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EDGE-FACE TOTAL CHROMATIC NUMBER OF HALIN GRAPHS*

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Abstract. We show that the edge-face total chromatic number of Halin graphs with maximum vertex degree Δ not less than 4 is equal to $\max\{5, \Delta\}$. For the cases of $\Delta = 4$ and 5, we provide a proper 5-edge-face total coloring algorithm.

Key words. Halin graphs, edge-face total chromatic number

AMS subject classification. 05C15

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1. Introduction.

A *Halin* graph G is a plane graph consisting of a plane embedding of a tree T of order at least 4 containing no vertex of degree 2, and of a cycle C^* connecting all end vertices of T. The tree T is called the *characteristic tree* of G, and C^* is called the *adjoint cycle* of G. Vertices and edges on the cycle C^* are called the *outer vertices* and *outer edges*, respectively. Other vertices and edges are called *inner vertices* and *inner edges*, respectively. A path consisting of inner edges is called an *inner path.* The face incident with all outer vertices and outer edges is called the *outer face* and is denoted by f_0. All other faces are called *inner faces.* Note that an inner face is bounded by one outer edge and an inner path. Two end vertices of the characteristic tree of a Halin graph are called *neighboring vertices* if they are linked by an edge of the adjoint cycle C^*. Two inner faces are *neighbors* of each other, or *neighboring faces*, if they are incident with a common outer vertex. Thus, two neighboring faces must be adjacent to each other, but adjacent inner faces are not necessarily neighbors. Furthermore, the "neighboring" relation produces a cyclic ordering of the outer vertices and also of the inner faces. Graph theory notation and terminology not defined in this paper is as described in [1].

DEFINITION 1.1. *A proper* k-edge-face total *coloring of a loopless plane graph G is an assignment of k colors $1, 2, \ldots, k$ to all edges and faces in $E \cup F$ such that no two adjacent or incident elements have the same color. A graph G is* k-edge-face total colorable *if there exists a* k-edge-face total coloring *on G. Moreover,*

$$\chi_{ef}(G) = \min\{k | G \textit{ is } k\textit{-edge-face total colorable}\}$$

is called the edge-face total chromatic *number of G.*

In 1995, Lin, Hu, and Zhang [5] proved that any plane graph of maximum degree not exceeding 3 is 6-edge-face total colorable. Chang et al. [3] also showed that $\chi_{ef}(G) = 6$ if G is a 2-connected outerplanar graph with $\Delta = 6$. The study of the edge-face total chromatic number of planar graphs was started in this period. In 2000, we obtained in [4] that the edge-face total chromatic number of 3-regular Halin graphs is equal to 4 or 5. We also provided the sufficient and necessary condition to characterize 3-regular Halin graphs with the edge-face total chromatic number equal to 4 [2]. As

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the maximum vertex degree of Halin graphs is at least 3, we will finish the problem, in this paper, by proving that the edge-face chromatic number is equal to $\max\{5, \Delta(G)\}$ for any Halin graph G with maximum vertex degree $\Delta(G) \geq 4$. We also give an algorithm for proper 5-edge-face total colorings of Halin graphs with $\Delta = 4$ and 5.

2. Lower bound.

LEMMA 2.1. *If G is a Halin graph with $\Delta(G) \geq 4$, then $\chi_{ef}(G) \geq \max\{5, \Delta(G)\}$.*

Proof. As $\chi_{ef}(G) \geq \Delta(G)$, we need only show that when $\Delta(G) = 4$, $\chi_{ef}(G) \geq 5$. In the following, we try to use four colors c_1, c_2, c_3, and c_4 to color the outer face and the incident faces and edges of a vertex v of degree 4. Without loss of generality, we may put color c_1 on the outer face, and the four incident edges of v are given colors c_1, c_2, c_3, and c_4, respectively, in the clockwise direction. Since all inner faces are adjacent to the outer face, the inner face incident with the two edges colored with c_2 and c_3 must be colored c_4. Similarly, the face incident with the edges colored with c_3 and c_4 must be colored c_2. The face incident with the edges colored with c_1 and c_4 is also adjacent to a face colored with c_2; hence, this face must be colored c_3.

