

On Wiener numbers of polygonal nets

SHIU, Wai Chee; Lam, Peter Che Bor; Poon, Kin Keung

Published in:
Discrete Applied Mathematics

DOI:
[10.1016/S0166-218X\(01\)00317-1](https://doi.org/10.1016/S0166-218X(01)00317-1)

Published: 15/10/2002

[Link to publication](#)

Citation for published version (APA):
SHIU, W. C., Lam, P. C. B., & Poon, K. K. (2002). On Wiener numbers of polygonal nets. *Discrete Applied Mathematics*, 122(1-3), 251-261. [https://doi.org/10.1016/S0166-218X\(01\)00317-1](https://doi.org/10.1016/S0166-218X(01)00317-1)

General rights

Copyright and intellectual property rights for the publications made accessible in HKBU Scholars are retained by the authors and/or other copyright owners. In addition to the restrictions prescribed by the Copyright Ordinance of Hong Kong, all users and readers must also observe the following terms of use:

- Users may download and print one copy of any publication from HKBU Scholars for the purpose of private study or research
- Users cannot further distribute the material or use it for any profit-making activity or commercial gain
- To share publications in HKBU Scholars with others, users are welcome to freely distribute the permanent publication URLs

On Wiener Numbers of Polygonal Nets*

Wai Chee SHIU[†], Peter Che Bor LAM[†] and Kin Keung POON^{†‡}

Abstract

The Wiener number of a connected graph is equal to the sum of distances between all pairs of its vertices. In this paper, we shall generalize the elementary cuts method to homogeneous n -gonal nets and give a formula to calculate the Wiener numbers of irregular convex triangular hexagons.

Key words and phrases : Graph, distance, polygonal net, Wiener number.

AMS 2000 MSC : 05C12

1 Introduction

An important invariant of connected graphs is the *Wiener number* (or *Wiener index*) W , which is equal to the sum of distances between all pairs of vertices of the respective graph. American physico-chemist Harold Wiener first examined the quantity W in 1947 and 1948 [24–28]. He conceived this index in an attempt to formulate a mathematical model capable of describing molecular shapes. Wiener, and after him numerous researchers reported the existence of correlation between W and a variety of physico-chemical properties of alkanes. For recent reviews on this matter and references to previous work in this area, please see [7–10, 16]. The Wiener number was extensively studied also in the mathematical literature (see, for instance, [4–6, 13, 17–23, 29]). The generalization of the Wiener number, can be found in [3] and [11].

Despite large number of works on the theory of the Wiener number, some basic problems remain unsolved. For example, there is no known recursive method to calculate W of general graphs, especially for polycyclic graphs. In 1995, Shiu *et al.* ([18] published in 1997) introduced the “method of wall”, which leads to recursive expression of the Wiener number of benzenoid hydrocarbons (i.e. hexagonal nets) [18–22]. In 1995, Klavzar, Gutman and Mohar [14] introduced the “elementary cuts method”. It was demonstrated and exemplified how the method work in [12]. Proof of this method can be found in [14] or [23]. By using this method, the Wiener number of irregular hexagonal nets can be found efficiently [23].

Both the elementary cuts method and the method of wall cannot be applied to polycyclic graphs with odd-sided regular polygons, called *cells*. In this paper, we would like to generalize the elemen-

*Partially supported by Research Grant Council Grant, Hong Kong.

[†]Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong.

[‡]Present address is Department of Mathematics, Hong Kong Institute of Education, Tai Po, NT, Hong Kong.

2 Definitions and main results

In this paper, all undefined notations and terminologies are referred to the book [1]. In the following, a plane graph G is called a *homogeneous n -gonal net* or simply an *n -net* if each bounded face of G is a regular n -side polygon. We first define some homogeneous 3-gonal nets, which we call *triangular nets*.

Suppose $\{H_1, H_2, \dots, H_h\}$, $\{R_1, R_2, \dots, R_r\}$ and $\{L_1, L_2, \dots, L_l\}$ are three sets of parallel lines lying on the same plane, where $h, r, l \geq 2$. Moreover the following conditions are satisfied:

- (a) All H -lines run horizontally (East-West). All R -lines run towards North-East (or South-West) cutting an H -line at an angle of $2\pi/3$. All L -lines run towards North-West (or South-East) cutting an H -line at an angle of $2\pi/3$.
- (b) Two successive lines in the same set have unit distance between them when measured along another sets of lines.
- (c) Each line has at least one point which is the intersection of three lines (called *triple intersection point*).

