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PLASMON RESONANCE WITH FINITE FREQUENCIES: A VALIDATION OF THE QUASI-STATIC APPROXIMATION FOR DIAMETRICALLY SMALL INCLUSIONS*

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Abstract. We study resonance for the Helmholtz equation with a finite frequency in a plasmonic material of negative dielectric constant in two and three dimensions. We show that the quasi-static approximation is valid for diametrically small inclusions. In fact, we quantitatively prove that if the diameter of an inclusion is small compared to the loss parameter, then resonance occurs exactly at eigenvalues of the Neumann–Poincaré operator associated with the inclusion.

Key words. Neumann-Poincaré operator, eigenvalues, Helmholtz equation, finite frequency, plasmon resonance, quasi-static limit

AMS subject classifications. 35R30, 35C20

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1. Introduction.

Resonance phenomena are often observed in nanoscale particles whose material has a negative dielectric permittivity with a large wavelength in comparison with particle dimensions both experimentally and numerically [22]. It is known that such a resonance occurs only at certain frequencies. Noble metals such as gold and silver show a negative permittivity [26] and are called plasmonic materials. Recently, there has been considerable interest in the plasmon resonance and its various applications, including invisibility cloaking, biomedical imaging, and medical therapy; see, e.g., [1, 2, 3, 7, 9, 12, 13, 16, 18, 20, 21, 22, 23, 25] and references therein.

It is known (see, e.g., [7, 12]) that in the quasi-static limit the plasmon resonance occurs at the eigenvalues of the Neumann–Poincaré (NP) operator associated with the inclusion. To be more precise, let D be a bounded simply connected domain in $\mathbb{R}^d$ $(d = 2, 3)$ whose boundary ∂D is $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha < 1$. Suppose that D is occupied with a plasmonic material which has a dielectric constant $\epsilon_c + i\delta$, where $\epsilon_c < 0$ and $\delta > 0$ is the dissipation, and that the matrix $\mathbb{R}\setminus\overline{D}$ has a dielectric constant $\epsilon_m > 0$. Hence, the distribution of the dielectric constant is given by

$$\epsilon_D = \begin{cases} \epsilon_c + i\delta & \text{in } D, \\ \epsilon_m & \text{in } \mathbb{R}^d \setminus \overline{D}. \end{cases} \tag{1.1}$$

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The dielectric equation in the quasi-static limit is given by

$$\nabla \cdot \epsilon_D \nabla u_\delta = f. \tag{1.2}$$

It is proved (e.g., [7]) that when the source f is given by the polarizable dipole $a \cdot \nabla \delta_z$, the resonance occurs exactly when $\lambda(\epsilon_c/\epsilon_m)$ is an eigenvalue of the NP operator associated with D (see the next section for the definition and spectral properties of the NP operator); in other words, $\|\nabla u_\delta\|_{L^2(D)} \to \infty$ as $\delta \to 0$. Here,

$$\lambda(t) := \frac{t+1}{2(t-1)}. \tag{1.3}$$

When $\lambda(\epsilon_c/\epsilon_m)$ is an eigenvalue of the NP operator, ϵ_c/ϵ_m is called the plasmon eigenvalue [12].

In this paper, we consider plasmon resonance for the Helmholtz operator $\nabla \cdot \epsilon_D \nabla + \omega_0^2$, when D is a diametrically small inclusion such as a nanoscale particle. Here ω_0 represents the nonzero (but fixed) frequency, and the parameters ϵ_c and δ are determined by ω_0. We show that if the diameter s of D is much smaller than the dissipation parameter δ, then the resonance may occur exactly when $\lambda(\epsilon_c/\epsilon_m)$ is an eigenvalue of the NP operator on D, like the quasi-static limit case. So the result of this paper can be regarded as a validation of the quasi-static approximation for diametrically small inclusions. It is worth mentioning that a different validation of quasi-static approximation is proved in [3] by showing that the small volume asymptotic expansion of the far field for the Maxwell system holds away from the eigenvalues of the NP operator.

To describe results of this paper in a quantitative manner, we consider the solution v_δ to the Helmholtz equation

$$\nabla \cdot \epsilon_D \nabla v_\delta + \omega_0^2 v_\delta = a \cdot \nabla \delta_z \quad \text{in } \mathbb{R}^d,$$

where $a \in \mathbb{R}^d$ is a constant vector and δ_z is the Dirac mass at $z \in \mathbb{R}^d \setminus \overline{D}$. Suppose that D is of the form $D = s\Omega$, where Ω is a reference domain and s is a small number representing the diameter of D. Let $u_\delta(x) := v_\delta(sx)$. Then, u_δ satisfies

$$\nabla \cdot \epsilon_\Omega \nabla u_\delta + s^2 \omega_0^2 u_\delta = s^{1-d} a \cdot \nabla \delta_{s^{-1}z} \quad \text{in } \mathbb{R}^d, \tag{1.4}$$

where

$$\epsilon_\Omega = \begin{cases} \epsilon_c + i\delta & \text{in } \Omega, \\ \epsilon_m & \text{in } \mathbb{R}^d \setminus \overline{\Omega}. \end{cases} \tag{1.5}$$

The solution u_δ satisfies the Sommerfeld radiation condition

$$\left| \frac{\partial u_\delta}{\partial r} - i\omega \varepsilon_m^{-1/2} u_\delta \right| \le C r^{-(d+1)/2} \quad \text{as} \quad r = |x| \to \infty. \tag{1.6}$$

We also consider the equation

$$\nabla \cdot \epsilon_\Omega \nabla u_\delta + s^2 \omega_0^2 u_\delta = a \cdot \nabla \delta_z \quad \text{in } \mathbb{R}^d, \tag{1.7}$$

which can be regarded as the low-frequency case.

We characterize the resonance by the blow-up of $\|\nabla u_\delta\|_{L^2(\Omega)}$:

$$\|\nabla u_\delta\|_{L^2(\Omega)} \to \infty \text{ as } \delta \to +0, \tag{1.8}$$

where u_δ is the solution to (1.7). We show that if s is much smaller than δ, more precisely, if $s\delta^{-1} \ll 1$ in three dimensions, and $s^2|\ln s|\delta^{-1} \ll 1$ in two dimensions, then (1.8) may take place only when $\lambda(\epsilon_c/\epsilon_m)$ is an eigenvalue of the NP operator on Ω. Moreover, if $\lambda(\epsilon_c/\epsilon_m)$ is an eigenvalue, we obtain a quantitative estimate

$$\|\nabla u_\delta\|_{L^2(\Omega)} \approx \delta^{-1} \quad \text{as } \delta \to +0 \tag{1.9}$$

for most z's (the location of the dipole source) for the low-frequency case (see Theorems 5 and 7 for precise statements). For the diametrically small inclusions we show (1.9) occurs in three dimensions if $a = z/|z|$ and the far field pattern of the single layer potential of the eigenmode does not vanish at $z/|z|$. We also prove weaker resonance occurs in two dimensions. See Theorems 8 and 9.

The result of this paper when $\lambda(\epsilon_c/\epsilon_m)$ is not an eigenvalue of the NP operator may be regarded as stability of the solution. There is extensive work on the well-posedness and stability of the Helmholtz equation with sign-changing coefficients: by the boundary integral method [11], by T-coercivity [8], and by the reflection method [24]. We refer to the last mentioned paper for an informative list of references on this direction of research. We also mention that the quasi-static (zero-frequency) case with a general source has been considered in [25], namely, $\nabla \cdot \epsilon_D \nabla u = f$ when D is the unit disk and $\epsilon_m = -\epsilon_c$ so that $\lambda(\epsilon_c/\epsilon_m) = 0$, which is the only eigenvalue of the NP operator. In that paper the source functions f with which the resonance takes place are characterized, and it is proved that the field blows up everywhere at the rate of δ^{-1}.

The rest of this paper is organized as follows. In section 2 we review spectral properties of the NP operator. Section 3 derives a necessary asymptotic formula for the Helmholtz operator at low frequencies and estimates for the H^1-norm of the solution. The results for the low-frequency case in three and two dimensions are presented and proved in subsections 4.1 and 4.2, respectively. The main results for diametrically small inclusions are given in section 5.