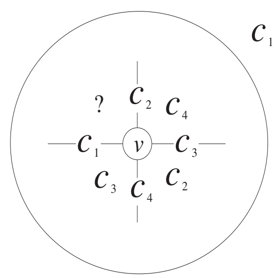


FIG. 1. *G is not 4-edge-face total colorable when $\Delta(G) = 4$.*

Now, we can check that the remaining uncolored incident face to v is not only incident with edges colored with c_1 and c_2 but also adjacent to faces colored with c_3 and c_4 (see Figure 1); therefore, it has been verified that a Halin graph with $\Delta = 4$ is not 4-edge-face total colorable. ∎

3. Halin graphs with $4 \leq \Delta \leq 5$.

Suppose G is a Halin graph and $4 \leq \Delta(G) \leq 5$. We shall give an algorithm for constructing a 5-edge-face total coloring of G. By Lemma 2.1, it follows that $\chi_{ef}(G) = 5$.

Let v be any inner vertex of a Halin graph G. The *contiguous elements* of v are comprised of the following: its incident edges, its incident faces, and outer edges incident with its incident faces.

Our algorithm consists in successively choosing inner vertices of G and coloring their contiguous elements that have not already been colored. After each such choice and coloring, we *label* that vertex (a process distinct from naming it as v, u, etc.) so that we have a record of which inner vertices have had their contiguous elements colored.

We begin the algorithm by choosing an inner vertex of maximum degree. This vertex is denoted by v in Figure 2, which depicts the coloring that we give to the contiguous elements in each of the cases $d(v) = 4, 5$.

Each step of the algorithm, after the first, consists of choosing an unlabeled inner vertex (if one exists) adjacent to a labeled vertex and coloring the contiguous elements. Since the graph T_I induced by the inner vertices is a subtree of the characteristic tree

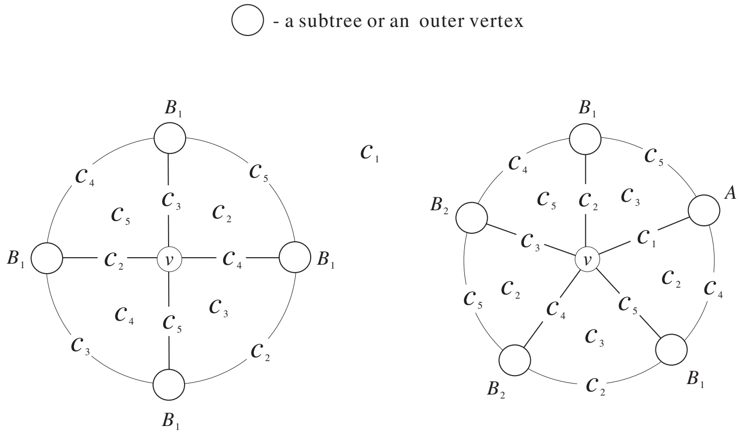


FIG. 2. *After coloring the contiguous elements of v, each subtree (or outer vertex) is surrounded by pattern A, B_1, or B_2 as indicated (see Figure 3).*

T, these steps will eventually label every inner vertex and color every face and edge of G, since each such element is contiguous with at least one inner vertex.

To specify the coloring process at each step, let u be an unlabeled inner vertex adjacent to a labeled vertex v. Note that since T_I is a tree, v is the unique labeled vertex adjacent to u, and so the edge uv is the only element contiguous with u that is already colored. Then u is a root of a subtree T_u of T.

We shall now describe a coloring process that ensures that, after each step of the algorithm, at any unlabeled u adjacent to a labeled v, the subtree T_u is surrounded by one of the patterns A, B_1, B_2, or C shown in Figure 3 (where p, q, r, and s are the distinct colors 2, 3, 4, and 5, respectively, in some order, the order varying with the instance of the pattern).