From the sets of lines $\{H_1, H_2, \dots, H_h\}$, $\{R_1, R_2, \dots, R_r\}$ and $\{L_1, L_2, \dots, L_l\}$, a graph is obtained by taking all triple intersection points as vertices and all unit line-segments joining two triple intersection points as edges. This graph is called a *convex triangulation mesh*, and is denoted by $\Delta_{h,r,l}$. Note that there are non-isomorphic convex triangulation meshes having the same parameters [15].

Let G be a plane graph. The *boundary of G* , denoted by ∂G , is the boundary of the unbounded face of G . A convex triangulation mesh $\Delta_{h,r,l}$ is called a *triangular trapezium* if its boundary forms a trapezium. In this case, $r = l$ and the length of the longer base and the height are $r - 1$ and $h - 1$ respectively. We denote this triangular trapezium by $T_{h-1,r-1}$. When $h = r$, $T_{h-1,r-1}$ becomes a *triangular triangle* which is simply denoted by T_{h-1} . For $n \geq 1$, $T_{1,n}$ is called a *($2n - 1$)-triangular chain* and is denoted by I_{2n-1} .

$\Delta_{h,r,l}$ is called a *triangular parallelogram* if its boundary forms a parallelogram. In this case, $|r - l| = h - 1$ and $\Delta_{h,r,l}$ is isomorphic to $\Delta_{h,l,r}$. In the following, we shall always assume that $l > r$ and denote the triangular parallelogram $\Delta_{h,r,h+r-1}$ by $P_{h-1,r-1}$, which is sometimes called a *($h - 1$) \times ($r - 1$)-triangular parallelogram*. For $n \geq 1$, $P_{1,n}$ is called a *$2n$ -triangular chain*, denoted by I_{2n} .

Definition 2.1: A graph obtained by merging the longer base of $T_{p,n}$ and $T_{q,n}$ with two bases of $P_{m,n}$ forming a convex 6-sided polygon, is called an *irregular convex triangular hexagon* and denoted by $H_{q,p,m,n}$ (see Figure 1), where $0 \leq p, q \leq n$ and $m \geq 0$. By reflection it is easy to see that $H_{q,p,m,n} \cong H_{p,q,m,n}$. Therefore, we always assume that $q \geq p$.

Note that $H_{q,0,0,n} = T_{q,n} \cong T_{n,q}$; $H_{n,0,0,n} = T_n$; $H_{0,0,m,n} = P_{m,n} \cong P_{n,m}$; $H_{1,0,0,n} = I_{2n-1}$ and $H_{0,0,1,n} = I_{2n}$. Some triangular nets defined above can be seen in Figure 1.

