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ON ANOMALOUS LOCALIZED RESONANCE FOR THE ELASTOSTATIC SYSTEM*

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Abstract. We consider the anomalous localized resonance due to a plasmonic structure for the elastostatic system in $\mathbb{R}^2$. The plasmonic structure takes a general core-shell-matrix form with the metamaterial located in the shell. If there is no core, we show that resonance occurs for a very broad class of sources. If the core is nonempty and of an arbitrary shape, we show that there exists a critical radius such that resonance occurs for a certain class of sources lying within the critical radius, whereas resonance does not occur for a certain class of sources lying outside the critical radius. Our argument is based on a variational technique by making use of the primal and dual variational principles for the elastostatic system, along with the construction of suitable test functions.

Key words. anomalous localized resonance, plasmonic material, elastostatics

AMS subject classifications. 35B34, 74E99, 74J20

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1. Introduction.

Recently, there has been growing interest in studying the resonance phenomena for an inhomogeneous body possessing negative material parameters and their connection to invisibility cloaking. The term "anomalous localized resonance" is used in the physics literature [22, 25], and those negative materials are referred to as "plasmonic materials." The mathematical principle lies in the fact that the ellipticity of the governing PDE system is lost in the limiting case as the loss parameter approaches zero. The limiting (nonelliptic) partial differential operator (PDO) possesses an infinite dimensional kernel, called *perfect plasmon waves* in the literature (cf. [16]). Then anomalous resonance occurs associated with the aforementioned infinite dimensional kernel. Moreover, the resonance is localized in the sense that as it occurs, the highly oscillatory field (namely, the solution of the concerned PDE system) develops with the associated energy divergent to infinity in a specific region as the loss goes to zero, whereas the field converges to a smooth field outside that region. Another surprisingly interesting feature of the anomalous resonance is that it strongly depends on the location of the source/forcing term. Furthermore, cloaking effects are found to be associated with the anomalous localized resonance in the sense that inhomogeneous materials or certain sources near the aforementioned plasmonic structure cause virtually invisible perturbations of the fields that would have been observed in the absence of the structure. We refer to [23] and the references therein for more relevant discussion on the cloaking mainly from the physical perspectives, and we also refer to [3, 16] for the relevant discussion mainly from a mathematical point of view. In what follows, we simply refer to anomalous resonance as resonance, and we use the acronym ALR to denote anomalous localized resonance. If there is a

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cloaking effect associated with the ALR, we call it CALR, which stands for cloaking due to ALR.

The plasmonic resonances including ALR and CALR have been extensively investigated for optical parameter distributions, respectively modeled by the Laplace equation [2, 3, 4, 8, 11, 12, 13, 16, 19, 21, 23, 24, 26, 27, 28], the Helmholtz equation [5, 6, 10, 14], and the Maxwell system [7]. In the literature, there are two approaches that have been proposed for analyzing the resonance behaviors: one is based on the analysis of the spectral properties of the Neumann–Poincaré operator via the layer potential theory [3] and the other one is based on variational arguments via the use of primal and dual variational principles for the corresponding PDE systems [16]. The spectral approach initiated in [3] by Ammari et al. for the Laplace equation can yield certain accurate characterizations of the ALR as well as its connection to the invisibility cloaking. The spectral approach has been extensively used in characterizing the ALR and CALR for the Laplace equation and the Helmholtz system [2, 4, 8, 10, 13]. However, the spectral approach requires accurate spectral information of the Neumann–Poincaré operator and is mainly restricted to spherical and elliptical geometries. The variational approach initiated in [16] by Kohn et al. for the Laplace equation can deal with general geometries in an elegant and concise way but can be used only to show the resoance results. It is also well suited to establish the dependence of the resonance on the location of the forcing/source term. The variational approach was followed in [19] to show the resonance results for the Laplace equation in three dimensions.

In a very recent work by Ando et al. [9], the spectral theory of the Neumann–Poincaré operator was extended from the Laplace equation to the Lamé system governing the elastostatics. Using the spectral approach, the authors also established the resonance and CALR results in the spherical and elliptic geometries for the elastostatic system. In this paper, we aim to establish the resonance results for the elastostatic system via the variational approach. In what follows, we first present the mathematical setup of our study.

Let $\mathbf{C}(x) := (\mathrm{C}_{ijkl}(x))_{i,j,k,l=1}^N$, $x \in \mathbb{R}^N$ with $N = 2, 3$, be a four-rank tensor such that

$$\mathrm{C}_{ijkl}(x) := \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\lambda, \mu \in \mathbb{C}$ are real-valued functions and δ is the Kronecker delta. $\mathbf{C}(x)$ describes an isotropic elasticity tensor distributed in the space, where λ and μ are called the Lamé constants. For a regular elastic material, it is required that the Lamé constants satisfy the strong convexity condition,

$$\mu > 0 \qquad \text{and} \qquad N\lambda + 2\mu > 0. \tag{1.2}$$

The existence of exotic elastic materials with negative stiffness was shown in [15] and [18], which we shall generally refer to as plasmonic materials in the current article. We write $\mathbf{C}_{\lambda,\mu}$ to specify the dependence of the elastic tensor on the Lamé parameters λ and μ. Let Σ and Ω be bounded domains in $\mathbb{R}^N$ with connected Lipschitz boundaries such that $\Sigma \Subset \Omega$. Consider an elastic parameter distribution $\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}$ given with

$$\big(\widetilde{\lambda}(x), \widetilde{\mu}(x)\big) = \big(A(x) + \mathrm{i}\delta\big)(\lambda, \mu), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $\delta \in \mathbb{R}_+$ denotes a loss parameter; (λ, μ) are two constants satisfying the strong convexity condition (1.2); and $A(x)$ has a matrix-shell-core character in the sense that

$$A(x) = \begin{cases} +1, & x \in \Sigma, \\ c, & x \in \Omega\backslash\overline{\Sigma}, \\ +1, & x \in \mathbb{R}^N\backslash\overline{\Omega}, \end{cases} \tag{1.4}$$

where c is constant that will be specified later. In principle, c will be negative-valued so that (1.3) yields a plasmonic structure. Let $\mathbf{f}$ be an $\mathbb{R}^N$-valued function that is compactly supported outside Ω satisfying

$$\int_{\mathbb{R}^N} \mathbf{f}(x)\ dV(x) = 0. \tag{1.5}$$

$\mathbf{f}$ signifies an elastic source/forcing term. Let $\mathbf{u}_\delta(x) \in \mathbb{C}^N$, $x \in \mathbb{R}^N$, denote the displacement field in the space that is occupied by the elastic configuration $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f})$ as described above. In the quasi-static regime, $\mathbf{u}_\delta(x) \in H^1_{\mathrm{loc}}(\mathbb{R}^N)^N$ verifies the following Lamé system:

$$\begin{cases} \mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{u}_\delta(x) = \mathbf{f}(x), \quad x \in \mathbb{R}^N, \\ \mathbf{u}_\delta|_- = \mathbf{u}_\delta|_+, \quad \partial_{\boldsymbol{\nu}_{\widetilde{\lambda},\widetilde{\mu}}}\mathbf{u}_\delta|_- = \partial_{\boldsymbol{\nu}_{\widetilde{\lambda},\widetilde{\mu}}}\mathbf{u}_\delta|_+ \quad \text{on} \ \ \partial\Sigma \cup \partial\Omega, \\ \mathbf{u}_\delta(x) = \mathcal{O}\big(\|x\|^{-1}\big) \quad \text{as} \ \ \|x\| \to +\infty, \end{cases} \tag{1.6}$$

where the partial differential operator $\mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}}$ is given as

$$\mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{u}_\delta := \nabla\cdot\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}\nabla^s\mathbf{u}_\delta = \widetilde{\mu}\Delta\mathbf{u}_\delta + (\widetilde{\lambda}+\widetilde{\mu})\nabla\nabla\cdot\mathbf{u}_\delta \tag{1.7}$$

with ∇^s defined to be the symmetric gradient

$$\nabla^s\mathbf{u}_\delta := \frac{1}{2}(\nabla\mathbf{u}_\delta + \nabla\mathbf{u}_\delta^T)$$

and T signifying the matrix transpose. In (1.6), the conormal derivative (or traction) is defined by

$$\partial_{\boldsymbol{\nu}_{\widetilde{\lambda},\widetilde{\mu}}}\mathbf{u}_\delta = \frac{\partial\mathbf{u}_\delta}{\partial\boldsymbol{\nu}_{\widetilde{\lambda},\widetilde{\mu}}} := \widetilde{\lambda}(\nabla\cdot\ \mathbf{u}_\delta)\boldsymbol{\nu} + \widetilde{\mu}(\nabla\mathbf{u}_\delta + \nabla\mathbf{u}_\delta^T)\boldsymbol{\nu} \quad \text{on} \ \ \partial\Sigma \text{ or } \partial\Omega, \tag{1.8}$$

where $\boldsymbol{\nu}$ denotes the exterior unit normal to $\partial\Sigma/\partial\Omega$, and the $\pm$ signify the traces taken from outside and inside of the domain Σ/Ω, respectively.

Next, for $\mathbf{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^N)^N$ and $\mathbf{v} \in H^1_{\mathrm{loc}}(\mathbb{R}^N)^N$, we introduce

$$\mathbf{P}_{\lambda,\mu}(\mathbf{u},\mathbf{v}) := \int_{\mathbb{R}^N} \left[\lambda(\nabla\cdot\mathbf{u})\overline{(\nabla\cdot\mathbf{v})}(x) + 2\mu\nabla^s\mathbf{u} : \overline{\nabla^s\mathbf{v}}(x)\right] dV(x), \tag{1.9}$$

where, and also in what follows, for two matrices $\mathbf{A} = (a_{ij})_{i,j=1}^N$ and $\mathbf{B} = (b_{ij})_{i,j=1}^N$,

$$\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^N a_{ij}b_{ij}.$$

For the solution $\mathbf{u}_\delta$ to (1.6), we define

$$\mathbf{E}_\delta(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}},\mathbf{f}) := \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\mathbf{u}_\delta,\mathbf{u}_\delta). \tag{1.10}$$

It is remarked that $\mathbf{E}_\delta$ defined above is the imaginary part of

$$\frac{1}{2}\int_{\mathbb{R}^N} \overline{\nabla^s \mathbf{u}_\delta} : \mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}} \nabla^s \mathbf{u}_\delta \, dV,$$

which represents the elastic energy of the system (1.6) (cf. (1.48)–(1.49) in [1]). Hence, $\mathbf{E}_\delta$ signifies the energy dissipation of the elastostatic system (1.6).

Definition 1.1. *The configuration* $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}},\mathbf{f})$ *in* (1.6) *is said to be* resonant *if*

$$\lim_{\delta\to+0}\mathbf{E}_\delta(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}},\mathbf{f}) = +\infty, \tag{1.11}$$

and it is said to be weakly resonant *if*

$$\limsup_{\delta\to+0}\mathbf{E}_\delta(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}},\mathbf{f}) = +\infty. \tag{1.12}$$

Following the variational spirit of the study on plasmonic resonance in electrostatics in [16], we develop a variational study on the plasmonic resonance in elastostatics. The main results obtained in the present paper can be summarized as follows:

1. We establish the primal and dual variational principles for the elastostatic system (1.6) and apply them to the study of the plasmonic resonance.
2. We derive a plasmonic structure of the form (1.3)–(1.4) such that the corresponding PDO $\mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}}$ possesses an infinite dimensional kernel and hence plasmonic resonance can be induced.
3. For the plasmonic structure in item 2, if the core Σ is empty, it is shown that resonance occurs for a very broad class of sources. If the core is nonempty and of an arbitrary shape, it is shown that there exists a critical radius such that resonance occurs for a certain class of sources lying within the critical radius, whereas resonance does not occur for a certain class of sources lying outside the critical radius. For the latter nonresonance case, it is assumed that the core Σ is a disk. That is, we can establish the dependence on the source location of the plasmonic resonance in the case with a nonempty core.

