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THE *hp* DISCONTINUOUS GALERKIN METHOD FOR DELAY DIFFERENTIAL EQUATIONS WITH NONLINEAR VANISHING DELAY*

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Abstract. We present the *hp*-version of the discontinuous Galerkin method for the numerical solution of delay differential equations with nonlinear vanishing delays and derive error bounds that are explicit in the time steps, the degrees of the approximating polynomials, and the regularity properties of the exact solutions. It is shown that the *hp* discontinuous Galerkin method exhibits exponential rates of convergence for smooth solutions on uniform meshes, and for nonsmooth solutions on geometrically graded meshes. The theoretical results are illustrated by various numerical examples.

Key words. pantograph delay differential equations, nonlinear vanishing delay, discontinuous Galerkin method, *hp*-version, spectral and exponential accuracy

AMS subject classifications. Primary, 65L03, 65L06; Secondary, 65L60, 65L70

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1. Introduction.

The aim of this paper is to study the optimal convergence properties of the *hp*-version of the discontinuous Galerkin (DG) method for delay differential equations (DDEs) with vanishing delays:

$$\begin{cases} u'(t) = a(t)u(t) + b(t)u(\theta(t)) + f(t), & t \in J := [0,T], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where the (smooth) delay function θ is subject to the conditions

(D1) $\theta(0) = 0$ and $\theta(t) < t$ for $t > 0$,

(D2) $\min_{t\in J} \theta'(t) =: q_0 > 0$,

and where the given functions a, b, f are at least continuous on J (more precise conditions will be stated in sections 3 and 4).

This class of DDEs has been studied extensively, both analytically and numerically, since the early 1970s and the early 1990s, respectively, mostly for (linear) proportional delay functions $\theta(t) = qt$ $(0 < q < 1)$. DDEs corresponding to this delay are known as *pantograph equations* (cf. Iserles [10]). The monographs by Bellen

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and Zennaro [3] and by Brunner [4], as well as the survey paper [5] convey good pictures of the present state of the art in the numerical analysis of these problems (and related more general functional integral and integro-differential equations) with vanishing delay functions.

The DG method for initial-value problems in ordinary differential equations (and time stepping in partial differential equations) was first proposed in [11] as a nonstandard finite element method for the numerical solution of neutron transport problems. Subsequently, a number of important contributions established the mathematical foundation (attainable order of superconvergence; a priori and a posteriori error analysis; analysis of the *hp*-version) of DG schemes; see, for example, Delfour, Hager, and Trochu [9], Schötzau and Schwab [13, 14], the review paper [2], and the books [8] and [15]. The *hp*-DG method has also been successfully applied to Volterra integro-differential equations with smooth and nonsmooth solutions; see, for example, Brunner and Schötzau [7].

For delay differential equations (1.1) with vanishing delays, Ali, Brunner, and Tang [1] analyzed spectral methods and showed that they lead to spectral accuracy. The DG method (h-version) for delay differential equations with linear vanishing delay has recently been studied by Brunner, Huang, and Xie [6]: they proved that the global error of DG solutions in the space $S_m^{(-1)}(\mathcal{T}_h)$ of (discontinuous) piecewise polynomials of fixed degree $m \geq 0$ behaves like $\mathcal{O}(h^{m+1})$, while the optimal order at the mesh points is given by $\mathcal{O}(h^{m+2})$ (for $m \geq 1$). This is in sharp contrast to DG solutions for (1.1) with $b(t) \equiv 0$; here, the local order of convergence of DG solutions is $\mathcal{O}(h^{2m+2})$ (see [9]).

Naturally, there arises the question as to whether the *hp*-version of the DG method can be used to improve the above low local order of convergence and achieve exponential convergence if the solution u of (1.1) is sufficiently smooth. It is the aim of this paper to answer this question, both for smooth and nonsmooth u.

The outline of the paper is as follows. In section 2, we introduce the *hp*-DG method for (1.1) with nonlinear vanishing delays and then describe the computational form of the method. The main results of the error analysis of the *hp*-DG solution are stated in section 3 (for smooth solutions) and in section 4 (for nonsmooth solutions). In section 5 we report the results of various numerical experiments illustrating the theoretical results. Some concluding remarks regarding future work are given in the last section.

Throughout this paper we employ the following notation. For a given interval I, we denote by $L^p(I)$ $(1 \leq p \leq \infty)$ the Lebesgue space of p-integrable functions, endowed with the norm $\|\cdot\|_{I,p}$. The space $W^{m,p}(I)$ $(1 \leq p \leq \infty)$ is the Sobolev space of order $m \geq 1$ equipped with the usual norm $\|\cdot\|_{I,m,p}$. The index I will be dropped when $I = J = [0,T]$, the interval on which the DDE (1.1) is to be solved.

2. The *hp* discontinuous Galerkin method.

In this section, we introduce the *hp*-version of the DG method and describe the corresponding computational ("time-stepping") scheme.

2.1. Formulation of the *hp* discontinuous Galerkin method.

Let $\mathcal{T}_h := \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\}$ be a mesh for the given interval $J = [0,T]$, and set

$$I_n := (t_{n-1}, t_n), \quad h_n := t_n - t_{n-1} \quad (1 \leq n \leq N), \quad \text{and} \quad h := \max_{1 \leq n \leq N}\{h_n\}.$$

For a given degree vector

$$\mathbf{m}_N := \{m_1, m_2, \ldots, m_N\}$$

the corresponding DG finite element space will be

$$S_{\mathbf{m}_N}^{(-1)}(\mathcal{T}_h) := \{v \in L^2(J) : v|_{I_n} \in P_{m_n},\ 1 \le n \le N\}, \tag{2.1}$$

where $P_{m_n} = P_{m_n}(I_n)$ denotes the space of (real) polynomials on I_n of degree not exceeding m_n ($m_n \ge 0$). If $m_n = m$ for each of the subintervals I_n, we simply write $\mathbf{m}_N = m$ and $S_m^{(-1)}(\mathcal{T}_h)$.

At the nodes $\{t_n\}_{n=0}^N$, the left-hand and right-hand limits (i.e., the jump discontinuities) of elements $v \in S_{\mathbf{m}_N}^{(-1)}(\mathcal{T}_h)$ will play an important role in the DG method. They are defined as follows:

$$v_n^+ := \lim_{s\to 0, s>0} v(t_n + s), \quad 0 \le n \le N-1,$$
$$v_n^- := \lim_{s\to 0, s>0} v(t_n - s), \quad 1 \le n \le N.$$

The jump across an interior node t_n is given by $[v]_n := v_n^+ - v_n^-$.

