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Bounding the sum of powers of normalized Laplacian eigenvalues of a graph*

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Abstract Let G be a simple connected graph of order n. Its normalized Laplacian eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$. In this paper, new bounds on $S^*_\beta(G) = \sum_{i=1}^{n-1} \lambda_i^\beta$ $(\beta \neq 0, 1)$ are derived.

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1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Its *order* is $|V(G)|$, denoted by n, and its *size* is $|E(G)|$, denoted by m. For $v \in V(G)$, let $d(v)$ be the degree of v. The maximum and the minimum degrees of G are denoted by Δ and δ, respectively.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The *Laplacian* and *normalized Laplacian* matrices of G are defined as $L = L(G) = D(G) - A(G)$ and $\mathcal{L} = \mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$, respectively. Note that $L(G)$ and $\mathcal{L}(G)$ are symmetric positive semidefinite matrices and 0 is the smallest eigenvalue of $L(G)$ and $\mathcal{L}(G)$, respectively. Let

$$\mu_1(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0 \quad \text{and} \quad \lambda_1(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0$$

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be the Laplacian eigenvalues and the normalized Laplacian eigenvalues of G, respectively. When only one graph G is under consideration, we sometimes write μ_k and λ_k instead of $\mu_k(G)$ and $\lambda_k(G)$ for $1 \le k \le n$, respectively. For more information on the eigenvalues of $L(G)$ and $\mathcal{L}(G)$, can be found in [4–6, 8, 9].

For a connected graph G of order n and a real number $\beta \neq 0$, Zhou [16] defined the sum of the β-th power of the non-zero Laplacian eigenvalues of G as

$$S_\beta(G) = \sum_{i=1}^{n-1} \mu_i^\beta. \tag{1.1}$$

For basic properties on $S_\beta(G)$, including various upper and lower bounds, relations between $S_\beta(G)$ and the incidence energy, the Kirchhoff index and the Laplacian Estrada index, respectively, see [11, 16, 17].

Very recently, in full analogy with (1.1), Bozkurt *et al.* [2] defined the sum of the β-th power of the non-zero normalized Laplacian eigenvalues of a connected graph G of order n as

$$S_\beta^*(G) = \sum_{i=1}^{n-1} \lambda_i^\beta.$$

The cases $\beta = 0$ and $\beta = 1$ are trivial as $S_0^*(G) = n - 1$ and $S_1^*(G) = n$, respectively. Moreover, $2mS_{-1}(G)$ is equal to the degree Kirchhoff index $Kf^*(G)$ of G [7]. There is an interesting relation between $S_\beta^*(G)$ and the general Randić index, $R_\alpha(G)$, of G defined by

$$R_\alpha = R_\alpha(G) = \sum_{uv \in E(G)} (d(u)d(v))^\alpha,$$

where the summation is over all (unordered) edges uv in G, and $\alpha \neq 0$ is a fixed real number [3]. Note that $S_2^*(G)$ is equal to the trace of $\mathcal{L}^2$, from which it is well known that [6]

$$\sum_{i=1}^{n-1} \lambda_i^2 = n + 2R_{-1}(G).$$

For more basic properties on $S_\beta^*(G)$, including upper and lower bounds, see [1, 2]. And information on $R_{-1}(G)$ and its importance to the normalized Laplacian eigenvalues, see [6, 7].

In this paper, we derive some new bounds on $S_\beta(G)$ ($\beta \neq 0, 1$). Some of those recover or improve the previous bounds in [2]. More-over, we consider the case $\beta = -1$ which gives some new bounds on degree Kirchhoff index $Kf^*(G)$.

2 Preliminaries

For a connected graph G of order n, let

$$P = 1 + \sqrt{\frac{2R_{-1}}{n(n-1)}} \quad \text{and} \quad Q = \det(D(G)) = \prod_{u \in V(G)} d(u).$$

We begin with introducing some basic properties on the normalized Laplacian eigenvalues of a connected graph G.

