methods, 9 (2012), 1250048.

[4] M. Hadizadeh, F. Ghoreishi, and S. Pishbin, *Jacobi spectral solution for integral algebraic equations of index*-2, Appl. Numer. Math., 61 (2011), pp. 131–148.

[5] R. Lamour, R. März, and C. Tischendorf, *Differential-Algebraic Equations: A Projector Based Analysis*, Springer-Verlag, Berlin, 2013.

[6] H. Liang and H. Brunner, *Integral-algebraic equations: theory of collocation methods* I, SIAM J. Numer. Anal., 51 (2013), pp. 2238–2259.

[7] R. März, *Numerical methods for differential-algebraic equations*, Acta Numer., 1 (1991), pp. 141–198.

[8] R. März, *Differential algebraic equations anew*, Appl. Numer. Math., 42 (2002), pp. 315–335.

[9] S. Pishbin, *Optimal convergence results of piecewise polynomial collocation solutions for integral-algebraic equations of index*-3, J. Comput. Appl. Math., 279 (2015), pp. 209–224.

[10] B. Shiri, *Numerical solution of higher index nonlinear integral algebraic equations of Hessenberg type using discontinuous collocation methods*, Math. Model. Anal., 19 (2014), pp. 99–117.

[11] B. Shiri, S. Shahmorad, and G. Hojjati, *Convergence analysis of piecewise continuous collocation methods for higher index integral algebraic equations of Hessenberg type*, Int. J. Appl. Math. Comput. Sci., 23 (2013), pp. 341–355.