In the hp-DG method, we are looking for an approximate solution $U \in S_{\mathbf{m}_N}^{(-1)}(\mathcal{T}_h)$ for (1.1) such that

$$\begin{aligned}&\sum_{n=1}^N \int_{I_n} U'(t)v(t)dt + \sum_{n=1}^{N-1}[U]_n v_n^+ + U_0^+ v_0^+\\ &= u_0 v_0^+ + \sum_{n=1}^N \int_{I_n} [a(t)U(t) + b(t)U(\theta(t)) + f(t)]v(t)dt \quad \forall v \in S_{\mathbf{m}_N}^{(-1)}(\mathcal{T}_h).\end{aligned} \tag{2.2}$$

It is obvious that the exact solution u of (1.1) also satisfies (2.2); that is, we have

$$\begin{aligned}&\sum_{n=1}^N \int_{I_n} u'(t)v(t)dt + \sum_{n=1}^{N-1}[u]_n v_n^+ + u_0^+ v_0^+\\ &= u_0 v_0^+ + \sum_{n=1}^N \int_{I_n} [a(t)u(t) + b(t)u(\theta(t)) + f(t)]v(t)dt \quad \forall v \in S_{\mathbf{m}_N}^{(-1)}(\mathcal{T}_h).\end{aligned} \tag{2.3}$$

Hence, subtracting (2.2) from (2.3) and setting $e := u - U$, $U_0^- := u_0$, we obtain

$$\begin{aligned}B_{DG}(e,v) := &\sum_{n=1}^N \int_{I_n} \big(e'(t) - a(t)e(t) - b(t)e(\theta(t))\big)v(t)dt\\ &+ \sum_{n=1}^N [e]_{n-1} v_{n-1}^+ = 0 \quad \forall v \in S_{\mathbf{m}_N}^{(-1)}(\mathcal{T}_h).\end{aligned} \tag{2.4}$$

In other words, the DG error e has the orthogonality property

$$B_{DG}(e,v) = 0 \quad \forall v \in S_{\mathbf{m}_N}^{(-1)}(\mathcal{T}_h).$$

We also note that the DG method (2.2) can be interpreted, and thus formulated, as a time-stepping scheme. If U is known on the time intervals $I_k,\ 1 \le k \le n-1$, we find (cf. [13, 6]) $U|_{I_n} \in P_{m_n}(I_n)$ by solving

$$\begin{aligned} &\int_{I_n} U'(t)v(t)dt + U_{n-1}^+ v_{n-1}^+ \\ (2.5)\qquad &= U_{n-1}^- v_{n-1}^+ + \int_{I_n} [a(t)U(t) + b(t)U(\theta(t)) + f(t)]v(t)dt, \quad \forall v \in P_{m_n}(I_n). \end{aligned}$$

Here, we use again $U_0^- := u_0$.

We define $\mathrm{NRDOF}(S_{\mathbf{m}_N}^{(-1)}(\mathcal{T}_h)) := \sum_{n=1}^{N}(m_n + 1)$ as the number of degrees of freedom used for the time discretization. Since on each subinterval I_n the DG method amounts to an initial-value problem of size $m_n + 1$, NRDOF can be viewed as a crude measure of the cost of the discretization of (1.1).

Suppose that, for fixed n, $L_1(s), \ldots, L_{m_n+1}(s)$ are given basis functions for $P_{m_n}([0,1])$. We denote by $l_{n,1}(t), \ldots, l_{n,m_n+1}(t)$ the corresponding (local) basis functions on the subinterval I_n, obtained by transforming $L_1(s), \ldots, L_{m_n+1}(s)$ from $[0,1]$ to I_n with $s = (t - t_{n-1})/h_n$. On I_n $(n = 1, \ldots, N)$, the DG solution may be written as

$$(2.6)\qquad U_n(t) = \sum_{j=1}^{m_n+1} u_{n,j} l_{n,j}(t) = \sum_{j=1}^{m_n+1} u_{n,j} L_j\left(\frac{t - t_{n-1}}{h_n}\right),$$

where the unknown coefficients $u_{n,j}$ are determined by the (local) DG equation (2.5).

2.2. The exact form of the *hp*-DG equation. After selecting the basis functions, we can rewrite the *hp*-DG equation (2.5) as a system of linear algebraic equations for the vector

$$\mathbf{U}_n := (u_{n,1}, \ldots, u_{n,m_n+1})^T \in \mathbb{R}^{m_n+1}.$$

The structure of these N systems depends strongly on the delay term $\int_{I_n} b(t)U(\theta(t))v(t)dt$ and changes for each value of n as we pass from Phase I to Phase III (described below). To make this more precise, we first define

$$(2.7)\quad \mathbf{g}_1 := \Big(L_1(0), \ldots, L_{m_n+1}(0)\Big)^T,$$

$$(2.8)\quad \mathbf{f}_n := \left(\int_0^1 f(t_{n-1} + sh_n)L_1(s)ds, \ldots, \int_0^1 f(t_{n-1} + sh_n)L_{m_n+1}(s)ds\right)^T,$$

and introduce the matrices (in $\mathbb{R}^{(m_n+1)\times(m_n+1)}$)

$$(2.9)\qquad M := \left(\int_0^1 L_j'(t)L_i(t)dt + L_j(0)L_i(0)\right)_{1\le i,j\le m_n+1},$$

$$(2.10)\qquad A_n := \left(\int_0^1 a(t_{n-1} + sh_n)L_j(s)L_i(s)ds\right)_{1\le i,j\le m_n+1},$$

$$(2.11)\qquad G := \Big(L_j(1)L_i(0)\Big)_{1\le i,j\le m_n+1},$$

reflecting contributions corresponding to the nondelay terms in the DG equation.

The contributions of the delay terms $\int_{I_n} b(t)U(\theta(t))l_{n,i}(t)dt$ in the DG equation (2.5) to the coefficient matrices and the right-hand side of the systems of linear algebraic equations for $\mathbf{U}_n$ are governed by certain relationships between the value of n and the nonlinear delay function $\theta(t)$. To be more precise, the computational scheme passes through the three distinct phases described below.

- *Phase* I: $n = 1$. In this *initial phase* we have *complete overlap*: for any $t \in I_1$ the images $\theta(t)$ lie in I_1. Hence we define the matrix

$$B_1^I := \left(\int_0^1 b(sh_1) L_j \left(\frac{\theta(h_1 s)}{h_1} \right) L_i(s) ds \right)_{1 \le i,j \le m_1+1}. \tag{2.12}$$

The vector $\mathbf{U}_1$ is then the solution of the linear algebraic system

$$\left(M - h_1 A_1 - h_1 B_1^I \right) \mathbf{U}_1 = u_0 \mathbf{g}_1 + h_1 \mathbf{f}_1. \tag{2.13}$$

- *Phase* II: If $\theta(t_n) > t_{n-1}$ for $t \in I_n$, then we will encounter the *transition phase*. During this *partial overlap*: for some $t \in I_n$ the images $\theta(t)$ are still in I_n, while for other (smaller) $t \in I_n$ we have $\theta(t) \notin I_n$.
In this phase, there is an integer θ_n such that $\theta(t_{n-1}) \in I_{n-\theta_n}$. Let $s_0^* = 0$ and $0 < s_1^*, \ldots, s_{\theta_n}^* < 1$ satisfying $\theta(t_{n-1} + s_k^* h_n) = t_{n-\theta_n+k-1}$ for $(k = 1, \ldots \theta_n)$. Then we have

$$s_k^* := \frac{\theta^{-1}(t_{n-\theta_n+k-1}) - t_{n-1}}{h_n} \in (0,1), \quad k = 1, \ldots, \theta_n.$$

For $k = 1, \ldots, \theta_n$, we define

$$B_{n,k}^{II} := \left(\int_{s_{k-1}^*}^{s_k^*} b(t_{n-1} + sh_n) L_j \left(\frac{\theta(t_{n-1} + sh_n) - t_{n-\theta_n+k-2}}{h_{n-\theta_n+k-1}} \right) L_i(s) ds \right)_{1 \le i,j \le m_n+1}$$

and

$$B_n^{II} := \left(\int_{s_{\theta_n}^*}^1 b(t_{n-1} + sh_n) L_j \left(\frac{\theta(t_{n-1} + sh_n) - t_{n-1}}{h_n} \right) L_i(s) ds \right)_{1 \le i,j \le m_n+1}.$$