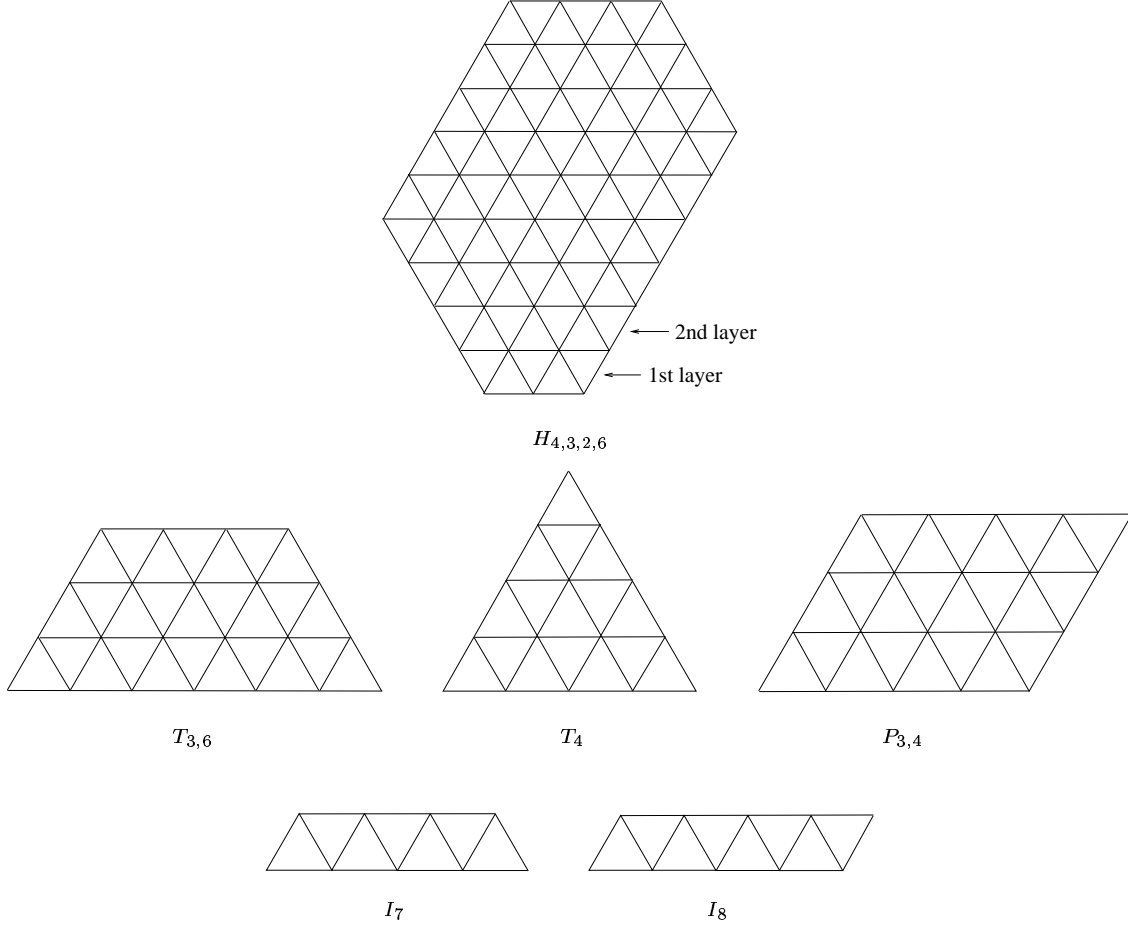


Figure 1: Some triangular nets.

We shall introduce the generalized elementary cut of a homogeneous polygonal net and use it to compute the Wiener numbers of irregular convex triangular hexagons.

Definition 2.2: Let H be an n -net and let e be an edge in a cell of H . We define the k^- -edge of e to be the k -th edge e' in the same cell counting in clockwise direction from e , whereas the k^+ -edge of e to be the k -th edge e'' in the same cell counting in anti-clockwise direction from e .

Remark: Suppose that G is an n -net. If $e \in E(G) \setminus \partial G$, then e is incident with exactly two cells of G . If $e \in \partial G$, then e is incident with exactly one cell of G and the unbounded face.

Definition 2.3 [Generalized elementary cut]:

Let G be an n -net and $s = \lfloor \frac{n}{2} \rfloor$. For any $e \in \partial G$, we shall define two generalized elementary cuts $C^+(e)$ and $C^-(e)$. A *cutline* for $C^+(e)$ is obtained by joining a number of line segments. The first of such line segments, say L^+ , links the mid-point M of e with the mid-point of e' , where e' is the s^+ -edge of e . If $e' \notin \partial G$, then e' belongs to another cell of G , say P . In P , draw a straight line segment L^- through the mid-point of e' and the mid-point of e'' , where e'' is the s^- -edge of e' . If $e'' \in \partial G$, then stop. Otherwise, continue the process with alternate orientations until it reaches the boundary (say at the point N).

The other generalized elementary cut, starting at the edge e with reverse the orientation, i.e. firstly, joining e to s^- -edge of e and continue with alternate orientation.

Finally, let C be a polygonal path joined by M and N followed by the line segments defined above. Note that by elementary geometry it is easy to see that C is a straight line. Please see Appendix for details. C with endpoints M and N is called a *generalized elementary cut pertaining to e* . Identify the generalized elementary cut C with the set of edges of G which are crossed by C . The set of all generalized elementary cuts of G is denoted by $\mathcal{C}(G)$.

Remarks: Note that if n is even, then $L^+ = L^-$ and hence the generalized elementary cut is the same as elementary cut described in [12] or [23].

Let C be a generalized elementary cut of G . Then $G - C$ consists of two components denoted by $G'(C)$ and $G''(C)$ respectively. Let e_1, e_2 be edges cut by C . Let m_i be the intersection point of e_i and C , $i = 1, 2$. Define the *distance* between e_1, e_2 , denoted by $\rho(e_1, e_2)$, to be the number of cells crossed by the line segment m_1m_2 . For example, if e_1 and e_2 lie in the same cell, then $\rho(e_1, e_2) = 1$.

Lemma 2.1 *Suppose G is an n -net. Let $C \in \mathcal{C}(G)$ and let $e_1 = u_1v_1, e_2 = u_2v_2$ be edges cut by C . Suppose u_1 and u_2 lie in $V(G'(C))$. Then there is a unique shortest (u_1, u_2) -path in G which lies in $G'(C)$. Moreover, this shortest path lies in the subgraph formed by the cells crossed by C .*

Proof: Proof by induction on $\rho(e_1, e_2)$. Suppose $\rho(e_1, e_2) = 1$. Let H be the cell that contains e_1 and e_2 . It is clear that there is a (u_1, u_2) -path, say P , of length $s = \lfloor \frac{n}{2} \rfloor$ lying in $G'(C) \cap H$. Let Q be a shortest (u_1, u_2) -path, then either the boundary of the subgraph $H \cup Q$ or $P \cup Q$ forms a regular polygon of size at most n . But this polygon contains another regular n -side polygon H . It is impossible unless $P = Q$.

