Collocation Methods for General Volterra Functional Integral Equations with Vanishing Delays

Xie, Hehu; Zhang, Ran; Brunner, Hermann

Published in:
SIAM Journal on Scientific Computing

DOI:
10.1137/100818595

Published: 01/12/2011

Document Version:
Publisher's PDF, also known as Version of record

SIAM J. SCI. COMPUT.
Vol. 33, No. 6, pp. 3303–3332

COLLOCATION METHODS FOR GENERAL VOLTERRA FUNCTIONAL INTEGRAL EQUATIONS WITH VANISHING DELAYS*

HEHU XIE†, RAN ZHANG‡, AND HERMANN BRUNNER§

Abstract. We analyze the existence, uniqueness, and regularity properties of solutions for general Volterra functional integral equations with the delay function $\theta(t)$ vanishing at the initial point of the given interval $[0,T]$ (with $\theta(t) = qt,\ 0 < q < 1$, representing an important special case). The focus of the paper is then on the the attainable order of convergence, and the question of possible superconvergence, for collocation solutions in certain piecewise polynomial spaces. Numerical experiments complement the theoretical convergence results.

Key words. Volterra functional integral equations, vanishing delays, pantograph-type delays, existence and regularity of solutions, collocation solutions, optimal order of convergence

AMS subject classifications. 65R20, 65Q20, 45D05

DOI. 10.1137/100818595

1. Introduction.

The aim of this paper is the study of the the optimal convergence properties of certain collocation solutions for Volterra functional integral equations (VFIEs) of the form

$$u(t) = b(t)u(\theta(t)) + f(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \qquad t \in I. \tag{1.1}$$

We shall be particularly interested, not least in order to highlight key ideas in the convergence analysis of collocation solutions, in the special cases

$$u(t) = b(t)u(\theta(t)) + f(t) + (\mathcal{V}_\theta u)(t), \qquad t \in I := [0,T], \tag{1.2}$$

and

$$u(t) = b(t)u(\theta(t)) + f(t) + (\mathcal{W}_\theta u)(t), \quad t \in I. \tag{1.3}$$

The Volterra delay integral operators $\mathcal{V}$, $\mathcal{V}_\theta$, and $\mathcal{W}_\theta$ (from $C(I) \to C(I)$) describing these VFIEs are defined by

$$(\mathcal{V}u)(t) := \int_0^t K_0(t,s)u(s)\,ds, \tag{1.4}$$

*Submitted to the journal's Methods and Algorithms for Scientific Computing section December 17, 2010; accepted for publication (in revised form) August 15, 2011; published electronically December 1, 2011.
http://www.siam.org/journals/sisc/33-6/81859.html

†LSEC and Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (hhxie@lsec.cc.ac.cn). This author's work was supported by the National Science Foundation of China (NSFC 10801062,11001259).

‡School of Mathematics, Jilin University, Changchun 130012, China (zhangran@jlu.edu.cn). This author's work was supported by NSFC (10801062, 11071102), Forefront of Science and Interdisciplinary Innovation Projects, Jilin University, the 985 project of Jilin University.

§Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C 5S7 Canada, and Department of Mathematics, Hong Kong Baptist University, Hong Kong, China (hbrunner@math.hkbu.edu.hk). This author's work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC Discovery Grant 9406) and by the Hong Kong Research Grants Council (HKBU 200207).

$$(\mathcal{V}_\theta u)(t) := \int_0^{\theta(t)} K_1(t,s)u(s)\,ds, \tag{1.5}$$

and

$$(\mathcal{W}_\theta u)(t) := \int_{\theta(t)}^{t} K(t,s)u(s)\,ds, \tag{1.6}$$

respectively, and the delay function θ is subject to the following conditions:

(D1) $\theta(0) = 0$, and θ is strictly increasing on the interval I;

(D2) $\theta(t) \le \bar{q}t,\ t \in I$, for some $\bar{q} \in (0,1)$;

(D3) $\theta \in C^d(I)$ for some $d \ge 0$.

We will refer to a θ that satisfies (D1) as a vanishing delay function (or, in short, a *vanishing delay*). (In section 7 we will briefly comment on vanishing delays that are not monotone on I.)

The linear case, $\theta(t) = qt = t - (1-q)t =: t - \tau(t)$ $(0 < q < 1)$ (proportional delay) is also known as the *pantograph* delay function (cf. [3]). Accordingly, we will refer to the corresponding VFIEs

$$u(t) = b(t)u(qt) + f(t) + \int_0^{qt} K(t,s)u(s)\,ds, \quad t \in I, \tag{1.7}$$

and

$$u(t) = b(t)u(qt) + \int_{qt}^{t} K(t,s)u(s)\,ds, \quad t \in I, \tag{1.8}$$

as pantograph VFIEs.

The given functions in (1.1)–(1.6) are assumed to be (at least) continuous on their respective domains; more precise conditions will be stated in section 2.

Volterra initiated the study of functional integral equations with vanishing proportional delays: in his 1897 paper [14] he investigated the existence and uniqueness of continuous solutions of the first-kind VFIE

$$\int_{qt}^{t} H(t,s)u(s)\,ds = g(t), \quad t \in I,$$

by employing its differentiated form,

$$H(t,t)u(t) - qH(t,qt)u(qt) + \int_{qt}^{t} \frac{\partial H(t,s)}{\partial t} u(s)\,ds = g'(t).$$

If the (continuous) kernel $H(t,s)$ is such that $H(t,t) \neq 0$ for all $t \in I$, then the above functional integral equation can be written in the form (1.3).

The special VFIE

$$u(t) = g(t) + \int_0^{\theta(t)} K(t,s)u(s)\,ds, \quad t \in I, \tag{1.9}$$

was studied by Andreoli [2], Volterra [15], and Chambers [10] for $\theta(t) = qt$, while existence and uniqueness results for more general vanishing delays satisfying (D1)–(D3) can be found in Brunner [6, Chap. 5] and Brunner and Hu [8]. The most general analysis of the functional integral equation (1.3) with general (nonlinear) vanishing delays and $b \not\equiv 0$ is due to Denisov and Lorenzi [11]; they deal in particular with the ill-posed nature of the problem.

In recent years the numerical solution of the VFIEs (1.1)–(1.3) with proportional delays qt $(0 < q < 1)$ has been studied in a number of papers:

(i) The 2005 paper [8] deals the attainable order of superconvergence in (iterated) collocation solutions for (1.1) when $b(t) \equiv 0$: it is shown that superconvergence at the points of a uniform mesh is possible only when $q = 1/2$.

(ii) The numerical solution of a nonlinear version of (1.1) with $b(t) \equiv 0$ was studied in [5]: the method combines analytical techniques (Picard iteration) with numerical quadrature (trapezoidal rule).

(iii) The paper [1] considers the VFIE (1.2) with $b(t) \equiv 0$ (but including multiple proportional delays); here, the spectral method is used to compute numerical approximations to the solution.

(iv) The functional equation (1.2) with $\mathcal{V}_\theta = 0$ is the subject of [9], where a complete analysis of the attainable order of global convergence of collocation solutions in discontinuous piecewise polynomial spaces is established. It was also conjectured that superconvergence at the mesh points is not possible. (We show in the present paper that this conjecture is true.)

We also note that a particular case of the VIFE (1.1) (constant kernels K_0 and K_1 in (1.4) and (1.5); $b(t) \equiv 0$) arises in a computationally convenient reformulation of the pantograph delay differential equation

$$u'(t) = au(t) + bu(qt), \quad t \in [0, T]; \qquad u(0) = u_0$$

(see Fox et al. [12, p. 292] and Iserles [13]). This functional equation plays a key role in the mathematical modeling of the dynamics of the overhead current collection system for an electric locomotive. Here, the VFIE was solved by an extension of the Lanczos τ method.

However, numerical solution of the VFIEs (1.1), (1.2), (1.3) with $b(t) \not\equiv 0$ has not yet been investigated. Since collocation methods in piecewise polynomial spaces are known to yield robust and highly accurate numerical schemes for standard VIEs and VFIEs (with $b(t) \equiv 0$), it is the aim of our paper to study the impact of the delay term $b(t)u(\theta(t))$ on the classical (super-)convergence results and the computational implementation of the schemes.

The paper is organized as follows. In section 2 we analyze the existence, uniqueness, and regularity of solutions to (1.1), highlighting the results for the VFIEs (1.2) and (1.3). Section 3 contains the detailed description of the collocation equations and their computational forms for the VFIEs (1.1) and (1.3). The attainable global order of convergence of collocation solutions in spaces of (discontinuous) piecewise polynomials is established in section 4. Here we also present analogous results for iterated collocation solutions; these include a proof of the above-mentioned conjecture that the iterated collocation solution cannot be locally superconvergent at the mesh points. Section 5 contains a wide range of numerical examples that confirm the results of section 4. The analysis of the convergence properties of collocation solutions in $S_m^{(0)}(I_h)$ (*globally continuous* piecewise polynomials) is much more challenging and is not yet understood. In section 6 provide computational evidence that collocation at the Gauss points leads to divergent collocation solutions. The paper concludes with a brief section describing possible future work.

2. Existence, uniqueness, and regularity of solutions. In the present section we analyze the solvability of the general VFIE (1.1) which we state again for the convenience of the reader:

$$u(t) = b(t)u(\theta(t)) + f(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \qquad t \in I. \tag{2.1}$$

Introducing the linear operator $\mathcal{K}:\ C(I) \to C(I)$ defined by

$$\mathcal{K}\varphi(t) := b(t)\varphi(\theta(t)) + (\mathcal{V}\varphi)(t) + (\mathcal{V}_\theta\varphi)(t), \quad t \in I, \tag{2.2}$$

the VFIE (2.1) can be written in the form

$$(\mathcal{I} - \mathcal{K})u = f, \tag{2.3}$$

where $\mathcal{I}$ denotes the identity operator.

The following theorem generalizes Volterra's 1897 classical result on the existence and uniqueness of solutions for the VFIE (1.3) with $\theta(t) = qt$ $(0 < q < 1)$.

THEOREM 2.1. *Assume that the given functions b, K_0, and K_1 in* (2.2) *satisfy*

(i) $f \in C(I)$, $K_0 \in C(D)$, *and* $K_1 \in C(D_\theta)$, *where*

$$D := \{(t,s) : 0 \le s \le t \le T\}, \quad D_\theta := \{(t,s) : 0 \le s \le \theta(t), t \in I\};$$

(ii) $b \in C(I)$ *and* $\|b\|_\infty < 1$*;*

(iii) $\theta(t)$ *is subject to the assumptions* (D1)–(D3).

Then for each $f \in C(I)$ there exists a unique function $u \in C(I)$ which solves the general VFIE (2.2) *on I.*

Proof. The above result is also related to an existence/uniqueness result in [11]: in that paper the authors study the particular VFIE (1.3) with general nonlinear delay function θ. Their proof techniques are readily extended to the more general VFIE (2.1) in which θ is subject to the conditions (D1)–(D3). We omit the details. ∎

Next, we present the regularity result for the solution of the general VFIE (2.1).

THEOREM 2.2. *Assume that the given functions in the VFIE* (2.3) *possess the following properties:*

(i) $f, b \in C^\nu(I)$, $K_0 \in C^\nu(D)$, *and* $K_1 \in C^\nu(D_\theta)$ *for some integer* $\nu \ge 1$*;*

(ii) $\|b\|_\infty < 1$*;*

(iii) *the delay function θ is subject to* (D1)–(D3), *with* $\theta \in C^\nu(I)$ *and* $\|\theta'\|_\infty < 1$.

Then the (unique) solution u of (2.3) *satisfies* $u \in C^\nu(I)$.

Proof. By Theorem 2.1, the solution u of the VFIE (2.1) is continuous whenever $f, b \in C(I)$, $K_0 \in C(D)$, and $K_1 \in C(D_\theta)$. Since now the given functions are smooth, formal differentiation of the VFIE leads to

$$\begin{aligned} u'(t) &= b'(t)u(\theta(t)) + b(t)\theta'(t)u'(\theta(t)) + f'(t) + K_0(t,t)u(t) \\ &\quad + \int_0^t \frac{\partial K_0(t,s)}{\partial t} u(s)\,ds + K_1(t,\theta(t))\theta'(t)u(\theta(t)) \\ &\quad + \int_0^{\theta(t)} \frac{\partial K_1(t,s)}{\partial t} u(s)\,ds. \end{aligned} \tag{2.4}$$

Defining

$$\begin{aligned} \tilde{f}(t) &:= b'(t)u(\theta(t)) + f'(t) + K_0(t,t)u(t) + K_1(t,\theta(t))\theta'(t)u(\theta(t)) \\ &\quad + \int_0^t \frac{\partial K_0(t,s)}{\partial t} u(s)\,ds + \int_0^{\theta(t)} \frac{\partial K_1(t,s)}{\partial t} u(s)\,ds \end{aligned}$$

and

$$\tilde{u}(t) := u'(t), \quad \tilde{b}(t) := b(t)\theta'(t),$$

we may write (2.4) as

$$\tilde{u}(t) = \tilde{b}(t)\tilde{u}(\theta(t)) + \tilde{f}(t), \quad t \in I. \tag{2.5}$$

Since $K_0 \in C^1(D)$ and $K_1 \in C^1(D_\theta)$, f, $b \in C^1(I)$ and $u \in C(I)$, it follows that $\tilde{f}$, $\tilde{b} \in C(I)$. Assumption (iii) and Theorem 2.2 in [9] thus imply that (2.5) has a unique solution $u \in C^1(I)$.