In this phase, $\mathbf{U}_n$ is given by the solution of the linear algebraic system

$$(M - h_n A_n - h_n B_n^{II}) \mathbf{U}_n = h_n \sum_{k=1}^{\theta_n} B_{n,k}^{II} \mathbf{U}_{n-\theta_n+k-1} + G \mathbf{U}_{n-1} + h_n \mathbf{f}_n. \tag{2.14}$$

- *Phase* III: If $\theta(t_n) \le t_{n-1}$ for $t \in I_n$, then we have reached the *pure delay phase*. Once we have reached this *pure delay phase*, we no longer have any overlap with I_n and the images $\theta(t)$ $(t \in I_n)$.
In this phase, there are two integers $\theta_{n,0}$ and $\theta_{n,1}$ $(\theta_{n,1} \le \theta_{n,0})$ such that $\theta(t_{n-1}) \in I_{n-\theta_{n,0}}$ and $\theta(t_n) \in I_{n-\theta_{n,1}}$. Let $s_0^* = 0$, $s_{\theta_{n,0}-\theta_{n,1}+1}^* = 1$, and $0 < s_1^*, \ldots, s_{\theta_{n,0}-\theta_{n,1}}^* < 1$ satisfying

$$\theta(t_{n-1} + s_k^* h_n) = t_{n-\theta_{n,0}+k-1} \text{ for } (k = 1, \ldots \theta_{n,0} - \theta_{n,1}).$$

Then we have

$$s_k^* := \frac{\theta^{-1}(t_{n-\theta_{n,0}+k-1}) - t_{n-1}}{h_n} \in (0,1), \quad k = 1, \ldots, \theta_{n,0} - \theta_{n,1}.$$

For $k = 1, \ldots, \theta_{n,0} - \theta_{n,1} + 1$ we define the matrices

$$B_{n,k}^{III} := \left(\int_{s_{k-1}^*}^{s_k^*} b(t_{n-1} + sh_n) \times L_j \left(\frac{\theta(t_{n-1} + sh_n) - t_{n-\theta_{n,0}+k-2}}{h_{n-\theta_{n,0}+k-1}} \right) L_i(s) ds \right)_{1 \le i,j \le m_n+1}.$$

In this phase $\mathbf{U}_n$ is the solution of the linear algebraic system

$$(M - h_n A_n)\mathbf{U}_n = h_n \sum_{k=1}^{\theta_{n,0}-\theta_{n,1}+1} B_{n,k}^{III} \mathbf{U}_{n-\theta_{n,0}+k-1} + G\mathbf{U}_{n-1} + h_n \mathbf{f}_n. \tag{2.15}$$

THEOREM 2.1. *Assume that the given functions a, b, and f in* (1.1) *are continuous on J and the (smooth) delay function θ is subject to the conditions* (D1) *and* (D2) *of section* 1. *Then there exists $\bar{h} > 0$ (depending on θ) such that for all $h \in (0, \bar{h})$, each of the linear algebraic systems* (2.13), (2.14), *and* (2.15) *possesses a unique solution $\mathbf{U}_n \in \mathbb{R}^{m_n+1}$, implying that each of the local representations* (2.6) *of the hp-DG solution is unique.*

For the proof of Theorem 2.1, please see [6, Theorem 2.1].

2.3. The fully discretized form of the *hp*-DG equation. The exact *hp*-DG equations derived in the previous section are in general not amenable to numerical computation since the integrals cannot be found analytically. Thus, another discretization step, employing appropriate quadrature rules, approximates the integrals occurring in the linear algebraic systems (2.13), (2.14), and (2.15).

To approximate the integrals, an $r_n + 1$-point quadrature formula with nodes $t_{n,j} := t_{n-1} + d_j h_n$ $(0 \le d_1 < \cdots < d_{r_n+1} \le 1)$ and weights ω_j $(j = 0, 1, \ldots, r_n + 1)$ is selected on the interval I_n. We denote the resulting fully discretized *hp*-DG solution in $S_{\mathbf{m}_N}^{(-1)}(\mathcal{T}_h)$ by $\hat{U}$ and write $\hat{U}_n$ for its restriction to the subinterval I_n. Similarly, denote the discretized form of $\mathbf{f}_n$, M, A_n by $\hat{\mathbf{f}}_n$, $\hat{M}$, $\hat{A}_n$, respectively. Then

$$\hat{\mathbf{f}}_n = \left(\sum_{\ell=1}^{r_n+1} \omega_\ell f(t_{n-1} + d_\ell h_n) L_1(d_\ell), \ldots, \sum_{\ell=1}^{r_n+1} \omega_\ell f(t_{n-1} + d_\ell h_n) L_{m_n+1}(d_\ell) \right)^T, \tag{2.16}$$

$$\hat{M} = \left(\sum_{\ell=1}^{r_n+1} \omega_\ell L_j'(d_\ell) L_i(d_\ell) + L_j(0) L_i(0) \right)_{1 \le i,j \le m_n+1}, \tag{2.17}$$

$$\hat{A}_n = \left(\sum_{\ell=1}^{r_n+1} \omega_\ell a(t_{n-1} + d_\ell h_n) L_j(d_\ell) L_i(d_\ell) \right)_{1 \le i,j \le m_n+1}. \tag{2.18}$$

In analogy to the exact *hp*-DG equations, the fully discretized *hp*-DG equations change their forms as n increases, as follows.

- *Phase* I: Denote the discretized form of B_1^I by $\hat{B}_1^I$. Then $\hat{B}_1^I$ has the form

$$\hat{B}_1^I = \left(\sum_{\ell=1}^{r_n+1} \omega_\ell b(d_\ell h_1) L_j \left(\frac{\theta(d_\ell h_1)}{h_1} \right) L_i(d_\ell) \right)_{1 \le i,j \le m_1+1}. \tag{2.19}$$

The discretized DG equation is

$$\left(\hat{M} - h_1 \hat{A}_1 - h_1 \hat{B}_1^I \right) \hat{\mathbf{U}}_1 = u_0 \mathbf{g}_1 + h_1 \hat{\mathbf{f}}_1. \tag{2.20}$$

The vector $\hat{\mathbf{U}}_1$ is then determined by the solution of the above linear algebraic system.

- *Phase* II: Denote the discretized form of $B^{II}_{n,k}$, B^{II}_n by $\hat{B}^{II}_{n,k}$, $\hat{B}^{II}_n$, respectively. Then

$$\hat{B}^{II}_{n,k} = \Bigg((s^*_k - s^*_{k-1}) \sum_{\ell=1}^{r_n+1} \omega_\ell b\big(t_{n-1} + (s^*_{k-1} + (s^*_k - s^*_{k-1})d_\ell)h_n\big) \times L_j\left(\frac{\theta\big(t_{n-1} + (s^*_{k-1} + (s^*_k - s^*_{k-1})d_\ell)h_n\big) - t_{n-\theta_n+k-2}}{h_{n-\theta_n+k-1}}\right) \times L_i\big(s^*_{k-1} + (s^*_k - s^*_{k-1})d_\ell\big)\Bigg)_{1\le i,j\le m_n+1}$$

and

$$\hat{B}^{II}_n = \Bigg((1 - s^*_{\theta_n}) \sum_{\ell=1}^{r_n+1} \omega_\ell b\big(t_{n-1} + (s^*_{\theta_n} + (1 - s^*_{\theta_n})d_\ell)h_n\big) \times L_j\left(\frac{\theta(t_{n-1} + (s^*_{\theta_n} + (1 - s^*_{\theta_n})d_\ell)h_n) - t_{n-1}}{h_n}\right) \times L_i\big(s^*_{\theta_n} + (1 - s^*_{\theta_n})d_\ell\big)\Bigg)_{1\le i,j\le m_n+1}.$$