The inductive step is easy to see. □

We have the following corollary.

Corollary 2.1 Suppose G is an n -net. Let $C \in \mathcal{C}(G)$ and let u, v be any two vertices of G . For any shortest (u, v) -path $P_{u,v}$, $|E(P_{u,v}) \cap C| = 0$, if u and v belong to the same component of $G - C$, otherwise, $|E(P_{u,v}) \cap C| = 1$.

Theorem 2.1 Let G be an n -net, where n is odd. Further assume that ∂G is connected. Then the Wiener number of G , $W(G)$ is

$$\frac{1}{2} \sum_{C \in \mathcal{C}(G)} |G'(C)||G''(C)| = \frac{1}{2} \sum_{C \in \mathcal{C}(G)} |G'(C)|\{|G| - |G'(C)|\}$$

where $|H|$ denotes the order of H .

Proof: Each edge $e \in E(G)$ belongs to exactly two generalized elementary cuts, since there are exactly two distinct directions which passing through e .

For $u, v \in V(G)$, let $P_{u,v}$ be a shortest (u, v) -path in G . Since each edge of G belongs to exactly two generalized elementary cuts, it follows that

$$W(G) = \sum_{\{u,v\} \in S} |E(P_{u,v})| = \frac{1}{2} \sum_{C \in \mathcal{C}(G)} \sum_{\{u,v\} \in S} |E(P_{u,v}) \cap C|,$$

where S is the set of all unordered pairs of $V(G)$. By Corollary 2.1, we have

$$\sum_{\{u,v\} \in S} |E(P_{u,v}) \cap C| = |G'(C)||G''(C)|,$$

and hence we have our conclusion. □

We acknowledge that V. Chepoi *et al.* [2] obtained the result on Wiener number of L_1 graphs (namely, they are isometrically embeddable into hypercube). The case of polygonal nets are independently discovered by us with different consideration. The notion of generalized elementary cut corresponds to alternating cut.

3 Wiener numbers of convex triangular nets

Let $G = H_{q,p,m,n}$. Then the order of G is $(n+1)(m+q+p+1) - \frac{1}{2}[p(p+1) + q(q+1)]$. We partition $\mathcal{C}(G)$ into three subsets $\mathcal{C}_H(G)$, $\mathcal{C}_L(G)$, and $\mathcal{C}_R(G)$, which are the sets of generalized elementary cuts whose cutlines are parallel to the H -lines, L -lines and R -lines, respectively. Then

$$W(G) = \frac{1}{2}(f_H + f_L + f_R)$$

where

$$\begin{aligned}
f_H &= \sum_{C \in \mathcal{C}_H(G)} |G'(C)|(|G| - |G'(C)|) = f(q, p, m, n); \\
f_L &= \sum_{C \in \mathcal{C}_L(G)} |G'(C)|(|G| - |G'(C)|); \\
f_R &= \sum_{C \in \mathcal{C}_R(G)} |G'(C)|(|G| - |G'(C)|).
\end{aligned}$$

We shall first compute f_H . Let C_z be the generalized elementary cut of G which passes through the z -th layer of G (the bottom layer is the first layer). Let $G'(C_z)$ be the lower component of $G - C_z$. Then $|G'(C_z)|$ is equal to

$$\begin{cases} \sum_{i=0}^{z-1} (n - q + 1 + i) & \text{if } 1 \leq z \leq q, \\ \sum_{i=0}^{q-1} (n - q + 1 + i) + \sum_{j=q}^{z-1} (n + 1) & \text{if } q \leq z \leq m + q, \\ \sum_{i=0}^{q-1} (n - q + 1 + i) + \sum_{j=q}^{m+q-1} (n + 1) + \sum_{k=m+q}^{z-1} (n + 1 + m + q - k) & \text{if } m + q \leq z \leq m + q + p. \end{cases}$$

After simplifying we have

$$|G'(C_z)| = \begin{cases} \frac{1}{2}(1 + 2n - 2q + z)z & \text{if } 1 \leq z \leq q, \\ \frac{1}{2}[2(n + 1)z - q(q + 1)] & \text{if } q \leq z \leq m + q, \\ \frac{1}{2}[(2n + 2m + 2q + 3 - z)z - 2q(m + q + 1) - m(m + 1)] & \text{if } m + q \leq z \leq m + q + p. \end{cases}$$

Let $f_1(q, p, m, n) = \sum_{z=1}^q |G'(C_z)|(|G| - |G'(C_z)|)$, $f_2(q, p, m, n) = \sum_{z=q+1}^{m+q} |G'(C_z)|(|G| - |G'(C_z)|)$ and $f_3(q, p, m, n) = \sum_{z=m+q+1}^{m+q+p} |G'(C_z)|(|G| - |G'(C_z)|)$. Then by symmetry, $f_3(q, p, m, n) = f_1(p, q, m, n)$.

