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Lam, Peter Che Bor; SHIU, Wai Chee; TONG, Chong Sze; Zhang, Zhong Fu

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# On The Equitable Chromatic Number of Complete $n$ -Partite Graphs<sup>1</sup>

Peter Che Bor Lam, Wai Chee Shiu, Chong Sze Tong  
Department of Mathematics, Hong Kong Baptist University  
Kowloon Tong, Hong Kong

and

Zhong Fu Zhang  
Lanzhou Railway Institute,  
Lanzhou, 730070, P.R.China

## Abstract

In this note, we derive an explicit formula for the equitable chromatic number of a complete  $n$ -partite graph  $K_{p_1, p_2, \dots, p_n}$ . Namely, if  $M$  is the largest integer such that

$$p_i \pmod{M} < \left\lceil \frac{p_i}{M} \right\rceil, \quad (i = 1, 2, \dots, n)$$

then

$$\chi_e(K_{p_1, p_2, \dots, p_n}) = \sum_{i=1}^n \left\lceil \frac{p_i}{M+1} \right\rceil,$$

where  $\chi_e(G)$  is the equitable chromatic number of graph  $G$ .

*Key words and phrases:* Complete  $n$ -partite graphs, Equitable chromatic number.

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## 1. Introduction

In this note, all graphs are simple and undirected. A graph  $G = (V, E)$  is said to be *equitably  $k$ -colorable*, if  $V$  may be partitioned into independent sets  $V_1, \dots, V_k$  such that for any  $i \neq j$ ,  $\|V_i\| - \|V_j\| \leq 1$ . The *equitable chromatic number* of  $G$ , denoted as  $\chi_e(G)$ , is defined as the smallest  $k$  such that  $G$  is equitably  $k$ -colorable.

In 1973, W. Meyer [5] proposed the conjecture:

$$\chi_e(G) \leq \Delta(G)$$

for simple graphs  $G$  which are neither complete graphs  $K_p$  nor odd cycles  $C_{2n+1}$ , where  $\Delta(G)$  denotes the maximal degree of  $G$ . In 1970, Hajnal and Szemerédi [3] proved that if  $k > \Delta(G)$ , then  $G$  is equitably  $k$ -colorable. In 1983, B. Bollobás and R.K. Guy verified Meyer's conjecture for trees [1]. Recently, K. W. Lih *et al* [2, 4] proved the validity of Meyer's conjecture for the cases when  $\Delta(G) \leq 3$ ,  $\Delta(G) \geq |V|/2$ , and  $G$  is a bipartite graph. Yap and Zhang [6–8] proved

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Meyer's conjecture for outerplanar graphs, for planar graphs  $G$  with  $\Delta(G) \geq 13$ , and for graphs  $G(V, E)$  with  $\Delta(G) \geq \frac{|V|}{3} + 1$ .

In this note, we derive an explicit formula for the equitable chromatic number of complete  $n$ -partite graphs ( $n \geq 2$ ).

## 2. Main Results

We shall first consider a related combinatoric problem. Let  $n$  natural numbers  $p_1, p_2, \dots, p_n$  be given. For each  $i$ ,  $1 \leq i \leq n$ , we decompose  $p_i$  into  $\lambda_i$  non-negative integers  $p_{ij}$ ,  $j = 1, \dots, \lambda_i$ , such that

$$p_i = \sum_{j=1}^{\lambda_i} p_{ij}$$

and  $|p_{ij} - p_{kl}| \leq 1$  for  $i, k = 1, \dots, n$ ,  $j = 1, \dots, \lambda_i$ , and  $l = 1, \dots, \lambda_k$ . In this way, the natural numbers  $p_1, \dots, p_n$  are said to be  $\lambda$ -equitably partitioned, where

$$\lambda = \sum_{i=1}^n \lambda_i.$$

The minimum value of  $\lambda$  for which  $p_1, \dots, p_n$  may be  $\lambda$ -equitably partitioned is called the *equitable partition number* of  $p_1, \dots, p_n$  and is denoted by  $e(p_1, \dots, p_n)$ .

**Lemma 1** *The equitable partition number of  $p_1, \dots, p_n$  is  $\sum_{i=1}^n \left\lceil \frac{p_i}{M+1} \right\rceil$ , where  $M$  is the largest integer such that*

$$p_i \pmod{M} < \left\lceil \frac{p_i}{M} \right\rceil$$

and  $0 \leq p_i \pmod{M} < M$ , for each  $i = 1, \dots, n$ .