In this phase, $\hat{\mathbf{U}}_n$ is given by the solution of the linear algebraic system

$$(\hat{M} - h_n\hat{A}_n - h_n\hat{B}^{II}_n)\hat{\mathbf{U}}_n = h_n \sum_{k=1}^{\theta_n} \hat{B}^{II}_{n,k}\hat{\mathbf{U}}_{n-\theta_n+k-1} + G\mathbf{U}_{n-1} + h_n\hat{\mathbf{f}}_n. \tag{2.21}$$

- *Phase* III: Denote the discretized form of $B^{III}_{n,k}$ by $\hat{B}^{III}_{n,k}$. Then

$$\hat{B}^{III}_{n,k} = \Bigg((s^*_k - s^*_{k-1}) \sum_{\ell=1}^{r_n+1} \omega_\ell b\big(t_{n-1} + (s^*_{k-1} + (s^*_k - s^*_{k-1})d_\ell)h_n\big) \times L_j\left(\frac{\theta\big(t_{n-1} + (s^*_{k-1} + (s^*_k - s^*_{k-1})d_\ell)h_n\big) - t_{n-\theta_{n,0}+k-2}}{h_{n-\theta_{n,0}+k-1}}\right) \times L_i\big(s^*_{k-1} + (s^*_k - s^*_{k-1})d_\ell\big)\Bigg)_{1\le i,j\le m_n+1}.$$

In this phase $\hat{\mathbf{U}}_n$ is the solution of the linear algebraic system

$$(\hat{M} - h_n\hat{A}_n)\hat{\mathbf{U}}_n = h_n \sum_{k=1}^{\theta_{n,0}-\theta_{n,1}+1} \hat{B}^{III}_{n,k}\hat{\mathbf{U}}_{n-\theta_{n,0}+k-1} + G\hat{\mathbf{U}}_{n-1} + h_n\hat{\mathbf{f}}_n. \tag{2.22}$$

In practical computations one always wants to use as few quadrature nodes as possible. This suggests the use of Gauss-type quadrature formulas since they lead to high accuracy for relatively few nodes. Since the matrix elements in the systems (2.20), (2.21), and (2.22) are bounded, the existence and uniqueness of the *hp*-DG solution for all sufficiently small $h > 0$ are readily proved. We omit the details but summarize the result in the following theorem.

THEOREM 2.2. *Under the conditions of Theorem* 2.1, *if the quadrature formula is such that* $\hat{M} = M$, *then there exists* $\bar{h} > 0$ *(depending on* θ*) such that for all* $h \in (0, \bar{h})$, *each of the linear algebraic systems* (2.20), (2.21), *and* (2.22) *possesses a unique solution* $\hat{\mathbf{U}}_n \in \mathbb{R}^{m_n+1}$, *implying that each of the local representations* (2.6) *of the hp-DG solution is uniquely defined.*

3. Error analysis: Smooth solutions. In this section we derive optimal a priori error estimates for the *hp*-DG solution for (1.1). In order to do so, we need to introduce an appropriate interpolation operator $\Pi_h: C[0,1] \to S^{(-1)}_{\mathbf{m}_N}(\mathcal{T}_h)$. In the framework of the discontinuous Galerkin method, the following interpolation operator is standard: in I_n it is defined by

$$\Pi_h u(t_n^-) := u(t_n^-); \tag{3.1}$$

$$\int_{I_n} (\Pi_h u) v \, dt := \int_{I_n} uv \, dt \quad \forall v \in P_{m_n-1}(I_n), \quad m_n \geq 1. \tag{3.2}$$

The following lemma describes the approximation properties of the interpolant $\Pi_h u$ (see [13] for details).

LEMMA 3.1. *Let* $I_n = (t_{n-1}, t_n), h_n = t_n - t_{n-1}, m_n \in \mathbb{N}_0$, *and* $u \in H^{s_{0,n}+1}(I_n)$ *for some* $s_{0,n} \geq 0$. *Then we have*

$$\|u - \Pi_h u\|^2_{I_n,\infty} \leq C \left(\frac{h_n}{2}\right)^{2s_n+1} \frac{\Gamma(m_n+1-s_n)}{\Gamma(m_n+1+s_n)} \|u\|^2_{I_n,s_n+1} \tag{3.3}$$

for any real s_n *with* $0 \leq s_n \leq \min(m_n, s_{0,n})$. *Furthermore, if* $u \in W^{s_{0,n}+1,\infty}(I_n)$, *there also holds*

$$\|u - \Pi_h u\|^2_{I_n,\infty} \leq C \left(\frac{h_n}{2}\right)^{2s_n+2} \frac{\Gamma(m_n+1-s_n)}{\Gamma(m_n+1+s_n)} \|u\|^2_{I_n,s_n+1,\infty} \tag{3.4}$$

for any real s_n *with* $0 \leq s_n \leq \min(m_n, s_{0,n})$.

THEOREM 3.2. *Assume*

(i) *the functions* a, b, f *describing the DDE* (1.1) *are continuous, and the delay function* θ *is subject to the conditions* (D1), (D2) *of section* 1*;*
(ii) u *is the exact solution of the initial-value problem for the DDE* (1.1)*;*
(iii) $U \in S^{(-1)}_{\mathbf{m}_N}(\mathcal{T}_h)$ *is the DG solution defined by* (2.5)*;*
(iv) $\mathcal{T}_h$ *is a mesh for* $J := [0, T]$ *and* h *is sufficiently small.*

Then the optimal global convergence estimate

$$\|u - U\|_\infty \leq C \log^{\frac{1}{2}} \big(\max\{|\mathbf{m}_N|, 2\}\big) \|u - \Pi_h u\|_\infty, \tag{3.5}$$

holds, where $|\mathbf{m}_N| := \max\{m_1, m_2, \ldots, m_n, \ldots, m_N\}$.

This result is a straightforward extension of Theorem 3.2 in [6]. We omit its proof but observe that the logarithmic factor on the right-hand side of (3.5) comes from [13, Lemma 3.1].

We see from Theorem 3.2 that the error of the *hp*-DG solution U can be transferred to the error of the interpolant $\Pi_h u$.

Combining Theorem 3.2 with Lemma 3.1, we are led to the following error estimate for the *hp*-DG solution of (1.1).

THEOREM 3.3. *Assume*

(i) $U \in S^{(-1)}_{\mathbf{m}_N}(\mathcal{T}_h)$ *is the hp-DG solution defined by* (2.5)*;*
(ii) u *is the exact solution of the initial-value problem for the DDE* (1.1) *satisfying that* $u|_{I_n} \in W^{s_{0,n}+1,\infty}(I_n)$ $(s_{0,n} \geq 0)$, $n = 1, \ldots, N$*;*
(iii) $\mathcal{T}_h$ *is a mesh for* $J := [0, T]$ *with sufficiently small mesh diameter* h.

Then we have

$$
\begin{aligned}
\|u-U\|_\infty^2 &\le C\log\big(\max\{|\mathbf{m}_N|,2\}\big) \\
&\quad \times \max_{1\le n\le N}\left\{\left(\frac{h_n}{2}\right)^{2s_n+2}\frac{\Gamma(m_n+1-s_n)}{\Gamma(m_n+1+s_n)}\|u\|^2_{I_n,s_n+1,\infty}\right\}, \qquad (3.6)
\end{aligned}
$$

for any real s_n with $0\le s_n\le \min\{s_{0,n},m_n\}$.