While writing this paper (after completing major work) we received the paper [6] from Habib Ammari. There an asymptotic formula for the solution similar to (4.11) is derived in three dimensions when there are multiple small inclusions, using the same method as in this paper (the spectral properties of the NP operator). Then the formula is used to study enhancement of scattering and absorption, and super-resonance by plasmonic particles. Here in this paper we use the asymptotic formula to show resonance quantified by (1.9).

2. Preliminaries. Let Ω be a bounded domain with the Lipschitz boundary in $\mathbb{R}^d$, $d = 2, 3$. Throughout this paper $H^s(\partial\Omega)$ denotes the L^2-Sobolev space on $\partial\Omega$ whose norm is expressed as $\|\cdot\|_s$. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing of $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. Let $H_0^{-1/2}(\partial\Omega)$ be the space of $\psi \in H^{-1/2}(\partial\Omega)$ satisfying $\langle \psi, 1 \rangle = 0$.

Let $\Gamma(x)$ be the fundamental solution to the Laplacian on $\mathbb{R}^d$, $d = 2, 3$:

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & d = 2, \\ -\dfrac{1}{4\pi} |x|^{-1}, & d = 3. \end{cases}$$

The single layer potential of $\varphi \in H^{-1/2}(\partial\Omega)$ for the Laplacian is defined by

$$\mathcal{S}[\varphi](x) = \int_{\partial\Omega} \Gamma(x - y)\varphi(y) d\sigma(y), \quad x \in \mathbb{R}^d.$$

It is well known (see, e.g., [4]) that the following jump formula holds:

$$\partial_\nu \mathcal{S}[\varphi]\big|_{\pm}(x) = (\pm 1/2 I + \mathcal{K}^*)[\varphi](x), \quad x \in \partial\Omega, \tag{2.1}$$

where $\mathcal{K}^*$ is the NP operator defined by

$$\mathcal{K}^*[\varphi](x) = \text{p.v.} \int_{\partial\Omega} \partial_{\nu_x}\Gamma(x-y)\varphi(y)d\sigma(y), \quad x \in \partial\Omega. \tag{2.2}$$

Here ∂_ν denotes the outward normal derivative, the subscripts $\pm$ the limits (to $\partial\Omega$) from outside and inside of Ω, respectively, and p.v. the Cauchy principal value.

It is proved in [17] (see also [14]) that the NP operator $\mathcal{K}^*$ can be symmetrized using Plemelj's symmetrization principle:

$$\mathcal{S}\mathcal{K}^* = \mathcal{K}\mathcal{S}. \tag{2.3}$$

In fact, if we define a new inner product on $H_0^{-1/2}(\partial\Omega)$ by

$$\langle \varphi, \psi \rangle_{\mathcal{H}^*} := -\langle \varphi, \mathcal{S}[\psi] \rangle, \tag{2.4}$$

where the right-hand side of (2.4) is well-defined since $\mathcal{S}$ maps $H^{-1/2}(\partial\Omega)$ into $H^{1/2}(\partial\Omega)$, then $\mathcal{K}^*$ is self-adjoint with respect to this inner product. Let $\mathcal{H}_0^*$ be the space $H_0^{-1/2}(\partial\Omega)$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^*}$ and $\| \cdot \|_{\mathcal{H}^*}$ be the induced norm. It is known (see [15]) that $\| \cdot \|_{\mathcal{H}^*}$ is equivalent to the norm $\|\cdot\|_{-1/2}$;

$$\|\varphi\|_{\mathcal{H}^*} \approx \|\varphi\|_{-1/2} \tag{2.5}$$

for $\varphi \in H_0^{-1/2}(\partial\Omega)$. Here and throughout this paper $A \lesssim B$ means $A \le CB$ for some constant C independent of parameters involved; $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$.

There is a nontrivial $\varphi_0 \in H^{-1/2}(\partial\Omega)$ such that

$$\mathcal{K}^*[\varphi_0] = \frac{1}{2}\varphi_0. \tag{2.6}$$

We note that $\mathcal{S}[\varphi_0]$ is constant, say, c_0, in Ω. In three dimensions, $c_0 \neq 0$, and hence $\mathcal{S} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is invertible. However, there are domains Ω in two dimensions such that $c_0 = 0$ (see [27]), which means $\mathcal{S}$ is not invertible in general. We introduce a variance of the single layer potential, denoted by $\tilde{\mathcal{S}}$, by $\tilde{\mathcal{S}} = \mathcal{S}$ if $c_0 \neq 0$, and if $c_0 = 0$, then

$$\tilde{\mathcal{S}}[\varphi] = \begin{cases} \mathcal{S}[\varphi] & \text{if } \langle \varphi, 1 \rangle = 0, \\ 1 & \text{if } \varphi = \varphi_0. \end{cases}$$

Then $\tilde{\mathcal{S}}$ is a bijection from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. Moreover, we have an extension of (2.3),

$$\tilde{\mathcal{S}}\mathcal{K}^* = \mathcal{K}\tilde{\mathcal{S}}, \tag{2.7}$$

which enables us to extend the inner product (2.4) to $H^{-1/2}(\partial\Omega)$:

$$\langle \varphi, \psi \rangle_{\mathcal{H}^*} := -\langle \varphi, \tilde{\mathcal{S}}[\psi] \rangle. \tag{2.8}$$

We denote by $\mathcal{H}^*$ the space $H^{-1/2}(\partial\Omega)$ equipped with the inner product (2.8). Then the symmetrization principle (2.7) makes $\mathcal{K}^*$ self-adjoint on $\mathcal{H}^*$. We emphasize that the norm equivalence (2.5) is valid for $\varphi \in H^{-1/2}(\partial\Omega)$.

The spectrum $\sigma(\mathcal{K}^*)$ of the NP operator on $\mathcal{H}^*$ lies in $(-1/2, 1/2]$. Moreover, $\mathcal{K}^*_{\partial\Omega}$ is a compact operator on $\mathcal{H}^*$ when $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$. Therefore, we have the spectral decomposition of $\mathcal{K}^*$ on $\mathcal{H}^*$:

$$\mathcal{K}^* = \frac{1}{2}\varphi_0 \otimes \varphi_0 + \sum_{n=1}^{\infty} \lambda_n \varphi_n \otimes \varphi_n = \sum_{n=0}^{\infty} \lambda_n \varphi_n \otimes \varphi_n, \tag{2.9}$$

where $\varphi_n \in \mathcal{H}^*$ is an eigenvector of $\mathcal{K}^*$ corresponding to the eigenvalue $\lambda_n \in \mathbb{R}$ (counting multiplicities), with $1/2 = \lambda_0 > |\lambda_1| \geq |\lambda_2| \cdots \geq |\lambda_n| \geq \cdots \to 0$ as $n \to \infty$. We note that $\{\varphi_n\}_{n=0}^{\infty}$ is chosen to be an orthonormal basis on $\mathcal{H}^*$ (φ_0 is normalized so that $\|\varphi_0\|_{\mathcal{H}^*} = 1$).

Define an inner product

$$\langle f, g\rangle_{\mathcal{H}} := -\langle f, \tilde{\mathcal{S}}^{-1}[g]\rangle \tag{2.10}$$

on $H^{1/2}(\partial\Omega)$, and denote by $\mathcal{H}$ the space $H^{1/2}(\partial\Omega)$ equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Then $\tilde{\mathcal{S}}$ is a unitary operator from $\mathcal{H}^*$ to $\mathcal{H}$, and hence $\{\tilde{\mathcal{S}}[\varphi_n],\ n = 0, 1, \ldots\}$ is an orthonormal basis of $\mathcal{H}$. Let $\mathcal{H}_0$ be the subspace of $\mathcal{H}$ spanned by $\{\tilde{\mathcal{S}}[\varphi_n],\ n = 1, \ldots\}$. Then $\mathcal{S} : \mathcal{H}_0^* \to \mathcal{H}_0$ is a bijection. We emphasize that the norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to $\|\cdot\|_{1/2}$.