Lemma 2.1 ([8]) *Let G be a connected graph of order n. Then $0 \le \lambda_i \le 2$ for $i = 1, 2, \ldots, n$. Moreover, $\lambda_1 = 2$ if and only if G is bipartite; $\lambda_1 \ge \frac{n}{n-1}$ with equality if and only if $G \cong K_n$.*

Lemma 2.2 (**[10]**) *Let G be a connected graph of order n. Then $\lambda_1 \geq P$. Moreover, the equality holds if and only if $G \cong K_n$.*

Lemma 2.3 (**[10]**) *Let G be a connected graph of order $n > 2$. Then*

(i) $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$ if and only if $G \cong K_n$;

(ii) $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_{p,n-p}$.

Lemma 2.4 (**[6]**) *Let G be a connected graph of order n with m edges. Then the number of spanning trees t of G is given as*

$$t = \frac{Q}{2m} \prod_{i=1}^{n-1} \lambda_i.$$

The following result concerning the relationship between the μ_i and λ_i for $1 \leq i \leq n$.

Lemma 2.5 (**[5,6]**) *Let G be a connected graph of order $n \geq 3$ with maximum degree Δ and minimum degree δ. Then for each $1 \leq i \leq n$,*

$$\frac{\mu_i}{\Delta} \leq \lambda_i \leq \frac{\mu_i}{\delta}.$$

Equality occurs in both bounds if and only if G is a regular graph.

Remark 2.1 *Note that for any connected graph G of order n, $\mu_1 \geq \Delta + 1$ with equality if and only if $\Delta = n-1$ [4]. This together with Lemma 2.5 imply that $\lambda_1 \geq \frac{\Delta+1}{\Delta}$, the equality holds if and only if $G \cong K_n$.*

Lemma 2.6 (**[6]**) *Let G be a connected graph of order $n \geq 3$ with maximum degree Δ and minimum degree δ. Then*

$$\frac{n}{2\Delta} \leq R_{-1}(G) \leq \frac{n}{2\delta}.$$

Equality occurs in both bounds if and only if G is a regular graph.

Remark 2.2 *Recall that $P = 1 + \sqrt{\frac{2R_{-1}}{n(n-1)}}$. This together with Lemma 2.6 imply that*

$$P = 1 + \sqrt{\frac{2R_{-1}}{n(n-1)}} \geq 1 + \sqrt{\frac{1}{(n-1)\Delta}} \geq \frac{n}{n-1} \quad as \quad \Delta \leq n-1.$$

Lemma 2.7 (**[13]**) *Let $a_i > 0$, $i = 1, 2, \ldots, p$, be p real numbers. Then*

$$p(A_p - G_p) \geq (p-1)(A_{p-1} - G_{p-1}), \tag{2.2}$$

where $A_p = \frac{\sum_{i=1}^{p} a_i}{p}$ and $G_p = \left(\prod_{i=1}^{p} a_i\right)^{1/p}$.

Lemma 2.8 (**[12]**) *For $a_1, a_2, \ldots, a_n \geq 0$ and $p_1, p_2, \ldots, p_n \geq 0$ such that $\sum_{i=1}^{n} p_i = 1$,*

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq n\lambda \left(\frac{1}{n} \sum_{i=1}^{n} a_i - \prod_{i=1}^{n} a_i^{1/n} \right), \tag{2.3}$$

where $\lambda = \min\{p_1, p_2, \ldots, p_n\}$. Moreover, the equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Some structural properties on graphs have exactly three distinct normalized Laplacian eigenvalues were explored in [6]. The following result concentrating the case of regular graphs.

Lemma 2.9 (**[6]**) *Let G be a connected regular graph of order n. Then G has exactly three distinct normalized Laplacian eigenvalues if and only if G is strongly regular.*

3 Main results

In this section, more bounds on $S_\beta^*(G)$ ($\beta \neq 0, 1$) are deduced. Some of those recover or improve the previous bounds in [2].

Let G be a connected graph with normalized Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$. For $1 \leq k \leq n-2$, let

$$M_k = \sum_{i=1}^{k} \lambda_i.$$

We begin with deriving some bounds on M_k. It is known that

$$M_k \geq \sum_{i=1}^{k} 1 = k \quad \text{for} \quad 1 \leq k \leq n-2, \tag{3.4}$$

which is a consequence of Schur's Theorem [14] stating that the spectrum of any symmetric, positive definite matrix majorizes its main diagonal. The following is an improvement of (3.4).