Due to the distinct physical and mathematical nature of the elastostatic system (1.6), we would need to develop technical and subtle modifications and extensions to the variational argument developed in [16] for the electrostatics; see also Remarks 3.1 and 3.5 for more relevant discussion. This distinguishes our present study from that in [16]. As also mentioned earlier, the resonance for the Lamé system (1.6) has been investigated in [9] in two dimensions. The results that we shall establish in this paper are distinguished from those in [9] in the following aspects:

1. The results in [9] were established by using the spectral approach, whereas in this paper we make use of the variational approach to establish our resonance results. As remarked earlier, the spectral argument can be used to establish both ALR and CALR results, but via the variational argument, we could only establish the resonance results. Nevertheless, the variational approach yields a different and alternative perspective to the plasmonic resonance for the elastostatic system.

2. In [9], it is always assumed that the core is empty, namely, $\Sigma = \emptyset$, whereas in this paper, Σ could be an empty set, or a nonempty set of an arbitrary shape. In [9], the exterior domain, namely, Ω, could be a disk or an ellipse, whereas in this paper, we take Ω to be a disk. Nevertheless, we would like to emphasize that our study can be extended to the case that Ω is an ellipse by following similar arguments.
3. In [9], the source term $\mathbf{f}$ is assumed to be a function compactly supported in $\mathbb{R}^2\backslash\overline{\Omega}$ in the case with the radial geometry and an elastic dipole in the case with the ellipse geometry, whereas in our study, we assume that $\mathbf{f}$ is distributed on a circular curve enclosing Ω. Our method also allows for more general distributional sources by using the principle of superpositions.
4. If the core Σ is an empty set, then in both [9] and the present paper, the resonance results are established for generic sources lying outside the plasmonic structure. Moreover, the resonance does not depend on the source location. If the exterior domain Ω is an ellipse with $\Sigma = \emptyset$, then ALR and CALR results were established in [9] for a dipole source term. The ALR and CALR results depend on the location of the dipole source. In our study, by assuming that Ω is a disk and Σ is nonempty, we establish the resonance result with the accurate characterization of the dependence on the source location.

The rest of the paper is organized as follows. In section 2, we establish the primal and dual variational principles for the elastostatic system. Section 3 is devoted to the resonance and nonresonance results. We conclude our study in section 4 with some discussion.

2. Variational principles for the elastostatic system.

In this section, we establish the primal and dual variational principles for the elastostatic system (1.6), which shall play a critical role in our subsequent resonance study. Throughout the present section, we assume that the source term $\mathbf{f} = (f_i)_{i=1}^N \in H^{-1}(\mathbb{R}^N)^N$ in (1.6) with a compact support and a zero average in the sense that

$$\langle f_i, \mathbf{1}\rangle = 0, \quad i = 1, 2, \ldots, N, \tag{2.1}$$

where $\mathbf{1} : \mathbb{R}^N \to \mathbb{R}$ is the constant function, $\mathbf{1}(x) = 1$ for all $x \in \mathbb{R}^N$.

2.1. Preliminaries for the elastostatic system.

We collect some preliminary knowledge on the elastostatic system (1.6). Those are standard results, but we cannot find a convenient reference. In what follows, we let B_R with $R \in \mathbb{R}_+$ denote a central ball of radius R in $\mathbb{R}^N$. Throughout, without loss of generality, we assume that there exists $R_0 \in \mathbb{R}_+$ such that $\text{supp}(\mathbf{f}) \subset B_{R_0}$. Let $\mathbf{u}_\delta \in H^1_{\text{loc}}(\mathbb{R}^N)^N$ satisfy the Lamé system, namely, $\mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{u}_\delta = \mathbf{f}$. For a Lipschitz domain $D \subset \mathbb{R}^N$ and any test function $\mathbf{v} \in H^1_{\text{loc}}(\mathbb{R}^N)^N$, we recall the following Green's formula:

$$\int_{\partial D} \overline{\mathbf{v}} \cdot \frac{\partial \mathbf{u}_\delta}{\partial \boldsymbol{\nu}_{\widetilde{\lambda},\widetilde{\mu}}}\, ds(x) = \int_D \overline{\mathbf{v}} \cdot \mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{u}_\delta\, dV(x) + \mathbf{P}_{\widetilde{\lambda},\widetilde{\mu}}(\mathbf{u}_\delta, \mathbf{v}), \tag{2.2}$$

where $\mathbf{P}_{\widetilde{\lambda},\widetilde{\mu}}(\mathbf{u}_\delta, \mathbf{v})$ is given by (1.9) with the integration domain replaced by D. Using (2.2), the weak solution $\mathbf{u}_\delta \in H^1_{\text{loc}}(\mathbb{R}^N)^N$ to (1.6) is given in the sense that

$$\mathbf{P}_{\widetilde{\lambda},\widetilde{\mu}}(\mathbf{u}_\delta, \mathbf{v}) = -\langle \mathbf{f}, \overline{\mathbf{v}}\rangle, \tag{2.3}$$

where $\mathbf{v} \in H^1_{\text{loc}}(\mathbb{R}^N)^N$, compactly supported in B_R for any $R \geq R_0$. In order to have a functional analytic framework for the variational formulation, we introduce the following function space:

$$\mathcal{S} := \left\{ \mathbf{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^N)^N;\ \nabla \mathbf{u} \in L^2(\mathbb{R}^N)^{N\times N} \ \text{ and } \int_{B_{R_0}} \mathbf{u} = 0 \right\}. \tag{2.4}$$

$\mathcal{S}$ is a Banach space endowed with the following norm for $\mathbf{u} = (u_i)_{i=1}^N$:

$$\|\mathbf{u}\|_{\mathcal{S}} := \left(\int_{\mathbb{R}^N} \sum_{i=1}^N \|\nabla u_i\|^2 \, dV + \int_{B_{R_0}} \|\mathbf{u}\|^2 \, dV \right)^{1/2}. \tag{2.5}$$

Consider the following sesquilinear form, $\mathscr{B}(\cdot,\cdot) : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$, defined by

$$\mathscr{B}(\mathbf{u}, \mathbf{v}) := -\mathrm{i} \cdot \mathbf{P}_{\widetilde{\lambda},\widetilde{\mu}}(\mathbf{u}, \mathbf{v}). \tag{2.6}$$

It is straightforward to verify that $\mathscr{B}$ is bounded and, by using the Poincaré–Wirtinger inequality, coercive. Hence, by the Lax–Milgram theorem, one can show that there exists a unique solution $\mathbf{u}_\delta \in \mathcal{S}$ to the Lamé system (1.6). Finally, by using the fact that the Kelvin matrix of the fundamental solution $\mathbf{\Phi} = (\Phi_{ij})_{i,j=1}^N$ to the PDO $\mathcal{L}_{\lambda,\mu}$ is given by (cf. [17])

$$\Phi_{ij}(x) = \begin{cases} \dfrac{\alpha}{2\pi} \delta_{ij} \ln \|x\| - \dfrac{\beta}{2\pi} \dfrac{x_i x_j}{\|x\|^2} & \text{when } \ N = 2, \\[2ex] -\dfrac{\alpha}{4\pi} \dfrac{\delta_{ij}}{\|x\|} - \dfrac{\beta}{4\pi} \dfrac{x_i x_j}{\|x\|^3} & \text{when } \ N = 3, \end{cases} \tag{2.7}$$

where

$$\alpha = \frac{1}{2} \left(\frac{1}{\mu} + \frac{1}{2\mu + \lambda} \right) \quad \text{and} \quad \beta = \frac{1}{2} \left(\frac{1}{\mu} - \frac{1}{2\mu + \lambda} \right),$$

and $x = (x_i)_{i=1}^N \in \mathbb{R}^N$ and δ_{ij} is the Kronecker delta, one can show by direct calculations that the solution $\mathbf{u}_\delta \in \mathcal{S}$ to $\mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}} \mathbf{u}_\delta = \mathbf{f}$ possesses the asymptotic behavior, $\mathbf{u}_\delta(x) = \mathcal{O}(\|x\|^{-1})$ as $\|x\| \to +\infty$.

2.2. Primal and dual variational principles. We now establish the primal and dual variational principles for the elastostatic system (1.6). For a source force term $\mathbf{f} \in H^{-1}(\mathbb{R}^N)^N$ and for the solution $\mathbf{u}_\delta \in H^1_{\mathrm{loc}}(\mathbb{R}^N)^N : \mathbb{R}^N \to \mathbb{C}^N$, we set

$$\mathbf{u}_\delta = \mathbf{v}_\delta + \mathrm{i} \frac{1}{\delta} \mathbf{w}_\delta, \tag{2.8}$$

where $\mathbf{v}_\delta, \mathbf{w}_\delta \in H^1_{\mathrm{loc}}(\mathbb{R}^N)^N : \mathbb{R}^N \to \mathbb{R}^N$ satisfying $\mathbf{v}_\delta = \mathcal{O}(\|x\|^{-1})$ and $\mathbf{w}_\delta = \mathcal{O}(\|x\|^{-1})$ as $\|x\| \to +\infty$. By straightforward calculations, one can show that the elastostatic system (1.6) for $\mathbf{u}_\delta$ is equivalent to the following coupled system for the two real functions $\mathbf{v}_\delta$ and $\mathbf{w}_\delta$:

$$\mathcal{L}_{\lambda_A,\mu_A} \mathbf{v}_\delta - \mathcal{L}_{\lambda,\mu} \mathbf{w}_\delta = \mathbf{f}, \tag{2.9}$$

$$\mathcal{L}_{\lambda_A,\mu_A} \mathbf{w}_\delta + \delta^2 \mathcal{L}_{\lambda,\mu} \mathbf{v}_\delta = 0, \tag{2.10}$$

where

$$\big(\lambda_A(x), \mu_A(x)\big) := A(x)(\lambda, \mu), \quad x \in \mathbb{R}^N, \tag{2.11}$$

with A given in (1.4), and (λ, μ) are the two regular Lamé constants in (1.3). Furthermore, by direct computations, one has that

$$\begin{aligned}\mathbf{E}_\delta(\mathbf{u}_\delta) := \mathbf{E}_\delta(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f}) &= \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\mathbf{u}_\delta, \mathbf{u}_\delta)\\ &= \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta, \mathbf{v}_\delta) + \frac{1}{2\delta}\mathbf{P}_{\lambda,\mu}(\mathbf{w}_\delta, \mathbf{w}_\delta),\end{aligned} \tag{2.12}$$

where $\mathbf{E}_\delta$ is given in (1.10) and $\mathbf{P}_{\lambda,\mu}$ is given in (1.9).