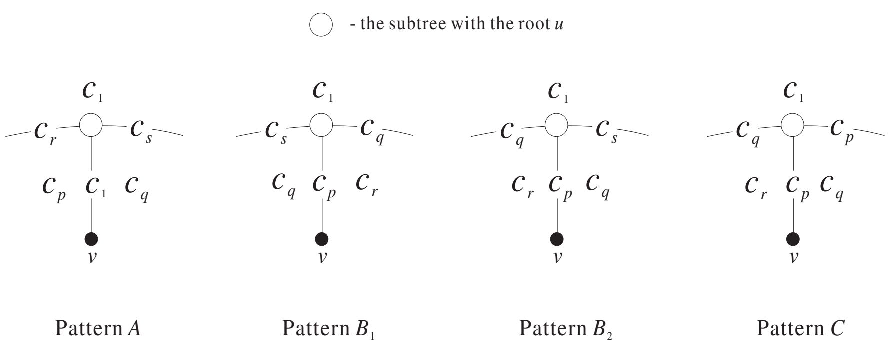


FIG. 3. *The possible patterns at each inner vertex adjacent to v after the step at which v is labeled.*

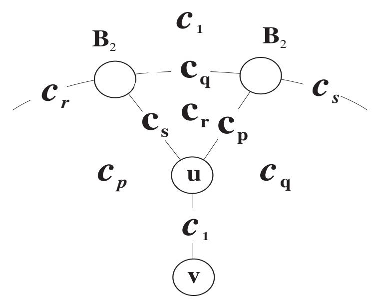
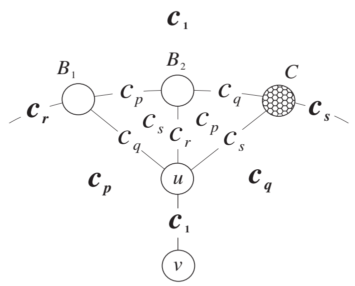
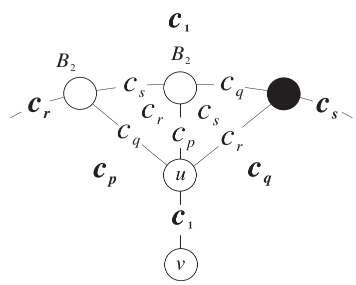
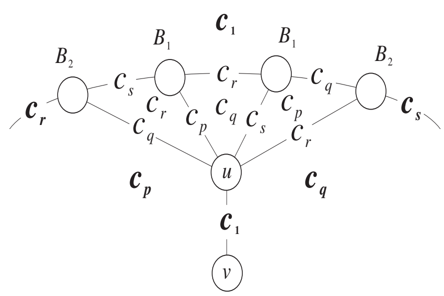
Note that, from Figure 2, if u is adjacent to the first labeled vertex, then the pattern is one of A, B_1, or B_2; thus the first step is valid.

Table 1 shows how the algorithm colors the contiguous elements of u if the pattern surrounding T_u is pattern A for each of the three possible values of $d(u)$. Note that, in the case $d(u) = 4$, the required coloring depends on whether the third vertex (other than v) adjacent to u is inner or outer.

Tables 2, 3, and 4 similarly show the required colorings if patterns B_1, B_2, or C, respectively, surround T_u.

TABLE 1

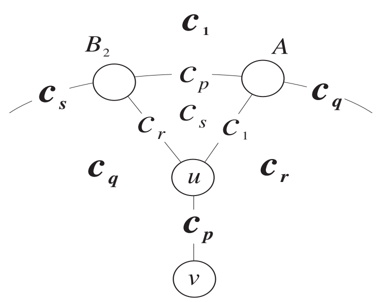
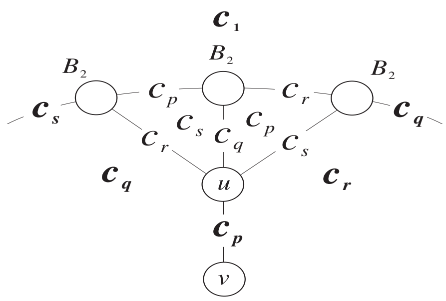
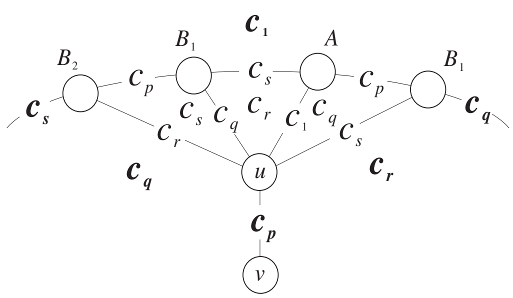
○ – an outer vertex or a subtree ● – an outer vertex ◍ – a subtree