After computation we have

$$\begin{aligned}
f_1(q, p, m, n) &= \frac{1}{60} \{q(1 + q)(18 + 20m + 40n + 50mn + 20n^2 + 30mn^2 + 10p + 35np + 30n^2p - 10p^2 - 15np^2 - 13q - 20mq - 20mnq + 10n^2q - 10pq - 20npq + 10p^2q - 7q^2 - 10nq^2 + 2q^3)\}. \\
f_2(q, p, m, n) &= \frac{1}{12} \{m(4 + 6m + 2m^2 + 8n + 12mn + 4m^2n + 4n^2 + 6mn^2 + 2m^2n^2 + 3p + 3mp + 9np + 9mnp + 6n^2p + 6mn^2p - 3p^2 - 3mp^2 - 3np^2 - 3mnp^2 + 3q + 3mq + 9nq + 9mnq + 6n^2q + 6mn^2q + 3pq + 12npq + 12n^2pq - 3p^2q - 6np^2q - 3q^2 - 3mq^2 - 3nq^2 - 3mnq^2 - 3pq^2 - 6npq^2 + 3p^2q^2)\}.
\end{aligned}$$

Then

$$\begin{aligned}
f(q, p, m, n) = & \frac{1}{60} \{ 20m + 30m^2 + 10m^3 + 40mn + 60m^2n + 20m^3n + 20mn^2 + 30m^2n^2 + 10m^3n^2 + \\
& 18p + 35mp + 15m^2p + 40np + 95mnp + 45m^2np + 20n^2p + 60mn^2p + 30m^2n^2p + \\
& 5p^2 - 15mp^2 - 15m^2p^2 + 40np^2 + 15mnp^2 - 15m^2np^2 + 30n^2p^2 + 30mn^2p^2 - 20p^3 - \\
& 20mp^3 - 10np^3 - 20mnp^3 + 10n^2p^3 - 5p^4 - 10np^4 + 2p^5 + 18q + 35mq + 15m^2q + \\
& 40nq + 95mnq + 45m^2nq + 20n^2q + 60mn^2q + 30m^2n^2q + 20pq + 15mpq + 70npq + \\
& 60mnpq + 60n^2pq + 60mn^2pq - 10p^2q - 15mp^2q - 30mnp^2q + 30n^2p^2q - 10p^3q - \\
& 20np^3q + 5q^2 - 15mq^2 - 15m^2q^2 + 40nq^2 + 15mnq^2 - 15m^2nq^2 + 30n^2q^2 + 30mn^2q^2 - \\
& 10pq^2 - 15mpq^2 - 30mnpq^2 + 30n^2pq^2 + 15mp^2q^2 - 30np^2q^2 + 10p^3q^2 - 20q^3 - 20mq^3 - \\
& 10nq^3 - 20mnq^3 + 10n^2q^3 - 10pq^3 - 20npq^3 + 10p^2q^3 - 5q^4 - 10nq^4 + 2q^5 \}.
\end{aligned}$$

Similar expression may be obtained for f_L and f_R after rotating G for $2\pi/3$ and $4\pi/3$ radians respectively. In fact,

$$\begin{aligned}
f_L = & \begin{cases} f(m+q, m+p, n-m-q-p, m+q+p) & \text{if } q+p \leq n-m, \\ f(n-p, n-q, m+q+p-n, n) & \text{if } q+p \geq n-m. \end{cases} \\
f_R = & \begin{cases} f(q, p, n-q-p, m+q+p) & \text{if } q+p \leq n, \\ f(n-p, n-q, q+p-n, m+n) & \text{if } q+p \geq n. \end{cases}
\end{aligned}$$

After some computation, we obtain the following formula for the Wiener number of an irregular convex triangular hexagon $H_{q,p,m,n}$.