Next, we introduce the energy functional

$$\mathbf{I}_\delta(\mathbf{v}, \mathbf{w}) := \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\mathbf{v}, \mathbf{v}) + \frac{1}{2\delta}\mathbf{P}_{\lambda,\mu}(\mathbf{w}, \mathbf{w}) \quad \text{for} \quad (\mathbf{v}, \mathbf{w}) \in \mathcal{S} \times \mathcal{S} \tag{2.13}$$

and consider the following optimization problem:

$$\begin{aligned}&\text{Minimize } \mathbf{I}_\delta(\mathbf{v}, \mathbf{w}) \text{ over all pairs } (\mathbf{v}, \mathbf{w}) \in \mathcal{S} \times \mathcal{S}\\ &\text{subject to the PDE constraint } \mathcal{L}_{\lambda_A,\mu_A}\mathbf{v} - \mathcal{L}_{\lambda,\mu}\mathbf{w} = \mathbf{f}.\end{aligned} \tag{2.14}$$

In what follows, we shall refer to (2.14) as the primal variational problem for the elastostatic system (1.6), or equivalently (2.9)–(2.10).

We have the following result about the primal problem (2.14).

Lemma 2.1. *The primal variational problem* (2.14) *is equivalent to the elastostatic problem* (1.6) *in the following sense:*

1. *The infimum*

$$\inf\left\{\mathbf{I}_\delta(\mathbf{v}, \mathbf{w});\, \mathcal{L}_{\lambda_A,\mu_A}\mathbf{v} - \mathcal{L}_{\lambda,\mu}\mathbf{w} = \mathbf{f}\right\} \tag{2.15}$$

is attainable at a pair $(\mathbf{v}_\delta, \mathbf{w}_\delta) \in \mathcal{S} \times \mathcal{S}$.

2. *The minimizing pair* $(\mathbf{v}_\delta, \mathbf{w}_\delta)$ *verifies that the function* $\mathbf{u}_\delta := \mathbf{v}_\delta + \mathrm{i}\delta^{-1}\mathbf{w}_\delta$ *is the unique solution to the elastostatic problem* (1.6).
3. *For the solution in* (2)*, the energies coincide, namely,*

$$\mathbf{E}_\delta(\mathbf{u}_\delta) = \mathbf{I}_\delta(\mathbf{v}_\delta, \mathbf{w}_\delta). \tag{2.16}$$

Proof. It is directly verified that $\mathbf{I}_\delta(\mathbf{v}, \mathbf{w})$ is a convex functional. Moreover, the PDE constraint in (2.14) yields a nonempty set. Indeed, for any $\hat{\mathbf{v}} \in C_c^\infty(\mathbb{R}^N)$ satisfying $\int_{B_{R_0}} \hat{\mathbf{v}} = 0$, we let $\hat{\mathbf{w}}$ be the weak solution to $\mathcal{L}_{\lambda,\mu}\hat{\mathbf{w}} = \mathcal{L}_{\lambda_A,\mu_A}\hat{\mathbf{v}} - \mathbf{f}$ given by

$$\hat{\mathbf{w}} = \mathbf{\Phi} * (\mathcal{L}_{\lambda_A,\mu_A}\hat{\mathbf{v}} - \mathbf{f}).$$

It is clear that $\hat{\mathbf{w}} \in \mathcal{S}$ and hence $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$ satisfies the PDE constraint in (2.14). Hence, the infimum of (2.15) is attainable; that is, there exists a pair $(\mathbf{v}_\delta, \mathbf{w}_\delta)$ such that

$$\mathbf{I}_\delta(\mathbf{v}_\delta, \mathbf{w}_\delta) \le \mathbf{I}_\delta(\mathbf{v}, \mathbf{w}) \text{ for all } (\mathbf{v}, \mathbf{w}) \in \mathcal{S} \times \mathcal{S} \text{ satisfying } \mathcal{L}_{\lambda_A,\mu_A}\mathbf{v} - \mathcal{L}_{\lambda,\mu}\mathbf{w} = \mathbf{f}. \tag{2.17}$$

Next we show that $\mathbf{u}_\delta := \mathbf{v}_\delta + \mathrm{i}\delta^{-1}\mathbf{w}_\delta$ is the solution to the elastostatic problem (1.6). As a minimizer of $\mathbf{I}_\delta$, the pair $(\mathbf{v}_\delta, \mathbf{w}_\delta)$ must verify the Euler–Lagrange equation,

$$\partial_\tau \mathbf{I}_\delta(\mathbf{v}_\delta + \tau\widetilde{\mathbf{v}}, \mathbf{w}_\delta + \tau\widetilde{\mathbf{w}})\big|_{\tau=0} = 0, \tag{2.18}$$

for every pair $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}) \in \mathcal{S} \times \mathcal{S}$ satisfying

$$\mathcal{L}_{\lambda_A,\mu_A}\widetilde{\mathbf{v}} - \mathcal{L}_{\lambda,\mu}\widetilde{\mathbf{w}} = 0. \tag{2.19}$$

Here, it is assumed that $\widetilde{\mathbf{v}}$ could be any element in $\mathcal{S}$ having a compact support. For the energy $\mathbf{I}_\delta$, this equation reads

$$\delta\mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta, \tilde{\mathbf{v}}) + \frac{1}{\delta}\mathbf{P}_{\lambda,\mu}(\mathbf{w}_\delta, \tilde{\mathbf{w}}) = 0. \tag{2.20}$$

With the help of Green's formula and (2.19), the last equation yields

$$\begin{aligned}
&-\delta\int_{\mathbb{R}^N}\mathcal{L}_{\lambda,\mu}\mathbf{v}_\delta\cdot\widetilde{\mathbf{v}}dV(x) - \frac{1}{\delta}\int_{\mathbb{R}^N}\mathbf{w}_\delta\cdot\mathcal{L}_{\lambda,\mu}\widetilde{\mathbf{w}}dV(x)\\
&= -\delta\int_{\mathbb{R}^N}\mathcal{L}_{\lambda,\mu}\mathbf{v}_\delta\cdot\widetilde{\mathbf{v}}dV(x) - \frac{1}{\delta}\int_{\mathbb{R}^N}\mathbf{w}_\delta\cdot\mathcal{L}_{\lambda_A,\mu_A}\widetilde{\mathbf{v}}dV(x)\\
&= -\frac{1}{\delta}\int_{\mathbb{R}^N}(\delta^2\mathcal{L}_{\lambda,\mu}\mathbf{v}_\delta + \mathcal{L}_{\lambda_A,\mu_A}\mathbf{w}_\delta)\cdot\widetilde{\mathbf{v}}dV(x) = 0,
\end{aligned} \tag{2.21}$$

which is the weak form of (2.10). As a solution of (2.9)–(2.10), the pair $(\mathbf{v}_\delta, \mathbf{w}_\delta)$ defines through $\mathbf{u}_\delta := \mathbf{v}_\delta + \mathrm{i}\delta^{-1}\mathbf{w}_\delta$ a solution to the original problem (1.6).

The uniqueness is a consequence of the fact that the original problem (1.6) possesses a unique solution. Finally, it is obvious that $\mathbf{E}_\delta(\mathbf{u}_\delta) = \mathbf{I}_\delta(\mathbf{v}_\delta, \mathbf{w}_\delta)$ from (2.12).

The proof is complete. ∎

Remark 2.1. By Lemma 2.1, it is readily seen that for the solution $\mathbf{u}_\delta$ to (1.6) and the energy $\mathbf{E}_\delta(\mathbf{u}_\delta)$ in (2.12), one has that

$$\mathbf{E}_\delta(\mathbf{u}_\delta) \le \mathbf{I}_\delta(\mathbf{v}, \mathbf{w}) \tag{2.22}$$

for the energy functional $\mathbf{I}_\delta$ defined in (2.13) and every pair $(\mathbf{v}, \mathbf{w})$ verifying the constraint specified in (2.14). For the elastic configuration $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f})$, we shall make use of the primal variational principle via (2.22) to show the nonresonance results by constructing suitable test functions $\mathbf{v}$ and $\mathbf{w}$.

We proceed to introduce the dual variational problem by defining the following energy functional:

$$\mathbf{J}_\delta(\mathbf{v}, \boldsymbol{\psi}) := \int_{\mathbb{R}^N}\mathbf{f}\cdot\boldsymbol{\psi} - \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\mathbf{v}, \mathbf{v}) - \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\boldsymbol{\psi}, \boldsymbol{\psi}) \text{ for } (\mathbf{v}, \boldsymbol{\psi}) \in \mathcal{S}\times\mathcal{S}. \tag{2.23}$$

Consider the following optimization problem:

$$\begin{aligned}
&\text{Maximize } \mathbf{J}_\delta(\mathbf{v}, \boldsymbol{\psi}) \text{ over all pairs } (\mathbf{v}, \boldsymbol{\psi}) \in \mathcal{S}\times\mathcal{S}\\
&\text{subject to the PDE constraint } \mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi} + \delta\mathcal{L}_{\lambda,\mu}\mathbf{v} = 0.
\end{aligned} \tag{2.24}$$

In what follows, we shall refer to (2.24) as the dual variational problem for the elastostatic system (1.6), or equivalently (2.9)–(2.10).

We have the following result about the dual problem (2.24).

LEMMA 2.2. *The dual variational problem* (2.24) *is equivalent to the elastostatic problem* (1.6) *in the following sense:*

1. *The supremum*

$$\sup\left\{\mathbf{J}_\delta(\mathbf{v}, \boldsymbol{\psi}); \mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi} + \delta\mathcal{L}_{\lambda,\mu}\mathbf{v} = 0\right\} \tag{2.25}$$

is attainable at a pair $(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta) \in \mathcal{S}\times\mathcal{S}$.

2. *The maximizing pair* $(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta)$ *verifies that the function* $\mathbf{u}_\delta := \mathbf{v}_\delta + \mathrm{i}\boldsymbol{\psi}_\delta$ *is the unique solution to the elastostatic problem* (1.6).
3. *For the solution in* (2), *the energies coincide, namely,*

$$\mathbf{E}_\delta(\mathbf{u}_\delta) = \mathbf{J}_\delta(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta). \tag{2.26}$$

Proof. We shall follow arguments similar to those for the proof of Lemma 2.1. The existence and uniqueness can be proved similarly to that of the primal variational problem.