For $\varphi \in \mathcal{H}^*$, we write

$$\hat{\varphi}(n) := \langle \varphi, \varphi_n\rangle_{\mathcal{H}^*}, \quad n = 0, 1, 2, \ldots, \tag{2.11}$$

so that

$$\varphi = \sum_{n=0}^{\infty} \hat{\varphi}(n)\varphi_n (= \hat{\varphi}(0)\varphi_0 + \varphi'), \quad \|\varphi\|^2_{\mathcal{H}^*} = |\hat{\varphi}(0)|^2 + \|\varphi'\|^2_{\mathcal{H}^*}. \tag{2.12}$$

For $f \in \mathcal{H}$, we define

$$\check{f}(n) := \langle f, \tilde{\mathcal{S}}[\varphi_n]\rangle_{\mathcal{H}}, \quad n = 0, 1, 2, \ldots, \tag{2.13}$$

so that

$$f = \sum_{n=0}^{\infty} \check{f}(n)\tilde{\mathcal{S}}[\varphi_n] (= \check{f}(0)\tilde{\mathcal{S}}[\varphi_0] + f'), \quad \|f\|^2_{\mathcal{H}} = |\check{f}(0)|^2 + \|f'\|^2_{\mathcal{H}}. \tag{2.14}$$

We refer to [7] and references therein for more details on the preliminaries presented in this section.

Finally, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from a Banach space X to a Banach space Y; in particular, $\mathcal{L}(X)$ is the space of bounded linear operators on a Banach space X.

3. Asymptotic expansion at low frequencies. Let $\omega = s\omega_0$ from now on to make notation short. A fundamental solution $\Gamma^\omega(x)$ to the Helmholtz operator $\Delta + \omega^2$ in $\mathbb{R}^d$ is a solution of

$$(\Delta + \omega^2)\Gamma^\omega = \delta_0, \tag{3.1}$$

where δ_0 is the Dirac function at 0. Among solutions to (3.1), we seek a solution satisfying the Sommerfeld radiation condition

$$\left|\frac{\partial \Gamma^\omega}{\partial r} - i\omega\Gamma^\omega\right| \le C r^{-(d+1)/2} \quad \text{as} \quad r = |x| \to \infty. \tag{3.2}$$

Then, it is given by

$$\Gamma^\omega(x) = \begin{cases} -\dfrac{i}{4} H_0^1(\omega\,|x|) & \text{if } d = 2, \\ -\dfrac{1}{4\pi}\dfrac{e^{i\omega|x|}}{|x|} & \text{if } d = 3, \end{cases} \tag{3.3}$$

where $H_0^1(z)$ is the Hankel function of the first kind of order 0.

For the subsequent use, we consider the asymptotic expansion of the fundamental solution $\Gamma^\omega(x)$ as $\omega \to +0$. When $d = 2$, we recall the behavior of the Hankel function $H_0^1(z)$ near $z = 0$ (see, e.g., [19]):

$$-\frac{i}{4} H_0^1(\omega\,|x|) = \frac{1}{2\pi}\ln|x| + \tau + \sum_{n=1}^{\infty} \left(b_n \ln(\omega\,|x|) + c_n\right)(\omega\,|x|)^{2n}, \tag{3.4}$$

where

$$b_n = \frac{(-1)^n}{2\pi}\frac{1}{2^{2n}(n!)^2}, \quad c_n = -b_n\left(\gamma - \ln 2 - \frac{\pi i}{2} - \sum_{j=1}^{n}\frac{1}{j}\right)$$

and

$$\tau = \frac{1}{2\pi}(\ln\omega + \gamma - \ln 2) - \frac{i}{4} \tag{3.5}$$

(γ is the Euler constant). So we have

$$\Gamma^\omega(x) = \Gamma(x) + \tau + \omega^2 \ln\omega K_2^\omega(x) \tag{3.6}$$

as $\omega \to +0$ (see also [5]). The definition of $K_2^\omega(x)$ is obvious. When $d = 3$, one can easily see that

$$-\frac{1}{4\pi}\frac{e^{i\omega|x|}}{|x|} = -\frac{1}{4\pi}\frac{1}{|x|} - \frac{i\omega}{4\pi}\sum_{n=1}^{\infty}\frac{(i\omega\,|x|)^{n-1}}{n!}, \tag{3.7}$$

which implies that

$$\Gamma^\omega(x) = \Gamma(x) + \omega K_3^\omega(x). \tag{3.8}$$

Let us observe a regularity property of the function $K_d^\omega(x)$ ($d = 2, 3$) for a later purpose. Let ω_1 be a small positive number. Then there is a constant C independent of $\omega \le \omega_1$ such that

$$\int_\Omega \int_{\partial\Omega} |\partial_x^\alpha K_d^\omega(x-y)|^2 d\sigma(y)\, dx \le C \tag{3.9}$$

for all $\alpha = (\alpha_1, \dots, \alpha_d)$ such that $|\alpha| \le 2$. Here ∂_x^α is the partial derivative with respect to x. Moreover, $\nabla K_3^\omega(x)$ gains ω and it holds that

$$\frac{1}{\omega}\int_\Omega \int_{\partial\Omega} |\partial_x^\alpha \nabla_x K_3^\omega(x-y)|^2 d\sigma(y)\, dx \le C \tag{3.10}$$

for all $|\alpha| \le 1$.

The single layer potential of $\varphi \in H^{-1/2}(\partial\Omega)$ for the Helmholtz operator $\Delta + \omega^2$ is defined by

$$\mathcal{S}^\omega[\varphi](x) = \int_{\partial\Omega} \Gamma^\omega(x-y)\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^d. \tag{3.11}$$

We note that $\mathcal{S}^\omega[\varphi](x)$ satisfies the Sommerfeld radiation condition (3.2) (see [5]). Let $\mathcal{R}_d^\omega$ $(d = 2, 3)$ be the integral operator defined by K_d^ω, namely,

$$\mathcal{R}_d^\omega[\varphi](x) = \int_{\partial\Omega} K_d^\omega(x-y)\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^d. \tag{3.12}$$

Then, we obtain from (3.6) and (3.8) that

$$\mathcal{S}^\omega = \begin{cases} \mathcal{S} + \tau\langle\cdot, 1\rangle + \omega^2 \ln\omega \mathcal{R}_2^\omega & \text{if } d = 2, \\ \mathcal{S} + \omega\mathcal{R}_3^\omega & \text{if } d = 3. \end{cases} \tag{3.13}$$

Analogously to (2.1), the following jump formula holds:

$$\partial_\nu \mathcal{S}^\omega[\varphi]\big|_\pm(x) = \left(\pm 1/2 I + (\mathcal{K}^\omega)^*\right)[\varphi](x), \quad x \in \partial\Omega, \tag{3.14}$$

where $(\mathcal{K}^\omega)^*$ is defined by

$$(\mathcal{K}^\omega)^*[\varphi](x) = \int_{\partial\Omega} \partial_{\nu_x}\Gamma^\omega(x-y)\varphi(y)d\sigma(y), \quad x \in \partial\Omega.$$

For $d = 2, 3$, let

$$\mathcal{Q}_d^\omega[\varphi](x) := \begin{cases} \partial_\nu \mathcal{R}_2^\omega[\varphi](x), & d = 2, \\ \frac{1}{\omega}\partial_\nu \mathcal{R}_3^\omega[\varphi](x), & d = 3, \end{cases} \quad x \in \partial\Omega. \tag{3.15}$$

Then, we have

$$(\mathcal{K}^\omega)^* = \begin{cases} \mathcal{K}^* + \omega^2 \ln\omega \mathcal{Q}_2^\omega & \text{if } d = 2, \\ \mathcal{K}^* + \omega^2 \mathcal{Q}_3^\omega & \text{if } d = 3. \end{cases} \tag{3.16}$$

We now investigate the mapping property of $\mathcal{R}_d^\omega$ and $\mathcal{Q}_d^\omega$. By the Cauchy–Schwarz inequality we see from (3.9) that

$$\|\mathcal{R}_d^\omega[\varphi]\|_{H^2(\Omega)} \le C\|\varphi\|_{L^2(\partial\Omega)}.$$

We also see from (3.10) that

$$\frac{1}{\omega}\|\nabla\mathcal{R}_3^\omega[\varphi]\|_{H^1(\Omega)} \le C\|\varphi\|_{L^2(\partial\Omega)}.$$

By the trace theorem, $\mathcal{R}_d^\omega$ maps $L^2(\partial\Omega)$ into $H^{3/2}(\partial\Omega)$, and $\mathcal{Q}_d^\omega$ maps $L^2(\partial\Omega)$ into $H^{1/2}(\partial\Omega)$. By duality, $\mathcal{R}_d^\omega$ maps $H^{-3/2}(\partial\Omega)$ into $L^2(\partial\Omega)$, and $H^{-1/2}(\partial\Omega)$ into $H^1(\partial\Omega)$ by interpolation. Likewise we see that $\mathcal{Q}_d^\omega$ maps $H^{-1/2}(\partial\Omega)$ into $L^2(\partial\Omega)$. We summarize these properties in the following lemma.