**Proof.** To achieve an equitable partition of the natural numbers  $p_1, p_2, \dots, p_n$ , each part  $p_{ij}$  must be of size  $M$  or  $M + 1$ , for some integer  $M$ . Suppose the number  $p_i$  is decomposed into  $x_i$  numbers  $M$  and  $y_i$  numbers  $M + 1$ . For each  $p_i$ , we write

$$\begin{aligned} p_i &= Mx_i + (M + 1)y_i, \\ &= M(x_i + y_i) + y_i, \end{aligned} \tag{1}$$

$$= (M + 1)(x_i + y_i) - x_i. \tag{2}$$

where  $x_i, y_i \geq 0$ .

Thus  $\lambda_i = x_i + y_i$ . Now if  $x_i \geq M + 1$ , then we may let  $x_i = a(M + 1) + x'_i$ , ( $a > 0$ ) and rewrite

$$\begin{aligned} p_i &= Mx_i + (M + 1)y_i \\ &= Mx'_i + (M + 1)(y_i + Ma) \end{aligned}$$

Hence the size of the partitioning can be decreased to  $\lambda_i = x'_i + y_i + Ma = x_i + y_i - a$ . Thus for the minimum partition, we may assume that  $x_i \leq M$ . Equation (2) then yields

$$\begin{aligned} \left\lceil \frac{p_i}{M + 1} \right\rceil &= x_i + y_i - \left\lfloor \frac{x_i}{M + 1} \right\rfloor \\ &= x_i + y_i. \end{aligned}$$

From equation (1), we have

$$p_i \pmod{M} = y_i \pmod{M} < y_i + x_i + \left\lceil \frac{y_i}{M} \right\rceil = \left\lceil \frac{p_i}{M} \right\rceil. \quad (3)$$

Clearly,  $e(p_1, p_2, \dots, p_n)$  is minimized when  $M$  is maximized. For given  $K_{p_1, \dots, p_n}$ , ( $p_i \neq 0$ ), we select the largest integer  $M$  such that (3) holds for  $i = 1, 2, \dots, n$ . Then

$$\begin{aligned} e(p_1, p_2, \dots, p_n) &= \min \sum_{i=1}^n \lambda_i \quad (\text{w.r.t. } p_i) \\ &= \sum_{i=1}^n (x_i + y_i) \\ &= \sum_{i=1}^n \left\lceil \frac{p_i}{M + 1} \right\rceil. \end{aligned}$$

■

Let us illustrate the proof with an example:  $e(3, 5, 9)$ . Clearly, the minimum set size  $M \leq 3$ . We start by testing the case  $M = 3$ . But,

$$p_2 \pmod{M} = 5 \pmod{3} = 2 \not\leq \left\lceil \frac{p_2}{M} \right\rceil = \left\lceil \frac{5}{3} \right\rceil = 2.$$

Next we try  $M = 2$ , and it is easy to check that (3) holds for  $p_1 = 3$ ,  $p_2 = 5$  and  $p_3 = 9$ .

Hence the equitable partition number  $e(3, 5, 9)$  is  $\left\lceil \frac{3}{3} \right\rceil + \left\lceil \frac{5}{3} \right\rceil + \left\lceil \frac{9}{3} \right\rceil = 6$ . Specifically, the 6 partitions are (3), (2+3), and (3+3+3).

**Corollary 2** *Suppose  $M$  is the largest integer such that*

$$p_i \pmod{M} < \left\lceil \frac{p_i}{M} \right\rceil \quad \text{for } i = 1, \dots, n.$$

*Then*

$$\chi_e(K_{p_1, p_2, \dots, p_n}) = \sum_{i=1}^n \left\lceil \frac{p_i}{M+1} \right\rceil.$$

### 3. Remarks

Let  $G = K_{p_1, p_2, \dots, p_n}$  be a complete  $n$ -partite graph, where  $p_1 \leq p_2 \leq \dots \leq p_n$ . The order of  $G$  is then  $N = p_1 p_2 \cdots p_n$ . The proof of Lemma 1 provides an efficient algorithm for the explicit calculation of the equitable partition number of  $p_1, p_2, \dots, p_n$  and hence the equitable chromatic numbers of  $G$ . Suppose  $M > p_1/2$  and  $M+1 \neq p_1$ . Then (3) does not hold for  $i = 1$ . Hence we need only to consider the case  $M \leq p_1/2$  or  $M+1 = p_1$ . Consequently, the total number of steps required to determine the largest  $M$  to satisfy (3) is approximately  $p_1 n/2$ .

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