The estimate in Theorem 3.3 is explicit in the time steps h, in the components m_n of the degree vector $\mathbf{m}_N$, and the regularity $s_{0,n}$ of the exact solution u.

By recalling a well-known property of the gamma function,

$$
\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)}=\frac{1}{n^{\beta-\alpha}}\left(1+O\Big(\frac{1}{n}\Big)\right)
$$

(see [12] for details), we obtain the following convergence rates for the h- and p-versions of the DG method.

Corollary 3.4. *Let $m_n=m$ and $U\in S_m^{(-1)}(\mathcal{T}_h)$. If $u\in W^{s_0+1,\infty}(J)$, then*

$$
\|u-U\|_\infty\le C\log^{\frac12}\big(\max\{m,2\}\big)\frac{h^{\min(s_0,m)+1}}{m^{s_0}}\|u\|_{s_0+1,\infty}\,. \qquad (3.7)
$$

The estimates in Corollary 3.4 are uniform in h and m, which shows that the hp-DG method converges either as the time steps are refined ($h\to 0$) or as m is increased ($m\to\infty$).

Denote the dimension of the DG space $S_m^{(-1)}(\mathcal{T}_h)$ (i.e., NRDOF) by

$$
N_R:=\mathrm{NRDOF}\big(S_m^{(-1)}(\mathcal{T}_h)\big)=\sum_{n=1}^{N}(m_n+1)\le N(|\mathbf{m}_N|+1),
$$

with $N_R=N(m+1)$ if $m_n=m$ for all n.

We assume $h<c\frac{T}{N}$, then $h<c\frac{T}{N}=c\frac{T(m+1)}{N(m+1)}\le c\frac{T(m+1)}{N_R}$.

- For the h-version DG method, where the approximation degree m is fixed, the rate of convergence is increased when we decrease the step size h; that is, we have

$$
\begin{aligned}
\|u-U\|_\infty &\le C\log^{\frac12}\big(\max\{m,2\}\big)\frac{T^{\min(s_0,m)+1}}{m^{s_0}}N^{-1-\min(s_0,m)} \\
&\le C\log^{\frac12}\big(\max\{m,2\}\big)\frac{1}{m^{s_0}}\left(\frac{(m+1)T}{N_R}\right)^{1+\min(s_0,m)} \\
&\le Cm\log^{\frac12}(m+2)T^{s_0+1}N_R^{-1-\min(s_0,m)}. \qquad (3.8)
\end{aligned}
$$

- For the p-version DG method, we have the convergence obtained by increasing the approximation degree $\mathbf{m}_N$ on a fixed mesh $\mathcal{T}_h$. We see from (3.6) that

$$
\begin{aligned}
\|u-U\|_\infty &\le C\log^{\frac12}\big(\max\{|\mathbf{m}_N|,2\}\big)\frac{\big(T(|\mathbf{m}_N|+1)\big)^{s_0+1}}{|\mathbf{m}_N|^{s_0}}N_R^{-1-s_0} \\
&\le C|\mathbf{m}_N|\log^{\frac12}(|\mathbf{m}_N|+2)T^{1+s_0}N_R^{-1-s_0}. \qquad (3.9)
\end{aligned}
$$

We know from (3.8)–(3.9) that for smooth solutions (for which s_0 is large) it is advantageous to increase the degree m rather than to reduce the step size h at fixed lower m. Equation (3.9) also shows that arbitrarily high algebraic convergence rates are possible if the approximation degree m is raised. This is called the spectral convergence.

In fact, the p-version of the DG method can converge exponentially if the exact solution u is analytic in J.

THEOREM 3.5. *Assume the exact solution u is analytic in J. Let $\mathbf{m}_n = m$ and U be the DG solution in $S_m(\mathcal{T}_h)$ on a fixed mesh $\mathcal{T}_h$. Then we have*

$$\|u - U\|_\infty \leq C \exp(-\alpha m), \tag{3.10}$$

with constants $C, \alpha > 0$ that are independent of m.

Proof. The desired result (3.10) follows from Theorem 3.2, Lemma 3.1, and standard approximation theory for analytic functions. ∎

4. Error analysis: Nonsmooth solutions. Suppose now that the forcing term $f(t)$ in the DDE (1.1) is no longer smooth but has the form

$$f(t) = f_1(t) + t^\beta f_2(t), \quad \beta \in (0,1), \tag{4.1}$$

with analytic functions f_1 and f_2. The (smooth) delay function θ is again subject to the conditions (D1)–(D2) of section 1, and the functions a and b are assumed to be also analytic. An analytic function g can be characterized by analyticity constants $C_g, d_g > 0$ and the growth conditions

$$|g^{(s)}(t)| \leq C_g d_g^s \Gamma(s+1), \qquad t \in [0,T], \quad s \geq 0. \tag{4.2}$$

(See [15, pp. 78–79] for details.)

The following result describes the analyticity properties of the exact solution u.

THEOREM 4.1 (see [4, 7]). *Let $\gamma = 1 + \beta$. Then there exist constants $C, d > 0$ depending only on the analyticity constants of a, b, f_1, and f_2, such that the solution u of* (1.1) *satisfies*

$$|u^{(s)}(t)| \leq C d^s \Gamma(s+1) t^{\gamma - s}, \quad t \in (0,T], \quad s \in \mathbb{N}. \tag{4.3}$$

Remark 4.2. The solution of (1.1) is equivalent to the following delay integral equation:

$$u(t) = g(t) + \int_0^t a(s)u(s)ds + \int_0^t b(s)u(\theta(s))ds$$

with

$$g(t) = \int_0^t \big(f_1(s) + s^\beta f_2(s)\big)ds.$$

Thus the proof can be given along the lines of the one in [7, Theorem 4.1].

We now show that the hp-version of the DG method for the nonsmooth case can also attain exponential rates of convergence. We start with the following definition.

DEFINITION 4.3 (see [7, 13]). *The basic geometric partition $\widehat{J}_{p,\sigma} = \{I_n\}_{n=1}^{p+1}$ of $\widehat{J} = (0,1)$ with grading factor $\sigma \in (0,1)$ and p levels of refinement is given by $t_0 = 0, t_n = \sigma^{p-n+1}, 1 \leq n \leq p+1$. Away from $t = 0$, i.e., for $2 \leq n \leq p+1$, the intervals $I_n \in \widehat{J}_{p,\sigma}$ satisfy*

$$h_n = t_n - t_{n-1} = \lambda t_{n-1}, \qquad \lambda := \sigma^{-1}(1-\sigma). \tag{4.4}$$

Definition 4.4 (see [7, 13]). *A geometric partition* $\mathcal{T}_{p,\sigma}$ *of* $(0,T)$ *with grading factor* $\sigma \in (0,1)$ *and* p *levels of refinement is obtained by first quasi-uniformly partitioning* $(0,T)$ *into intervals* $\{J_k\}_{k=1}^{K}$. *The first interval* $J_1 = (0,t_1)$ *near* $t=0$ *is then further subdivided into* $p+1$ *subintervals* $\{I_n\}_{n=1}^{p+1}$ *by linearly mapping the basic geometric mesh* $\widehat{J}_{p,\sigma}$ *(cf. Definition 4.3) onto* J_1.

We will fix the coarse intervals $\{J_k\}_{k=2}^{K}$ and achieve convergence by increasing the polynomial degrees.

