

## MASTER'S THESIS

# Real Space Renormalization Group Transformation from Machine Learning

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# Abstract

Machine learning (ML) is a powerful tool for extracting important features from high dimensional system and performing data compression. The process of capturing relevant features while eliminating irrelevant ones in ML algorithms, such as the neural networks, has a close connection to the renormalization group (RG). Motivated by this, we study the phase transition of the  $q$ -state Potts model using real-space RG transformation obtained from two types of machine learning: supervised and unsupervised learning. We also investigate the RG transformation with vacancy for the first-order phase transition. For supervised learning, we showed that the convolutional neural network (CNN) is capable of classifying phases of the Potts model at different  $q$ . Then, by modifying the structure of the CNN, we are able to obtain RG transformation by training the CNN to classify ordered and disordered Potts configurations. For unsupervised learning, we obtain RG transformation from the restricted Boltzmann machine (RBM) by training it on Potts configurations at transition temperature. Given the RG transformation, we can obtain the critical exponents and estimate the renormalized Hamiltonian using Monte Carlo renormalization group (MCRG) and pseudolikelihood estimation respectively.

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