Lemma 3.1 *Let G be a connected graph of order n. Then $M_k \geq \frac{nk}{n-1}$. The equality is attained for $G \cong K_n$.*

Proof. Note that

$$\frac{M_k}{k} = \frac{\sum_{i=1}^{k} \lambda_i}{k} \geq \frac{\sum_{i=k+1}^{n-1} \lambda_i}{n-1-k} = \frac{n - M_k}{n-1-k} \quad \text{as} \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \quad \text{and} \quad \sum_{i=1}^{n-1} \lambda_i = n.$$

Then it follows that $M_k \geq \frac{nk}{n-1}$. Moreover, it is easy to check that the equality holds when $G \cong K_n$. $\square$

In particular, when G is bipartite, we can find the following somewhat stronger bound.

Lemma 3.2 *Let G be a connected bipartite graph of order n. Then $M_k \geq k+1$. The equality is attained for $G \cong K_{p,n-p}$.*

Proof. Note that Lemma 2.1 implies that $\lambda_1 = 2$ when G is bipartite. Then

$$\frac{M_k - 2}{k-1} = \frac{\sum_{i=2}^{k} \lambda_i}{k-1} \geq \frac{\sum_{i=k+1}^{n-1} \lambda_i}{n-1-k} = \frac{n - M_k}{n-1-k} \quad \text{as} \quad \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} \quad \text{and} \quad \sum_{i=1}^{n-1} \lambda_i = n.$$

It follows that $M_k \geq k+1$. Moreover, it is easy to check that the equality holds when $G \cong K_{p,n-p}$. $\square$

We now turn to drive upper bounds on M_k.

Lemma 3.3 *Let G be a connected graph of order n. Then*

$$M_k \leq \frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}.$$

The equality is attained for $G \cong K_n$ or some strongly regular graphs. In particular, when $k = 1$, the equality holds if and only if $G \cong K_{p,n-p}$ or $G \cong K_n$.

Proof. Note that

$$\begin{aligned}
(n - M_k)^2 = \left(\sum_{i=k+1}^{n-1} \lambda_i\right)^2 &\leq (n-1-k)\left(\sum_{i=k+1}^{n-1} \lambda_i^2\right) \\
&= (n-1-k)\left(n + 2R_{-1} - \sum_{i=1}^{k} \lambda_i^2\right) \\
&\leq (n-1-k)\left(n + 2R_{-1} - \frac{M_k^2}{k}\right).
\end{aligned}$$

That is,

$$\frac{n-1}{k}M_k^2 - 2nM_k + n(k+1) - 2(n-1-k)R_{-1} \leq 0.$$

It follows that

$$M_k \leq \frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}.$$

Suppose that the equality holds. Then all above inequalities must be equalities. That is $\lambda_1 = \lambda_2 = \cdots = \lambda_k$ and $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{n-1}$, i.e., G has exactly three (or two) distinct normalized Laplacian eigenvalues. Then by Lemma 2.9, it is easy to check that if G is K_n or a suitable strongly regular graph, then the equality is attained. In particular, when $k = 1$, Lemma 2.3 implies that $G \cong K_n$ or $G \cong K_{p,n-p}$. This completes the proof. $\square$

Lemma 3.3 together with Lemmas 2.6 and 2.9 imply that,

Lemma 3.4 *Let G be a connected graph of order n with minimum degree δ. Then*

$$M_k \leq \frac{nk + \sqrt{\frac{nk(n-1-k)(n-1-\delta)}{\delta}}}{n-1}.$$

Moreover, the equality is attained for $G \cong K_n$ or some strongly regulars.