Lemma 4.5. *Set* $\gamma = 1+\beta$. *Let* $\mathcal{T}_{p,\sigma}$ *be a geometric mesh of* $(0,T)$ *with* $\{J_k\}_{k=1}^{K}$ *denoting the underlying quasi-uniform partition of* $(0,T)$ *and* $\{I_n\}_{n=1}^{p+1}$ *the geometric refinement of* J_1. *Then the solution* u *of* (1.1) *satisfies*

$$\|u\|_{I_1,1,\infty}^2 \le C, \tag{4.5}$$

and

$$\|u\|_{I_n,s+1,\infty}^2 \le Cd^{2s}\Gamma(2s+1)\sigma^{2(p-n+2)(\gamma-s-1)}, \quad 2\le n\le p+1, \tag{4.6}$$

$$\|u\|_{J_k,s+1,\infty}^2 \le Cd^{2s}\Gamma(2s+1), \qquad 2\le k\le K \tag{4.7}$$

for $s \ge 0$. *The constants* $C, d > 0$ *are independent of* n, p, *and* s.

Remark 4.6. We point out that the constants C and d in Lemma 4.5 depend on the underlying quasi-uniform partition $\{J_k\}_{k=1}^{K}$ of $\mathcal{T}_{p,\sigma}$.

Proof. This is a simple consequence of (4.3), Definitions 4.3 and 4.4, and properties of the gamma function. ∎

Definition 4.7 (see [7, 13]). *Let* $\mathcal{T}_{p,\sigma}$ *be a geometric mesh of* $(0,T)$ *with* $\{J_k\}_{k=1}^{K}$ *denoting the underlying quasi-uniform partition of* $(0,T)$ *and* $\{I_n\}_{n=1}^{p+1}$ *the geometric refinement of* J_1. *A degree vector* $\mathbf{m}_N$ *on* $\mathcal{T}_{p,\sigma}$ *is called linear with slope* $\mu > 0$ *if* $m_n = \lfloor \mu n \rfloor$ *on the geometrically refined elements* $\{I_n\}_{n=1}^{p+1}$ *and if* $m_k = \lfloor \mu(p+1)\rfloor$ *on the coarse elements* $J_k, 2\le k\le K$, *away from* $t=0$.

The next result establishes the exponential convergence rate of the DG method for the nonsmooth case.

Theorem 4.8. *Let* $\mathcal{T}_{p,\sigma}$ *be a geometric partition of* $(0,T)$ *with sufficiently small step size (guaranteeing the existence of a unique DG solution). Then there exists a slope* $\mu > 0$ *depending only on* σ, β, *the constants* C, *and* d *in Lemma 4.5, such that for all linear polynomial degree vectors* $\mathbf{m}_N$ *with slope* $\mu \ge \mu_0$, *the DG approximation* $U \in S_{\mathbf{m}_N}(\mathcal{T}_{p,\sigma})$ *satisfies the error estimate*

$$\|u-U\|_\infty \le C\exp(-\alpha N_R^{\frac{1}{2}}), \tag{4.8}$$

with constants C, $\alpha > 0$ *that are independent of* $N_R := \dim\big(S_{\mathbf{m}_N}(\mathcal{T}_{p,\sigma})\big)$.

The proof here is similar to the proof of Theorem 3.22 in [14] and Theorem 4.7 in [7]. For convenience, we give the proof here in our situation.

Proof. From Theorem 3.2 and Lemma 3.1, we have

$$\|u-U\|_\infty^2 \le C\max\Big\{\max_{1\le n\le p+1} e_{1,n},\ \max_{2\le k\le K} e_k\Big\}$$

with

$$e_{1,n} = \Big(\frac{h_n}{2}\Big)^{2s_n+2}\frac{\Gamma(m_n+1-s_n)}{\Gamma(m_n+1+s_n)}\big\|u\big\|_{I_n,s_n+1,\infty}^2, \quad 1\le n\le p+1,$$
$$e_k = \|u-\Pi_h u\|_\infty^2, \qquad 2\le k\le K,$$

and $0 < s_n \le \min\{s_{0,n}, m_n\}$. From (4.3), u is analytic away from $t=0$ and hence the regularity exponents s_n can be chosen arbitrary large for $n = 2,\dots,p+1$.

We first bound the errors $\{e_{1,n}\}$ on the geometrically refined intervals $\{I_n\}_{n=1}^{p+1}$. On the first element I_1 near $t = 0$, we select $s_1 = t_1 = 0$ and have from Lemma 4.5

$$e_{1,1} \le Ch_1^2 = C\sigma^{2p}.$$

Next, fix an element $I_n, 2 \le n \le p+1$, away from $t = 0$. From Lemma 4.5 and the definition of λ in (4.4), we obtain

$$\begin{aligned} e_{1,n} &\le C\Big(\frac{\lambda\sigma^{p-n+2}}{2}\Big)^{2s_n+2}\frac{\Gamma(m_n+1-s_n)}{\Gamma(m_n+1+s_n)} \\ &\quad\cdot(\sigma^{p-n+2})^{2(\gamma-s_n-1)}d^{2s_n}\Gamma(2s_n+1) \\ &= C\sigma^{(p-n+2)2\gamma}\Big((\lambda d)^{2s_n}\frac{\Gamma(m_n+1-s_n)}{\Gamma(m_n+1+s_n)}\Gamma(2s_n+1)\Big). \end{aligned}$$

Taking $s_n = \gamma_n m_n$ with $\gamma_n \in (0,1)$ and Stirling's formula leads to

$$e_{1,n} \le C\sigma^{(p-n+2)2\gamma} m_n^{\frac{1}{2}}\left((\lambda d)^{2\gamma_n}\Big(\frac{(1-\gamma_n)^{1-\gamma_n}}{(1+\gamma_n)^{1+\gamma_n}}\Big)\right)^{m_n}.$$

The function $f_{\lambda,d}(\gamma) = (\lambda d)^{2\gamma}\frac{(1-\gamma)^{1-\gamma}}{(1+\gamma)^{1+\gamma}}$ satisfies

$$0 < \inf_{0<\gamma<1} f_{\lambda,d}(\gamma) =: f_{\lambda,d}(\gamma_{\min}) < 1 \quad \text{with} \quad \gamma_{\min} = \frac{1}{\sqrt{1+\lambda^2 d^2}}.$$

Set $f_{\min} = f_{\min}(\lambda, d) =: f_{\lambda,d}(\gamma_{\min})$ and select $\gamma_n = \gamma_{\min}$ for $2 \le n \le p+1$. Hence, for $m_n = \lfloor \mu n \rfloor$, we have

$$\begin{aligned} e_{1,n} &\le C\sigma^{(p-n+2)2\gamma} m_n^{\frac{1}{2}} f_{\min}^{m_n} \le C\sigma^{(p-n+2)2\gamma}(\mu_n)^{\frac{1}{2}} f_{\min}^{\mu n} \\ &\le C\sigma^{2\gamma p}\big(\mu(p+1)\big)^{\frac{1}{2}}\Big(\sigma^{(-n+2)2\gamma} f_{\min}^{\mu n}\Big). \end{aligned}$$

Let

$$\mu \ge \max\left\{\frac{2\gamma\log(\sigma)}{\log(f_{\min})},\ 1\right\}. \tag{4.9}$$

Then, $f_{\min}^{\mu n} \le \sigma^{2\gamma p}$ and, consequently,

$$e_{1,n} \le C\sigma^{2\gamma p}\big(\mu(p+1)\big)^{\frac{1}{2}}(\sigma^{4\gamma}) \le C\sigma^{2\gamma p}\big(\mu(p+1)\big)^{\frac{1}{2}}, \quad n \ge 2.$$

Thus, we obtain for $1 \le n \le p+1$ the bound

$$e_{1,n} \le C\max\Big\{\sigma^{2p},\ \sigma^{2\gamma p}\big(\mu(p+1)\big)^{\frac{1}{2}}\Big\}. \tag{4.10}$$

Further, from standard approximation properties for analytic functions, we can bound the errors $\{e_k\}$ on the elements $\{J_k\}_{k=2}^{K}$ away from $t = 0$ as

$$e_k \le C\exp(-\alpha\lfloor\mu(p+1)\rfloor), \quad 2 \le k \le K, \tag{4.11}$$

with constants C and α that solely depend on the constants C and d in Lemma 4.5.