Continuing in a similar way, we readily verify that, under the regularity assumptions of Theorem 2.2, the general VFIE (2.1) possesses a (unique) solution $u \in C^\nu(I)$. This completes the proof. □

Remark. Theorem 2.2 extends the special regularity results contained in [6, sect. 5.1.3], [8] (for (2.1) with $b(t) \equiv 0$), and [9] (for (1.1) with $\mathcal{V}_\theta = 0$).

As a particular case of Theorem 2.2 we obtain the following result on the regularity of solutions to the VFIE (1.3).

COROLLARY 2.3. *Assume that the given functions in the VFIE* (1.3),

$$u(t) = b(t)u(\theta(t)) + f(t) + \int_{\theta(t)}^{t} K(t,s)u(s)\,ds, \quad t \in I,$$

satisfy the following:

(i) $f, b \in C^\nu(I)$ *and* $K \in C^\nu(\bar{D}_\theta)$ *for some integer* $\nu \geq 1$, *where*

$$\bar{D}_\theta = \{(t,s) : \theta(t) \leq s \leq t, t \in I\};$$

(ii) $\|b\|_\infty < 1$;

(iii) $\theta(t)$ *is subject to* (D1)–(D3), *with* $\theta \in C^\nu(I)$ *and* $\|\theta'\|_\infty < 1$.

Then the unique solution u *of the VFIE* (1.3) *possesses the regularity* $u \in C^\nu(I)$.

3. Collocation in $S_{m-1}^{(-1)}(I_h)$.

Let

$$I_h := \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\}$$

be a mesh on the given interval $I = [0,T]$, and set

$$e_n := (t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n \ \ (n = 0, 1, \ldots, N-1),$$

and $h := \max\{h_n : 0 \leq n < N\}$. In the following we shall be concerned with collocation solutions u_h lying in the piecewise polynomial space

$$S_{m-1}^{(-1)}(I_h) := \{v : \ v|_{e_n} \in \mathcal{P}_{m-1} \ (n = 0, \ldots, N-1)\},$$

with $m \geq 1$, and where $\mathcal{P}_{m-1} = \mathcal{P}_{m-1}(e_n)$ denotes the set of (real) polynomials on e_n of degree not exceeding $m-1$. Since the dimension of this linear space equals Nm, we choose the set of collocation points as

$$X_h := \{t_{n,i} := t_n + c_i h_n : 0 < c_1 < \cdots < c_m \leq 1 \ (n = 0, \ldots, N-1)\}.$$

Here, $\{c_i\}$ is a prescribed set of collocation parameters.

We are looking for $u_h \in S_{m-1}^{(-1)}(I_h)$ satisfying the collocation equations

$$u_h(t) = b(t)u_h(\theta(t)) + f(t) + \int_0^t K_0(t,s)u_h(s)\,ds + \int_0^{\theta(t)} K_1(t,s)u_h(s)\,ds, \quad t \in X_h. \tag{3.1}$$

For the special VFIE (1.3) (on which we will often focus in the following) the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ is determined by

$$u_h(t) = b(t)u_h(\theta(t)) + f(t) + \int_{\theta(t)}^t K(t,s)u_h(s)\,ds, \quad t \in X_h. \tag{3.2}$$

In analogy to VIEs without delay arguments, we also define the *iterated* collocation solution u_h^{it} associated with u_h (cf. [6]): in the case of the VFIE (1.3) it is given by

$$u_h^{it}(t) := b(t)u_h(\theta(t)) + f(t) + \int_{\theta(t)}^t K(t,s)u_h(s)\,ds, \quad t \in I. \tag{3.3}$$

We note in passing that $u_h^{it}(t) = u_h(t)$ whenever $t \in X_h$. Moreover, if the given functions are continuous, then $u_h^{it} \in C(I)$.

We shall return to u_h^{it} in section 4.4 when we address the question as to whether, similar to classical VIEs, u_h^{it} can exhibit a higher order of convergence (superconvergence), on I or at the mesh points I_h, for judiciously chosen collocation parameters $\{c_i\}$.

3.1. The collocation equations on uniform meshes: Location of $\theta(t_{n,i})$. If the given mesh is *uniform*, $I_h := \{t_n = nh : \ 0 \le n \le N, \ t_N = T\}$, and if the delay function θ satisfies conditions (D1)–(D3), then for a collocation point $t = t_{n,i} := t_n + c_i h \in X_h$ we may write

$$\theta(t_{n,i}) = \theta(t_n + c_i h) =: h\{q_{n,i} + \gamma_{n,i}\} = t_{q_{n,i}} + \gamma_{n,i}h, \tag{3.4}$$

where

$$q_{n,i} := \left\lfloor \frac{\theta(t_n + c_i h)}{h} \right\rfloor, \qquad \gamma_{n,i} := \frac{\theta(t_n + c_i h)}{h} - q_{n,i} \in [0,1). \tag{3.5}$$

Here, $\lfloor x \rfloor$ denotes the greatest integer not exceeding $x \in \mathbb{R}$. For $\theta(t) = qt$ $(0 < q < 1)$, (3.5) becomes

$$q_{n,i} = \lfloor q(n + c_i) \rfloor, \quad \gamma_{n,i} = q(n + c_i) - q_{n,i} \ \ (i = 1, \dots, m). \tag{3.6}$$

Thus, for a given uniform mesh I_h and $\theta = qt$ the location of the images $\theta(t_{n,i})$ of the collocation points $t_{n,i} \in e_n$ can be characterized as follows (see also [6, 9]).

Phase I: If $0 \le n < q^I := \lceil \frac{qc_1}{1-q} \rceil$, then

$$\theta(t_{n,i}) \in (t_n, t_{n+1}) \quad \text{for all} \ \ i = 1, \dots, m. \tag{3.7}$$

Here, $\lceil x \rceil$ is the smallest integer greater than or equal to x.

Phase II: If $q^I \leq n < q^{II} := \lceil \frac{qc_m}{1-q} \rceil$, then there exist integers $\kappa_n \in \{1, \ldots, m-1\}$ so that

$$qt_{n,i} \in \begin{cases} (t_{n-1}, t_n], & 1 \leq i \leq \kappa_n, \\ (t_n, t_{n+1}], & \kappa_n < i \leq m. \end{cases} \tag{3.8}$$

(Note that this transition phase may be empty, for example, if $m = 1$ or if $q \leq 1/2$.)

Phase III: If $q^{II} \leq n \leq N-1$, then

$$\theta(t_{n,i}) \leq t_n \quad \text{for all } \; i = 1, \ldots, m. \tag{3.9}$$

For more general (nonlinear) vanishing delay functions θ satisfying (D1)–(D3), the numbers q^I and q^{II} are defined in a similar (but implicit) manner.

Remark. We observe that the numbers q^I and q^{II} defining the above Phases I, II, III do *not* depend on n or h: they depend only on q and, respectively, on the collocation parameters c_1 and c_m.

3.2. The computational form of the collocation equations on uniform I_h. In order to obtain a computationally feasible form of the collocation equation (3.1), we express u_h on the subinterval e_n by the local Lagrange representation,

$$u_h(t_n + vh) = \sum_{j=1}^{m} L_j(v) U_{n,j}, \qquad v \in (0, 1] \; (0 \leq n \leq N-1), \tag{3.10}$$

with

$$L_j(v) := \prod_{k \neq j}^{m} \frac{v - c_k}{c_j - c_k} \quad \text{and} \quad U_{n,j} := u_h(t_n + c_j h).$$

As a consequence of the above notation, the collocation equation (3.1) at $t = t_{n,i}$ ($i = 1, \ldots, m$) becomes

$$\begin{aligned} U_{n,i} &= M_n(t_{n,i}) + h \int_0^{c_i} K_0(t_{n,i}, t_n + sh) u_h(t_n + sh) ds \\ &\quad + b(t_{n,i}) u_h(\theta(t_{n,i})) + f(t_{n,i}) + (\mathcal{V}_\theta u_h)(t_{n,i}). \end{aligned} \tag{3.11}$$

The lag term corresponding to the operator $\mathcal{V}$ is

$$M_n(t_{n,i}) := \int_0^{t_n} K_0(t_{n,i}, s) u_h(s) ds,$$

while that corresponding to the delay term $\mathcal{V}_\theta$ can be expressed in the form

$$\begin{aligned} (\mathcal{V}_\theta u_h)(t_{n,i}) &= \int_0^{t_{q_{n,i}}} K_1(t_{n,i}, s) u_h(s) ds \\ &\quad + h \int_0^{\gamma_{n,i}} K_1(t_{n,i}, t_{q_{n,i}} + sh) u_h(t_{q_{n,i}} + sh) ds. \end{aligned}$$

The above expressions for $M_n(t_{n,i})$ and $(\mathcal{V}_\theta u_h)(t_{n,i})$ now become

$$M_n(t_{n,i}) = h \sum_{l=0}^{n-1} \sum_{j=1}^{m} \left(\int_0^1 K_0(t_{n,i}, t_l + sh) L_j(s) ds \right) U_{l,j}$$

and

$$(\mathcal{V}_\theta u_h)(t_{n,i}) = h\sum_{l=0}^{q_{n,i}-1}\sum_{j=1}^{m}\left(\int_0^1 K_1(t_{n,i},t_l+sh)L_j(s)ds\right)U_{l,j} + h\sum_{j=1}^{m}\left(\int_0^{\gamma_{n,i}} K_1(t_{n,i},t_{q_{n,i}}+sh)L_j(s)ds\right)U_{q_{n,i},j}$$

$(i = 1,\ldots,m)$.

In order to obtain a concise notation for the linear algebraic equations corresponding to (3.11), we introduce the matrices and vectors

$$M_n := \left(\int_0^{c_i} K_0(t_{n,i},t_n+sh)L_j(s)ds\right)_{1\le i,j\le m},$$

$$\mathbf{F}_n := (f(t_n+c_1h),\ldots,f(t_n+c_mh))^T,$$

$$M_{n,l} := \left(\int_0^1 K_0(t_{n,i},t_l+sh)L_j(s)ds\right)_{1\le i,j\le m}, \qquad \mathbf{U}_n := (U_{n,1},\ldots,U_{n,m})^T.$$

The structure of these systems of linear algebraic equations for the vectors (3.11), $\mathbf{U}_n\in\mathbb{R}^m$ $(0\le n\le N-1)$, depends on the delay function θ and changes as n moves from Phase I to Phase III (cf. (3.7), (3.8), (3.9)). The details are given below.

Phase I (total overlap): $0\le n<q^I$. Here we have $\theta(t_{n,i})\in(t_n,t_{n+1})$ for $i=1,\ldots,m$. This is always true for $n=0$. The corresponding systems of linear algebraic equations (3.11) have the form

$$[\mathcal{I}_m - B_n^I(\theta) - h(M_n+N_n^I(\theta))]\mathbf{U}_n = \mathbf{F}_n + h\sum_{l=0}^{n-1}(M_{n,l}+N_{n,l}^I)\mathbf{U}_l, \tag{3.12}$$

where $\mathcal{I}_m$ denotes the identity matrix in $\mathbb{R}^{m\times m}$. The matrices $B_n^I(\theta), N_n^I(\theta)$, and $N_{n,l}^I(\theta)$ are given by

$$B_n^I(\theta) := (b(t_{n,i})L_j(\theta(t_{n,i})/h-n))_{1\le i,j\le m},$$

$$N_n^I(\theta) := \left(\int_0^{\gamma_{n,i}} K_1(t_{n,i},t_n+sh)L_j(s)ds\right)_{1\le i,j\le m},$$

and

$$N_{n,l}^I := \left(\int_0^1 K_1(t_{n,i},t_l+sh)L_j(s)ds\right)_{1\le i,j\le m} \quad (l<n),$$

respectively.

Phase II (partial overlap): $q^I\le n<q^{II}$. Note that this phase may be *empty* (for example, when $m=1$). If $q^{II}>q^I$, the system of linear algebraic equations (3.11) for the vector $\mathbf{U}_n$ becomes

$$[\mathcal{I}_m - \bar{B}_n^{II}(\theta) - hM_n]\mathbf{U}_n = \mathbf{F}_n + h\sum_{l=0}^{n-1}M_{n,l}\mathbf{U}_l + h\sum_{l=0}^{n-2}N_{n,l}^I\mathbf{U}_l + \bar{S}_{n-1}^{II}(\theta)\mathbf{U}_{n-1}, \tag{3.13}$$

where

$$\bar{B}_n^{II}(\theta) := \mathrm{diag}(\underbrace{0,\ldots,0}_{\kappa_n},1,\ldots,1)(B_n^I(\theta)+hN_n^I),$$

$$\bar{S}_{n-1}^{II}(\theta) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\kappa_n},0,\ldots,0)(B_{n-1}^{II}(\theta)+hN_{n-1}^{II}(\theta))$$
$$+\,\mathrm{diag}(\underbrace{0,\ldots,0}_{\kappa_n},1,\ldots,1)(hN_{n,n-1}^I),$$

with

$$B_{n-1}^{II}(\theta) := \Big(b(t_{n,i})L_j(\theta(t_{n,i})/h-(n-1))\Big)_{1\le i,j\le m},$$

$$N_{n-1}^{II}(\theta) := \left(\int_0^{\gamma_{n,i}} K_1(t_{n,i},t_{n-1}+sh)L_j(s)ds\right)_{1\le i,j\le m}.$$

Here, $\mathrm{diag}(d_1,\ldots,d_m)$ denotes the diagonal matrix in $\mathbb{R}^{m\times m}$ with diagonal elements $d_1,\ldots,d_m$. The integers κ_n are defined in (3.8).