Pattern	$d(u)$	Coloring
	3	
A	4	 
	5	

The result now follows from the fact that (as may easily be verified) in each of the cases $d(u) = 3, 4, 5$ in Tables 1–4, the coloring is such that the subtrees rooted at the unlabeled inner vertices adjacent to u are once again surrounded by one of the patterns of Figure 3. As already noted, since T_I is a tree, the process terminates with all faces and edges colored. The procedure is summarized as follows:

1. Color the outer face with c_1.
2. Choose a vertex v with $d(v) = \Delta$, and put colors to the contiguous elements of it as stated in Figure 2. Mark v as labeled.

TABLE 2

Pattern	$d(u)$	Coloring
B_1	3	
	4	
	5	

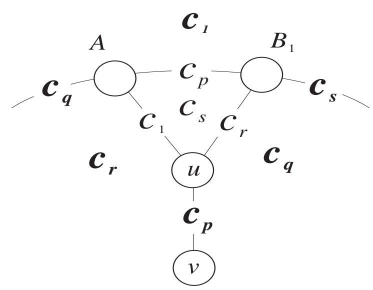
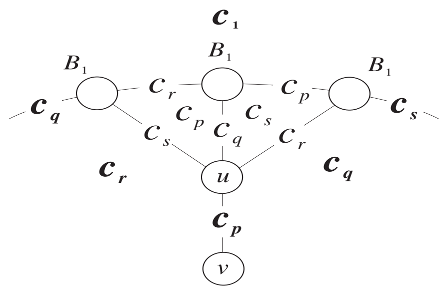
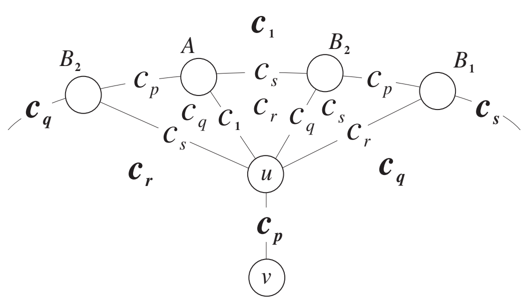
3. Choose an unlabeled inner vertex u which is incident to a labeled vertex v, and put colors to the uncolored contiguous elements of u following the schemes in the tables. Mark u as labeled.
4. If there are some unlabeled inner vertices remaining, go to step 3; otherwise stop.

THEOREM 3.1. *If G is a Halin graph with $\Delta(G) = 4$ or 5, then $\chi_{ef}(G) = 5$.*

4. Halin graphs with $\Delta \geq 6$. In this section, we shall show by induction on the number of inner vertices of G that $\chi_{ef}(G) = \Delta(G)$ when $\Delta(G) \geq 6$.

LEMMA 4.1. *Let T be the characteristic tree of a Halin graph G. There is an inner vertex v such that $d(v) - 1$ vertices adjacent to v are outer vertices.*

TABLE 3

Pattern	$d(u)$	Coloring
	3	
B_2	4	
	5	

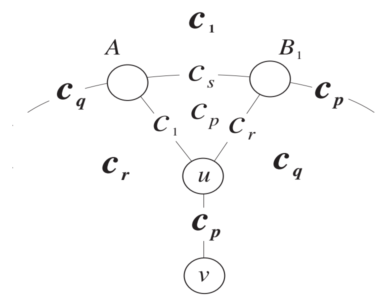
Proof. Suppose $D = v_0v_1v_2 \dots v_n$ is a diameter of the characteristic tree T of G. Clearly, all vertices adjacent to v_1 are end vertices of T except v_2. Thus, $d(v_1) - 1$ vertices adjacent to v_1 are outer vertices. ∎

THEOREM 4.2. *If G is a Halin graph with $\Delta(G) \geq 6$, then $\chi_{ef}(G) = \Delta(G)$.*