Combining the estimates in (4.10) and (4.11) yields

$$\|u-U\|_{\infty}^2 \le C\log\big(\max\{\lfloor\mu(p+1)\rfloor,2\}\big) \cdot \max\Big\{\sigma^{2p},\ \sigma^{2\gamma p}\big(\mu(p+1)\big)^{\frac{1}{2}},\ \exp(-\alpha\lfloor\mu(p+1)\rfloor)\Big\}.$$

Since we have

$$\log\big(\max\{\lfloor\mu(p+1)\rfloor,2\}\big) \cdot \max\Big\{\sigma^{2p},\ \sigma^{2\gamma p}\big(\mu(p+1)\big)^{\frac{1}{2}},\ \exp\big(-\alpha\lfloor\mu(p+1)\rfloor\big)\Big\} \le C\exp(-\alpha p),$$

as $p\to\infty$, and $N_R = \dim(S_{\mathbf{m}_N}(\mathcal{T}_{p,\sigma})) \le Cp^2$, the L^∞-error bound (4.8) follows. ∎

Remark 4.9. From a practical point of view, it may be more convenient to use a fixed polynomial degree m on a geometric partition $\mathcal{T}_{p,\sigma}$. In this case, exponential convergence results for all $\sigma\in(0,1)$ provided that m is proportional to the number of refinements, i.e., $m=\mu(p+1)$ with the slope parameter μ. Indeed, from the proof of Theorem 4.8, we also have

$$\|u-U\|_{\infty} \le C\max\big\{\sigma^{2p},\ m^{\frac{1}{2}}f_{\min}^m\big\} \le C\exp(-\alpha m) \le C\exp(-\alpha N_R^{\frac{1}{2}}).$$

Note that condition (4.9) on the slope is not necessary in this case.

5. Numerical experiments.

In this section, we present some numerical experiments to confirm our theoretical error estimates.

Example 5.1. Here we solve the DDE with proportional delay:

$$\begin{cases} u'(t) = bu(qt), & 0\le t\le T, \quad b\ne 0,\\ u(0)=u_0 \end{cases} \tag{5.1}$$

with $\theta(t)=qt, 0<q<1$. The exact solution is given by

$$u(t)=\sum_{j=0}^{\infty}\frac{q^{j(j-1)/2}}{j!}(bt)^j u_0,\ t\ge 0.$$

We choose $T=1$ and present the numerical results for the h-version and p-version of the DG method.

In Figure 1 we present the errors in $L^\infty(0,T)$ for the h-version of the DG method where uniform partitions and a fixed degree m are adopted. The results are shown for $m=0$, $m=1$, and $m=2$, respectively. The numerical results show that the slopes correspond to the algebraic rate of order $m+1$, thus confirming the h-version results in Corollary 3.4.

The performance of the p-version of the DG method is displayed in Figure 2. In the p-version, we increase the polynomial degree for fixed partitions with the time step size $h=1$, $h=0.5$, $h=0.25$, and $h=0.1$, respectively. For each of the fixed time partitions the results show that exponential rates of convergence are achieved, in agreement with the theoretical results in Theorem 3.5. As expected, the numerical results also show the smaller the underlying fixed time step, the smaller the errors.

Example 5.2. We use the hp-DG method to solve the DDE with nonlinear vanishing delay $\theta(t)=t^2$:

$$\begin{cases} u'(t) = au(t)+bu(t^2)+f(t), & t\in J=[0,1],\\ u(0)=0. \end{cases} \tag{5.2}$$

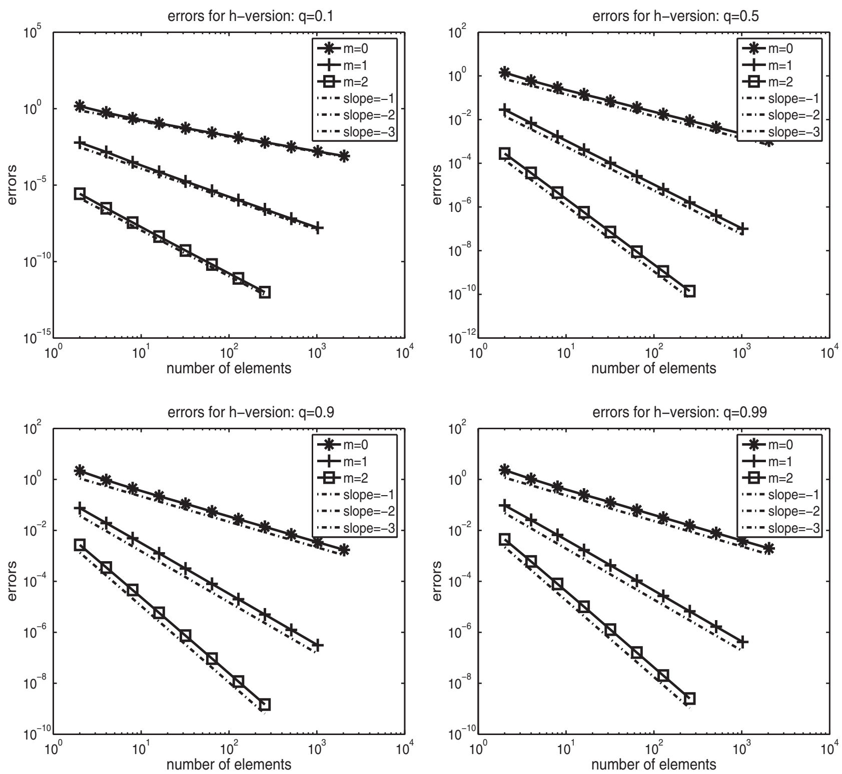

FIG. 1. *Errors for h-version of Example* 5.1.

We set $f(t) = \cos(t) - a\sin(t) - b\sin(t^2)$ to make the exact solution $u(t) = \sin(t)$ for any $a, b \in \mathbb{R}$.

In Figure 3, we present the h-version and p-version of the DG method by plotting the maximum-norm errors against number of elements (h-version) and polynomial degree (p-version), respectively.

For the h-version DG method, we choose the uniform partitions and fixed degree m. The results are shown for $m = 1$, $m = 2$, $m = 3$, $m = 4$, and $m = 5$, respectively. The slopes correspond to the algebraic convergence rates of order $m + 1$ which is predicted in Corollary 3.4.

In the p-version DG method, we increase the degree m on fixed partitions with time-step size $h = 1$, $h = 0.5$, $h = 0.25$, and $h = 0.1$, respectively. We see from Figure 3 that the p-version DG solutions exhibit exponential rates of convergence. This confirms the results in Theorem 3.5. We also observe that we achieve an error of 10^{-12} with fewer than 60 degrees of freedom ($N_R = 32$ with $h = 0.25$, i.e., $N = 4$) for the p-version DG solution, while the h-version of the DG solution with $m = 3$ requires a much higher degree of freedom ($N_R = 800$ with $N = 200$) in order to reach the same accuracy.