LEMMA 1. *For a given small positive number ω_1, there exists a constant C independent of $\omega \le \omega_1$ such that*

$$\|\mathcal{R}_d^\omega[\varphi]\|_1 \le C\|\varphi\|_{-1/2} \tag{3.17}$$

and

$$\|\mathcal{Q}_d^\omega[\varphi]\|_0 \le C\|\varphi\|_{-1/2} \tag{3.18}$$

for all $\varphi \in H^{-1/2}(\partial\Omega)$.

As a consequence we obtain the following proposition, which will be used in later sections.

PROPOSITION 2. *Let $\varphi \in \mathcal{H}^*$ and $\varphi = \varphi' + \hat{\varphi}(0)\varphi_0$ be its orthogonal decomposition where $\varphi' \in \mathcal{H}_0^*$. The following estimates hold:*

(i) *If $d = 2$, then*

$$\|\varphi'\|_{\mathcal{H}^*}^2 - |\omega \ln \omega|^2 |\hat{\varphi}(0)|^2 \lesssim \|\nabla \mathcal{S}^\omega[\varphi]\|_{L^2(\Omega)}^2 \lesssim \|\varphi'\|_{\mathcal{H}^*}^2 + |\omega \ln \omega|^2 |\hat{\varphi}(0)|^2. \tag{3.19}$$

(ii) *If $d = 3$, then*

$$\|\varphi'\|_{\mathcal{H}^*}^2 - |\omega||\hat{\varphi}(0)|^2 \lesssim \|\nabla \mathcal{S}^\omega[\varphi]\|_{L^2(\Omega)}^2 \lesssim \|\varphi'\|_{\mathcal{H}^*}^2 + |\omega||\hat{\varphi}(0)|^2. \tag{3.20}$$

Proof. We only prove (3.19) since, three-dimensional case can be proved in a similar way.

We have from Gauss's divergence theorem

$$\int_\Omega |\nabla \mathcal{S}^\omega[\varphi]|^2 \, dx + \int_\Omega \mathcal{S}^\omega[\varphi] \, \overline{\Delta \mathcal{S}^\omega[\varphi]} dx = \int_{\partial\Omega} \mathcal{S}^\omega[\varphi] \, \overline{\partial_\nu \mathcal{S}^\omega[\varphi]|_-} \, d\sigma. \tag{3.21}$$

Since $\Delta \mathcal{S}^\omega[\varphi] = -\omega^2 \mathcal{S}^\omega[\varphi]$, we have

$$\int_\Omega |\nabla \mathcal{S}^\omega[\varphi]|^2 \, dx = \omega^2 \int_\Omega |\mathcal{S}^\omega[\varphi]|^2 \, dx + \int_{\partial\Omega} \mathcal{S}^\omega[\varphi] \, \overline{\partial_\nu \mathcal{S}^\omega[\varphi]|_-} \, d\sigma. \tag{3.22}$$

One can see from (3.13) and Lemma 1 that

$$\int_\Omega |\mathcal{S}^\omega[\varphi]|^2 \, dx \lesssim |\ln \omega|^2 \, \|\varphi\|_{-1/2}^2 \lesssim |\ln \omega|^2 \, \|\varphi\|_{\mathcal{H}^*}^2 \tag{3.23}$$

since $|\tau| \lesssim |\ln \omega|$. The last inequality holds because of (2.5).

Using the jump formula (3.14) we have

$$\int_{\partial\Omega} \mathcal{S}^\omega[\varphi] \, \overline{\partial_\nu \mathcal{S}^\omega[\varphi]|_-} \, d\sigma = \int_{\partial\Omega} \mathcal{S}^\omega[\varphi] \, \overline{\left(-1/2I + (\mathcal{K}^\omega)^*\right)[\varphi]} \, d\sigma.$$

One then see from (3.13) and (3.16) that

$$\begin{aligned} &\int_{\partial\Omega} \mathcal{S}^\omega[\varphi] \, \overline{\partial_\nu \mathcal{S}^\omega[\varphi]|_-} \, d\sigma \\ &\quad = \int_{\partial\Omega} \mathcal{S}[\varphi] \, \overline{(-1/2I + \mathcal{K}^*)[\varphi]} \, d\sigma + \tau\langle\varphi, 1\rangle \int_{\partial\Omega} \overline{(-1/2I + \mathcal{K}^*)[\varphi]} \, d\sigma + \omega^2 \ln \omega E, \end{aligned} \tag{3.24}$$

where

$$E=\int_{\partial\Omega}\mathcal{R}_2^\omega[\varphi]\,\overline{\left(-1/2I+(\mathcal{K}^\omega)^*\right)[\varphi]}\,d\sigma+\int_{\partial\Omega}\mathcal{S}^\omega[\varphi]\,\overline{\mathcal{Q}_2^\omega[\varphi]}\,d\sigma+\tau\langle\varphi,1\rangle\int_{\partial\Omega}\overline{\mathcal{Q}_2^\omega[\varphi]}\,d\sigma.$$

Using (3.13) and Lemma 1 one can show that

$$|E|\le C|\ln\omega|\|\varphi\|^2_{\mathcal{H}^*} \tag{3.25}$$

for some constant C independent of $\omega\le\omega_1$. In fact, we have from (3.5)

$$\begin{aligned}|E|&\le\|\mathcal{R}_2^\omega[\varphi]\|_{1/2}\|(-1/2I+(\mathcal{K}^\omega)^*)[\varphi]\|_{-1/2}\\&\quad+\|\mathcal{S}^\omega[\varphi]\|_{1/2}\|\mathcal{Q}_2^\omega[\varphi]\|_{-1/2}+\tau\|\varphi\|_{-1/2}\|\mathcal{Q}_2^\omega[\varphi]\|_0\\&\le C|\ln\omega|\|\varphi\|^2_{-1/2}.\end{aligned}$$

Since $\mathcal{K}[1]=1/2$, we have

$$\int_{\partial\Omega}\overline{\left(-1/2I+\mathcal{K}^*\right)[\varphi]}\,d\sigma=\int_{\partial\Omega}\left(-1/2I+\mathcal{K}\right)[1]\,\overline{\varphi}\,d\sigma=0. \tag{3.26}$$

On the other hand, since $\mathcal{K}^*[\varphi_0]=1/2\varphi_0$, we have

$$\int_{\partial\Omega}\mathcal{S}[\varphi]\,\overline{\left(-1/2I+\mathcal{K}^*\right)[\varphi]}\,d\sigma=\int_{\partial\Omega}\mathcal{S}[\varphi']\,\overline{\left(-1/2I+\mathcal{K}^*\right)[\varphi']}\,d\sigma.$$

Using $\varphi'=\sum_{n=1}^\infty\hat{\varphi}(n)\varphi_n$, we have

$$\int_{\partial\Omega}\mathcal{S}[\varphi]\,\overline{\left(-1/2I+\mathcal{K}^*\right)[\varphi]}\,d\sigma=\sum_{n,m=1}^\infty\left(-1/2+\lambda_l\right)\hat{\varphi}(n)\overline{\hat{\varphi}(m)}\int_{\partial\Omega}\mathcal{S}[\varphi_j]\,\overline{\varphi_l}\,d\sigma.$$