Remark 3.1 *When $k = 1$, we then have that $\lambda_1 \leq \frac{n+\sqrt{\frac{n(n-2)(n-1-\delta)}{\delta}}}{n-1}$. It should be pointed out that this bound is always nontrivial for $\delta \geq \lceil \frac{n}{2} \rceil$, since $\frac{n+\sqrt{\frac{n(n-2)(n-1-\delta)}{\delta}}}{n-1} \leq 2$ for $\delta \geq \lceil \frac{n}{2} \rceil$.*

In particular, when G is bipartite, Lemma 2.1 implies that $\lambda_1(G) = 2$. Hence, using the same argument as that in the proof of Lemma 3.3, we have

Lemma 3.5 *Let G be a connected bipartite graph of order n. Then for $1 \leq k \leq n-2$,*

$$M_k \leq k + 1 + \frac{\sqrt{2(k-1)(n-1-k)(n-2)(R_{-1}-1)}}{n-2}.$$

The equality holds for $1 \leq k \leq n-2$ when $G \cong K_{p,n-p}$.

Remark 3.2 *When $k = 2$, we then have $\lambda_2(G) \leq 1 + \frac{\sqrt{2(n-3)(n-2)(R_{-1}-1)}}{n-2}$ for any bipartite graph G.*

In what follows, we give bounds on $S_\beta^*(G)$ $(\beta \neq 0, 1)$.

Theorem 3.6 *Let G be a connected graph of order $n \geq 2$ and k $(1 \leq k \leq n-2)$ be a positive integer.*

(i) If $0 < \beta < 1$, then

$$S_\beta^*(G) \leq k^{1-\beta}\left(\frac{nk}{n-1}\right)^\beta + (n-1-k)\left(\frac{n}{n-1}\right)^\beta.$$

The equality holds if and only if $G \cong K_n$.

(ii) If $\beta > 1$, then

$$S_\beta^*(G) \geq k^{1-\beta}\left(\frac{nk}{n-1}\right)^\beta + (n-1-k)\left(\frac{n}{n-1}\right)^\beta.$$

The equality holds if and only if $G \cong K_n$.

(iii) If $\beta < 0$, then

$$\begin{aligned} S^*_\beta(G) &\leq k^{1-\beta}\left(\frac{nk+\sqrt{(n-1-k)k[2(n-1)R_{-1}-n]}}{n-1}\right)^\beta \\ &\quad + (n-1-k)^{1-\beta}\left(\frac{n(n-1-k)-\sqrt{(n-1-k)k[2(n-1)R_{-1}-n]}}{n-1}\right)^\beta. \end{aligned}$$

The equality is attained for $G \cong K_n$ or some strongly regular graphs.

Proof. (i) By power mean inequality with $0 < \beta < 1$, we then have

$$\left(\frac{\sum_{i=1}^k \lambda_i^\beta}{k}\right)^{1/\beta} \leq \frac{M_k}{k}, \quad \text{i.e.,} \quad \sum_{i=1}^k \lambda_i^\beta \leq k^{1-\beta}M_k^\beta,$$

with equality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_k$.

Similarly,

$$\sum_{i=k+1}^{n-1} \lambda_i^\beta \leq (n-1-k)^{1-\beta}(n-M_k)^\beta \quad \text{as} \quad M_k + \sum_{i=k+1}^{n-1}\lambda_i = n,$$

with equality holds if and only if $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{n-1}$.

Therefore,

$$\begin{aligned} S^*_\beta(G) = \sum_{i=1}^{n-1}\lambda_i^\beta &= \sum_{i=1}^{k}\lambda_i^\beta + \sum_{i=k+1}^{n-1}\lambda_i^\beta \\ &\leq k^{1-\beta}M_k^\beta + (n-1-k)^{1-\beta}(n-M_k)^\beta. \end{aligned}$$

Let

$$f(x) := k^{1-\beta}x^\beta + (n-1-k)^{1-\beta}(n-x)^\beta, \ x \geq \frac{nk}{n-1}.$$

Note that

$$f'(x) = \beta\left[\left(\frac{x}{k}\right)^{\beta-1} - \left(\frac{n-x}{n-1-k}\right)^{\beta-1}\right] \leq 0 \quad \text{as} \quad 0 < \beta < 1, \, x \geq \frac{nk}{n-1}.$$