Next we prove that $\mathbf{u}_\delta := \mathbf{v}_\delta + \mathrm{i}\boldsymbol{\psi}_\delta$ is the solution of the original problem (1.6). As a maximizer of $\mathbf{J}_\delta$, the pair $(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta)$ must verify the Euler–Lagrange equation,

$$\partial_\tau \mathbf{J}_\delta(\mathbf{v}_\delta + \tau\widetilde{\mathbf{v}}, \boldsymbol{\psi}_\delta + \tau\widetilde{\boldsymbol{\psi}})|_{\tau=0} = 0 \tag{2.27}$$

for every pair $(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\psi}}) \in \mathcal{S} \times \mathcal{S}$ satisfying

$$\mathcal{L}_{\lambda_A,\mu_A}\widetilde{\boldsymbol{\psi}} + \delta\mathcal{L}_{\lambda,\mu}\widetilde{\mathbf{v}} = 0. \tag{2.28}$$

Here, it is assumed that $\widetilde{\boldsymbol{\psi}}$ could be any element in $\mathcal{S}$ having a compact support. For the energy functional $\mathbf{J}_\delta$, this equation is equivalent to

$$\int_{\mathbb{R}^N} \mathbf{f}\cdot\tilde{\boldsymbol{\psi}} - \delta\mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta, \tilde{\mathbf{v}}) - \delta\mathbf{P}_{\lambda,\mu}(\boldsymbol{\psi}_\delta, \tilde{\boldsymbol{\psi}}) = 0. \tag{2.29}$$

Using Green's formula and (2.28), together with straightforward calculations, one has from (2.29) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \mathbf{f}\cdot\widetilde{\boldsymbol{\psi}}\, dV(x) + \delta\int_{\mathbb{R}^N} \mathbf{v}_\delta\cdot\mathcal{L}_{\lambda,\mu}\widetilde{\mathbf{v}}\, dV(x) + \delta\int_{\mathbb{R}^N} \boldsymbol{\psi}_\delta\cdot\mathcal{L}_{\lambda,\mu}\widetilde{\boldsymbol{\psi}}\, dV(x)\\
&=\int_{\mathbb{R}^N} \mathbf{f}\cdot\widetilde{\boldsymbol{\psi}}\, dV(x) - \int_{\mathbb{R}^N} \mathbf{v}_\delta\cdot\mathcal{L}_{\lambda_A,\mu_A}\widetilde{\boldsymbol{\psi}}\, dV(x) + \delta\int_{\mathbb{R}^N} \mathcal{L}_{\lambda,\mu}\boldsymbol{\psi}_\delta\cdot\widetilde{\boldsymbol{\psi}}\, dV(x)\\
&=\int_{\mathbb{R}^N} \mathbf{f}\cdot\widetilde{\boldsymbol{\psi}}\, dV(x) - \int_{\mathbb{R}^N} \mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}_\delta\cdot\widetilde{\boldsymbol{\psi}}\, dV(x) + \delta\int_{\mathbb{R}^N} \mathcal{L}_{\lambda,\mu}\boldsymbol{\psi}_\delta\cdot\widetilde{\boldsymbol{\psi}}\, dV(x)\\
&=\int_{\mathbb{R}^N} (\mathbf{f} - \mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}_\delta + \delta\mathcal{L}_{\lambda,\mu}\boldsymbol{\psi}_\delta)\cdot\widetilde{\boldsymbol{\psi}}\, dV(x) = 0.
\end{aligned} \tag{2.30}$$

By (2.30), we conclude that the pair $(\mathbf{v}_\delta, \mathbf{w}_\delta) := (\mathbf{v}_\delta, \delta\boldsymbol{\psi}_\delta)$ is a weak solution of (2.9)–(2.10), and thus $\mathbf{u}_\delta := \mathbf{v}_\delta + i\boldsymbol{\psi}_\delta$ is a solution to the original problem (1.6).

Finally, we verify that $\mathbf{E}_\delta(\mathbf{u}_\delta) = \mathbf{J}_\delta(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta)$. By using Green's formula again, we have

$$\begin{aligned}
&\mathbf{E}_\delta(\mathbf{u}_\delta) - \mathbf{J}_\delta(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta)\\
&=\frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta, \mathbf{v}_\delta) + \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\boldsymbol{\psi}_\delta, \boldsymbol{\psi}_\delta) - \int_{\mathbb{R}^N} \mathbf{f}\cdot\boldsymbol{\psi}_\delta + \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta, \mathbf{v}_\delta) + \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\boldsymbol{\psi}_\delta, \boldsymbol{\psi}_\delta)\\
&=\delta\mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta, \mathbf{v}_\delta) + \delta\mathbf{P}_{\lambda,\mu}(\boldsymbol{\psi}_\delta, \boldsymbol{\psi}_\delta) - \int_{\mathbb{R}^N} \mathbf{f}\cdot\boldsymbol{\psi}_\delta\\
&=-\delta\int_{\mathbb{R}^N} \mathbf{v}_\delta\cdot\mathcal{L}_{\lambda,\mu}\mathbf{v}_\delta - \delta\int_{\mathbb{R}^N} \boldsymbol{\psi}_\delta\cdot\mathcal{L}_{\lambda,\mu}\boldsymbol{\psi}_\delta - \int_{\mathbb{R}^N} \mathbf{f}\cdot\boldsymbol{\psi}_\delta\\
&=\int_{\mathbb{R}^N} \mathbf{v}_\delta\cdot\mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi}_\delta - \delta\int_{\mathbb{R}^N} \mathcal{L}_{\lambda,\mu}\boldsymbol{\psi}_\delta\cdot\boldsymbol{\psi}_\delta - \int_{\mathbb{R}^N} \mathbf{f}\cdot\boldsymbol{\psi}_\delta\\
&=\int_{\mathbb{R}^N} (\mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}_\delta - \delta\mathcal{L}_{\lambda,\mu}\boldsymbol{\psi}_\delta - \mathbf{f})\cdot\boldsymbol{\psi}_\delta\, dV(x) = 0.
\end{aligned} \tag{2.31}$$

The proof is complete. ∎

Remark 2.2. By Lemma 2.2, it is readily seen that for the solution $\mathbf{u}_\delta$ to (1.6) and the energy $\mathbf{E}_\delta(\mathbf{u}_\delta)$ in (2.12), one has that

$$\mathbf{E}_\delta(\mathbf{u}_\delta) \geq \mathbf{J}_\delta(\mathbf{v}, \boldsymbol{\psi}) \tag{2.32}$$

for the energy functional $\mathbf{J}_\delta$ defined in (2.23) and every pair $(\mathbf{v}, \boldsymbol{\psi})$ verifying the constraint specified in (2.24). For the elastic configuration $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f})$, we shall make use of the dual variational principle via (2.32) to show the resonance results by constructing suitable test functions $\mathbf{v}$ and $\boldsymbol{\psi}$.

3. Plasmonic resonances for the elastostatic system. In this section, we are in a position to present the resonance results for the Lamé system (1.6) with the elastic configuration $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f})$ in two dimensions. Henceforth, we assume that the source term $\mathbf{f}(x)$ is of the following form:

$$\mathbf{f} = \mathbf{F}\mathcal{H}^1 \lfloor \partial B_q, \quad \mathbf{F} : \partial B_q \to \mathbb{R}^2, \quad \mathbf{F} \in L^2(\partial B_q)^2, \quad q \in \mathbb{R}_+, \tag{3.1}$$

and

$$\int_{\partial B_q} \mathbf{F} \, d\mathcal{H}^1 = 0. \tag{3.2}$$

Moreover, we let the exterior domain Ω for the plasmonic structure (1.4) be taken to be B_R with a fixed $R \in \mathbb{R}_+$.

3.1. Perfect plasmon elastic waves. As discussed in Remarks 2.1 and 2.2, in order to show the resonance and nonresonance results by using the variational principles, we need to construct suitable trial functions. Those functions are referred to as *perfect plasmon elastic waves* and are contained in the following lemma. Starting here and throughout the rest of the paper, we make use of the polar coordinates $x = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$.

LEMMA 3.1. *Consider the PDE for a function $\boldsymbol{\psi} \in H^1_{\mathrm{loc}}(\mathbb{R}^2)^2 : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:*

$$\begin{cases} \mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi} = 0, \\ \boldsymbol{\psi}|_- = \boldsymbol{\psi}|_+, \quad \partial_{\boldsymbol{\nu}_{\lambda_A,\mu_A}}\boldsymbol{\psi}|_- = \partial_{\boldsymbol{\nu}_{\lambda_A,\mu_A}}\boldsymbol{\psi}|_+ \quad \text{on} \quad \partial B_R, \\ \boldsymbol{\psi}(x) = \mathcal{O}(\|x\|^{-1}) \quad \text{as} \quad \|x\| \to \infty, \end{cases} \tag{3.3}$$

where (λ_A, μ_A) is given of the form (2.11) *with*

$$A(x) = \begin{cases} c, & \|x\| \leq R, \\ +1, & \|x\| > R. \end{cases} \tag{3.4}$$

If

$$c := -\frac{\lambda + \mu}{\lambda + 3\mu}, \tag{3.5}$$

then there exist nontrivial solutions $\boldsymbol{\psi} = \widehat{\boldsymbol{\psi}}_k$, $k = 1, 2, \ldots$, which are given as follows:

$$\widehat{\boldsymbol{\psi}}_k(x) := \begin{cases} \begin{bmatrix} r^k\cos(k\theta) \\ -r^k\sin(k\theta) \end{bmatrix}, & r \leq R, \\ R^{2k}\begin{bmatrix} r^{-k}\cos(k\theta) + k\alpha(r^2 - R^2)\frac{1}{r^{k+2}}\cos((k+2)\theta) \\ -r^{-k}\sin(k\theta) + k\alpha(r^2 - R^2)\frac{1}{r^{k+2}}\sin((k+2)\theta) \end{bmatrix}, & r > R, \end{cases} \tag{3.6}$$

or

$$(3.7)\quad \widehat{\boldsymbol{\psi}}_k(x) := \begin{cases} \begin{bmatrix} r^k \sin(k\theta) \\ r^k \cos(k\theta) \end{bmatrix}, & r \le R, \\ R^{2k} \begin{bmatrix} r^{-k} \sin(k\theta) + k\alpha(r^2 - R^2)\frac{1}{r^{k+2}} \sin((k+2)\theta) \\ r^{-k} \cos(k\theta) - k\alpha(r^2 - R^2)\frac{1}{r^{k+2}} \cos((k+2)\theta) \end{bmatrix}, & r > R, \end{cases}$$

where

$$(3.8)\quad \alpha = -c.$$

If

$$(3.9)\quad c = -\frac{\lambda + 3\mu}{\lambda + \mu},$$

there also exist nontrivial solutions $\boldsymbol{\psi} = \widehat{\boldsymbol{\psi}}_k$*,* $k = 2, 3, \ldots$*, which are given as follows:*

$$(3.10)\quad \widehat{\boldsymbol{\psi}}_k(x) := \begin{cases} \begin{bmatrix} r^k \cos(k\theta) - k\alpha(r^2 - R^2) r^{k-2} \cos((k-2)\theta) \\ r^k \sin(k\theta) + k\alpha(r^2 - R^2) r^{k-2} \sin((k-2)\theta) \end{bmatrix}, & r \le R, \\ R^{2k} \begin{bmatrix} r^{-k} \cos(k\theta) \\ r^{-k} \sin(k\theta) \end{bmatrix}, & r > R, \end{cases}$$

or

$$(3.11)\quad \widehat{\boldsymbol{\psi}}_k(x) := \begin{cases} \begin{bmatrix} -r^k \sin(k\theta) + k\alpha(r^2 - R^2) r^{k-2} \cos((k-2)\theta) \\ r^k \cos(k\theta) + k\alpha(r^2 - R^2) r^{k-2} \sin((k-2)\theta) \end{bmatrix}, & r \le R, \\ R^{2k} \begin{bmatrix} -r^{-k} \sin(k\theta) \\ r^{-k} \cos(k\theta) \end{bmatrix}, & r > R, \end{cases}$$

where α *is also given of the form* (3.8)*.*

Furthermore, one has

$$(3.12)\quad \mathbf{P}_{\lambda,\mu}(\widehat{\boldsymbol{\psi}}_k, \widehat{\boldsymbol{\psi}}_k) = 8k\pi \frac{\mu(\lambda + 2\mu)}{\lambda + 3\mu} R^{2k-1}.$$

Proof. The lemma can be verified by straightforward (but lengthy and tedious) computations. ∎

Remark 3.1. Here, it is easily seen from (1.2) with $N = 2$ that c defined in (3.5) is negatively valued. Hence, the PDO $\mathcal{L}_{\lambda_A,\mu_A}$ is a nonelliptic operator and the perfect plasmon waves belong to its kernel. Clearly, the plasmon constant c in (3.5) or (3.9), and the associated perfect plasmon waves shall play a critical role in our subsequent study of the plasmonic resonance in elastostatics. We would like to emphasize that the construction of those critical ingredients is not as straightforward as that in [16] for the electrostatics. In what follows, we briefly discuss the general procedure that leads us to Lemma 3.1.