Theorem 3.1

1. If $q+p \leq n-m$, then the Wiener number of $H_{q,p,m,n}$ is

$$\begin{aligned}
& \frac{1}{120} \{ 36m + 60m^2 + 25m^3 - m^5 + 40n + 130mn + 125m^2n + 40m^3n + 5m^4n + 60n^2 + 140mn^2 + \\
& 90m^2n^2 + 10m^3n^2 + 20n^3 + 40mn^3 + 20m^2n^3 + 14p + 60mp + 45m^2p - 5m^4p + 70np + 150mnp + \\
& 90m^2np + 20m^3np + 110n^2p + 150mn^2p + 30m^2n^2p + 40n^3p + 40mn^3p + 5p^2 - 10mp^2 - 30m^2p^2 - \\
& 10m^3p^2 + 30n^2p^2 + 20n^3p^2 - 10p^3 - 30mp^3 - 10m^2p^3 - 20np^3 - 20n^2p^3 - 5p^4 - 10mp^4 + 10np^4 - \\
& 4p^5 + 14q + 60mq + 45m^2q - 5m^4q + 70nq + 150mnq + 90m^2nq + 20m^3nq + 110n^2q + 150mn^2q + \\
& 30m^2n^2q + 40n^3q + 40mn^3q + 40pq + 60mpq - 20m^3pq + 80npq + 120mnpq + 60m^2npq + 120n^2pq + \\
& 60mn^2pq + 40n^3pq + 10p^2q - 30mp^2q - 30m^2p^2q - 20p^3q - 20mp^3q - 10p^4q + 5q^2 - 10mq^2 - \\
& 30m^2q^2 - 10m^3q^2 + 30n^2q^2 + 20n^3q^2 + 10pq^2 - 30mpq^2 - 30m^2pq^2 - 10q^3 - 30mq^3 - 10m^2q^3 - \\
& 20nq^3 - 20n^2q^3 - 20pq^3 - 20mpq^3 - 5q^4 - 10mq^4 + 10nq^4 - 10pq^4 - 4q^5 \}.
\end{aligned}$$

2. If $n-m \leq q+p \leq n$, then the Wiener number of $H_{q,p,m,n}$ is

$$\begin{aligned}
& \frac{1}{120} \{ 40m + 60m^2 + 20m^3 + 36n + 130mn + 140m^2n + 40m^3n + 60n^2 + 125mn^2 + 90m^2n^2 + 20m^3n^2 + \\
& 25n^3 + 40mn^3 + 10m^2n^3 + 5mn^4 - n^5 + 18p + 60mp + 30m^2p + 70np + 180mnp + 90m^2np + 95n^2p + \\
& 150mn^2p + 60m^2n^2p + 40n^3p + 20mn^3p + 5n^4p + 5p^2 - 25mp^2 - 30m^2p^2 + 15np^2 - 30m^2np^2 + \\
& 30n^2p^2 + 30mn^2p^2 + 10n^3p^2 - 15p^3 - 30mp^3 - 20np^3 - 20mnp^3 - 10n^2p^3 - 5p^4 - 5mp^4 + 5np^4 - 3p^5 + \\
& 18q + 60mq + 30m^2q + 70nq + 180mnq + 90m^2nq + 95n^2q + 150mn^2q + 60m^2n^2q + 40n^3q + 20mn^3q +
\end{aligned}$$

$$5n^4q + 40pq + 30mpq + 110npq + 120mnpq + 120n^2pq + 120mn^2pq + 20n^3pq - 5p^2q - 30mp^2q - 60mnp^2q + 30n^2p^2q - 20p^3q - 20np^3q - 5p^4q + 5q^2 - 25mq^2 - 30m^2q^2 + 15nq^2 - 30m^2nq^2 + 30n^2q^2 + 30mn^2q^2 + 10n^3q^2 - 5pq^2 - 30mpq^2 - 60mnpq^2 + 30n^2pq^2 + 30mp^2q^2 - 30np^2q^2 + 10p^3q^2 - 15q^3 - 30mq^3 - 20nq^3 - 20mnq^3 - 10n^2q^3 - 20pq^3 - 20npq^3 + 10p^2q^3 - 5q^4 - 5mq^4 + 5nq^4 - 5pq^4 - 3q^5\}.$$

3. If $n \leq q + p$, then the Wiener number of $H_{q,p,m,n}$ is

$$\frac{1}{120}\{40m + 60m^2 + 20m^3 + 32n + 130mn + 140m^2n + 40m^3n + 60n^2 + 125mn^2 + 90m^2n^2 + 20m^3n^2 + 30n^3 + 40mn^3 + 10m^2n^3 + 5mn^4 - 2n^5 + 22p + 60mp + 30m^2p + 70np + 180mnp + 90m^2np + 80n^2p + 150mn^2p + 60m^2n^2p + 40n^3p + 20mn^3p + 10n^4p + 5p^2 - 25mp^2 - 30m^2p^2 + 30np^2 - 30m^2np^2 + 30n^2p^2 + 30mn^2p^2 - 20p^3 - 30mp^3 - 20np^3 - 20mnp^3 - 5p^4 - 5mp^4 - 2p^5 + 22q + 60mq + 30m^2q + 70nq + 180mnq + 90m^2nq + 80n^2q + 150mn^2q + 60m^2n^2q + 40n^3q + 20mn^3q + 10n^4q + 40pq + 30mpq + 140npq + 120mnpq + 120n^2pq + 120mn^2pq - 20p^2q - 30mp^2q - 60mnp^2q + 60n^2p^2q - 20p^3q - 40np^3q + 5q^2 - 25mq^2 - 30m^2q^2 + 30nq^2 - 30m^2nq^2 + 30n^2q^2 + 30mn^2q^2 - 20pq^2 - 30mpq^2 - 60mnpq^2 + 60n^2pq^2 + 30mp^2q^2 - 60np^2q^2 + 20p^3q^2 - 20q^3 - 30mq^3 - 20nq^3 - 20mnq^3 - 20pq^3 - 40npq^3 + 20p^2q^3 - 5q^4 - 5mq^4 - 2q^5\}.$$