Since $\int_{\partial\Omega}\mathcal{S}[\varphi_n]\,\overline{\varphi_m}\,d\sigma=-\langle\varphi_n,\varphi_m\rangle_{\mathcal{H}^*}=-\delta_{nm}$ (the Kronecker's delta), we have

$$\int_{\partial\Omega}\mathcal{S}[\varphi]\,\overline{\left(-1/2I+\mathcal{K}^*\right)[\varphi]}\,d\sigma=\sum_{n=1}^\infty\left(\lambda_n-1/2\right)|\hat{\varphi}(n)|^2.$$

So we have

$$\left|\int_{\partial\Omega}\mathcal{S}[\varphi]\,\overline{\left(-1/2I+\mathcal{K}^*\right)[\varphi]}\,d\sigma\right|\approx\|\varphi'\|^2_{\mathcal{H}^*}. \tag{3.27}$$

Combining (3.24)–(3.27) we obtain

$$\|\varphi'\|^2_{\mathcal{H}^*}-|\omega\ln\omega|^2\|\varphi\|^2_{\mathcal{H}^*}\lesssim\left|\int_{\partial\Omega}\mathcal{S}^\omega[\varphi]\,\overline{\partial_\nu\mathcal{S}^\omega[\varphi]|_-}\,d\sigma\right|\lesssim\|\varphi'\|^2_{\mathcal{H}^*}+|\omega\ln\omega|^2\|\varphi\|^2_{\mathcal{H}^*},$$

which together with (3.21) and (3.23) yields (3.19). ∎

4. Analysis of resonance for small frequencies. In this section we investigate resonance using (1.7). From now on, we assume that $\epsilon_m=1$ without loss of generality.

Set $k_m=\omega(=s\omega_0)$ and

$$k_c^2=\frac{\omega^2}{\epsilon_c+i\delta},\quad\Re k_c>0,\quad\Im k_c<0.$$

Since

$$k_c = \omega\,(\epsilon_c + i\delta)^{-1/2} \simeq -i\frac{\omega}{\sqrt{|\epsilon_c|}}\left(1 - i\frac{\delta}{2\epsilon_c}\right),$$

we assume for simplicity

$$k_c = -i\frac{\omega}{\sqrt{|\epsilon_c|}}\left(1 - i\frac{\delta}{2\epsilon_c}\right). \tag{4.1}$$

Then the problem (1.7) can be written as

$$\begin{cases} \Delta u_\delta + k_c^2 u_\delta = 0 & \text{in } \ \Omega, \\ \Delta u_\delta + \omega^2 u_\delta = a \cdot \nabla \delta_z & \text{in } \ \mathbb{R}^d \setminus \overline{\Omega}, \\ u_\delta|_- - u_\delta|_+ = 0 & \text{on } \ \partial\Omega, \\ (\epsilon_c + i\delta)\,\partial_\nu u_\delta|_- - \partial_\nu u_\delta|_+ = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{4.2}$$

under the Sommerfeld radiation condition (1.6).

Let

$$F_z(x) := \int_{\mathbb{R}^d} \Gamma^\omega(x-y) a \cdot \nabla \delta_z(y) dy = a \cdot \nabla_x \Gamma^\omega(x-z). \tag{4.3}$$

Then, the solution u_δ can be represented as

$$u_\delta(x) = \begin{cases} \mathcal{S}^{k_c}[\varphi_\delta](x), & x \in \Omega, \\ F_z(x) + \mathcal{S}^\omega[\psi_\delta](x), & x \in \mathbb{R}^d \setminus \Omega, \end{cases} \tag{4.4}$$

for some $\varphi_\delta, \psi_\delta \in \mathcal{H}^*$. In view of transmission conditions on $\partial\Omega$ (the third and fourth conditions in (4.2)), $(\varphi_\delta, \psi_\delta)$ should solve the following system of integral equations:

$$\begin{cases} \mathcal{S}^{k_c}[\varphi_\delta] - \mathcal{S}^\omega[\psi_\delta] = F_z \\ (\epsilon_c + i\delta)\partial_\nu \mathcal{S}^{k_c}[\varphi_\delta]|_- - \partial_\nu \mathcal{S}^\omega[\psi_\delta]|_+ = \partial_\nu F_z \end{cases} \quad \text{on } \partial\Omega. \tag{4.5}$$

Let $X := \mathcal{H}^* \times \mathcal{H}^*$ and $Y := \mathcal{H} \times \mathcal{H}^*$, and define an operator $A_\delta^s : X \to Y$ by

$$A_\delta^s = \begin{bmatrix} \mathcal{S}^{k_c} & -\mathcal{S}^\omega \\ (\epsilon_c + i\delta)\partial_\nu \mathcal{S}^{k_c}|_- & -\partial_\nu \mathcal{S}^\omega|_+ \end{bmatrix}. \tag{4.6}$$

Then we can rewrite (4.5) as

$$A_\delta^s \begin{bmatrix} \varphi_\delta \\ \psi_\delta \end{bmatrix} = \begin{bmatrix} F_z \\ \partial_\nu F_z \end{bmatrix}. \tag{4.7}$$

The solvability of (4.5) is equivalent to the invertibility of A_δ^s. We will investigate the behavior of the norm $(A_\delta^s)^{-1}$ as $\delta \to +0$.

4.1. Three dimensions. We deal with the three-dimensional case first since it is simpler.

We split A_δ^s into two parts: $A_\delta^s = A_\delta + T_\delta^s$, where

$$A_\delta = \begin{bmatrix} \mathcal{S} & -\mathcal{S} \\ (\epsilon_c + i\delta)(-1/2I + \mathcal{K}^*) & -(1/2I + \mathcal{K}^*) \end{bmatrix}. \tag{4.8}$$

Then we can infer from (3.13), (3.16), and Lemma 1 that

$$\|T_\delta^s\|_{\mathcal{L}(X,Y)} \lesssim \omega. \tag{4.9}$$

LEMMA 3. *For $f \in \mathcal{H}$ and $g \in \mathcal{H}^*$, the solution to*

$$A_\delta \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{4.10}$$

is given by

$$\varphi = \sum_{n=0}^{\infty} \frac{\hat{g}(n) - (1/2 + \lambda_n)\check{f}(n)}{(\epsilon_c - 1)(\lambda_n - \lambda(\epsilon_c)) + i\delta(\lambda_n - \frac{1}{2})} \varphi_n \tag{4.11}$$

and

$$\psi = \varphi - \mathcal{S}^{-1}[f]. \tag{4.12}$$

Proof. Equation (4.10) can be written as

$$\begin{cases} \mathcal{S}[\varphi] - \mathcal{S}[\psi] = f \\ (\epsilon_c + i\delta)(-1/2I + \mathcal{K}^*)[\varphi] - (1/2I + \mathcal{K}^*)[\psi] = g \end{cases} \quad \text{on } \partial\Omega.$$

Since $\mathcal{S} : \mathcal{H}^* \to \mathcal{H}$ is invertible in three dimensions, we obtain (4.11) from the first equation above. Substituting this into the second equation, we obtain

$$\big(- 1/2(\epsilon_c + i\delta + 1)I + (\epsilon_c + i\delta - 1)\mathcal{K}^*\big)[\varphi] = g - (1/2I + \mathcal{K}^*)\mathcal{S}^{-1}[f].$$

We then use the spectral decomposition (2.12) to obtain

$$\varphi = \sum_{n=0}^{\infty} \frac{a_n}{-1/2(\epsilon_c + i\delta + 1) + (\epsilon_c + i\delta - 1)\lambda_n} \varphi_n,$$

where

$$a_n = \hat{g}(n) - \langle (1/2I + \mathcal{K}^*)\mathcal{S}^{-1}[f], \varphi_n \rangle_{\mathcal{H}^*}.$$

Since $f = \sum_{j=0}^{\infty} \check{f}(j)\mathcal{S}[\varphi_j]$, we have

$$\langle (1/2I + \mathcal{K}^*)\mathcal{S}^{-1}[f], \varphi_n \rangle_{\mathcal{H}^*} = \sum_{j=0}^{\infty} \check{f}(j) \langle (1/2I + \mathcal{K}^*)[\varphi_j], \varphi_n \rangle_{\mathcal{H}^*} = (1/2 + \lambda_n)\check{f}(n).$$

This completes the proof. □

As a consequence of Lemma 3 we obtain the following corollary.