Phase III (pure delay phase): $q^{II}\le n\le N-1$. For these values of n there is no longer any overlap of the images $\theta(t_{n,i})$ and the interval (t_n,t_{n+1}) (cf. (3.9)): we now have $\theta(t_n+c_ih)\le t_n$ for $i=1,\ldots,m$. To be more precise, for each value of n there exist integers $\nu_n\in\{1,\ldots,m\}$ and $\theta_n<n-1$ so that

$$q_{n,i}=\theta_n \ \ (i\le\nu_n) \quad \text{and} \quad q_{n,i}=\theta_n+1 \ \ (i>\nu_n).$$

Hence, it is readily seen that the linear systems corresponding to the collocation equations assume the form

$$[\mathcal{I}_m-hM_n]\mathbf{U}_n=\mathbf{F}_n+h\sum_{l=0}^{n-1}M_{n,l}\mathbf{U}_l+h\sum_{l=0}^{\theta_n-1}N_{n,l}^I\mathbf{U}_l$$
$$+\,\bar{S}_{\theta_n}^{III}(\theta)\mathbf{U}_{\theta_n}+\hat{S}_{\theta_n+1}^{III}(\theta)\mathbf{U}_{\theta_n+1}, \tag{3.14}$$

with matrices

$$\bar{S}_{\theta_n}^{III}(\theta) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)(B_{\theta_n}^{III}(\theta)+hN_{\theta_n}^{III}(\theta))$$
$$+\,\mathrm{diag}(\underbrace{0,\ldots,0}_{\nu_n},1,\ldots,1)(hN_{n,\theta_n}^I),$$

$$\hat{S}_{\theta_n+1}^{II}(\theta) := \mathrm{diag}(\underbrace{0,\ldots,0}_{\nu_n},1,\ldots,1)(B_{\theta_n+1}^{III}(\theta)+hN_{\theta_n+1}^{III}(\theta)),$$

where

$$B_{\theta_n}^{III} := \Big(b(t_{n,i})L_j(\theta(t_{n,i})/h-\theta_n)\Big)_{1\le i,j\le m},$$

$$N_{\theta_n}^{III} := \left(\int_0^{\gamma_{n,i}} K_1(t_{n,i},t_{q_n}+sh)L_j(s)ds\right)_{1\le i,j\le m},$$

and

$$B_{\theta_n+1}^{III} := \Big(b(t_{n,i})L_j(\theta(t_{n,i})-(\theta_n+1))\Big)_{1\le i,j\le m},$$

$$N_{\theta_n+1}^{III} := \left(\int_0^{\gamma_{n,i}} K_1(t_{n,i},t_{\theta_n+1}+sh)L_j(s)ds\right)_{1\le i,j\le m}.$$

3.3. The collocation equations for the VFIE (1.3). We briefly describe the structure of the collocation equations for VFIE corresponding to the special delay integral operator $\mathcal{W}_\theta$ defined by (1.6), assuming that $\theta(t) = qt$ $(0 < q < 1)$. We first observe that $(\mathcal{W}_\theta u_h)(t_{n,i})$ can be written as

$$
\begin{aligned}
(\mathcal{W}_\theta u_h)(t_{n,i}) &= h\int_{\gamma_{n,i}}^{1} K(t_{n,i}, t_{q_{n,i}} + sh)u_h(t_{q_{n,i}} + sh)ds \\
&\quad + \int_{t_{q_{n,i}+1}}^{t_n} K(t_{n,i}, s)u_h(s)ds \\
&\quad + h\int_0^{c_i} K(t_{n,i}, t_n + sh)u_h(t_n + sh)ds \\
&= h\sum_{j=1}^{m}\left(\int_{\gamma_{n,i}}^{1} K(t_{n,i}, t_{q_{n,i}} + sh)L_j(s)ds\right)U_{q_{n,i},j} \\
&\quad + h\sum_{l=q_{n,i}+1}^{n-1}\sum_{j=1}^{m}\left(\int_0^1 K(t_{n,i}, t_l + sh)L_j(s)ds\right)U_{l,j} \\
&\quad + h\sum_{j=1}^{m}\left(\int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(s)ds\right)U_{n,j},
\end{aligned}
\tag{3.15}
$$

with $q_{n,i}$ and $\gamma_{n,i}$ defined in (3.6). Thus, the collocation equation for (1.3),

$$
u_h(t) = b(t)u_h(\theta(t)) + f(t) + (\mathcal{W}_\theta u_h)(t), \quad t \in X_h,
\tag{3.16}
$$

yields the following systems of linear algebraic equations for $\mathbf{U}_n \in \mathbb{R}^m$. We use the notation

$$
\begin{aligned}
C_n &:= \left(\int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(s)ds\right)_{1\le i,j\le m}, \\
\bar{C}_n^I(\theta) &:= \left(\int_{\gamma_{n,i}}^{c_i} K(t_{n,i}, t_n + sh)L_j(s)ds\right)_{1\le i,j\le m}, \\
C_n^{(l)} &:= \left(\int_0^1 K(t_{n,i}, t_l + sh)L_j(s)ds\right)_{1\le i,j\le m}, \\
\tilde{C}_n^{(l)}(\theta) &:= \left(\int_{\gamma_{n,i}}^1 K(t_{n,i}, t_l + sh)L_j(s)ds\right)_{1\le i,j\le m}.
\end{aligned}
$$

Phase I (total overlap): $0 \le n < q^I$. The analogue of (3.12) is given by

$$
[\mathcal{I}_m - B_n^I(\theta) - h\bar{C}_n^I(\theta)]\mathbf{U}_n = \mathbf{F}_n.
\tag{3.17}
$$

Here, $\bar{C}_n^I(\theta)$ is formally equivalent to $M_n + N_n^I(\theta)$ in (3.12) when $K_1 = -K_0 =: -K$.

Phase II (partial overlap): $q^I \le n < q^{II}$.

If this phase is not empty, the systems of linear algebraic equations (3.13) for $\mathbf{U}_n$ become

$$
(\mathcal{I}_m - h\bar{C}_n^{II}(\theta))\mathbf{U}_n = \mathbf{F}_n + h\bar{S}_{n-1}^{II}(\theta)\mathbf{U}_{n-1},
\tag{3.18}
$$

with

$$\bar{C}_n^{II}(\theta) := \mathrm{diag}(\underbrace{1,\dots,1}_{\kappa_n},0,\dots,0)C_n + \mathrm{diag}(\underbrace{0,\dots,0}_{\kappa_n},1,\dots,1)(\bar{C}_n^I + B_n^I(\theta)),$$

$$\tilde{S}_{n-1}^{II}(\theta) := \mathrm{diag}(\underbrace{1,\dots,1}_{\kappa_n},0,\dots,0)\hat{C}_n^{(n-1)}(\theta).$$

Phase III (pure delay phase): $q^{II} \le n \le N-1$. Here we have (cf. (3.14))

$$(\mathcal{I}_m - hC_n)\mathbf{U}_n = \mathbf{F}_n + \left[\bar{S}_{\theta_n}^{III}(\theta)\mathbf{U}_{\theta_n} + S_{\theta_n+1}^{III}(\theta)\mathbf{U}_{\theta_n+1} + h\sum_{l=\theta_n+2}^{n-1} C_n^{(l)}\mathbf{U}_l\right], \tag{3.19}$$

with

$$\bar{S}_{\theta_n}^{III}(\theta) := \mathrm{diag}(\underbrace{1,\dots,1}_{\nu_n},0,\dots,0)(B_{\theta_n}^{III}(\theta) + h\tilde{C}_n^{(\theta_n)}(\theta)),$$

$$S_{\theta_n+1}^{III} := \mathrm{diag}(\underbrace{1,\dots,1}_{\nu_n},0,\dots,0)C_n^{(\theta_n+1)} + \mathrm{diag}(\underbrace{0,\dots,0}_{\nu_n},1,\dots,1)(B_{\theta_n+1}^{III}(\theta) + h\tilde{C}_n^{(\theta_n+1)}(\theta)).$$

Once the above systems of linear algebraic equations have been solved, the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for $t = t_n + vh$ ($v \in (0,1]$, $0 \le n \le N-1$) is given by the local representation (3.10).

We can then also compute the *iterated* collocation solution u_h^{it} (cf. (3.3)) at this point: its value is determined by

$$\begin{aligned} u_h^{it}(t_n + vh) &= b(t_n+vh)u_h(\theta(t_n+vh)) + f(t_n+vh) \\ &\quad + \int_{\theta(t_n+vh}^{t_n+vh} K(t_n+vh, u_h(s)\,ds \\ &= b(t_n+vh)u_h(\theta(t_n+vh)) + f(t_n+vh) \\ &\quad + \int_{t_{q_n(v)}+\gamma_n(v)}^{t_n+vh} K(t_n+vh, s)u_h(s)\,ds, \end{aligned}$$

where, for $v \in [0,1]$ and $\theta(t) = qt$ ($0 < q < 1$),

$$q_n(v) := \lfloor q(n+v)\rfloor, \quad \gamma_n(v) := q(n+v) - q_n(v).$$

The (fully discretized) computational form of u_h^{it} will be considered at the end of section 3.5.

3.4. Existence and uniqueness of the collocation solution.

THEOREM 3.1. *Let* f, b, K_0, K_1 *in* (2.2) *be subject to the assumptions stated in Theorem* 2.1, *and assume that* $u_h \in S_{m-1}^{(-1)}(I_h)$ *is the collocation solution to the delay VFIE*

$$u(t) = b(t)u(\theta(t)) + f(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \quad t \in I$$

(see (3.1) *and* (3.10)*). Then there exists an* $\bar{h} > 0$ *(depending on* θ*) so that for all* $h \in (0, \bar{h})$ *each of the linear algebraic systems* (3.12), (3.13), *and* (3.14) *possesses*

a unique solution $\boldsymbol{U}_n$. Thus, the collocation solution u_h exists and is unique for all uniform meshes with diameters $h < \bar{h}$.

Proof. The proof is based on an existence and uniqueness result for the collocation equations corresponding to the special functional equation $u(t) = b(t)u(\theta(t)) + f(t)$ (i.e. (3.1) with $\mathcal{V} = \mathcal{V}_\theta = 0$).

LEMMA 3.2. (see [9]) *Assume that*

(i) f *and* b *in* $u(t) = b(t)u(\theta(t)) + f(t)$ *are continuous on* I*;*

(ii) $\theta(t)$ *satisfies* (D1)–(D3) *with* $d \geq 2$, *and* $\|\theta'\|_\infty < 1$*;*

(iii) $\|b\|_\infty < 1$*;*

(iv) *the meshes* I_h *underlying the collocation space* $S_{m-1}^{(-1)}(I_h)$ *are uniform ones. Then there exists an* $\bar{h} > 0$ *(depending on* θ*) such that for all* $h \in (0, \bar{h})$ *the linear algebraic systems in Phases* I, II, *and* III,

$$[\mathcal{I}_m - B_n^I(\theta)]\,\boldsymbol{U}_n = \boldsymbol{F}_n,$$

$$[\mathcal{I}_m - \mathrm{diag}(\underbrace{0,\ldots,0}_{\kappa_n},1,\ldots,1)B_n^I(\theta)]\,\boldsymbol{U}_n = \boldsymbol{F}_n + \mathrm{diag}(\underbrace{1,\ldots,1}_{\kappa_n},0,\ldots,0)B_{n-1}^{II}(\theta)\,\boldsymbol{U}_{n-1},$$

and

$$\boldsymbol{U}_n = \boldsymbol{F}_n + \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)B_{\theta_n}^{III}(\theta)\,\boldsymbol{U}_{\theta_n} + \mathrm{diag}(\underbrace{0,\ldots,0}_{\nu_n},1,\ldots,1)B_{\theta_n+1}^{III}(\theta)\,\boldsymbol{U}_{\theta_n+1},$$

are unique for all meshes I_h *with diameters* $h \in (0, \bar{h})$.

We will prove Theorem 3.1 first for the linear algebraic systems (3.12) of Phase I. According to Lemma 3.2 there exists an $\bar{h} > 0$ such that, for all $h \in (0, \bar{h})$, the coefficient matrix of the collocation scheme (3.12), $\mathcal{I}_m - B_n^I(\theta)$, is nonsingular. It thus follows that

$$|\mathcal{I}_m - B_n^I(\theta) - h(M_n + N_n^I(\theta)) - (\mathcal{I}_m - B_n^I(\theta))\|_\infty \leq h\|M_n + N_n^I(\theta)\|_\infty.$$

This means that $\mathcal{I}_m - B_n^I(\theta) - h(M_n + N_n^I(\theta))$ is nonsingular for all sufficiently small mesh diameters h, assuring the unique solvability of the linear algebraic systems (3.12).

The proof of the analogous statement for the linear algebraic systems (3.13) in Phase II can be carried out in a very similar way.

For Phase III it is obvious that the linear algebraic systems (3.14) are uniquely solvable for all sufficiently small $h > 0$, since the matrices $\mathcal{I}_m - hM_n$ do not depend on θ. ∎

The existence and uniqueness of the collocation solution for the VIFE

$$u(t) = b(t)u(\theta(t)) + f(t) + (\mathcal{W}_\theta u)(t), \quad t \in I, \tag{3.20}$$

can be established by a straightforward modification of the above arguments. We omit the details but summarize the result in the following theorem.