We seek a solution to (3.3) using the ansatz as a single-layer potential as follows (cf. [1]):

$$(3.13)\quad \boldsymbol{\psi} = \mathbf{S}[\boldsymbol{\varphi}](x) := \int_{\partial B_R} \boldsymbol{\Phi}(x - y)\boldsymbol{\varphi}(y)\, ds(y), \quad x \in \mathbb{R}^2,$$

where $\mathbf{\Phi}$ is the Kelvin matrix given in (2.7), and $\boldsymbol{\varphi} \in H^{-1/2}(\partial B_R)^2$. There holds the following jump relationship of the conormal derivative of the single layer potential in (3.13) (cf. [1]):

$$\frac{\partial}{\partial \boldsymbol{\nu}_{\lambda_A,\mu_A}} \mathbf{S}[\boldsymbol{\varphi}]\big|_{\pm} = \left(\pm\frac{1}{2}I + \mathbf{K}^*\right)[\boldsymbol{\varphi}] \quad \text{on} \quad \partial B_R, \tag{3.14}$$

where

$$\mathbf{K}^*[\boldsymbol{\varphi}] = \text{p.v.} \int_{\partial B_R} \frac{\partial}{\partial \boldsymbol{\nu}_{\lambda_A,\mu_A}(x)} \mathbf{\Phi}(x-y)\boldsymbol{\varphi}(y)ds(y) \quad x \in \partial B_R \tag{3.15}$$

and p.v. stands for the Cauchy principle value. Using the transmission condition across ∂B_R for $\boldsymbol{\psi}$ (see (3.3)), along with the help of (3.14), one can show that

$$\mathbf{K}^*[\boldsymbol{\varphi}] = \frac{c+1}{2(c-1)}\boldsymbol{\varphi}. \tag{3.16}$$

Clearly, in order to have a nontrivial $\boldsymbol{\psi}$ of the form (3.13), $(c+1)/(2(c-1))$ must be an eigenvalue of the Neumann–Poincaré operator $\mathbf{K}^*$, and $\boldsymbol{\varphi}$ is the corresponding eigenvector. Using the results in section 3.3 of [9] about the spectra of the operator $\mathbf{K}^*$, along with straightforward but lengthy and tedious calculations, one can derive the plasmon constant c as well as the corresponding perfect plasmon waves in Lemma 3.1.

Remark 3.2. For subsequent use, we remark that if in Lemma 3.1,

$$A(x) = \begin{cases} +1, & \|x\| \le R, \\ c, & \|x\| > R, \end{cases} \tag{3.17}$$

then by straightforward calculations, one can verify that there exist perfect plasmon elastic waves $\widehat{\boldsymbol{\psi}}_k(x)$, $k = 2, 3, \dots$, given by

$$\widehat{\boldsymbol{\psi}}_k(x) := \begin{cases} \begin{bmatrix} r^k \cos(k\theta) - k\alpha(r^2 - R^2) r^{k-2} \cos((k-2)\theta) \\ r^k \sin(k\theta) + k\alpha(r^2 - R^2) r^{k-2} \sin((k-2)\theta) \end{bmatrix}, & r \le R; \\ R^{2k} \begin{bmatrix} r^{-k}\cos(k\theta) \\ r^{-k}\sin(k\theta) \end{bmatrix}, & r > R. \end{cases} \tag{3.18}$$

Next we give the Fourier series expression of the source term $\mathbf{f}$. Suppose that the source $\mathbf{f}$ is real-valued and supported at distance q from origin, given in (3.1) and (3.2); then it can be represented as follows:

$$\mathbf{f} = \sum_{k=1}^{\infty} (\beta_k \mathbf{f}_{1,k}^q + \gamma_k \mathbf{f}_{2,k}^q + \xi_k \mathbf{f}_{3,k}^q + \eta_k \mathbf{f}_{4,k}^q), \tag{3.19}$$

where

$$\mathbf{f}_{1,k}^q = \begin{bmatrix} \cos(k\theta) \\ \sin(k\theta) \end{bmatrix} \mathcal{H}^1 \lfloor \partial B_q, \tag{3.20}$$

$$\mathbf{f}_{2,k}^q = \begin{bmatrix} \cos(k\theta) \\ -\sin(k\theta) \end{bmatrix} \mathcal{H}^1 \lfloor \partial B_q, \tag{3.21}$$

$$\mathbf{f}_{3,k}^q = \begin{bmatrix} -\sin(k\theta) \\ \cos(k\theta) \end{bmatrix} \mathcal{H}^1 \lfloor \partial B_q, \tag{3.22}$$

$$\mathbf{f}_{4,k}^q = \begin{bmatrix} \sin(k\theta) \\ \cos(k\theta) \end{bmatrix} \mathcal{H}^1 \lfloor \partial B_q, \tag{3.23}$$

and

$$\beta_k = \int_{\partial B_q} \mathbf{f} \cdot \mathbf{f}_{1,k}^q \, ds, \quad \gamma_k = \int_{\partial B_q} \mathbf{f} \cdot \mathbf{f}_{2,k}^q \, ds, \tag{3.24}$$

$$\xi_k = \int_{\partial B_q} \mathbf{f} \cdot \mathbf{f}_{3,k}^q \, ds, \quad \eta_k = \int_{\partial B_q} \mathbf{f} \cdot \mathbf{f}_{4,k}^q \, ds. \tag{3.25}$$

In order to simplify the exposition, in our subsequent study, we shall always assume that $\xi_k = \eta_k \equiv 0$. That is, we exclude the presence of the modes $\mathbf{f}_{3,k}$ and $\mathbf{f}_{4,k}$ in the source term, and hence instead of the general form (3.19), we shall consider a force term of the following form:

$$\mathbf{f} = \sum_{k=1}^{\infty} (\beta_k \mathbf{f}_{1,k}^q + \gamma_k \mathbf{f}_{2,k}^q). \tag{3.26}$$

However, it is emphasized that all of the resonance and nonresonance results in the present paper still hold with the presence of the modes $\mathbf{f}_{3,k}$ and $\mathbf{f}_{4,k}$, by following completely similar arguments with necessary modifications.

3.2. Resonance with no core. We first consider the case that there is no core with $\Sigma = \emptyset$ in the plasmonic structure (1.3)–(1.4). In this case, we have that the elastic configuration $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f})$ is always resonant in the sense as follows.

THEOREM 3.1. *Consider the elastic configuration* $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f})$, *where* $\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}$ *is described in* (1.3)–(1.4) *with c given in* (3.5) *and* $\Omega = B_R$ *for a certain* $R \in \mathbb{R}_+$. *Let* $\mathbf{f}$ *be given by* (3.26) *with* $\gamma_k \neq 0$ *for some* $k \in \mathbb{N}$, *representing the force supported at a distance* $q > R$. *Assume that there is no core; that is,* $\Sigma = \emptyset$. *Then the configuration is resonant, i.e.,* $\mathbf{E}_\delta(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f}) \to +\infty$ *as* $\delta \to +0$.

Proof. We shall make use of the dual variational principle for its proof. Fix the radii R, q and consider an arbitrary sequence $\delta = \delta_j \to +0$ as $j \to +\infty$. Our aim is to find a sequence $(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta)$, satisfying the constraint $\mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi}_\delta + \delta\mathcal{L}_{\lambda,\mu}\mathbf{v}_\delta = 0$ of (2.24) and such that $\mathbf{J}_\delta(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta) \to +\infty$. We choose

$$\mathbf{v}_\delta \equiv 0, \tag{3.27}$$

$$\boldsymbol{\psi}_\delta :\equiv \tau_\delta \widehat{\boldsymbol{\psi}}_k, \tag{3.28}$$

where $\widehat{\boldsymbol{\psi}}_k$ is given by (3.6) and $\tau_\delta \in \mathbb{R}$ is to be chosen below. Thus the pair $(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta)$ satisfies the PDE constraint in (2.24). With the help of (2.32), the definition of $\mathbf{J}_\delta$, the orthogonality of Fourier series, and $\gamma_k \neq 0$ for some $k \in \mathbb{N}$, we can obtain

$$\begin{aligned} \mathbf{E}_\delta(\mathbf{u}_\delta) \geq \mathbf{J}_\delta(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta) = \mathbf{J}_\delta(0, \boldsymbol{\psi}_\delta) &= \int \mathbf{f} \cdot \boldsymbol{\psi}_\delta - \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\boldsymbol{\psi}_\delta, \boldsymbol{\psi}_\delta) \\ &= \int_{\partial B_q} \gamma_k \tau_\delta q^{-k} R^{2k}(\cos^2(k\theta) + \sin^2(k\theta)) - \frac{\delta}{2}|\tau_\delta|^2 \mathbf{P}_{\lambda,\mu}(\widehat{\boldsymbol{\psi}}_k, \widehat{\boldsymbol{\psi}}_k) \\ &= 2\pi q \gamma_k \tau_\delta q^{-k} R^{2k} - (\delta|\tau_\delta|^2) 4k\pi \frac{\mu(\lambda + 2\mu)}{\lambda + 3\mu} R^{2k-1}. \end{aligned} \tag{3.29}$$

Choosing $\tau_\delta \to +\infty$ with $\delta|\tau_\delta|^2 \to +0$ as $\delta \to +0$, we obtain $\mathbf{E}_\delta(\mathbf{u}_\delta) \to +\infty$ for $\delta \to +0$.

The proof is complete. ∎

Remark 3.3. In Theorem 3.1, we assume $\gamma_k \neq 0$ for some $k \in \mathbb{N}$; that is, there is at least a mode $\mathbf{f}_{2,k}$ presented in the force term $\mathbf{f}$. If we assume that in the source term (3.26), there is a coefficient $\beta_k \neq 0$ for some $k \geq 2$, then by following a completely similar argument, together with the modification of the plasmon constant c in (1.4) to be

$$c := -\frac{\lambda + 3\mu}{\lambda + \mu}, \tag{3.30}$$

one can draw a similar conclusion to Theorem 3.1 about the resonance.

3.3. Resonance with a core of an arbitrary shape. In this section, we consider a nonradial geometry with a core, $\Sigma \subset B_1$, of an arbitrary shape. We shall show that resonance occurs and moreover the resonance of the configuration depends on the location of the source term $\mathbf{f}$.