COROLLARY 4. *Suppose that $\epsilon_c \neq -1$, and let (φ, ψ) be the solution of* (4.10). *Then the following hold for sufficiently small δ:*

(i) $\|A_\delta^{-1}\|_{\mathcal{L}(Y,X)} \lesssim \delta^{-1}$.

(ii) *If $\lambda(\epsilon_c) \neq \lambda_n$ for any n, then $\|A_\delta^{-1}\|_{\mathcal{L}(Y,X)} \le C$ for some C depending on ϵ_c.*

(iii) *If $\lambda(\epsilon_c) = \lambda_n$ for some $n \neq 0$, then $\|\varphi'\|_{\mathcal{H}^*} \gtrsim |a_n|\delta^{-1}$, where $a_n = \hat{g}(n) - (1/2 + \lambda_n)\check{f}(n)$.*

Proof. Since

$$\frac{1}{|(\epsilon_c - 1)(\lambda_n - \lambda(\epsilon_c)) + i\delta(\lambda_n - \frac{1}{2})|} \lesssim \delta^{-1},$$

we have from (4.11) that

$$\|\varphi\|_{\mathcal{H}^*}^2 \lesssim \delta^{-2} \sum_{n=0}^{\infty} |\hat{g}(n) - (1/2 + \lambda_n)\check{f}(n)|^2 \lesssim \delta^{-2}(\|f\|_{\mathcal{H}}^2 + \|g\|_{\mathcal{H}^*}^2).$$

We have from (4.12) that

$$\|\psi\|^2_{\mathcal{H}^*} \lesssim \delta^{-2}(\|f\|^2_{\mathcal{H}} + \|g\|^2_{\mathcal{H}^*}).$$

This proves (i).

Since $\epsilon_c \neq -1$, $\lambda(\epsilon_c) \neq 0$. If $\lambda(\epsilon_c) \neq \lambda_n$ for any n, then $|\lambda_n - \lambda(\epsilon_c)| \geq C$ for some $C > 0$. We emphasize that C depends on ϵ_c. So we have

$$\frac{1}{|(\epsilon_c - 1)(\lambda_n - \lambda(\epsilon_c)) + i\delta(\lambda_n - \frac{1}{2})|} \lesssim 1,$$

and hence

$$\|\varphi\|^2_{\mathcal{H}^*} \lesssim \sum_{n=0}^{\infty} |\hat{g}(n) - (1/2 + \lambda_n)\check{f}(n)|^2 \lesssim \|f\|^2_{\mathcal{H}} + \|g\|^2_{\mathcal{H}^*}$$

and

$$\|\psi\|^2_{\mathcal{H}^*} \lesssim \|f\|^2_{\mathcal{H}} + \|g\|^2_{\mathcal{H}^*}.$$

This proves (ii).

If $\lambda(\epsilon_c) = \lambda_n$ for some $n \neq 0$, then we have

$$\frac{1}{|(\epsilon_c - 1)(\lambda_n - \lambda(\epsilon_c)) + i\delta(\lambda_n - \frac{1}{2})|} \gtrsim \delta^{-1}.$$

Therefore we have

$$\|\varphi'\|_{\mathcal{H}^*} \geq |\hat{\varphi}(n)| \gtrsim \delta^{-1}|a_n|.$$

This completes the proof. □

The following is the main theorem of this paper in three dimensions.

THEOREM 5. *Suppose $d = 3$ and assume*

$$s\delta^{-1} \leq c \tag{4.13}$$

for sufficiently small c. Let u_δ be the solution to (1.7).

(i) *If $\lambda(\epsilon_c) \neq \lambda_n$ for any n, then there is C independent of δ (C may depend on ϵ_c) such that*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \leq C. \tag{4.14}$$

(ii) *If $\lambda(\epsilon_c) = \lambda_n$ for some $n \neq 0$, let z be such that $a \cdot \nabla\mathcal{S}[\varphi_n](z) \neq 0$. Then*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \approx \delta^{-1} \quad \text{as } \delta \to +0. \tag{4.15}$$

Proof. We still assume $\epsilon_m = 1$. Since $A^s_\delta = A_\delta + T^s_\delta = A_\delta(I + A^{-1}_\delta T^s_\delta)$, it follows from (4.7) that

$$\Phi_\delta = (I + A^{-1}_\delta T^s_\delta)^{-1} A^{-1}_\delta[\mathbf{F}],$$

where

$$\Phi_\delta = \begin{bmatrix} \varphi_\delta \\ \psi_\delta \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} F_z \\ \partial_\nu F_z \end{bmatrix}.$$

We see from (4.9) and Corollary 4(i) that

$$\|A^{-1}_\delta T^s_\delta\|_{\mathcal{L}(X)} \lesssim \delta^{-1}s.$$

So, if $s\delta^{-1}$ is sufficiently small, then we have

$$\|\Phi_\delta - A_\delta^{-1}[\mathbf{F}]\|_X \lesssim \delta^{-1} s \|A_\delta^{-1}[\mathbf{F}]\|_X. \tag{4.16}$$

If $\lambda(\epsilon_c) \neq \lambda_n$ for any n, then we infer from Corollary 4(ii) that

$$\|\Phi_\delta\|_X \le C\|\mathbf{F}\|_Y.$$

Note that

$$\|\mathbf{F}\|_Y = \|F_z\|_{\mathcal{H}^*} + \|\partial_\nu F_z\|_{\mathcal{H}} \le C. \tag{4.17}$$

So, we have from (3.20) and (4.4)

$$\|\nabla u_\delta\|_{L^2(\Omega)} = \|\nabla \mathcal{S}^{k_c}[\varphi_\delta]\|_{L^2(\Omega)} \lesssim \|\varphi_\delta\|_{\mathcal{H}^*} \le C$$

regardless of δ.

Suppose that $\lambda(\epsilon_c) = \lambda_n$ for some $n \neq 0$. Let $A_\delta^{-1}[\mathbf{F}] = (\varphi_1, \psi_1)^T$. Then Corollary 4(iii) shows that

$$\|\varphi_1'\|_{\mathcal{H}^*} \gtrsim |a_n|\delta^{-1},$$

where

$$a_n = (\widehat{\partial_\nu F_z})(n) - (1/2 + \lambda_n)\check{F}_z(n). \tag{4.18}$$

It then follows from (4.16) that

$$\|\varphi_\delta'\|_{\mathcal{H}^*} \gtrsim \|\varphi_1'\|_{\mathcal{H}^*} - \delta^{-1} s \|A_\delta^{-1}[\mathbf{F}]\|_X \gtrsim |a_n|\delta^{-1}$$

if $a_n \neq 0$ for sufficiently small δ. Thus we obtain from (3.20) that

$$\|\nabla u_\delta\|_{L^2(\Omega)} = \|\nabla \mathcal{S}^{k_c}[\varphi_\delta]\|_{L^2(\Omega)} \gtrsim |a_n|\delta^{-1} - s|\hat{\varphi}_\delta(0)|. \tag{4.19}$$

We now show that $|\hat{\varphi}_\delta(0)|$ is bounded, and $a_n \neq 0$ for generic z's. For that purpose we write A_δ^ω as $A_\delta^\omega = (I + T_\delta^s A_\delta^{-1})A_\delta$ so that (4.7) takes the form

$$A_\delta[\Phi_\delta] = (I + T_\delta^s A_\delta^{-1})^{-1}[\mathbf{F}]. \tag{4.20}$$