THEOREM 3.3. *Assume that* b *(with* $\|b\|_\infty < 1$*),* f*, and* K *in* (3.20), (1.4) *are continuous on their respective domains, and let the collocation solution* $u_h \in S_{m-1}^{(-1)}(I_h)$ *be defined by* (3.16) *and* (3.10)*. Then there exists an* $\bar{h} > 0$ *(depending on* θ*) such that for all* $h \in (0, \bar{h})$ *the linear algebraic systems* (3.17), (3.18), *and* (3.19) *all possess* *unique solutions* $\boldsymbol{U}_n$*. Thus, for any such uniform mesh* I_h *the collocation solution* u_h *for* (3.20) *exists and is unique.*

3.5. The fully discretized collocation equations for the VFIE (1.3). The collocation equations that we derived in the previous section are in general not yet in a form amenable to numerical computation, since the memory terms corresponding to the Volterra integral operators $\mathcal{V}$, $\mathcal{V}_\theta$, and $\mathcal{W}_\theta$ can in general not be evaluated explicitly. Thus, another discretization step, employing appropriate quadrature rules, is needed to approximate the integrals in the matrices occurring in the linear algebraic systems (3.12)–(3.14) and (3.17)–(3.19).

To approximate these integrals we will choose interpolatory m-point quadrature rules whose abscissas are based on the m collocation parameters $\{c_i\}$. This will make sure that the order of the resulting quadrature errors will (at least) match the order of convergence of the exact collocation solution, either globally (on I) or at the mesh points I_h.

Owing to limitation of space, we will describe this in detail only for the collocation equation corresponding to the (in our view, most interesting) VFIE (3.20), where the limits of integration in $\mathcal{W}_\theta$ are $\theta(t)$ and t. Denoting by $\hat{\mathcal{W}}_\theta$ the discretized delay integral operator $\mathcal{W}_\theta$, the resulting collocation solution $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$ is defined by the *discretized collocation equation*

$$\hat{u}_h(t) = b(t)\hat{u}_h(\theta(t)) + f(t) + (\hat{\mathcal{W}}_\theta \hat{u}_h)(t), \qquad t \in X_h. \tag{3.21}$$

In analogy to (3.10), its local representation on e_n is given by

$$\hat{u}_h(t_n + vh) = \sum_{j=1}^{m} L_j(v)\hat{U}_{n,j}, \quad v \in (0,1], \tag{3.22}$$

with $\hat{U}_{n,j} := \hat{u}_h(t_n + c_j h)$. The vectors $\hat{\mathbf{U}}_n := (\hat{U}_{n,1}, \dots, \hat{U}_{n,m})^T$ are the solutions of the discretized versions of the linear algebraic systems (3.11)–(3.13). To be more precise, we denote the approximations to the terms

$$\begin{aligned}
(Q_n^{(l)} u_h)(t) &:= \int_0^1 K(t, t_l + sh)u_h(t_l + sh)ds \quad (l < n),\\
(Q_n u_h)(t) &:= \int_0^v K(t, t_n + sh)u_h(t_n + sh)ds\\
&= v\int_0^1 K(t, t_n + svh)u_h(t_n + svh)ds \qquad (v = c_i),
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{Q}_n^{(l)} u_h)(t) &:= \int_v^1 K(t, t_l + sh)u_h(t_l + sh)ds\\
&= (1-v)\int_0^1 K(t, t_l + vh + (1-v)sh)u_h(t_l + vh + (1-v)sh)ds,
\end{aligned}$$

$$\begin{aligned}
(\bar{Q}_n^I u_h)(t) &:= \int_v^w K(t, t_n + sh)u_h(t_n + sh)ds\\
&= (w-v)\int_0^1 K(t, t_n + vh + (w-v)sh)u_h(t_n + vh + (w-v)sh)ds
\end{aligned}$$

(i.e. the elements of the matrices

$$C_n^{(l)} := \left(\int_0^1 K(t_{n,i}, t_l + sh) L_j(s) ds\right)_{1\le i,j\le m},$$
$$\tilde{C}_n^{(l)}(\theta) := \left(\int_{\gamma_{n,i}}^1 K(t_{n,i}, t_l + sh) L_j(s) ds\right)_{1\le i,j\le m},$$
$$C_n := \left(\int_0^{c_i} K(t_{n,i}, t_n + sh) L_j(s) ds\right)_{1\le i,j\le m},$$
$$\bar{C}_n^I(\theta) := \left(\int_{\gamma_{n,i}}^{c_i} K(t_{n,i}, t_n + sh) L_j(s) ds\right)_{1\le i,j\le m},$$

in (3.12), (3.13), (3.14)) by

$$(\hat{Q}_n^{(l)} u_h)(t) := \sum_{k=1}^m b_k K(t, t_l + c_k h) U_{l,k} \quad (l < n), \tag{3.23}$$

$$\begin{aligned}(\hat{Q}_n u_h)(t) &:= v \sum_{k=1}^m b_k K(t, t_n + v c_k h) u_h(t_n + v c_k h) \\ &= v \sum_{j=1}^m \left[\sum_{k=1}^m b_k K(t, t_n + v c_k h) L_j(c_k v)\right] U_{n,j},\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}(\hat{\hat{Q}}_n^{(l)} u_h)(t) &:= (1-v) \sum_{k=1}^m b_k K(t, t_l + vh + (1-v) c_k h) u_h(t_l + vh + (1-v) c_k h) \\ &= (1-v) \sum_{j=1}^m \sum_{k=1}^m b_k K(t, t_l + vh + (1-v) c_k h) L_j(v + (1-v) c_k) U_{l,j},\end{aligned} \tag{3.25}$$

$$\begin{aligned}(\hat{\hat{Q}}_n^I u_h)(t) &:= (w-v) \sum_{k=1}^m b_k K(t, t_n + vh + (w-v) c_k h) u_h(t_n + vh + (w-v) c_k h) \\ &= (w-v) \sum_{j=1}^m \sum_{k=1}^m b_k K(t, t_n + vh + (w-v) c_k h) L_j(v + (w-v) c_k) U_{n,j},\end{aligned} \tag{3.26}$$

respectively, where $t = t_n + vh \in e_n$ and $b_k := \int_0^1 L_k(s) ds$.

The fully discretized collocation equation (3.21) is then obtained from the exact collocation equation (3.16) by replacing the integrals by the above quadrature approximations, disregarding the quadrature errors induced by this secondary discretization process. Therefore, the fully discretized linear algebraic systems corresponding to the exact systems (3.17)–(3.19) assume the following forms.

Phase I (total overlap): $0 \le n < q^I$. We have

$$[\mathcal{I}_m - B_n^I(\theta) - h\hat{\bar{C}}_n^I(\theta)]\hat{\mathbf{U}}_n = \mathbf{F}_n, \tag{3.27}$$

with

$$\hat{\hat{C}}_n^I(\theta) := \left((c_i - \gamma_{n,i}) \sum_{k=1}^m b_k K(t, t_n + \gamma_{n,i}h + (c_i - \gamma_{n,i})c_k h) L_j(\gamma_{n,i} + (c_i - \gamma_{n,i})c_k) \right)_{1 \le i,j \le m}.$$

Phase II (partial overlap): $q^I \le n < q^{II}$. If this phase is not empty, then

$$(\mathcal{I}_m - h\hat{\hat{C}}_n^{II}(\theta))\hat{\mathbf{U}}_n = \mathbf{F}_n + h\hat{\hat{S}}_{n-1}^{II}(\theta)\hat{\mathbf{U}}_{n-1}, \tag{3.28}$$

where

$$\hat{\hat{C}}_n^{II}(\theta) := \mathrm{diag}(\underbrace{1,\dots,1}_{\kappa_n},0,\dots,0)\hat{C}_n + \mathrm{diag}(\underbrace{0,\dots,0}_{\kappa_n},1,\dots,1)(\hat{\hat{C}}_n^I(\theta) + B_n^I(\theta)),$$

$$\hat{\hat{S}}_{n-1}^{II}(\theta) := \mathrm{diag}(\underbrace{1,\dots,1}_{\kappa_n},0,\dots,0)\hat{C}_n^{(n-1)}(\theta),$$

with

$$\hat{C}_n := \left(c_i \sum_{k=1}^m b_k K(t, t_n + c_i c_k h) L_j(c_k c_i) \right)_{1 \le i,j \le m},$$

$$\hat{C}_n^{(l)} := \left(\sum_{k=1}^m b_k K(t, t_l + c_k h) L_j(c_k) \right)_{1 \le i,j \le m} \quad (l < n).$$

Phase III (pure delay phase): $q^{II} \le n \le N-1$).

$$(\mathcal{I}_m - h\hat{C}_n)\hat{\mathbf{U}}_n = \mathbf{F}_n + \hat{\hat{S}}_{\theta_n}^{III}(\theta)\hat{\mathbf{U}}_{\theta_n} + \hat{S}_{\theta_n+1}^{III}(\theta)\hat{\mathbf{U}}_{\theta_n+1} + h\sum_{l=\theta_n+2}^{n-1}\hat{C}_n^{(l)}\hat{\mathbf{U}}_l, \tag{3.29}$$

where

$$\hat{\hat{S}}_{\theta_n}^{III}(\theta) := \mathrm{diag}(\underbrace{1,\dots,1}_{\nu_n},0,\dots,0)(B_{\theta_n}^{III}(\theta) + h\hat{\hat{C}}_n^{(\theta_n)}(\theta)),$$

$$\hat{S}_{\theta_n+1}^{III} := \mathrm{diag}(\underbrace{1,\dots,1}_{\nu_n},0,\dots,0)\hat{C}_n^{(\theta_n+1)} + \mathrm{diag}(\underbrace{0,\dots,0}_{\nu_n},1,\dots,1)(B_{\theta_n+1}^{III}(\theta) + h\hat{\hat{C}}_n^{(\theta_n+1)}(\theta)),$$

with

$$\hat{\hat{C}}_n^{(l)}(\theta) := \left((1 - \gamma_{n,i}) \sum_{k=1}^m b_k K(t, t_l + \gamma_{n,i}h + (1 - \gamma_{n,i})c_k h) L_j(\gamma_{n,i} + (1 - \gamma_{n,i})c_k) \right)_{1 \le i,j \le m}.$$

The equations (3.22) and (3.27)–(3.29) describe a *continuous* m-stage *implicit Runge–Kutta–Volterra method* for the VFIE (3.20).

Since the fully discrete linear algebraic systems (3.27)–(3.29) possess the same structure as the exact systems (3.17)–(3.19), and since the matrix elements in the latter systems are bounded, the existence of a unique collocation solution $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$

for all sufficiently small $h > 0$ is readily proved. We omit the details but summarize the result in the following theorem.

THEOREM 3.4. *Let the assumptions of Theorem* 3.1 *hold. Then there is an* $\hat{h} > 0$ *such that for any uniform mesh* I_h *with* $h \in (0, \hat{h})$ *the fully discretized collocation equation* (3.21) *defines a unique collocation solution* $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$, *which is defined by the unique solutions* $\hat{\boldsymbol{U}}_n$ *of the linear algebraic systems* (3.27), (3.28), (3.29) *and the local representation* (3.22).

At the end of section 3.3 we introduced the *iterated* collocation solution u_h^{it} corresponding to the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$: for $t = t_n + vh$ its value is given by

$$u_h^{it}(t_n + vh) = b(t_n + vh)u_h(\theta(t_n + vh)) + f(t_n + vh) + \int_{\theta(t_n+vh)}^{t_n+vh} K(t_n + vh, s)u_h(s)\,ds, \qquad v \in [0, 1]. \tag{3.30}$$

In many applications one is often interested only in the values of u_h^{it} at (some of) the mesh points (e.g., at $t_N = T$). Thus, for a point $t = t_{n+1}$ (i.e., for $v = 1$ and $0 \le n \le N-1$) we may write $q(t_{n+1}) = t_{q_n} + \gamma_n h$ (assuming that $\theta(t) = qt,\ 0 < q < 1$), where

$$q_n := \lfloor q(n+1) \rfloor, \qquad \gamma_n := q(n+1) - q_n.$$

Thus, the delay integral term in (3.30) can be written as

$$(\mathcal{W}_\theta u_h)(t_{n+1}) = \int_{t_n+\gamma_n h}^{t_{n+1}} K(t_{n+1}, s)u_h(s)\,ds$$

if $\underline{q_n = n}$ ($\Rightarrow\ \theta(t_{n+1}) \in [t_n, t_{n+1})$), or as

$$(\mathcal{W}_\theta u_h)(t_{n+1}) = \int_{t_{q_n}+\gamma_n h}^{t_n} K(t_{n+1}, s)u_h(s)\,ds + \int_{t_n}^{t_{n+1}} K(t_{n+1}, s)u_h(s)\,ds$$

if $\underline{q_n < n}$ ($\Rightarrow\ \theta(t_{n+1}) \in [t_{q_n}, t_{q_n+1})$).

The discretized delay integral term $(\hat{\mathcal{W}}_\theta \hat{u}_h)(t_{n+1})$ in the *fully discretized* version

$$\hat{u}_h^{it}(t_{n+1}) := b(t_{n+1})\hat{u}_h(\theta(t_{n+1})) + f(t_{n+1}) + (\hat{\mathcal{W}}_\theta \hat{u}_h)(t_{n+1}) \tag{3.31}$$

of (3.30) is then obtained by resorting to the quadrature approximations described at the beginning of this section. We leave the details to the reader.

4. Convergence analysis.

We now analyze the attainable global order of convergence (on I) of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the general VFIE (1.1) in section 4.1; in section 4.2 we will establish analogous results for the iterated collocation solution u_h^{it}. In the following theorems we adopt, in order to avoid cumbersome notation, the convention that the constant C in the various error estimates denotes a generic constant. Its values will depend on the collocation parameters $\{c_i\}$, the delay function θ, and on bounds for the derivatives of u; they will not depend on h.