By considering some special cases, we have the following theorems.

Theorem 3.2 The Wiener number of I_n is

$$\begin{cases} \frac{1}{24}(n+1)(n+3)(2n+7) & \text{for odd } n, \\ \frac{1}{24}(n+2)(n+4)(2n+3) & \text{for even } n. \end{cases}$$

Theorem 3.3 The Wiener number of a triangular triangle T_m is

$$\frac{1}{40}m(m+1)(m+2)(m+3)(2m+3).$$

Since the Wiener number is invariant under rotation and reflection, so we may simply assume that $m \leq n$ for the triangular parallelogram $P_{m,n}$ and triangular trapezium $T_{m,n}$.

Theorem 3.4 The Wiener number of $P_{m,n}$, where $m \leq n$, is

$$\frac{1}{120}(m+1)\{-m^4 + (5n+1)m^3 + (10n^2+35n+24)m^2 + (20n^3+80n^2+90n+36)m + 20n(n+1)(n+2)\}.$$

Theorem 3.5 The Wiener number of $T_{m,n}$, where $m \leq n$, is

$$\frac{1}{120}(m+1)\{-4m^4 + (10n-1)m^3 - (20n^2+30n+9)m^2 + (20n^3+50n^2+30n+14)m + 20n(n+1)(n+2)\}.$$

Remarks: The method always works in the case of a plane graph formed by n -side regular polygons with ∂G is connected. If a plane graph consisting of n -gons that are not regular, then some elementary cuts may cross itself (see Figure 2). In general, no recursive method is known for calculating the Wiener number of general odd polycyclic graphs.

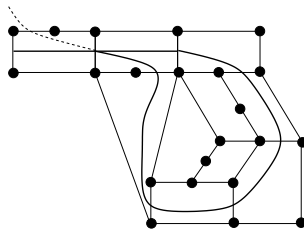


Figure 2

Appendix

Consider a homogeneous n -gonal net G , without loss of generality, we can regard G as a net consisting of two polygons P_1 and P_2 only.

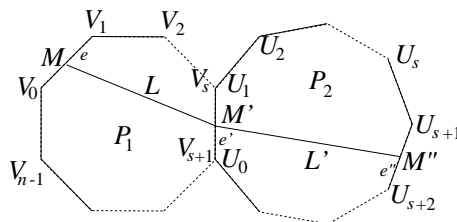


Figure 3

Starting with the edge $e \in P_1$ and counting with s^+ -step, where $s = \lfloor \frac{n}{2} \rfloor$ (assume), let M and M' be the mid-point of $V_1V_0 = e$ and $V_sV_{s+1} = e'$ respectively and L be the line segment that joining M and M' . By symmetry, $\angle V_1MM' = \angle V_sM'M$. In the polygon P_2 , relabeling the vertices V_{s+1} to U_0 and V_s to U_1 (see the diagram). Starting with the edge $e' \in P_2$ and counting with s^- -step, let L' be the line segment joining M' and M'' , where M'' is the mid-point of $U_{s+1}U_{s+2} = e''$. By symmetry, $\angle U_0M'M'' = \angle U_{s+2}M''M$. In addition, the polygon $MV_1V_2 \dots V_sM'$ is congruent to the polygon $M'U_0U_{n-1} \dots U_{s+2}M''$ and hence $\angle V_sM'M = \angle V_{s+1}M'M''$. In conclusion, $C = L \cup L'$ is a straight line. \square

References

- [1] J. A Bondy and U. S. R. Murty, Graph theory with applications, *American Elsevier*, New York, (1976).
- [2] V. Chepoi, M. Deza and V. Grishukhin, Clin d'oeil on L_1 -embeddable planar graphs, *Discrete Appl. Math.*, **80** (1997), 3-19.
- [3] A.A. Dobrynin and I. Gutman, Solving problems connected with distances in graphs, *Graph Theory Notes of New York*, **XXVIII (5)** (1994), 21-23.