Then $f(x)$ is a decreasing function on $x \geq \frac{nk}{n-1}$. Moreover, note that $M_k \geq \frac{nk}{n-1}$ from Lemma 3.1. Hence we have

$$S^*_\beta(G) = f(M_k) \leq f\left(\frac{nk}{n-1}\right) = k^{1-\beta}\left(\frac{nk}{n-1}\right)^\beta + (n-1-k)\left(\frac{n}{n-1}\right)^\beta.$$

Suppose that the equality holds. Then all above inequalities must be equalities. That is $\lambda_1 = \lambda_2 = \cdots = \lambda_k$, $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{n-1}$ and $M_k = \frac{nk}{n-1}$. These imply that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \frac{n}{n-1} \quad \text{as} \quad \sum_{i=1}^{n-1}\lambda_i = n.$$

Hence, Lemma 2.3 implies that $G \cong K_n$.

Conversely, one can easily check that the equality holds when $G \cong K_n$.

(ii) When $\beta > 1$, using power mean inequality, form (i), we then have

$$S^*_\beta(G) \geq k^{1-\beta}M_k^\beta + (n-1-k)^{1-\beta}(n-M_k)^\beta.$$

This together with the fact that $f(x)$ is a increasing function on $x \geq \frac{nk}{n-1}$ as $\beta > 1$. Hence

$$S_\beta^*(G) = f(M_k) \geq f\left(\frac{nk}{n-1}\right) = k^{1-\beta}\left(\frac{nk}{n-1}\right)^\beta + (n-1-k)\left(\frac{n}{n-1}\right)^\beta.$$

Similarly, the equality holds if and only if $G \cong K_n$.

(iii) Recall that $f(x)$ is a increasing function on $x \geq \frac{nk}{n-1}$ as $\beta < 0$. From Lemmas 3.1 and 3.3, we then have

$$\frac{nk}{n-1} \leq M_k \leq \frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}$$

Hence,

$$\begin{aligned} S_\beta^*(G) &\leq f\left(\frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}\right) \\ &= k^{1-\beta}\left(\frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}\right)^\beta \\ &\quad + (n-1-k)^{1-\beta}\left(\frac{n(n-1-k) - \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}\right)^\beta. \end{aligned}$$

Moreover, by Lemma 2.9, it is easy to check that the equality is attained for $G \cong K_n$ or some strongly regular graphs. □

Recall that $\lambda_1 = 2$ when G is bipartite. Combining Lemmas 3.2 and 3.5, the same approach can be used to find the following bounds for bipartite graphs.

Theorem 3.7 *Let G be a connected bipartite graph of order $n \geq 2$ and k $(1 \leq k \leq n-2)$ be a positive integer.*

(i) If $0 < \beta < 1$, then

$$S_\beta^*(G) \leq k^{1-\beta}(k+1)^\beta + (n-1-k).$$

The equality holds if and only if $G \cong K_{p,n-p}$.

(ii) If $\beta > 1$, then

$$S_\beta^*(G) \geq k^{1-\beta}(k+1)^\beta + (n-1-k).$$

The equality holds if and only if $G \cong K_{p,n-p}$.

(iii) If $\beta < 0$, then

$$\begin{aligned} S_\beta^*(G) &\leq k^{1-\beta}\left(k+1+\frac{\sqrt{2(k-1)(n-1-k)(n-2)(R_{-1}-1)}}{n-2}\right)^\beta \\ &\quad + (n-1-k)^{1-\beta}\left(n-k-1-\frac{\sqrt{2(k-1)(n-1-k)(n-2)(R_{-1}-1)}}{n-2}\right)^\beta. \end{aligned}$$

The equality is attained for $G \cong K_{p,n-p}$.