THEOREM 3.2. *Consider the elastic configuration* $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f})$, *where* $\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}$ *is described in* (1.3)–(1.4) *with* c *given in* (3.5), *and* $\Omega = B_R$ *for a certain* $R > 1$ *and* $\Sigma \subset B_1$ *with a connected Lipschitz boundary* $\partial\Sigma$. *Consider the elastostatic system* (1.6) *with* $\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}$ *described above. Then for every radius* $R < q < R^* := R^{3/2}$, *there exists a source* $\mathbf{f}$ *of the form* (3.26) *supported at a distance* q *from the origin, such that the configuration* $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f})$ *is resonant.*

Proof. We fix $R < q < R^*$ and a sequence $\delta = \delta_j \to +0$ and consider a source term $\mathbf{f}$ given by (3.26). Our aim is to find a sequence $(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta)$, satisfying the PDE constraint $\mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi}_\delta + \delta\mathcal{L}_{\lambda,\mu}\mathbf{v}_\delta = 0$ in (2.24) and such that $\mathbf{J}_\delta(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta) \to +\infty$.

We choose

$$\boldsymbol{\psi}_\delta :\equiv \tau_\delta \widehat{\boldsymbol{\psi}}_{k_\delta}, \tag{3.31}$$

where $\widehat{\boldsymbol{\psi}}_k$ is given by (3.6). The numbers $k = k_\delta \in \mathbb{N}$ and $\tau_\delta \in \mathbb{R}$ will be properly chosen below. For $\boldsymbol{\psi}_\delta$, it is apparent that $\mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi}_\delta \neq 0$ along the core interface $\partial\Sigma \subset B_1$. In order to satisfy the PDE constraint we define $\mathbf{v}_\delta$ to be the solution of $-\delta\mathcal{L}_{\lambda,\mu}\mathbf{v}_\delta = \mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi}_\delta$. Since $-\mathcal{L}_{\lambda,\mu}$ is an elliptic PDO, by the standard elliptic estimates one can arrive at the following estimate:

$$\delta\mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta, \mathbf{v}_\delta) \leq C_{\lambda,\mu}\delta^{-1}\|\mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi}_\delta\|^2_{H^{-1}(\mathbb{R}^2)^2} \leq C_{\lambda,\mu}\delta^{-1}\tau_\delta^2 k_\delta, \tag{3.32}$$

where, and also in what follows, $C_{\lambda,\mu}$ denotes a generic positive constant. Indeed, one has by direct calculations that

$$\mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta, \mathbf{v}_\delta) = \int_{\mathbb{R}^2} (\lambda|\nabla\cdot\mathbf{v}_\delta|^2 + 2\mu|\nabla^s\mathbf{v}_\delta|^2) \leq C_{\lambda,\mu}\|\mathbf{v}_\delta\|^2_{H^1(\mathbb{R}^2)^2}. \tag{3.33}$$

Since $\mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi}_\delta = 0$ outside B_1, we choose $\boldsymbol{\omega} \in H_0^1(B_1)^2$ with $\|\boldsymbol{\omega}\|_{H_0^1(B_1)^2} = 1$ and we then have

$$\begin{aligned}
&\int_{\mathbb{R}^2} (\mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi}_\delta)\cdot\boldsymbol{\omega} = \int_{B_1} (\mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi}_\delta)\cdot\boldsymbol{\omega} \\
&= A(\lambda+\mu)\int_{B_1}(\nabla\cdot\boldsymbol{\psi}_\delta)(\nabla\cdot\boldsymbol{\omega}) + A\mu\int_{B_1}\left((\nabla\psi_\delta^1)\cdot(\nabla\omega_1) + (\nabla\psi_\delta^2)\cdot(\nabla\omega_2)\right) \\
&\leq |A(\lambda+\mu)|\|\nabla\cdot\boldsymbol{\psi}_\delta\|_{L^2(B_1)} + |A\mu|\left(\|\nabla\psi_\delta^1\|_{L^2(B_1)^2} + \|\nabla\psi_\delta^2\|_{L^2(B_1)^2}\right) \\
&\leq C_{\lambda,\mu}\tau_\delta k_\delta^{1/2},
\end{aligned} \tag{3.34}$$

where

$$\boldsymbol{\psi}_\delta = \begin{bmatrix} \psi_\delta^1 \\ \psi_\delta^2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}. \tag{3.35}$$

It remains to calculate the energy $\mathbf{J}_\delta(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta)$. We choose k_δ to be the smallest integer such that

$$R^{-k_\delta} < \delta. \tag{3.36}$$

We also note that one must have $R^{-k_\delta+1} \geq \delta$ since k_δ is the smallest integer fulfilling (3.36). With the help of (2.32), we have

$$\begin{aligned} \mathbf{E}_\delta(\mathbf{u}_\delta) \geq \mathbf{J}_\delta(\mathbf{v}_\delta, \boldsymbol{\psi}_\delta) &= \int \mathbf{f} \cdot \boldsymbol{\psi}_\delta - \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta, \mathbf{v}_\delta) - \frac{\delta}{2}\mathbf{P}_{\lambda,\mu}(\boldsymbol{\psi}_\delta, \boldsymbol{\psi}_\delta) \\ &\geq c_0\gamma_{k_\delta}\tau_\delta q^{-k_\delta} R^{2k_\delta} - C_{\lambda,\mu}\delta^{-1}\tau_\delta^2 k_\delta - C_{\lambda,\mu}\delta\tau_\delta^2 k_\delta R^{2k_\delta} \\ &\geq \tau_\delta R^{k_\delta}\left(c_0\gamma_{k_\delta}\left(\frac{R}{q}\right)^{k_\delta} - C_{\lambda,\mu}\frac{1}{(\delta R^{k_\delta})}\tau_\delta k_\delta - C_{\lambda,\mu}\tau_\delta k_\delta(\delta R^{k_\delta})\right). \end{aligned} \tag{3.37}$$

The choice of $R^{k_\delta} < \delta$ with $1 < \delta R^{k_\delta} \leq R$ ensures that the last two contributions are of comparable order. We then find, for some $C_{\lambda,\mu} > 0$,

$$\mathbf{E}_\delta(\mathbf{u}_\delta) \geq \tau_\delta R^{k_\delta}\left(c_0\gamma_{k_\delta}\left(\frac{R}{q}\right)^{k_\delta} - C_{\lambda,\mu}\tau_\delta k_\delta\right). \tag{3.38}$$

We choose τ_δ to be

$$\tau_\delta = \frac{1}{2C_{\lambda,\mu}k_\delta}c_0\gamma_{k_\delta}\left(\frac{R}{q}\right)^{k_\delta}, \tag{3.39}$$

and then from (3.38) and (3.39) we readily have that

$$\mathbf{E}_\delta(\mathbf{u}_\delta) \geq \tau_\delta R^{k_\delta}\left(\frac{1}{2}c_0\gamma_{k_\delta}\left(\frac{R}{q}\right)^{k_\delta}\right) = \frac{1}{4C_{\lambda,\mu}k_\delta}(c_0\gamma_{k_\delta})^2\left(\frac{R^3}{q^2}\right)^{k_\delta}. \tag{3.40}$$

By the assumption, $q < R^*$, and if the sequence of the Fourier coefficients γ_k of the source term $\mathbf{f}$ decays not very quickly (ensuring that the right-hand side (RHS) term of (3.40) goes to infinity as $\delta \to +0$), we easily see from (3.40) that $\mathbf{E}_\delta(\mathbf{u}_\delta) \to +\infty$ as $\delta \to +0$.

This proof is complete. ∎

Remark 3.4. Similar to Remark 3.3, if one chooses the plasmon constant c in (1.4) to be

$$c := -\frac{\lambda + 3\mu}{\lambda + \mu},$$

then by following a completely similar argument to the proof of Theorem 3.2, one can show that if the Fourier coefficients β_k of the source term $\mathbf{f}$ decay not very quickly, then ALR occurs. It is also worth mentioning that the nonquickly decaying condition on the Fourier coefficients of the source term required in the proof of Theorem 3.2 might be mainly a technical condition due to the corresponding mathematical argument. Indeed, we conducted numerical experiments showing that without the decaying condition, one can still have the resonance result for a generic source term. We shall report those findings in a forthcoming computational paper [20].

3.4. Nonresonance in the radial case. In section 3.3, we showed that for certain source/force terms lying within the critical radius R^*, resonance occurs. In this section, we shall show that for a certain source term lying outside the critical radius, resonance does not occur. To that end, we would consider our study in the radial geometry by assuming that the core $\Sigma = B_1$.

THEOREM 3.3. *Consider the elastic configuration* $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f})$*, where* $\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}$ *is described in* (1.3)–(1.4) *with c given in* (3.5) *and* $\Omega = B_R$ *for a certain* $R > 1$ *and* $\Sigma = B_1$. *Consider the elastostatic system* (1.6) *with* $\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}$ *describe above. Let the source* $\mathbf{f}$ *be given by* (3.26) *with* $\{\sqrt{k}\beta_k\} \in l^2(\mathbb{N}, \mathbb{R})$*, and*

$$\frac{\beta_k}{\gamma_{k-2}} = -\frac{q^2(\lambda + 3\mu)}{(k-2)(\lambda+\mu)(q^2 - R^2)} \qquad k > 2; \quad \beta_k = 0, \quad k = 1, 2. \tag{3.41}$$

Then for any $q > R^* := R^{3/2}$*, the configuration* $(\mathbf{C}_{\widetilde{\lambda},\widetilde{\mu}}, \mathbf{f})$ *is nonresonant.*

Proof. We make use of the primal variational principle to show the nonresonance result. We shall construct the test function $(\mathbf{v}_\delta, \mathbf{w}_\delta)$, satisfying the constraint

$$\mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}_\delta - \mathcal{L}_{\lambda,\mu}\mathbf{w}_\delta = \mathbf{f} \tag{3.42}$$

such that the energy along this sequence, $\mathbf{I}_\delta(\mathbf{v}_\delta, \mathbf{w}_\delta)$ remains bounded. To that end, our strategy is to decompose the source $\mathbf{f}$ into a low-frequency part and a high-frequency part such that

$$\begin{gathered} \mathbf{f} = \mathbf{f}^{\text{low}} + \mathbf{f}^{\text{high}}, \\ \mathbf{f}^{\text{low}} := \sum_{k=3}^{k^*}(\beta_k \mathbf{f}^q_{1,k} + \gamma_{k-2}\mathbf{f}^q_{2,k-2}), \quad \mathbf{f}^{\text{high}} := \sum_{k=k^*+1}^{\infty}(\beta_k \mathbf{f}^q_{1,k} + \gamma_{k-2}\mathbf{f}^q_{2,k-2}), \end{gathered} \tag{3.43}$$

where $\mathbf{f}^q_{1,k}$ and $\mathbf{f}^q_{2,k}$ are given by (3.20) and (3.21), respectively, and k^* will be chosen to depend on δ. Indeed, we shall choose $k^* = k^*(\delta)$ to be the smallest integer such that $R^{-k^*} > \delta$, and this will be explicitly specified again in what follows. We then construct $(\mathbf{v}_\delta, \mathbf{w}_\delta)$ with $\mathbf{v}_\delta = \mathbf{v}^{\text{low}}_\delta + \mathbf{v}^{\text{high}}_\delta$ and $\mathbf{w}_\delta = \mathbf{w}^{\text{high}}_\delta$ as follows:

$$\mathbf{v}^{\text{low}}_\delta \quad \text{satisfies} \quad \mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}^{\text{low}}_\delta = \mathbf{f}^{\text{low}}, \tag{3.44}$$

$$\mathbf{v}^{\text{high}}_\delta \quad \text{satisfies} \quad \mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}^{\text{high}}_\delta|_{\partial B_q(0)} = \mathbf{f}^{\text{high}}, \tag{3.45}$$

$$\mathbf{w}^{\text{high}}_\delta \quad \text{satisfies} \quad -\mathcal{L}_{\lambda,\mu}\mathbf{w}^{\text{high}}_\delta = -\mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}^{\text{high}}_\delta + \mathbf{f}^{\text{high}}. \tag{3.46}$$

This construction yields $(\mathbf{v}_\delta, \mathbf{w}_\delta)$, which satisfies the constraint (3.42) of the primal problem (2.14). Furthermore, we shall show that with an appropriate choice of the cutoff integer $k^* = k^*(\delta)$ in (3.43), $\mathbf{I}_\delta(\mathbf{v}_\delta, \mathbf{w}_\delta)$ remains bounded as $\delta \to +0$.