4.1. Attainable order of convergence of u_h on I.

THEOREM 4.1. *Assume that the given functions* f, b, K_0, K_1, θ *in the VFIE* (2.3) *are at least* m *times continuously differentiable on their respective domains. If* $\|b\|_\infty < 1$ *and if* θ *is subject to the conditions* (D1)–(D3), *then for all sufficiently small* $h > 0$,

$$\|u - u_h\|_\infty \le C(\|(\mathcal{I} - P_h)f\|_\infty + \|(\mathcal{I} - P_h)\mathcal{K}u\|_\infty), \tag{4.1}$$

where the operator $\mathcal{K}:\ C(I) \to C(I)$ *is defined by*

$$(\mathcal{K}u)(t) := b(t)u(\theta(t)) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \tag{4.2}$$

and $P_h : C(I) \mapsto S_{m-1}^{-1}(I_h)$ *is the normal Lagrange type of piecewise polynomial interpolation operator corresponding to the collocation points* X_h *(defined by the parameters* $\{c_i\}$*). The constant* C *is independent of* h *but depends on the collocation parameters* $\{c_i\}$ *and* θ*.*

If the exact solution satisfies $u \in W^{m,\infty}(I)$*, we obtain the convergence estimate*

$$\|u - u_h\|_\infty \le Ch^m \|u\|_{m,\infty}, \tag{4.3}$$

with

$$\|y\|_{m,\infty} := \max_{0\le j\le m} \left(\sup_{t\in I} \left| \frac{d^j y(t)}{dt^j} \right| \right).$$

The constant C *depends on the collocation parameters* $\{c_i\}$ *and on* θ*, but not on* h*.*

Proof. The operator forms for the VFIE (2.2) and its collocation equation (3.1) are, respectively,

$$\begin{cases} u = f + \mathcal{K}u, \\ u_h = P_h f + P_h \mathcal{K} u_h. \end{cases}$$

Using the unique solvability of the VFIE and its collocation equation (for all sufficiently small $h > 0$), we may write

$$\begin{cases} u = (\mathcal{I} - \mathcal{K})^{-1} f, \\ u_h = (\mathcal{I} - P_h\mathcal{K})^{-1} P_h f. \end{cases}$$

Hence, the collocation error can be written as

$$\begin{aligned} &u - u_h \\ &\quad = (\mathcal{I} - \mathcal{K})^{-1} f - (\mathcal{I} - P_h\mathcal{K})^{-1} P_h f \\ &\quad = (\mathcal{I} - \mathcal{K})^{-1} (f - P_h f) + (\mathcal{I} - \mathcal{K})^{-1} P_h f - (\mathcal{I} - P_h\mathcal{K})^{-1} P_h f \\ &\quad = (\mathcal{I} - \mathcal{K})^{-1} (f - P_h f) + (\mathcal{I} - P_h\mathcal{K})^{-1} (\mathcal{K} - P_h\mathcal{K})(\mathcal{I} - \mathcal{K})^{-1} P_h f \\ &\quad = (\mathcal{I} - \mathcal{K})^{-1} (f - P_h f) + (\mathcal{I} - P_h\mathcal{K})^{-1} (\mathcal{K} - P_h\mathcal{K})(\mathcal{I} - \mathcal{K})^{-1} (P_h f - f) \\ &\qquad + (\mathcal{I} - P_h\mathcal{K})^{-1} (\mathcal{K} - P_h\mathcal{K})(\mathcal{I} - \mathcal{K})^{-1} f, \end{aligned}$$

and this leads to the desired estimate (4.1):

$$\|u - u_h\|_\infty \le C(\|(\mathcal{I} - P_h)f\|_\infty + \|(\mathcal{I} - P_h)\mathcal{K}u\|_\infty),$$

where the constant C depends on the norms of the operators $(\mathcal{I} - \mathcal{K})^{-1}$ and $(\mathcal{I} - P_h\mathcal{K})^{-1}$. Hereafter in this paper, the symbol C denotes a constant (that may attain different values at its different occurrences) which is independent of h.

If $u \in W^{m,\infty}(I)$, the error estimates for the interpolation operator P_h,

$$\|(\mathcal{I} - P_h)f\|_\infty \le Ch^m \|f\|_{m,\infty} \le Ch^m \|u\|_{m,\infty}$$

and

$$\|(\mathcal{I} - P_h)\mathcal{K}u\|_\infty \le Ch^m \|\mathcal{K}u\|_{m,\infty} \le Ch^m \|u\|_{m,\infty},$$

allow us to derive the (optimal) estimate

$$\|u - u_h\|_\infty \leq Ch^m \|u\|_{m,\infty}.$$

This establishes (4.3). ∎

Similar convergence estimates hold for the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to the VFIE (1.3) corresponding to the Volterra operator $\mathcal{W}_\theta$. They can be established by a simple adaptation of the above proof, replacing $\mathcal{V}$ and $\mathcal{V}_\theta$ in the definition (4.2) of the operator $\mathcal{K}$ by $\mathcal{W}_\theta$.

4.2. Attainable order of convergence of u_h^{it} on I. Since there exist many global superconvergence results for Volterra functional differential and integral equations with proportional delays (Brunner [6, 7], Brunner and Hu [8]), it is natural for us to investigate whether global superconvergence of order $p \geq m+1$ (on I) is possible for the iterated collocation solution u_h^{it} corresponding to the collocation solution $u_h \in S_{m-1}^{(-1)}$ (I_h for the *general* VFIE (2.2) (and hence for (1.1) and (1.2)). Here we show by some simple analysis that this is not possible.

THEOREM 4.2. *Assume that functions f, b (with $b \not\equiv 0$, $\|b\|_\infty < 1$) and θ in the VFIE* (1.3) *are arbitrarily smooth on their respective domains, and let $K \equiv 0$. If θ is subject to the conditions* (D1)–(D3), *then for all sufficiently small $h > 0$ the collocation solution u_h^{it} defined by* (3.3) *satisfies*

$$\|u - u_h^{it}\|_\infty \leq Ch^m, \tag{4.4}$$

where the exponent m cannot in general be replaced by $m+1$, regardless of the choice of the collocation parameters $\{c_i\}$.

Proof. Using (3.2) with $K(t,s) \equiv 0$ and $|b(t)| > 0$ ($t \in I$), it follows that

$$u(t) - u_h^{it}(t) = b(t)[u(\theta(t)) - u_h(\theta(t))], \qquad t \in I.$$

We thus have the estimate

$$|u(t) - u_h^{it}(t)| \leq \|b\|_\infty |u(\theta(t)) - u_h(\theta(t))|, \quad t \in I,$$

where $\theta(t) \in I$. Since b does not depend on h, and since the global order of convergence of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ cannot exceed m (recall section 4.1), we obtain the assertion of Theorem 4.2. ∎

Remark. We observe that, regardless of the choice of the collocation parameters $\{c_i\}$, the value p in

$$\|u - u_h^{it}\|_\infty \leq Ch^p$$

cannot exceed m. This is in contrast to VFIEs (1.1)–(1.3) with $b(t) \equiv 0$: in this case, p can attain the value $p = m+1$, for example when the collocation points are the Gauss points.

4.3. Convergence estimates for discrete collocation solutions. The convergence estimates of Theorems 4.1 and 4.2 remain valid (but with different error constants C in (4.3) and (4.4)) for the *discrete* collocation solution $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$ defined by the fully discretized collocation equation (3.31). We summarize the results in the following theorem.

THEOREM 4.3. *Under the assumptions of Theorem* 4.1, *and for all sufficiently small mesh diameters $h > 0$, the discrete collocation solution $\hat{u}_h$ (determined by* (3.22) *and* (3.27)–(3.29)*) for* (1.2) *satisfies*

$$\|u - \hat{u}_h\|_\infty \le C(\|(\mathcal{I} - P_h)f\|_\infty + \|(\mathcal{I} - P_h)\mathcal{K}u\|_\infty),$$

where the operator $\mathcal{K}: C(I) \to C(I)$ *is defined by*

$$\mathcal{K}u := b(t)u(\theta(t)) + (\mathcal{W}_\theta u)(t),$$

and P_h *is the Lagrange interpolation operator corresponding to the collocation points* X_h*.*

If the exact solution u *of* (1.2) *is in* $W^{m,\infty}(I)$*, then the optimal order of convergence of* $\hat{u}_h$ *is described by*

$$\|u - \hat{u}_h\|_\infty \le Ch^m \|u\|_{m,\infty}.$$

The proof uses the fact that the quadrature errors induced by the interpolatory m-point quadrature rules employed in the derivation of the fully discretized systems of linear algebraic equations (3.27)–(3.29) are of $\mathcal{O}(h^m)$. We omit the details.

Remark. Under the assumptions of Theorem 4.1, and for all sufficiently small mesh diameters $h > 0$, the fully discretized collocation solution $\hat{u}_h$ for (2.2) also satisfies

$$\|u - \hat{u}_h\|_\infty \le C(\|(\mathcal{I} - P_h)f\|_\infty + \|(\mathcal{I} - P_h)\mathcal{K}u\|_\infty),$$

where the operator $\mathcal{K}: C(I) \to C(I)$ is defined by

$$\mathcal{K}u := b(t)u(\theta(t)) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t).$$

Similarly, if the exact solution u of (2.2) is in $W^{m,\infty}(I)$, then the optimal order of convergence of $\hat{u}_h$ can be described by

$$\|u - \hat{u}_h\|_\infty \le Ch^m \|u\|_{m,\infty}.$$

An analogous result holds for the discrete *iterated* collocation solution $\hat{u}_h^{it}$ (cf. (3.31) and Theorem 4.3).

4.4. Attainable order of u_h^{it} on I_h. It has been shown in [6, sect. 5.3.3] and [8] that the iterated collocation solution u_h^{it} corresponding to $u_h \in S_{m-1}^{(-1)}(I_h)$ for VFIEs (1.1) and (1.2) with $b(t) \equiv 0$ exhibits *local superconvergence* of order $m + 2$ at the points of uniform meshes if the collocation parameters $\{c_i\}$ are chosen in a special way (e.g., as the Gauss points) and if $q = 1/2$.

For the general VFIEs (1.1), (1.4) with $b(t) \not\equiv 0$ we now show that this local superconvergence result for u_h^{it} no longer holds.

THEOREM 4.4. *Let the assumptions stated in Theorem* 4.3 *hold. Then the iterated collocation solution* u_h^{it} *corresponding to the collocation solution* $u_h \in S_{m-1}^{(-1)}(I_h)$ *cannot be locally superconvergent of order* $p^* \ge m+1$ *at all the points of a uniform mesh* I_h*.*

Proof. It follows from the proof of Theorem 4.2 (section 4.2) that if $t = t_n$ is a mesh point, then

$$|u(t_n) - u_h^{it}(t_n)| \le |b(t_n)||u(\theta(t_n)) - u_h(\theta(t_n))| \quad (1 \le n \le N),$$

where $\|b\|_\infty < 1$. While $\theta(t_n)$ may be a mesh point when $\theta(t) = qt$ $(0 < q < 1)$ for *some* n and special values of q (e.g., for $q = 1/r,\ r \in \mathbb{N},\ r \ge 2$), it is clearly not possible to have

$$|u(t_n) - u_h^{it}(t_n)| \le Ch^{p^*}, \quad p^* \ge m + 1,$$

for *all* $n = 1, \dots, N$ (recall Theorems 4.1 and 4.2). ∎

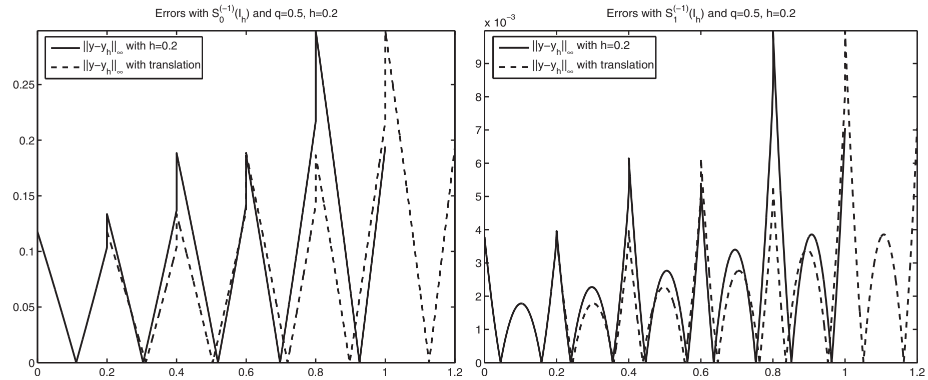

FIG. 1. *Example* 5.1: *The local superconvergence points investigation on the uniform mesh with* $h = 0.2$.

We illustrate the result of Theorem 4.4 by means of a representative example. Here we solve the problem in Example 5.1 below with piecewise constant space $S_0^{(-1)}(I_h)$ (with collocation parameter $c_1 = 0.5$) and piecewise linear space $S_1^{(-1)}(I_h)$ (with the collocation parameters $c_1 = \frac{3-\sqrt{3}}{6}$, $c_2 = \frac{3+\sqrt{3}}{6}$). In order to investigate the local superconvergence points, we compare the errors $\|u - u_h\|_\infty$ on the uniform mesh I_h and the same error $\|u - u_h\|_\infty$ but with a step size translation. For our illustration we choose $q = 0.5$; the corresponding numerical results are displayed in Figure 1.

Since if there exist local superconvergence points (Gauss points) for problem (1.2), the superconvergence points should coincide after a step size translation. However, as the error plotted in Figure 1 shows, the superconvergence points are separated from each other. So we see that there are no classical (local) superconvergence points in collocation solutions for the VFIE (1.2). We now show the absence of superconvergence for u_h^{it} by means of the simple example

$$u(t) = b(t)u(\theta(t)) + f(t), \qquad t \in I := [0, T]. \tag{4.5}$$

where $b(t) = 0.5$. The numerical results are given for $u_h \in S_0^{(-1)}(I_h)$, with $c_1 = 0.5$, and for $u_h \in S_1^{(-1)}(I_h)$, with $c_1 = \frac{3-\sqrt{3}}{6}$, $c_2 = \frac{3+\sqrt{3}}{6}$. The function $f(t)$ is chosen such that the exact solution is $u(t) = t$ (for $S_0^{(-1)}(I_h)$) and $u(t) = t^2$ (for $S_1^{(-1)}(I_h)$).