Proof. Suppose G is a Halin graph with $\Delta(G) = k$ and $k \geq 6$. It is obvious that $\chi_{ef}(G) \geq k$; therefore, we shall show in the following that G is k-edge-face total colorable by induction on the number of inner vertices of G. It is clear that when G has only one inner vertex v, all other k vertices of G are outer vertices. The characteristic tree of G is a *star*, and G is called a *wheel*. Let the outer vertices of G in the clockwise direction be $v_1, v_2, \dots, v_k$, the inner face incident with v_i and v_{i+1} for $i = 1, 2, \dots, k$ be f_i, where $v_{k+1} = v_1$. We can color the edges and faces of G with the following steps:

1. Color the outer face with c_1, and, for $i = 1, 2, \dots, k$, color the edge v_iv with c_i.

Table 4

<table>
<tr><th>Pattern</th><th>$d(u)$</th><th>Coloring</th></tr>
<tr><td rowspan="3">C</td><td>3</td><td>
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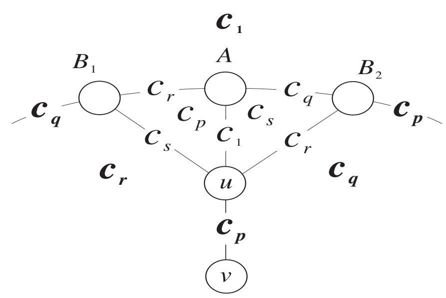
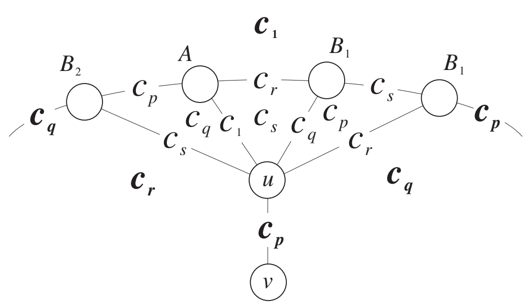
<tr><td>4</td><td>
</td></tr>
<tr><td>5</td><td>
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</table>

2. Color f_i with c_{i+2} for $i = 1, 2, \ldots, k-2$, and color f_{k-1} and f_k with c_3 and c_2, respectively.
3. For $i = 1, 2, \ldots, k-3$, color v_iv_{i+1} with c_{i+3}, color both $v_{k-2}v_{k-1}$ and v_kv_1 with c_3, and color $v_{k-1}v_k$ with c_2 as shown in Figure 4.

Now, let $n > 1$ and make the inductive assumption that any Halin graph H with $\Delta(H) \geq 6$ is $\Delta(H)$-edge-face colorable if the number of inner vertices of H is less than n. Let G be a Halin graph with $\Delta(G) = k \geq 6$ and G have n inner vertices. Lemma 4.1 tells us that there is an inner vertex v of G such that $d(v) - 1$ adjacent vertices of v are outer vertices. Let v_0 be the outer vertex preceding v_1, and $v_{d(v)}$ the outer vertex following $v_{d(v)-1}$, again in the clockwise direction. Now let G' be the graph formed by contracting the edges $vv_1, vv_2, \ldots, vv_{d(v)-1}, v_1v_2, v_2v_3, \ldots, v_{d(v)-2}v_{d(v)-1}$ so that $v, v_1, \ldots, v_{d(v)-1}$ are merged into a single vertex (see Figure 5). Then G' has $n-1$ inner vertices and $\Delta(G') \leq k$.