Next, we construct $\mathbf{v}^{\text{low}}_\delta$. First, we present the base function $\hat{\mathbf{v}}_k$ for our construction, which can be represented as follows:

$$\hat{\mathbf{v}}_k(x) = \begin{cases} \hat{\mathbf{v}}^{(c)}_k(x), & r \le 1, \\ \hat{\mathbf{v}}^{(s)}_k(x), & 1 < r \le R, \\ \hat{\mathbf{v}}^{(f)}_k(x), & R < r \le q, \\ \hat{\mathbf{v}}^{(e)}_k(x), & r > q, \end{cases} \qquad k = 3, 4, 5, \ldots, \tag{3.47}$$

with

(3.48)

$$\hat{\mathbf{v}}_k^{(c)}(x) := \begin{bmatrix} r^k\cos(k\theta) - k\alpha(r^2-1)r^{k-2}\cos((k-2)\theta) \\ r^k\sin(k\theta) + k\alpha(r^2-1)r^{k-2}\sin((k-2)\theta) \end{bmatrix}, \qquad r \le 1,$$

(3.49)

$$\hat{\mathbf{v}}_k^{(s)}(x) := \begin{bmatrix} r^{-k}\cos(k\theta) \\ r^{-k}\sin(k\theta) \end{bmatrix}, \qquad 1 < r \le R,$$

(3.50)

$$\hat{\mathbf{v}}_k^{(f)}(x) := R^{-2k}\begin{bmatrix} r^k\cos(k\theta) - k\alpha(r^2-R^2)r^{k-2}\cos((k-2)\theta) \\ r^k\sin(k\theta) + k\alpha(r^2-R^2)r^{k-2}\sin((k-2)\theta) \end{bmatrix}, \; R < r \le q,$$

(3.51)

$$\hat{\mathbf{v}}_k^{(e)}(x) := \left(\frac{q}{R}\right)^{2k}\left(\begin{bmatrix} r^{-k}\cos(k\theta) \\ r^{-k}\sin(k\theta) \end{bmatrix}\right.$$
$$\left. + c_1 k\begin{bmatrix} (k-2)\alpha(r^2-q^2)r^{-k}\cos(k\theta) + r^{-(k-2)}\cos((k-2)\theta) \\ (k-2)\alpha(r^2-q^2)r^{-k}\sin(k\theta) - r^{-(k-2)}\sin((k-2)\theta) \end{bmatrix}\right), \; r > q,$$

where

$$c_1 = \frac{\alpha(R^2-q^2)}{q^4}$$

and α is given in (3.8). We note that $\hat{\mathbf{v}}_k$ have the following properties:

1. $\hat{\mathbf{v}}_k$ is continuous on all $\mathbb{R}^2$.
2. $\hat{\mathbf{v}}_k$ satisfies $\mathcal{L}_{\lambda_A,\mu_A}\hat{\mathbf{v}}_k = 0$ for $x \in \mathbb{R}^2\backslash\partial B_q$.
3. Along ∂B_q, $\hat{\mathbf{v}}_k$ has a jump in its normal flux:

(3.52)
$$\left[\frac{\partial\hat{\mathbf{v}}_k}{\partial\boldsymbol{\nu}_{\lambda_A,\mu_A}}\right]_{\partial B_q} = -c_2 k q^k R^{-2k}\left(q^2(\lambda+3\mu)\right)\begin{bmatrix} \cos(k\theta) \\ \sin(k\theta) \end{bmatrix}$$
$$+ c_2 k q^k R^{-2k}\left((k-2)(\lambda+\mu)(q^2-R^2)\right)\begin{bmatrix} \cos((k-2)\theta) \\ -\sin((k-2)\theta) \end{bmatrix},$$

where $[\cdot]$ denote the jump of the normal flux and

(3.53)
$$c_2 = \frac{4\mu(\lambda+2\mu)}{q^3(\lambda+3\mu)^2}.$$

By using the properties listed above, we shall verify that with an appropriate constant multiple $\tau_k\hat{\mathbf{v}}_k$, one has

(3.54)
$$\mathcal{L}_{\lambda_A,\mu_A}(\tau_k\hat{\mathbf{v}}_k)|_{\partial B_q} = \beta_k\mathbf{f}_{1,k}^q + \gamma_{k-2}\mathbf{f}_{2,k-2}^q.$$

Indeed, we choose τ_k such that

(3.55)
$$\tau_k\cdot\left[-c_2 k q^k R^{-2k}\left(q^2(\lambda+3\mu)\right)\right] = \beta_k$$

and

(3.56)
$$\tau_k\cdot\left[c_2 k q^k R^{-2k}\left((k-2)(\lambda+\mu)(q^2-R^2)\right)\right] = \gamma_{k-2}.$$

With the help of (3.41), one has by direct calculations that

$$\tau_k := \frac{\beta_k}{-c_2 k \left(q^2(\lambda+3\mu)\right)} q^{-k} R^{2k}. \tag{3.57}$$

Now we set

$$\mathbf{v}_\delta^{\mathrm{low}} := \sum_{k=3}^{k^*} \tau_k \hat{\mathbf{v}}_k \tag{3.58}$$

with τ_k given in (3.57). By combining (3.54)–(3.57), along with straightforward calculations, one can directly verify that $\mathbf{v}_\delta^{\mathrm{low}}$ defined in (3.58) satisfies (3.44).

After constructing $\mathbf{v}_\delta^{\mathrm{low}}$, we next give the construction of $\mathbf{v}_\delta^{\mathrm{high}}$ and $\mathbf{w}_\delta^{\mathrm{high}}$, respectively, in (3.45) and (3.46). Similar to the construction of $\mathbf{v}_\delta^{\mathrm{low}}$ via a certain base function $\hat{\mathbf{v}}_k$ in (3.47), the construction of the function $\mathbf{v}_\delta^{\mathrm{high}}$ shall also be constructed from certain base functions $\widehat{\mathbf{V}}_k$ for $k = 3, 4, 5, \ldots$. Those functions do not fulfill $-\mathcal{L}_{\lambda_A,\mu_A}\mathbf{v} = 0$ on ∂B_1 or ∂B_R, but they are small along these curves. We introduce $\widehat{\mathbf{V}}_k$ as follows:

$$\widehat{\mathbf{V}}_k(x) = \begin{cases} \widehat{\mathbf{V}}_k^{(i)}(x), & r \le q, \\ \widehat{\mathbf{V}}_k^{(o)}(x), & r > q, \end{cases} \qquad k = 3, 4, 5, \ldots, \tag{3.59}$$

with

$$\begin{aligned} \widehat{\mathbf{V}}_k^{(i)}(x) := {} & c_3 \begin{bmatrix} r^k \cos(k\theta) - k\alpha(r^2-q^2) r^{k-2}\cos((k-2)\theta) \\ r^k \sin(k\theta) + k\alpha(r^2-q^2) r^{k-2}\sin((k-2)\theta) \end{bmatrix} \\ & + c_4 k \begin{bmatrix} r^{k-2}\cos((k-2)\theta) \\ -r^{k-2}\sin((k-2)\theta) \end{bmatrix}, \qquad r \le q, \end{aligned} \tag{3.60}$$

$$\begin{aligned} \widehat{\mathbf{V}}_k^{(o)}(x) := {} & c_4 k q^{2(k-2)} \begin{bmatrix} r^{-(k-2)}\cos((k-2)\theta) + (k-2)\alpha(r^2-q^2) r^{-k}\cos(k\theta) \\ -r^{-(k-2)}\sin((k-2)\theta) + (k-2)\alpha(r^2-q^2) r^{-k}\sin(k\theta) \end{bmatrix} \\ & + c_3 q^{2k} \begin{bmatrix} r^{-k}\cos(k\theta) \\ r^{-k}\sin(k\theta) \end{bmatrix}, \qquad r > q, \end{aligned} \tag{3.61}$$

where

$$c_3 = -(\lambda + 3\mu) \quad \text{and} \quad c_4 = (\lambda+\mu)(q^2 - R^2).$$

Recall that $A(x) = 1$ in a neighborhood of $\|x\| = q$. One can show by direct calculations that the jump of the normal flux of $\widehat{\mathbf{V}}_k$ in (3.59) on ∂B_q is given by

$$\begin{aligned} \left[\frac{\partial \widehat{\mathbf{V}}_k}{\partial \boldsymbol{\nu}_{\lambda_A,\mu_A}}\right]_{\partial B_q(0)} = & - c_5 k q^k \left(q^2(\lambda+3\mu)\right) \begin{bmatrix} \cos(k\theta) \\ \sin(k\theta) \end{bmatrix} \\ & + c_5 k q^k \left((k-2)(\lambda+\mu)(q^2-R^2)\right) \begin{bmatrix} \cos((k-2)\theta) \\ -\sin((k-2)\theta) \end{bmatrix}, \end{aligned} \tag{3.62}$$

where

$$c_5 = -\frac{4\mu(\lambda+2\mu)}{q^3(\lambda+3\mu)}.$$

Therefore if we set

$$\mathbf{v}_\delta^{\text{high}} := \sum_{k>k^*} \tau_k \widehat{\mathbf{V}}_k, \quad \tau_k = \frac{\beta_k}{-c_5 k\left(q^2(\lambda+3\mu)\right)} q^{-k}, \tag{3.63}$$

then by using (3.41) and (3.62), one can show by direct calculations that (3.45) is satisfied:

$$\mathcal{L}_{\lambda_A,\mu_A} \mathbf{v}_\delta^{\text{high}}|_{\partial B_q} = \mathbf{f}^{\text{high}}. \tag{3.64}$$

We emphasize that $\mathbf{v}_\delta^{\text{high}}$ is not a solution to (3.64) on all of $\mathbb{R}^2$ due to the normal fluxes at $\|x\| = 1$ and $\|x\| = R$. In order to construct a solution to (3.64) on $\mathbb{R}^2$, we introduce $\mathbf{w}_\delta^{\text{high}}$ as follows:

$$\begin{aligned} -\mathcal{L}_{\lambda,\mu} \mathbf{w}_\delta^{\text{high}} &= -\mathcal{L}_{\lambda_A,\mu_A} \mathbf{v}_\delta^{\text{high}} + \mathbf{f}^{\text{high}} \\ &= -\sum_{k>k^*} \tau_k \left[\frac{\partial \mathbf{v}_\delta^{\text{high}}}{\partial \boldsymbol{\nu}_{\lambda_A,\mu_A}}\right]_{\partial B_1} - \sum_{k>k^*} \tau_k \left[\frac{\partial \mathbf{v}_\delta^{\text{high}}}{\partial \boldsymbol{\nu}_{\lambda_A,\mu_A}}\right]_{\partial B_R}. \end{aligned} \tag{3.65}$$

Clearly, $\mathbf{w}_\delta^{\text{high}}$ satisfies (3.46). With the test functions being ready, it remains to calculate the energy $\mathbf{I}_\delta(\mathbf{v}_\delta, \mathbf{w}_\delta)$ for the choice of $\mathbf{v}_\delta = \mathbf{v}_\delta^{\text{low}} + \mathbf{v}_\delta^{\text{high}}$ and $\mathbf{w}_\delta = \mathbf{w}_\delta^{\text{high}}$. In this step, we shall choose an appropriate cut-off frequency, $k^* = k^*(\delta)$, to ensure that $\mathbf{I}_\delta(\mathbf{v}_\delta, \mathbf{w}_\delta)$ remains uniformly bounded as $\delta \to +0$.