- [4] A.A. Dobrynin and I. Gutman, Wiener index for trees and graphs of hexagonal systems (in Russian), *Diskret. Anal. Issled. Oper. (Novosibirsk) Ser. 2*, **5(2)** (1998), 34-60.
- [5] A.A. Dobrynin, R. Entringer and I. Gutman, Wiener index of trees: theory and applications, *Acta Appl. Math.*, to appear.
- [6] R. C. Entringer, D. E. Jackson, D. A. Snyder, Distance in graphs, *Czech. Math. J.*, **26** (1976), 283-296.
- [7] I. Gutman and O. E. Polansky, Wiener numbers of polyacenes and related benzenoid molecules, *Commun. Math. Chem. (MATCH)*, **20** (1986), 115-123.
- [8] I. Gutman, Wiener numbers of benzenoid hydrocarbons: two theorems, *Chem. Phys. Lett.*, **136** (1987), 134-136.
- [9] I. Gutman, S. Markovic, D. Lukovic, V. Radivojevic and S. Rancic, On Wiener numbers of benzenoid hydrocarbons, *Coll. Sci. Papers Fac. Sci. Kragujevac*, **8** (1987), 15-34.
- [10] I. Gutman, J. W. Kennedy and L. V. Quintas, Wiener numbers of random benzenoid chains, *Chem. Phys. Lett.*, **173** (1990), 403-408.
- [11] I. Gutman, Y. N. Yeh, S. L. Lee and Y. L. Luo, Recent results in the theory of the Wiener number, *Indian J. Chem.*, **32A** (1993), 651-661.
- [12] I. Gutman and S. Klavzar, A method for calculating Wiener numbers of benzenoid hydrocarbons, *Models in Chem.*, **133(4)**, (1996), 389-399.
- [13] F. Harary, Status and contrastatus, *Sociometry*, **22** (1959), 23-43.
- [14] S. Klavzar, I. Gutman and B. Mohar, Labeling of benzenoid systems which reflects the vertex-distance relations, *J. Chem. Inf. Comput. Sci.*, **35** (1995), 590-593.
- [15] P. C. B. Lam, W. C. Shiu and W. H. Chan, On the bandwidth of convex triangulation meshes, *Discrete Math.*, **173** (1997), 285-289.
- [16] Z. Mihalic, D. Veljan, D. Amic, S. Nikolic, D. Plavsic and D. Trinajstic, The distance matrix in chemistry, *J. Math. Chem. Eng.*, **11** (1992), 223-258.
- [17] J. Plesnik, On the sum of all distances in a graph or digraph, *J. Graph Theory*, **8** (1984), 1-21.
- [18] W. C. Shiu, P. C. B. Lam, The Wiener number of hexagonal nets, *Discrete Appl. Math.*, **73** (1997), 101-111.

- [19] W. C. Shiu, P. C. B. Lam and I. Gutman, Wiener number of hexagonal parallelograms, *Graph Theory Notes of New York*, **XXX:6** (1996), 21-25.
- [20] W. C. Shiu, C. S. Tong and P. C. B. Lam, Wiener number of some polycyclic graphs, *Graph Theory Notes of New York*, **XXXII:2** (1997), 10-15.
- [21] W. C. Shiu, P. C. B. Lam and I. Gutman, Wiener number of hexagonal bitrapeziums and trapeziums, *Bull. Acad. Serbe Sci. Arts. Cl. Sci. Math. Natur.*, **22** (1997), 9-25.
- [22] W. C. Shiu, C. S. Tong and P. C. B. Lam, Wiener number of hexagonal jagged-rectangles, *Discrete Appl. Math.*, **80** (1997), 83-96.
- [23] W. C. Shiu and P. C. B. Lam, Wiener numbers of pericondensed benzenoid molecule systems, *Congr. Numer.*, **126** (1997) 113-124.
- [24] H. Wiener, Structural determination of Paraffin boiling points, *J. Amer. Chem. Soc.*, **69** (1947), 17-20.
- [25] H. Wiener, Correlation of heats of isomerization and differences in heats of vaporization of isomers among the paraffin hydrocarbons, *J. Amer. Chem. Soc.*, **69** (1947), 2636-2638.
- [26] H. Wiener, Influence of interatomic forces on paraffin properties, *J. Chem. Phys.*, **15** (1947), 766-766.
- [27] H. Wiener, Vapor pressure-temperature relationships among the branched paraffin hydrocarbons, *J. Phys. Chem.*, **52** (1948), 425-430.
- [28] H. Wiener, Relation of physical properties of the isomeric alkanes to molecular structure, *J. Phys. Chem.*, **52** (1948), 1082-1089.
- [29] Y. N. Yeh, I. Gutman, On the sum of all distances in composite graphs, *Discrete Appl. Math.*, **135** (1994), 359-365.