Let $(I + T_\delta^s A_\delta^{-1})^{-1}[\mathbf{F}] = (f, g)^T$. Then since $\|T_\delta^s A_\delta^{-1}\|_{\mathcal{L}(Y)} \lesssim \delta^{-1}s$, we have $\|f\|_{\mathcal{H}} + \|g\|_{\mathcal{H}^*}$ is bounded. Since $\lambda_0 = 1/2$, we have according to (4.11)

$$|\hat{\varphi}_\delta(0)| = \left| \frac{\hat{g}(0) - \check{f}(0)}{(\epsilon_c - 1)(\frac{1}{2} - \lambda(\epsilon_c))} \right| \le C.$$

According to (4.18) we have

$$\begin{aligned} a_n &= \langle \partial_\nu F_z, \varphi_n \rangle_{\mathcal{H}^*} - (1/2 + \lambda_n)\langle F_z, \mathcal{S}[\varphi_n]\rangle_{\mathcal{H}} \\ &= -\langle \partial_\nu F_z, \mathcal{S}[\varphi_n]\rangle + (1/2 + \lambda_n)\langle F_z, \varphi_n\rangle \\ &= \omega^2 \int_\Omega F_z\, \mathcal{S}[\varphi_n]\, dx - \langle F_z, \partial_\nu \mathcal{S}[\varphi_n]\big|_-\rangle + (1/2 + \lambda_n)\langle F_z, \varphi_n\rangle \\ &= \omega^2 \int_\Omega F_z\, \mathcal{S}[\varphi_n]\, dx + \langle F_z, \varphi_n\rangle. \end{aligned} \tag{4.21}$$

Since $F_z(x) = a \cdot \nabla_x \Gamma^\omega(x-z)$, we have

$$\langle F_z, \varphi_n \rangle = -a \cdot \nabla \mathcal{S}^\omega[\varphi_n](z).$$

By (3.3) we have

$$\nabla \mathcal{S}^\omega[\varphi_n](z) = \nabla \mathcal{S}[\varphi_n](z) + O(\omega),$$

and hence

$$a_n = a \cdot \nabla \mathcal{S}[\varphi_n](z) + O(\omega).$$

Note that $a \cdot \nabla \mathcal{S}[\varphi_n](z)$ is a harmonic function in $z \in \mathbb{R}^3 \setminus \overline{\Omega}$. So it cannot be zero for z in an open set. We choose z so that $a \cdot \nabla \mathcal{S}[\varphi_n](z) \neq 0$, and then $a_n \neq 0$ if ω is sufficiently small. Thus we have

$$\|\nabla u_\delta\|_{L^2(\Omega)} \gtrsim \delta^{-1}. \tag{4.22}$$

This completes the proof. ∎

4.2. Two dimensions. In two dimensions we decompose A_δ^s in (4.6) into three parts: $A_\delta^s = A_\delta + B^s + T_\delta^s$, where A_δ is defined by (4.8) and

$$B^s = \begin{bmatrix} \tau^{k_c} \langle \cdot, 1 \rangle & -\tau \langle \cdot, 1 \rangle \\ 0 & 0 \end{bmatrix}. \tag{4.23}$$

Here, τ^{k_c} is defined by

$$\tau^{k_c} = (1/2\pi)(\ln k_c + \gamma - \ln 2) - i/4, \tag{4.24}$$

and τ is defined by (3.5). We emphasize that

$$|\tau^{k_c}| \sim -\ln \omega, \quad |\tau| \sim -\ln \omega. \tag{4.25}$$

We have from (3.13), (3.16), and Lemma 1

$$\|T_\delta^s\|_{\mathcal{L}(X,Y)} \lesssim |s^2 \ln s|. \tag{4.26}$$

Unlike the three-dimensional case, $A_\delta : X \to Y$ may not be invertible since $\mathcal{S} : \mathcal{H}^* \to \mathcal{H}$ is not invertible in general. Instead we prove that $A_\delta + B^s : X \to Y$ is invertible. In fact, we obtain the following lemma.

Lemma 6. *The operator $A_\delta + B^s : X \to Y$ is invertible. For $(f,g)^T \in Y$, the solution $(\varphi, \psi)^T$ to the equation*

$$(A_\delta + B^s) \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

is given by

$$\varphi = \varphi' + \hat{\varphi}(0)\varphi_0 = \varphi' + \frac{\check{f}(0)\tilde{\mathcal{S}}[\varphi_0] - \hat{g}(0)\left(\mathcal{S}[\varphi_0] + \tau\langle \varphi_0, 1\rangle\right)}{\mathcal{S}[\varphi_0] + \tau^{k_c}\langle \varphi_0, 1\rangle}\varphi_0 \tag{4.27}$$

and

$$\psi = \varphi' - \mathcal{S}^{-1}[f'] - \hat{g}(0)\varphi_0, \tag{4.28}$$

where

$$\varphi' = \sum_{n=1}^{\infty} \frac{\hat{g}(n) - (\frac{1}{2} + \lambda_n)\check{f}(n)}{(\epsilon_c - 1)(\lambda_n - \lambda(\epsilon_c)) + i\delta(\lambda_n - \frac{1}{2})}\varphi_n. \tag{4.29}$$

Before proving Theorem 6, we emphasize that $\mathcal{S}[\varphi_0]$ is constant $(= c_0)$. If $c_0 \neq 0$, then $\tilde{\mathcal{S}}[\varphi_0] = \mathcal{S}[\varphi_0] = c_0$, and

$$\langle \varphi_0, 1 \rangle = c_0^{-1} \langle \varphi_0, \mathcal{S}[\varphi_0] \rangle = c_0^{-1}.$$

So we have

$$\hat{\varphi}(0) = \frac{c_0 \check{f}(0) - (c_0 + c_0^{-1}\tau)\hat{g}(0)}{c_0 + c_0^{-1}\tau^{k_c}}.$$

If $c_0 = 0$, then $\tilde{\mathcal{S}}[\varphi_0] = 1$, and $\langle \varphi_0, 1 \rangle = 1$. So we have

$$\hat{\varphi}(0) = \frac{\check{f}(0) - \tau \hat{g}(0)}{\tau^{k_c}}.$$

Proof of Lemma 6. Let

$$f = f' + \check{f}(0)\tilde{\mathcal{S}}[\varphi_0] \quad g = g' + \hat{g}(0)\varphi_0$$

be orthogonal decompositions in $\mathcal{H}$ and $\mathcal{H}^*$, so that $f' \in \mathcal{H}_0$ and $g' \in \mathcal{H}_0^*$.

Since $\mathcal{S} : \mathcal{H}_0^* \to \mathcal{H}_0$ is invertible, one can see as in Lemma 3 that the solution to

$$A_\delta \begin{bmatrix} \varphi' \\ \psi' \end{bmatrix} = \begin{bmatrix} f' \\ g' \end{bmatrix}$$

is given by (4.29) and

$$\psi' = \varphi' - \mathcal{S}^{-1}[f'].$$

Since $(-1/2I + \mathcal{K}^*)[\varphi_0] = 0$ and $\varphi', \psi' \in \mathcal{H}_0^*$, we can see that

$$\begin{aligned}(A_\delta + B^s) \begin{bmatrix} \varphi' + c\varphi_0 \\ \varphi' - \mathcal{S}^{-1}[f'] + d\varphi_0 \end{bmatrix} &= \begin{bmatrix} f' \\ g' \end{bmatrix} + (A_\delta + B^s) \begin{bmatrix} c\varphi_0 \\ d\varphi_0 \end{bmatrix} \\ &= \begin{bmatrix} f' \\ g' \end{bmatrix} + \begin{bmatrix} c\left(\mathcal{S}[\varphi_0] + \tau^{k_c}\langle \varphi_0, 1\rangle\right) - d\left(\mathcal{S}[\varphi_0] + \tau\langle \varphi_0, 1\rangle\right) \\ -d\varphi_0 \end{bmatrix}.\end{aligned}$$

So we solve

$$c\left(\mathcal{S}[\varphi_0] + \tau^{k_c}\langle \varphi_0, 1\rangle\right) - d\left(\mathcal{S}[\varphi_0] + \tau\langle \varphi_0, 1\rangle\right) = \check{f}(0)\tilde{\mathcal{S}}[\varphi_0], \quad -d = \hat{g}(0)$$

for c, d to have (4.27) and (4.28). This completes the proof. ∎

We can obtain from Lemma 6 results similar to those in Corollary 4 for two dimensions. We then obtain the following theorem for two dimensions.