The numerical results are displayed in Figure 2. The figure clearly shows that the convergence order of u_h^{it} is the same as that of u_h, and that there is no local superconvergence at the mesh points for u_h^{it}.

5. Numerical experiments. In this section, we apply the collocation method described in section 3 to a number of VFIEs corresponding to different choices of the data b, f, K_0, K_1, and θ. The numerical results given below confirm the theoretical optimal convergence estimates derived in section 4.

Example 5.1. The VFIE

$$\begin{aligned} u(t) &= f(t) + b(t)u(qt) + \int_0^{qt} K(t,s)u(s)ds, \\ & t \in I := [0,1] \quad (0 < q < 1), \end{aligned} \tag{5.1}$$

with $K(t,s) = a/q$, $b(t) = \frac{1}{2}\sin t$, and $f(t) = e^t - \frac{1}{2}\sin t e^{qt} - \frac{a(e^{qt}-1)}{q}$, has the exact solution $u(t) = \mathrm{e}^t$.

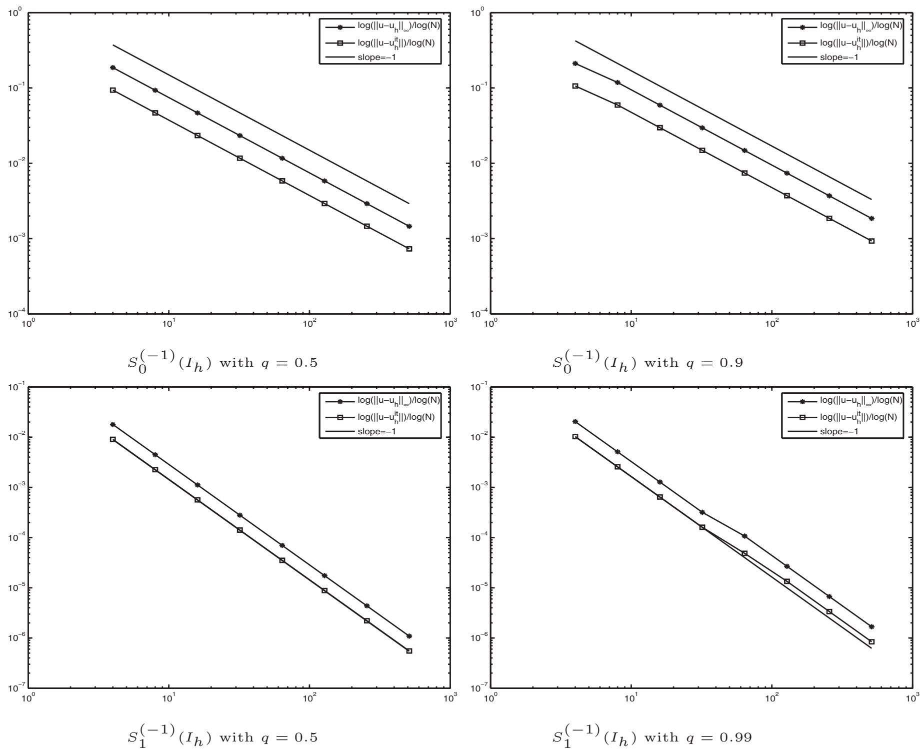

FIG. 2. *Global superconvergence investigation of problem* (4.5).

In our numerical experiments we choose $q = 0.1, q = 0.5, q = 0.9$, and $q = 0.99$, and we use the piecewise polynomial spaces $S_{m-1}^{(-1)}(I_h)$ $(m = 1, 2, 3)$ with the collocation parameters given by the Gauss points ($m = 1$: $c_1 = 0.5$; $m = 2$: $c_1 = \frac{3-\sqrt{3}}{6}$, $c_2 = \frac{3+\sqrt{3}}{6}$; $m = 3$: $c_1 = \frac{5-\sqrt{15}}{10}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{5+\sqrt{15}}{10}$) and with uniform meshes I_h. The results for the piecewise constant approximation are shown in Figure 3. It is easy to see that the numerical convergence order is 1.

Figure 4 shows the numerical results for the piecewise linear approximation. The numerical results confirm that the attainable order of convergence is 2. Similarly, Figure 5 shows the numerical results for the piecewise quadratic approximation; we observe an order of convergence of 3.

Example 5.2. Consider the VFIE (1.2) with $b(t)$ and $K(t, s)$ as in Example 5.1. We choose $f(t)$ so that its exact solution is $u(t) = \mathrm{e}^{-t}$. It is approximated in the collocation spaces $S_{m-1}^{(-1)}(I_h)$ $(m = 2, 3)$ with the collocation parameters given by the Radau II points ($m = 2$: $c_1 = 1/3$, $c_2 = 1$; $m = 3$: $c_1 = (4-\sqrt{6})/10$, $c_2 = (4+\sqrt{6})/10$, $c_3 = 1$).

Figure 6 shows the numerical results for the piecewise linear approximation. The numerical results confirm that the attainable order of convergence is 2. Similarly, Figure 7 shows the numerical results for the piecewise quadratic approximation; we observe an order of convergence of 3.

Example 5.3. Consider the VFIE (1.2) with

$$b(t) = te^{-t},$$
$$K_0 = K_1 = e^{-\gamma(t-s)}.$$

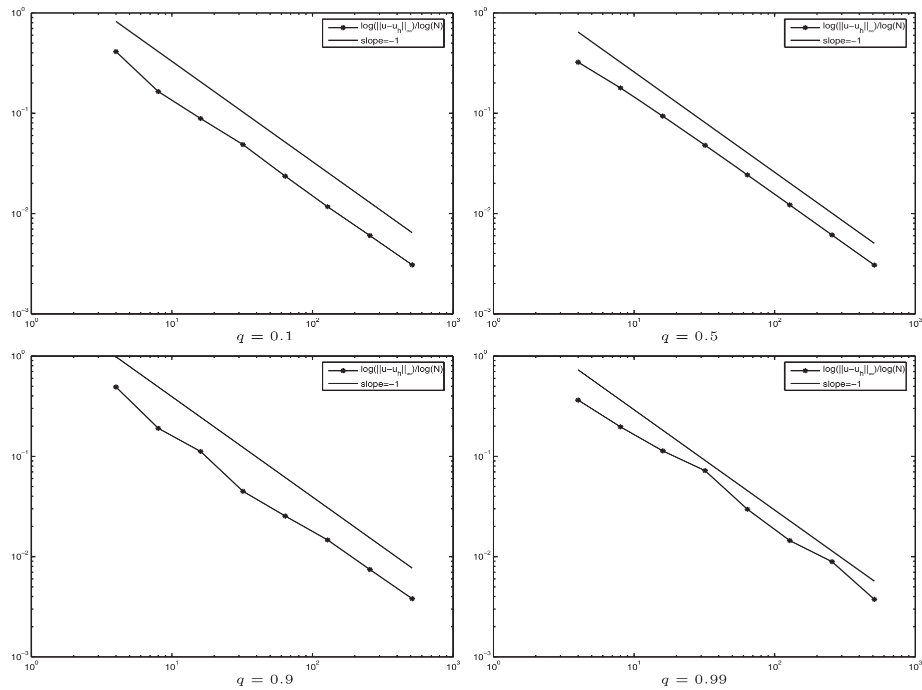

FIG. 3. *Example* 5.1: *The errors for* $S_0^{(-1)}(I_h)$.

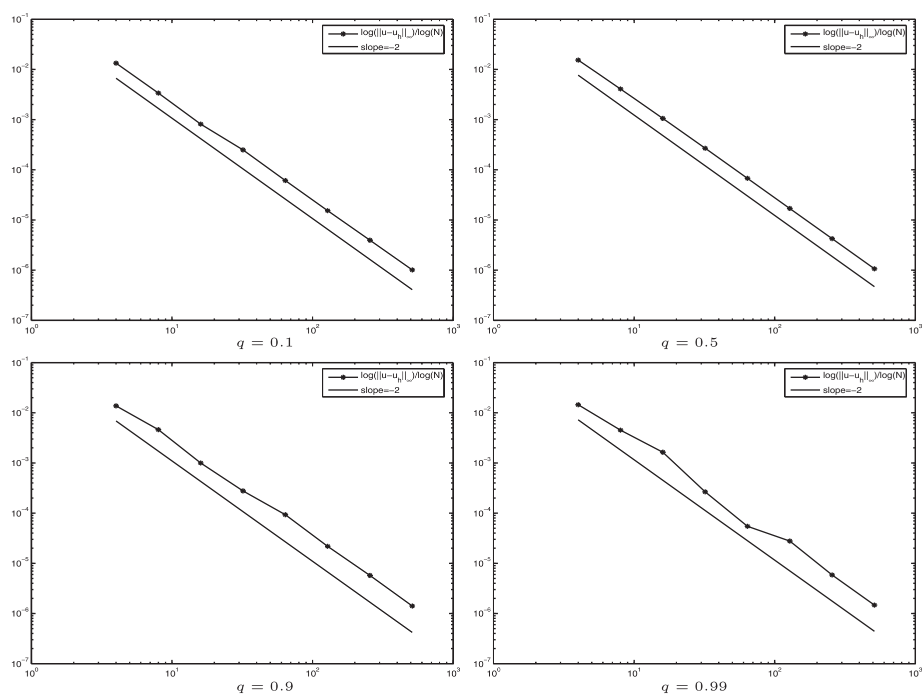

FIG. 4. *Example* 5.1: *The errors for* $S_1^{(-1)}(I_h)$.

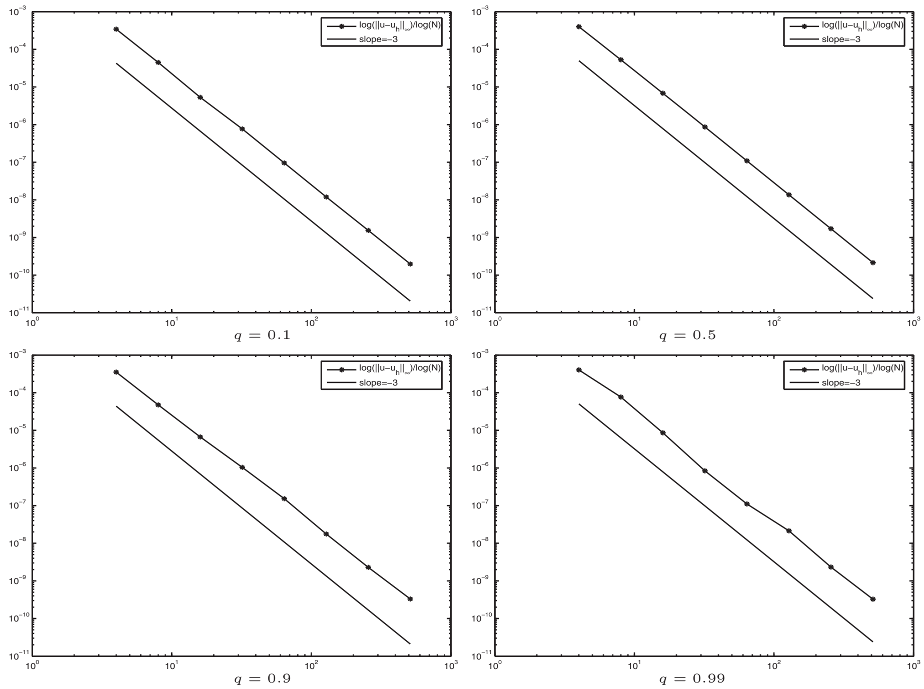

FIG. 5. *Example* 5.1: *The errors for* $S_2^{(-1)}(I_h)$.

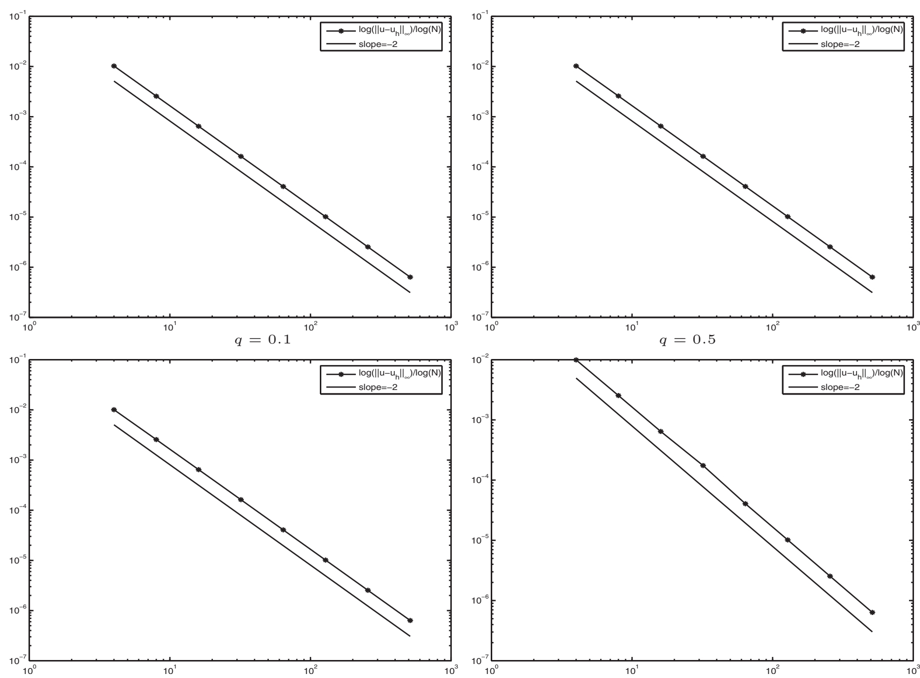

FIG. 6. *Example* 5.2: *The errors for* $S_1^{(-1)}(I_h)$.