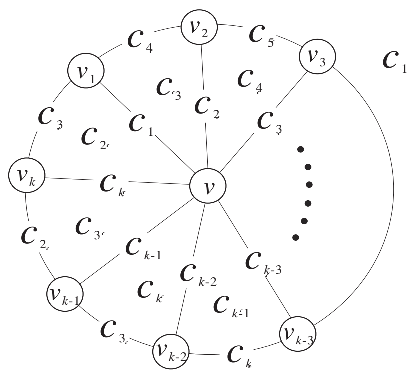

FIG. 4. *A wheel with maximum degree $k \geq 6$ is k-edge-face total colorable.*

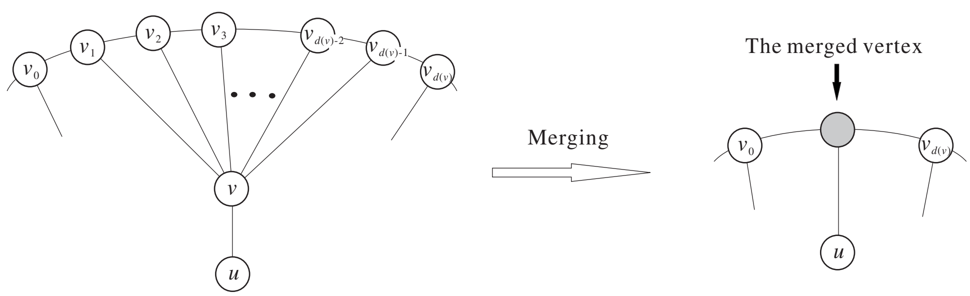

FIG. 5. *Merging v with its adjacent outer vertices.*

From the induction hypothesis, G' is k-edge-face colorable. G can be restored from extracting the merged vertex in G'. Let u be the inner vertex and v_0 and $v_{d(v)}$ be the outer vertices adjacent to the merged vertex in G'. Clearly, u is adjacent to v, and v_0 and $v_{d(v)}$ are adjacent to v_1 and $v_{d(v)}$, respectively. Let f'_i be the inner faces incident with v_iv_{i+1} for $i = 0, 1, \ldots, d(v) - 1$ as shown in Figure 6.

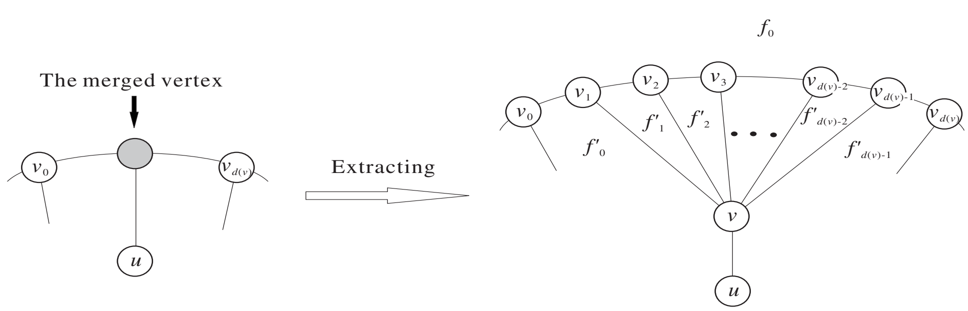

FIG. 6. *Restoring G by extracting the merged vertex.*

Let c^* be a k-edge-face total coloring of G'. A proper k-edge-face total coloring of G can be obtained by extending c^* to the inner edges $vv_1, vv_2, \ldots, vv_{d(v)-1}$, the inner faces $f'_1, f'_2, \ldots, f'_{d(v)-2}$, and the outer edges $v_1v_2, v_2v_3, \ldots, v_{d(v)-2}v_{d(v)-1}$.

We first consider $d(v) \geq 4$. We may put $c^*(v_{d(v)-2}v_{d(v)-1}) = c^*(f'_{d(v)-1})$ and choose proper colors for the remaining faces and edges in the following order:

$$vv_{d(v)-1}, vv_{d(v)-2}, vv_1, vv_2, \ldots, vv_{d(v)-3}, v_1v_2, v_2v_3, \ldots, v_{d(v)-3}v_{d(v)-2}, f'_1, f'_2, \ldots, f'_{d(v)-2}.$$

It can be easily checked that when coloring each face or edge, there are not more than five colors on its incident faces and adjacent edges. Thus, there is at least one color left as a possible option.

If $d(v) = 3$, we put $c^*(v_1v_2) = c^*(f'_2)$ and choose another color for v_0v_1 if $c^*(v_0v_1) = c^*(v_1v_2)$. We finally choose colors for vv_1, vv_2, and then f'_1.

Hence, G is k-edge-face total colorable. ∎

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