In what follows, we shall make use of the following orthogonality relations that can be directly verified:

$$\mathbf{P}_{\lambda,\mu}(\hat{\mathbf{v}}_k, \hat{\mathbf{v}}_{k'}) = 0 \quad \text{and} \quad \mathbf{P}_{\lambda,\mu}(\widehat{\mathbf{V}}_k, \widehat{\mathbf{V}}_{k'}) = 0 \quad \text{for} \quad k \neq k', \tag{3.66}$$

where $\hat{\mathbf{v}}_k$ and $\widehat{\mathbf{V}}_k$, $k = 3, 4, 5, \ldots$, are the base functions constructed in (3.47) and (3.59), respectively. Now, we first have by straightforward calculations that

$$\delta \mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta^{\text{low}}, \mathbf{v}_\delta^{\text{low}}) \leq C_{\lambda,\mu} \delta \sum_{k\leq k^*} k|\beta_k|^2 \left(\frac{R^2}{q}\right)^{2k} \max\left\{1, \left(\frac{q}{R^2}\right)^k\right\}^2. \tag{3.67}$$

For the case when $q \geq R^2$, we easily have from (3.67) that

$$\delta \mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta^{\text{low}}, \mathbf{v}_\delta^{\text{low}}) \leq C_{\lambda,\mu} \delta \sum_{k\leq k^*} k|\beta_k|^2 \leq C_{\lambda,\mu} \delta, \tag{3.68}$$

which is obvious bounded. The other case when $R^* < q < R^2$ is much more subtle. It is noted that the estimate (3.67) can be simplified in the case when $R^* < q < R^2$ to be

$$\delta \mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta^{\text{low}}, \mathbf{v}_\delta^{\text{low}}) \leq C_{\lambda,\mu} \delta \sum_{k\leq k^*} k|\beta_k|^2 \left(\frac{R^2}{q}\right)^{2k^*}. \tag{3.69}$$

In what follows, we shall show that with a special choice of k^*, the RHS of (3.69) can be bounded. The energy of $\mathbf{v}_\delta^{\text{high}}$ is easy to control:

$$\delta \mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta^{\text{high}}, \mathbf{v}_\delta^{\text{high}}) \leq C_{\lambda,\mu} \delta \sum_{k} k|\beta_k|^2 \leq C_{\lambda,\mu}. \tag{3.70}$$

Next, we study the energy due to $\mathbf{w}_\delta$, and with the help of (3.63), along with the standard elliptic estimate, we have

$$
\begin{aligned}
\frac{1}{\delta}\mathbf{P}_{\lambda,\mu}(\mathbf{w}_\delta,\mathbf{w}_\delta) &\le C\frac{1}{\delta}\|-\mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}_\delta^{\mathrm{high}}+\mathbf{f}^{\mathrm{high}}\|^2_{H^{-1}(\mathbb{R}^2)^2}\\
&\le C\frac{1}{\delta}\sum_{k>k^*}|\tau_k|^2R^{2k}k \le C\sum_{k>k^*}|\beta_k|^2\frac{1}{\delta}\left(\frac{R}{q}\right)^{2k^*}.
\end{aligned} \tag{3.71}
$$

Here, it is noted that in (3.71), the second inequality can be obtained by a duality argument as follows. Since $-\mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}_\delta^{\mathrm{high}}+\mathbf{f}^{\mathrm{high}}=0$ outside B_R from (3.65), we choose $\boldsymbol{\omega}\in H_0^1(B_R)^2$ with $\|\boldsymbol{\omega}\|_{H_0^1(B_R)^2}=1$ to have that

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(-\mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}_\delta^{\mathrm{high}}+\mathbf{f}^{\mathrm{high}}\right)\cdot\boldsymbol{\omega}=\int_{B_R}\left(-\mathcal{L}_{\lambda_A,\mu_A}\mathbf{v}_\delta^{\mathrm{high}}\right)\cdot\boldsymbol{\omega}\\
&\le C\left(\int_{B_R}(\nabla\cdot\mathbf{v}_\delta^{\mathrm{high}})(\nabla\cdot\boldsymbol{\omega})+\int_{B_R}\left((\nabla\mathbf{v}_{\delta,1}^{\mathrm{high}})\cdot(\nabla\omega_1)+(\nabla\mathbf{v}_{\delta,2}^{\mathrm{high}})\cdot(\nabla\omega_2)\right)\right)\\
&\le C\left(\|\nabla\cdot\mathbf{v}_\delta^{\mathrm{high}}\|_{L^2(B_R)}+\|\nabla\mathbf{v}_{\delta,1}^{\mathrm{high}}\|_{L^2(B_R)^2}+\|\nabla\mathbf{v}_{\delta,2}^{\mathrm{high}}\|_{L^2(B_R)^2}\right)\\
&\le C\sum_{k>k^*}\tau_kR^kk^{1/2},
\end{aligned} \tag{3.72}
$$

where

$$
\mathbf{v}_\delta^{\mathrm{high}}=\begin{bmatrix}\mathbf{v}_{\delta,1}^{\mathrm{high}}\\ \mathbf{v}_{\delta,2}^{\mathrm{high}}\end{bmatrix}\quad\text{and}\quad\boldsymbol{\omega}=\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}. \tag{3.73}
$$

Clearly, (3.72) readily implies the second inequality in (3.71). Now we are in a position to balance the RHSs of the bounds in (3.69) and (3.71) by choosing k^* such that

$$
\delta\left(\frac{R^2}{q}\right)^{2k^*}\sim\frac{1}{\delta}\left(\frac{R}{q}\right)^{2k^*}; \tag{3.74}
$$

namely, we choose $k^*=k^*(\delta)$ to be the smallest integer with $R^{-k^*}<\delta$ such that

$$
\delta\le R^{-k^*+1}\quad\text{and}\quad\frac{1}{\delta}\le R^{k^*}. \tag{3.75}
$$

Combining (3.75) with (3.69) and (3.71), we obtain

$$
\delta\mathbf{P}_{\lambda,\mu}(\mathbf{v}_\delta^{\mathrm{low}},\mathbf{v}_\delta^{\mathrm{low}})\le C_{\lambda,\mu}\sum_{k\le k^*}k|\beta_k|^2\left(\frac{R^3}{q^2}\right)^{k^*(\delta)} \tag{3.76}
$$

and

$$
\frac{1}{\delta}\mathbf{P}_{\lambda,\mu}(\mathbf{w}_\delta,\mathbf{w}_\delta)\le C\sum_{k>k^*}|\beta_k|^2\left(\frac{R^3}{q^2}\right)^{k^*(\delta)}. \tag{3.77}
$$

Hence, if $q>R^*=R^{3/2}$, $\mathbf{I}_\delta(\mathbf{v}_\delta,\mathbf{w}_\delta)$ is bounded as $\delta\to+0$. Therefore, by (2.22), the elastic configuration is nonresonant.

The proof is complete. ∎

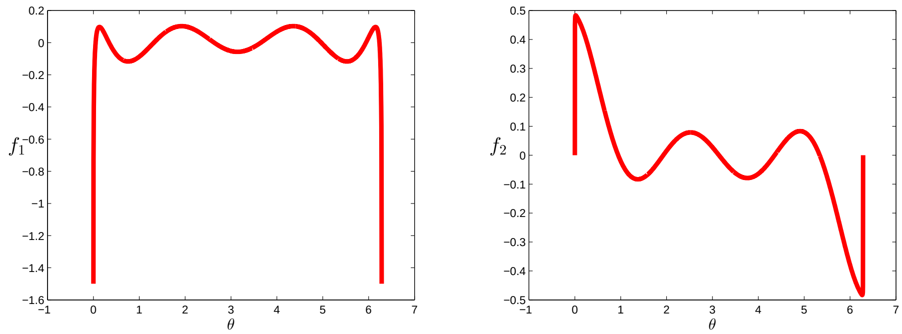

FIG. 1. *A source function* $\mathbf{f} = (f_1, f_2)'$ *as described in Theorem* 3.3 *with* $\lambda = 1, \mu = 2, R = 2, q = 4$ *and,* $\beta_1 = 0$, $\beta_2 = 0$, $\beta_k = k^{-2}$, *and* $\gamma_{k-2} = \big(-\frac{(k-2)(\lambda+\mu)(q^2-R^2)}{q^2(\lambda+3*\mu)}\big)\beta_k,\ k \geq 3$.

Remark 3.5. In Theorem 3.3, we show that for certain sources lying outside the critical radius, resonance does not occur, which in combination with Theorem 3.2 reveals the interesting feature that the resonance strongly depends on the location of the source term. In Theorem 3.3, it is required that the Fourier coefficients of the source $\mathbf{f}$ satisfy the relationship (3.41). This condition is mainly needed in the construction of the test function $\hat{\mathbf{v}}_\delta^{\mathrm{low}}$ in (3.54)–(3.58). This is sharply different from Proposition 4.1 in [16], which shows that resonance does not occur in electrostatics for generic sources lying outside the critical radius. The investigation of nonresonance or resonance for sources not fulfilling the condition (3.41) and lying outside the critical radius is fraught with significant difficulties. This signifies one of the distinct features of our present study on the plasmonic resonance in elastostatics, which also distinguishes our results from those in [16] for the electrostatics. The same remark applies to the nonradial geometry. In [16], a nonresonance result is also given in the noncircular geometry. We shall further investigate those important and interesting issues, particularly first from a numerical point of view in [20]. Finally, as an illustrative example, we present the plotting of a specific source term $\mathbf{f}$ in Figure 1, which fulfils the requirement in Theorem 3.3.

4. Concluding remarks. In this paper, we consider the plasmonic resonance for the elastostatics in $\mathbb{R}^2$. We propose an elastic plasmon structure which takes the core-shell-matrix form. Using the variational approach, we establish both the resonance and nonresonance results for the proposed elastic plasmon structure. We also show that the plasmonic resonance strongly depends on the location of the source term. This paper not only provides a different and alternative treatment of the plasmonic resonance for the elastostatic system in $\mathbb{R}^2$ from the one in [9] using a spectral approach, but also extends the variational study in [16] for the Laplace equation to the much more challenging elastostatic system. The natural continuation of the present study is the three-dimensional plasmonic resonance for the elastostatic system. For that purpose, the first significant challenge one would be confronted with is the construction of an appropriate three-dimensional plasmonic structure and the derivation of the corresponding perfect plasmon waves, as in Lemma 3.1 in the current article. We shall investigate this and other interesting and challenging issues (cf. Remarks 3.1 and 3.5) in our future work.

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