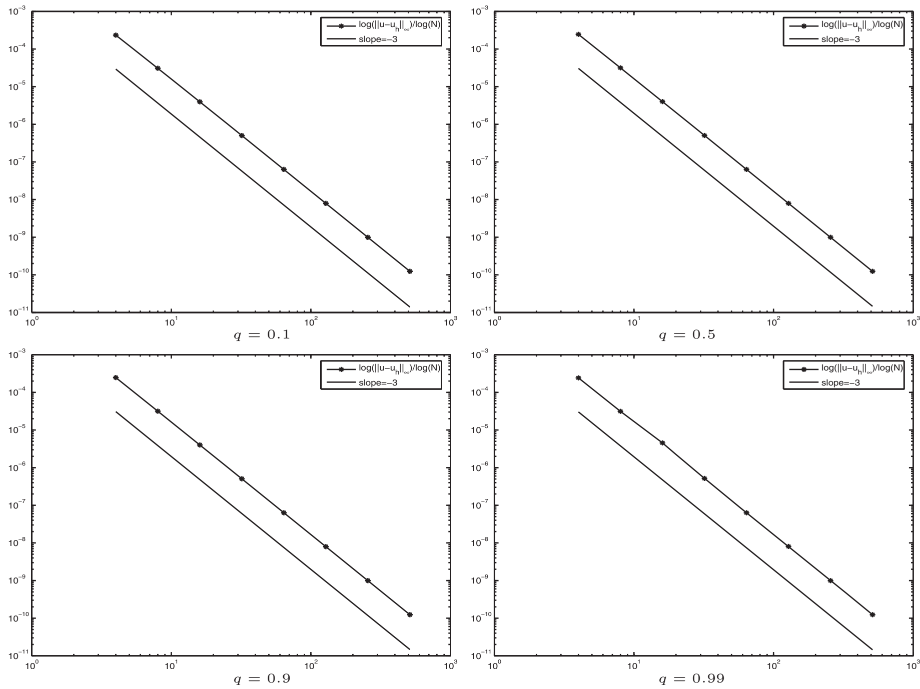

FIG. 7. *Example* 5.2: *The errors for* $S_2^{(-1)}(I_h)$.

We choose the corresponding function f such that the exact solution is $u(t) = \beta t e^{-\gamma t}$ ($t \in [0,1]$). In our computations, we choose $\beta = 1, 10, 50, 100$ and $\gamma = 1, 10, 50, 100$. The graphs of the corresponding exact solutions are shown in Figure 8, while the error plots in Figure 9 reflect the behavior of the collocation solution in $S_2^{(-1)}(I_h)$ and confirm that the attainable order is indeed 3.

Example 5.4. Here, we consider the VFIE

$$\int_{\theta(t)}^{t} K(t,s)u(s)ds = g(t), \qquad t \in [0,1], \quad g(0) = 0, \tag{5.2}$$

with $\theta(t) = qt$, $K(t,s) = e^{-10(t-s)}$ and $g = \frac{e^t - e^{10t-9qt}}{11}$. Its exact solution is $u(t) = \mathrm{e}^t$. Under these conditions, (5.2) can be transformed into the equivalent second-kind equation

$$K(t,t)u(t) = qK(t,qt)u(qt) - \int_{\theta(t)}^{t} \frac{\partial K(t,s)}{\partial t} u(s)ds + g'(t), \quad t \in [0,1]. \tag{5.3}$$

We use the piecewise quadratic space $S_2^{(-1)}(I_h)$ (with the collocation parameters $c_1 = \frac{5-\sqrt{15}}{10}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{5+\sqrt{15}}{10}$) on a uniform mesh I_h. The results are shown in Figure 10. It is easy to see that the convergence order reaches 3.

Example 5.5. The VFIE

$$u(t) = f(t) + b(t)u(\theta(t)) + \int_0^{\theta(t)} K(t,s)u(s)ds, \qquad t \in I := [0,1] \ \ (0 < q < 1), \tag{5.4}$$

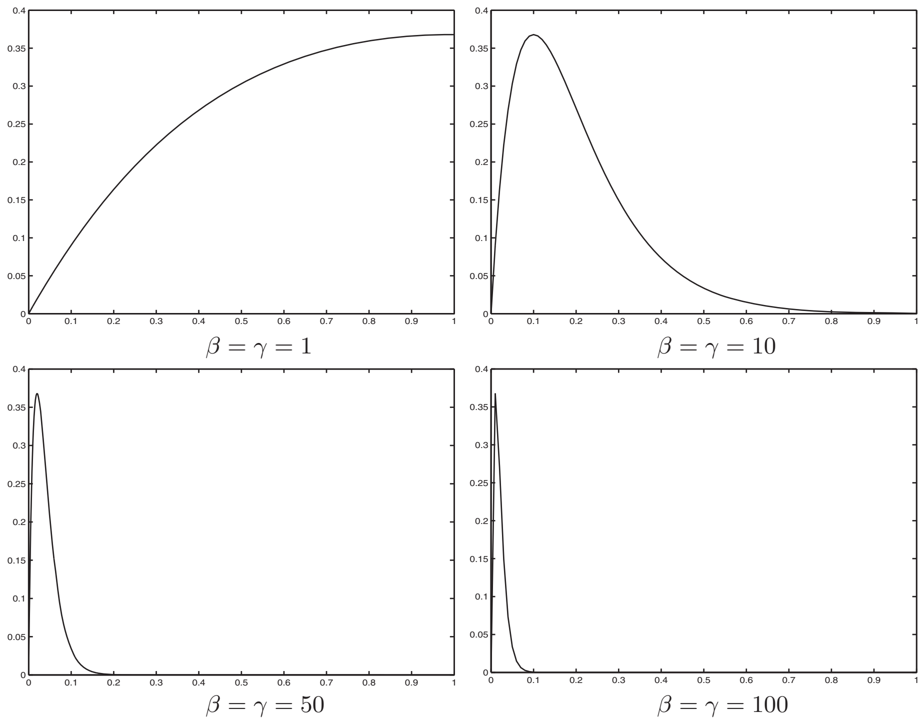

FIG. 8. *Example* 5.3: *The figures for exact solutions* $y(t) = \beta t e^{-\gamma t}$.

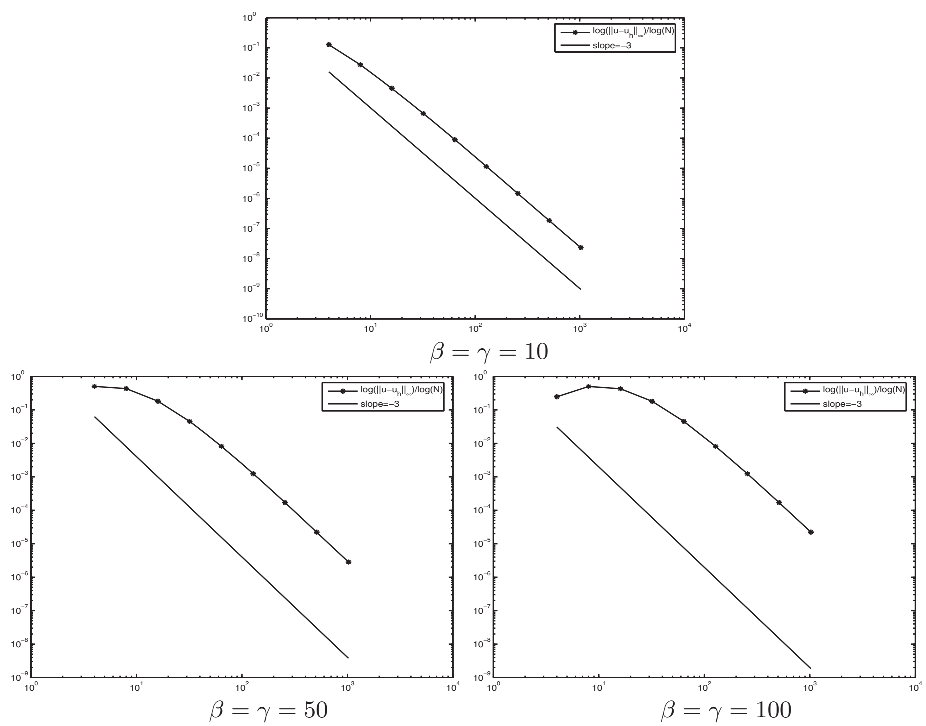

FIG. 9. *Example* 5.3: *The error figures for* $b(t) = te^{-t}$, $u(t) = \beta t e^{-\gamma t}$, *and* $q = 0.99$.

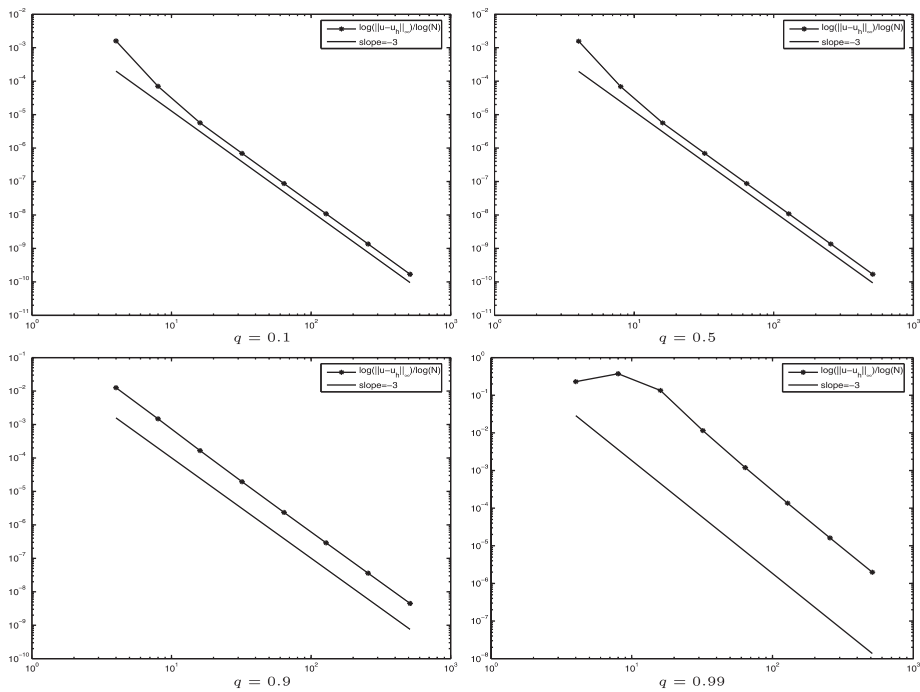

FIG. 10. *Example* 5.4: *The errors for* $S_2^{(-1)}(I_h)$.

with *nonlinear* delay function $\theta(t) = 1/2\arctan(t)$, $K(t,s) = a$, $b(t) = \frac{1}{2}\sin t$, and $f(t) = e^t - \frac{1}{2}\sin(t)e^{1/2\arctan(t)} - a(e^{1/2\arctan(t)} - 1)$, possesses the exact solution $u(t) = \mathrm{e}^t$.

In our numerical implementation, we use the collocation spaces $S_0^{(-1)}(I_h)$ $(m = 1)$ with the collocation parameter $c_1 = 0.5$; $S_1^{(-1)}(I_h)$ $(m = 2)$ with the collocation parameters $c_1 = \frac{3-\sqrt{3}}{6}$, $c_2 = \frac{3+\sqrt{3}}{6}$; and $S_2^{(-1)}(I_h)(m = 3)$ with the collocation parameters $c_1 = \frac{5-\sqrt{15}}{10}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{5+\sqrt{15}}{10}$ on the uniform mesh I_h. Figure 11 exhibits the behavior of the resulting collocation solutions. Clearly, the attained convergence orders are 1, 2, and 3, respectively.

6. Collocation in continuous piecewise polynomial spaces.

In this section we show numerically that collocation solutions for the VFIEs (1.1), (1.2) in spaces of *continuous* piecewise polynomials do not converge uniformly to the exact solutions when the value of $q \in (0,1)$ in the delay function $\theta(t) = qt$ is close to 1.

Example 6.1. We again solve (5.1) of Example 5.1, but now in the space $S_m^{(0)}(I_h)$ of *globally continuous* piecewise polynomials. In our numerical implementation, we choose $q = 0.1$, $q = 0.5$, $q = 0.9$, and $q = 0.99$, with $m = 1,2,3$. The collocation parameters $\{c_i\}$ are the Gauss points ($m = 1$: $c_1 = 1/2$; $m = 2$: $c_1 = \frac{3-\sqrt{3}}{6}$, $c_2 = \frac{3+\sqrt{3}}{6}$; $m = 3$: $c_1 = \frac{5-\sqrt{15}}{10}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{5+\sqrt{15}}{10}$) on a uniform mesh I_h.

The results for the approximation in $S_1^{(0)}(I_h)$ are shown in Figure 12. We observe that the collocation solutions in $S_1^{(0)}(I_h)$ are unstable when $q = 0.9$ and $q = 0.99$.

Figure 13 reflects the numerical results for the approximation in $S_2^{(0)}(I_h)$. We see that the collocation solutions in $S_2^{(0)}(I_h)$ are again unstable when $q = 0.9$ and $q = 0.99$.