THEOREM 7. *Suppose $d = 2$ and assume*

$$s^2 |\ln s| \delta^{-1} \leq c \tag{4.30}$$

for sufficiently small c. Let u_δ be the solution to (1.7).

(i) *If $\lambda(\epsilon_c) \neq \lambda_n$ for any n, then there is C independent of δ (may depend on ϵ_c/ϵ_m) such that*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \leq C. \tag{4.31}$$

(ii) *If $\lambda(\epsilon_c) = \lambda_n$ for some $n \neq 0$, let z be such that $a \cdot \nabla \mathcal{S}[\varphi_n](z) \neq 0$. Then*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \approx \delta^{-1} \quad \textit{as } \delta \to +0. \tag{4.32}$$

Proof. We write

$$A_\delta^s = A_\delta + B^s + T_\delta^s = (A_\delta + B^s)\left(I + (A_\delta + B^s)^{-1} T_\delta^s\right) \tag{4.33}$$

and follow the same lines of the proof for Theorem 5. One thing we need to check is that $|\hat{\varphi}_\delta(0)|$ is bounded. To do that it suffices to show that $|\hat{\varphi}(0)|$ is bounded, where φ is the solution expressed in (4.27). Note that

$$|\hat{\varphi}(0)| = \left| \frac{\check{f}(0)\tilde{\mathcal{S}}[\varphi_0] - \hat{g}(0)\left(\mathcal{S}[\varphi_0] + \tau\langle \varphi_0, 1\rangle\right)}{\mathcal{S}[\varphi_0] + \tau^{k_c}\langle \varphi_0, 1\rangle} \right| \lesssim \frac{|\tau|}{|\tau^{k_c}|} \lesssim 1.$$

This completes the proof. ∎

5. Analysis of resonance for small inclusions. In this section we investigate resonance using (1.4). In this case F_z defined in (4.3) becomes

$$F_z(x) = s^{1-d} a \cdot \nabla_x \Gamma^\omega (x - s^{-1} z), \quad d = 2, 3. \tag{5.1}$$

THEOREM 8. *Suppose $d = 3$ and assume that* (4.13) *holds. Let u_δ be the solution to* (1.4).

(i) *If $\lambda(\epsilon_c) \neq \lambda_n$ for any n, then there is C independent of δ such that*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \le C. \tag{5.2}$$

(ii) *Suppose that $\lambda(\epsilon_c) = \lambda_n$ for some $n \neq 0$. Let $a = \widehat{z} := z/|z|$ and u_n^∞ be the far field pattern of $u_n(x) = \mathcal{S}[\varphi_n](x)$, where φ_n is the eigenfunction corresponding to λ_n. If $u_n^\infty(\widehat{z}) \neq 0$, then*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \approx \delta^{-1} \quad \text{as } \delta \to +0. \tag{5.3}$$

Proof. We see from (3.3) that in three dimensions

$$|\nabla_x \Gamma^\omega (x - y)| \lesssim |y|^{-2} + \omega |y|^{-1} \tag{5.4}$$

for all $x \in \Omega$ and $|y| \to \infty$. Thus we obtain from (5.1)

$$|F_z(x)| \lesssim s^{-2} |\nabla_x \Gamma^\omega (x - s^{-1} z)| \le C. \tag{5.5}$$

We also have

$$|\nabla F_z(x)| \lesssim s. \tag{5.6}$$

Thus we have $\|F_z\|_{H^1(\Omega)} \le C$. It then follows from the trace theorem that $\|F_z\|_{1/2} \le C$. We also have $\|\partial_\nu F_z\|_{-1/2} \le C$. It then follows that $\|\mathbf{F}\|_Y \le C$. We then follow the proof of Theorem 5(i) to derive (i).

According to (4.21) a_n in this case is given by

$$a_n = -s^{-2} a \cdot \nabla \mathcal{S}^\omega[\varphi_n](s^{-1} z) + \omega^2 \int_\Omega F_z\, \mathcal{S}[\varphi_n]\, dx. \tag{5.7}$$

It is well known (see [10, Theorem 2.5]) that

$$u_n(y) = \mathcal{S}^\omega[\varphi_n](y) = \frac{e^{i\omega r}}{r}\left[u_n^\infty(\widehat{y}) + O(r^{-1})\right],$$

where $r = |y|$. So, we have

$$\partial_r u_n(y) = \left[-\frac{e^{i\omega r}}{r^2} + i\omega \frac{e^{i\omega r}}{r} \right] u_n^\infty(\widehat{y}) + O(r^{-3}).$$

Since $\omega = s\omega_0$ and $a = z/|z|$, we have

$$a_n = \left[\frac{e^{i\omega_0|z|}}{|z|^2} - i\omega_0 \frac{e^{i\omega_0|z|}}{|z|} \right] u_n^\infty(\widehat{z}) + O(s).$$

This proves (ii). ◻

The two-dimensional case is different because of the different far field behavior of the solution. We obtain the following theorem in two dimensions.

THEOREM 9. *Suppose $d = 2$ and assume that* (4.30) *holds. Let u_δ be the solution to* (1.4).

(i) *If $\lambda(\epsilon_c) \neq \lambda_n$ for any n, then there is C independent of δ such that*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \leq C. \tag{5.8}$$

(ii) *Suppose that $\lambda(\epsilon_c) = \lambda_n$ for some $n \neq 0$. Let $a = \widehat{z} := z/|z|$. If $u_n^\infty(\widehat{z}) \neq 0$, then*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \gtrsim s^{1/2}\delta^{-1} \quad \text{as } \delta \to +0. \tag{5.9}$$

For example, if $s^{2-\epsilon} \leq \delta \leq s^{1/2+\epsilon}$ for some $\epsilon > 0$, then resonance occurs, namely,

$$\|\nabla u_\delta\|_{L^2(\Omega)} \to \infty \quad \text{as } \delta \to +0.$$

Proof. Recall that $\Gamma^\omega(x - s^{-1}z) = -\frac{i}{4} H_0^1(\omega|x - s^{-1}z|)$. Since $\omega = s\omega_0$, there are positive constants C_1 and C_2 such that

$$C_1 \leq \omega|x - s^{-1}z| \leq C_2$$

for all s small enough. So, we have

$$|a \cdot \nabla_x \Gamma^\omega(x - s^{-1}z)| = \frac{1}{4} \left| (H_0^1)'(\omega|x - s^{-1}z|) \right| \frac{\omega|(x - s^{-1}z) \cdot a|}{|x - s^{-1}z|} \lesssim s$$

for all $x \in \Omega$ as $s \to 0$. Thus we have $|F_z(x)| \leq C$. One can also see that $|\nabla F_z(x)| \lesssim s$ for all $x \in \Omega$. So, following arguments in the proof of Theorem 8, we prove (i).

In two dimensions a_n is given by

$$a_n = -s^{-1} a \cdot \nabla \mathcal{S}^\omega[\varphi_n](s^{-1}z) + \omega^2 \int_\Omega F_z \, \mathcal{S}[\varphi_n] \, dx.$$

It is known (see [10, (3.63)]) that

$$u_n(y) = \mathcal{S}^\omega[\varphi_n](y) = \frac{e^{i\omega r}}{\sqrt{r}} \left[u_n^\infty(\widehat{y}) + O(r^{-1}) \right],$$

where $r = |y|$. So, we have

$$\partial_r u_n(y) = \left[-\frac{1}{2} \frac{e^{i\omega r}}{r^{3/2}} + i\omega \frac{e^{i\omega r}}{r^{1/2}} \right] u_n^\infty(\widehat{y}) + O(r^{-5/2}),$$

and hence

$$a_n = s^{1/2} \left[\frac{1}{2} \frac{e^{i\omega_0|z|}}{|z|^{3/2}} - i\omega_0 \frac{e^{i\omega_0|z|}}{|z|^{1/2}} \right] u_n^\infty(\widehat{z}) + O(s^{3/2}).$$

This proves (ii). ◻

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