Example 5.3. In the third example we consider the DDE with nonlinear vanishing delay and nonsmooth solution:

$$\begin{cases} u'(t) = au(t) + bu(\theta(t)) + f(t), & t \in J = [0, 1], \\ u(0) = 0. \end{cases} \tag{5.3}$$

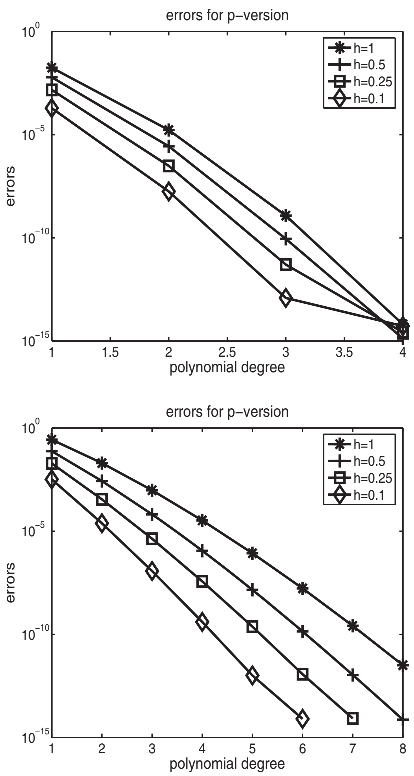

FIG. 2. *Errors for the p-version of Example* 5.1.

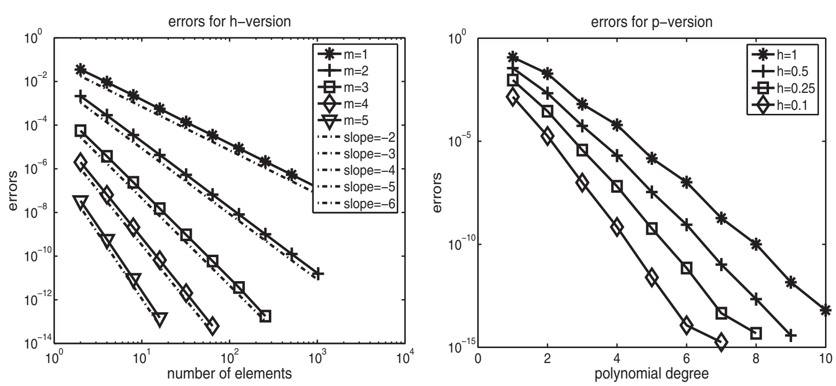

FIG. 3. *Errors for Example* 5.2.

We set $f(t)$ to make the exact solution $u(t) = t^{1.5}\exp(-t)$. Here we chose the delay function $\theta(t) = \arctan(t)$. Thus the function $f(t)$ has the form (4.1) with $\beta = 0.5$.

The solution is analytic in $J/\{0\}$ and has a singularity of the form (4.2) near $t = 0$. Globally, the solution only satisfies $u \in W^{1.5,\infty}(J)$ and $u \notin W^{2,\infty}(J)$.

In Figure 4 we present results for the h- and p-versions. In the h-version on uniform time partition, we see an optimal convergence rate of 1.5 for $m = 1$, whereas

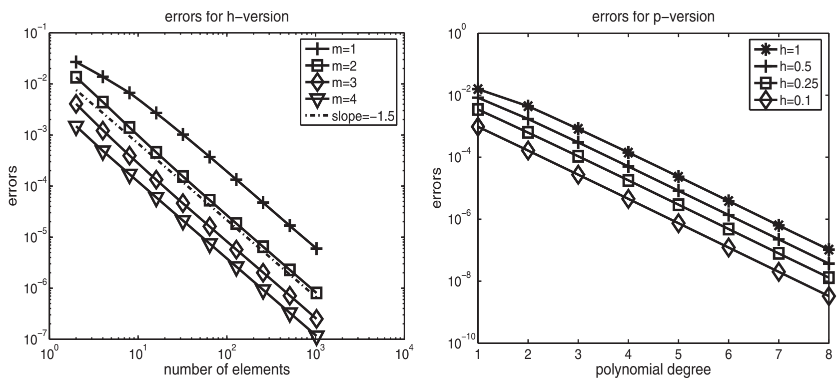

FIG. 4. *Errors of h-version and p-version for Example* 5.3 *with* $\theta(t) = \arctan(t)$.

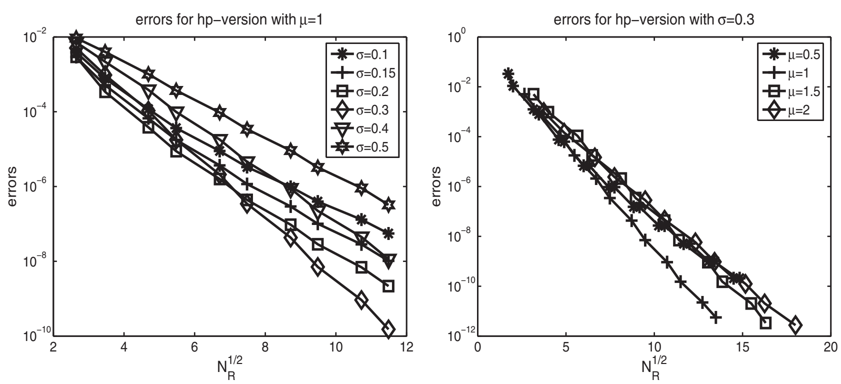

FIG. 5. *Errors of hp-version for Example* 5.3 *with* $\theta(t) = \arctan(t)$.

for higher orders m the optimal rates are no longer achieved. Since the solution has a singularity at $t = 0$, the p-version of the DG method can only be expected to yield algebraic rates of convergence, in contrast to the tests in Examples 5.1 and 5.2. Algebraic convergence behavior is indeed observed in Figure 4, where we increase the polynomial degree m on the same time partition as above.

Next we consider the performance of the hp-version of the DG method on the geometric partition $\mathcal{T}_{p,\sigma}$ as in Definition 4.4 and we use linearly increasing polynomial degrees as described in Definition 4.7.

In the left of Figure 5, we display the errors against the square root of the number of degrees of freedom in the underlying discretization space, for $\mu = 1$ and various values of the grading factor σ. The straight curves indicate exponential convergence for each grading factor σ, as predicted by Theorem 4.8. It can further be seen that the grading $\sigma = 0.3$ gives the best results; for example, they are several orders of magnitude better than those for $\sigma = 0.5$. In the right of Figure 5, we show the convergence curves for $\sigma = 0.3$ and several values of the slope parameter μ. The exponential convergence rates are less sensitive to variations in this parameter and good results are obtained for $\mu = 1$.

6. Concluding remarks. The following three problems remain to be addressed in future research work:

1. the choice of suitable values of σ, μ and the degree vector $\mathbf{m}_N$;
2. the analysis of the hp-DG method for Volterra functional integro-differential equations of the forms

$$u'(t) = a(t)u(t) + b(t)u(\theta(t)) + (V_\theta u)(t) + f(t), \qquad t \in J,$$

or

$$u'(t) = a(t)u(t) + b(t)u(\theta(t)) + (W_\theta u)(t) + f(t), \qquad t \in J,$$

with Volterra delay integral operators

$$(V_\theta u)(t) := \int_0^{\theta(t)} K(t,s)u(s)ds$$

and

$$(W_\theta u)(t) := \int_{\theta(t)}^{t} K(t,s)u(s)ds.$$

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