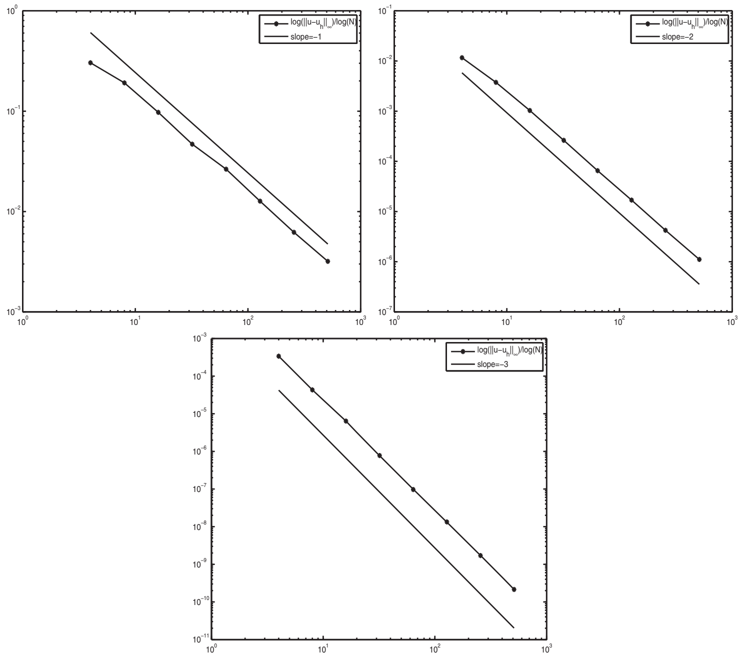

FIG. 11. *Example* 5.5: *The errors for* $S_0^{(-1)}(I_h)$, $S_1^{(-1)}(I_h)$, *and* $S_2^{(-1)}(I_h)$ *of the nonlinear case.*

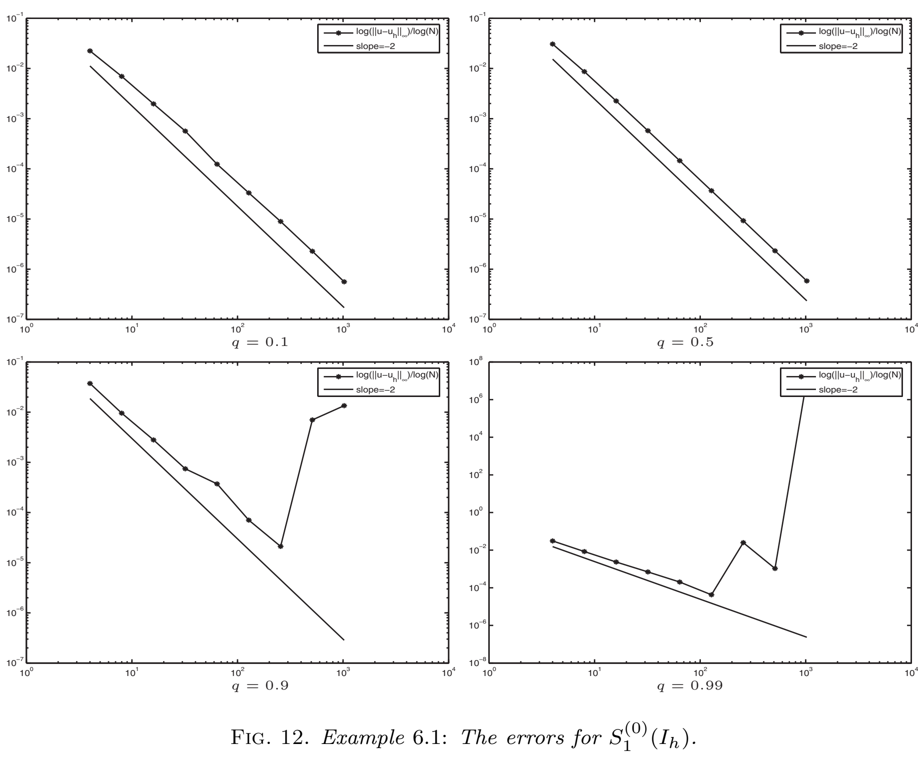

FIG. 12. *Example* 6.1: *The errors for* $S_1^{(0)}(I_h)$.

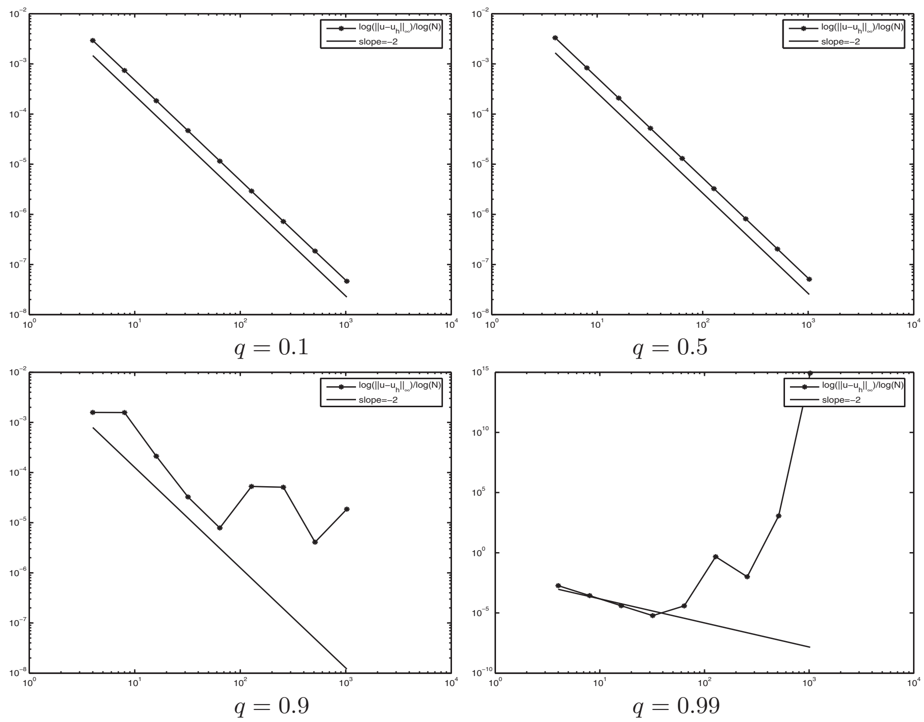

FIG. 13. *Example* 6.1: *The errors for* $S_2^{(0)}(I_h)$.

The results for the approximation in $S_3^{(0)}(I_h)$ are displayed in Figure 14. It can be observed that collocation in $S_3^{(0)}(I_h)$ is unstable when $q = 0.9$ and $q = 0.99$.

This motivates us to state the following conjecture.

CONJECTURE 6.1. *Let* $u_h \in S_m^{(0)}(I_h)$ *be the collocation solution for the VFIE* (1.2) *or* (1.3), *using uniform meshes* I_h *and collocation points* X_h *based on the Gauss points* $\{c_i\}$. *Then there exists a* $q^* \in (0,1)$ *so that*

$$\|u - u_h\|_\infty \nrightarrow 0 \quad as \quad h \to 0,$$

whenever $q \in (q^*, 1)$.

7. Future work. (I) One of the key assumptions underlying the analysis presented in sections 3 amd 4 is the condition in (D1) (see sections 1) that the vanishing delay function θ be *strictly increasing* on $[0, T]$ (with the prototype given by the proportional delay function $\theta(t) = qt,\ 0 < q < 1$). What happens when the delay function θ is *not monotone*, for example when θ is given by the (highly) oscillatory function

$$\theta(t) = q_1 t + (q_2 - q_1) t \sin^2(\omega t), \tag{7.1}$$

with $0 < q_1 < q_2 < 1$ and $\omega \geq 1$? In particular, how does one (efficiently) implement the collocation method in $S_{m-1}^{(-1)}(I_h)$ (including the discretization of the integrals arising in the analogue of the collocation equations (3.12)–(3.14) (for the general VIE (1.1)) or (3.17)–(3.19) (for the VFIE (1.4)), especially if $\omega \gg 1$ in (7.1)?

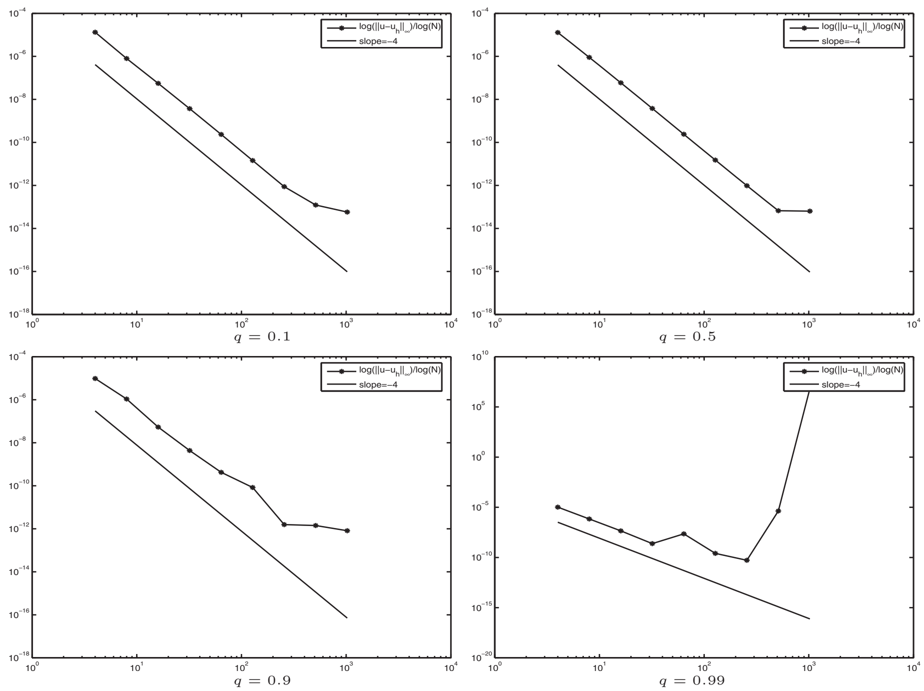

FIG. 14. *Example* 6.1: *The errors for* $S_3^{(0)}(I_h)$.

(II) The optimal convergence properties of collocation solutions in spaces of continuous piecewise polynomials for *neutral* Volterra functional integro-differential equations of the form

$$u'(t) = a(t)u(t)+b(t)u(\theta(t))+c(t)u'(\theta(t))+\int_{\theta(t)}^{t}[K_1(t,s)u(s)+K_2(t,s)u'(s)]\,ds, \tag{7.2}$$

with $c(t) \not\equiv 0$ and $K_2(t,s) \not\equiv 0$, have not yet been studied. Note that (7.2) contains the generalized pantograph equation (cf. [13], [4]),

$$u'(t) = a(t)u(t) + b(t)u(qt) + c(t)u'(qt),$$

as a special case. The (super-)convergence analysis of collocation methods is, to the authors' knowledge, open even for this delay differential equation.

Acknowledgments. The authors are grateful to the two referees for their constructive criticisms and valuable comments, which led to a more concise form of our paper.

REFERENCES

[1] I. Ali, H. Brunner, and T. Tang, *Spectral methods for pantograph-type differential and integral equations with multiple delays*, Front. Math. China, 4 (2009), pp. 49–61.
[2] G. Andreoli, *Sulle equazioni integrali*, Rend. Circ. Mat. Palermo, 37 (1914), pp. 76–112.
[3] A. Iserles, *On the generalized pantograph functional differential equation*, European J. Appl. Math., 4 (1993), pp. 1–38.
[4] A. Bellen and M. Zennaro, *Numerical Methods for Delay Differential Equations*, Oxford University Press, Oxford, UK, 2003.

[5] A.M. Bica and H. Oros, *Combined new numerical method for Volterra integral equations of pantograph type*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 16 (2009), suppl. S1, pp. 162–168.

[6] H. Brunner, *Collocation Methods for Volterra Integral and Related Function Equations*, Cambridge University Press, Cambridge, UK, 2004.

[7] H. Brunner, *Current work and open problems in the numerical analysis of Volterra functional equations with vanishing delays*, Front. Math. China, 4 (2009), pp. 49–61.

[8] H. Brunner and Q. Hu, *Optimal superconvergence of iterated collocation solutions for Volterra integral equations with vanishing delays*, SIAM J. Numer. Anal., 43 (2005), pp. 1934–1949.

[9] H. Brunner, H.H. Xie, and R. Zhang, *Analysis of collocation solutions for a class of functional equations with proportional delays*, IMA J. Numer. Anal., 31 (2011), pp. 698–718.

[10] L.G. Chambers, *Some properties of the functional equation* $\phi(x) = f(x) + \int_0^{\lambda x} g(x, y, \phi(y))dy$, Internat. J. Math. Math. Sci., 14 (1990), pp. 27–44.

[11] A.M. Denisov and A. Lorenzi, *Existence results and regularisation techniques for severely ill-posed integrofunctional equations*, Boll. Un. Mat. Ital. B (7), 11 (1997), pp. 713–731.

[12] L. Fox, D.F. Mayers, J.R. Ockendon, and A.B. Tayler, *On a functional differential equation*, J. Inst. Math. Appl., 8 (1971), pp. 271–307.

[13] A. Iserles, *On the generalized pantograph functional differential equation*, European J. Appl. Math., 4 (1993), pp. 1–38.

[14] V. Volterra, *Sopra alcune questioni di inversione di integrali definite*, Ann. Mat. Pura Appl., 25 (1897), pp. 139–178.

[15] V. Volterra, *Leçons sur les équations intégrales*, Gauthier-Villars, Paris, 1913.