Note that bounds on the degree Kirchhoff index $Kf^*(G)$ can be easily derived by bounds on $S_{-1}^*(G)$. Then we have

Corollary 3.8 *Let G be a connected graph of order n with m edges. Then for positive integer k $(1 \le k \le n-2)$, we have*

$$Kf^*(G) \le \frac{2m(n-1)k^2}{nk+\sqrt{(n-1-k)k[2(n-1)R_{-1}-n]}} + \frac{2m(n-1)(n-1-k)^2}{n(n-1-k)-\sqrt{(n-1-k)k[2(n-1)R_{-1}-n]}}.$$

Corollary 3.9 *Let G be a connected bipartite graph of order n with m edges. Then for positive integer k $(1 \le k \le n-2)$, we have*

$$\begin{aligned} Kf^*(G) \quad \le \quad & \frac{2m(n-2)k^2}{(n-2)(k+1)+\sqrt{2(k-1)(n-1-k)(n-2)(R_{-1}-1)}} \\ & + \frac{2m(n-2)(n-1-k)^2}{(n-2)(n-1-k)-\sqrt{2(k-1)(n-1-k)(n-2)(R_{-1}-1)}}. \end{aligned}$$

Theorem 3.10 *Let G be a connected graph of order $n \ge 3$ with m edges and t spanning trees, and let β be a real number with $\beta \ne 0, 1$. Then for any real number $k \ge 0$, we have*

$$S_\beta^*(G) \ge P^\beta + \frac{(k+1)(n-2)}{P^{\frac{\beta}{(k+1)(n-2)}}}\left(\frac{2mt}{Q}\right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} - \frac{k}{k+1}\left(\frac{2mt}{Q}\right)^{\beta/(n-1)}. \tag{3.5}$$

Moreover, the equality holds if and only if $G \cong K_n$.

Proof. Putting in (2.3) with $a_i = \lambda_i^\beta$ for $i = 1, 2, \ldots, n-1$, $p_1 = \frac{k}{(k+1)(n-1)}$ and $p_i = \frac{(k+1)n-(2k+1)}{(k+1)(n-1)(n-2)}$ for $i = 2, \ldots, n-1$, where $k \ge 0$ is a real number. Then by Lemma 2.8, we have

$$\begin{aligned} & \frac{k\lambda_1^\beta}{(k+1)(n-1)} + \frac{(k+1)n-(2k+1)}{(k+1)(n-1)(n-2)}\sum_{i=2}^{n-1}\lambda_i^\beta - \lambda_1^{\frac{k\beta}{(k+1)(n-1)}}\prod_{i=2}^{n-1}\lambda_i^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} \\ \ge \quad & \frac{k}{(k+1)(n-1)}\sum_{i=1}^{n-1}\lambda_i^\beta - \frac{k}{k+1}\prod_{i=1}^{n-1}\lambda_i^{\beta/(n-1)}. \end{aligned}$$

It follows that

$$S_\beta^*(G) \ge \lambda_1^\beta + \frac{(k+1)(n-2)}{\lambda_1^{\frac{\beta}{(k+1)(n-2)}}}\left(\frac{2mt}{Q}\right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} - \frac{k}{k+1}\left(\frac{2mt}{Q}\right)^{\beta/(n-1)}.$$

Let

$$f(x) := x^\beta + \frac{(k+1)(n-2)}{x^{\frac{\beta}{(k+1)(n-2)}}}\left(\frac{2mt}{Q}\right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta}.$$

Then solving

$$f'(x) = \beta\left[x^{\beta-1} - \left(\frac{2mt}{Q}\right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} x^{\frac{-\beta}{(k+1)(n-2)}-1}\right] \ge 0,$$

we can see that $f(x)$ is an increasing function for $x \ge \left(\frac{2mt}{Q}\right)^{1/(n-1)}$ whether $\beta > 0$ or $\beta < 0$. Note that by Lemmas 2.2 and 2.4, and Remark 2.2, we have

$$\lambda_1 \ge P \ge \frac{n}{n-1} = \frac{\sum_{i=1}^{n-1}\lambda_i}{n-1} \ge \left(\prod_{i=1}^{n-1}\lambda_i\right)^{1/(n-1)} = \left(\frac{2mt}{Q}\right)^{1/(n-1)}.$$

Then, by Lemma 2.2, we have

$$\begin{aligned} S^*_\beta(G) &= f(\lambda_1) - \frac{k}{k+1}\left(\frac{2mt}{Q}\right)^{\beta/(n-1)} \\ &\geq f(P) - \frac{k}{k+1}\left(\frac{2mt}{Q}\right)^{\beta/(n-1)} \\ &= P^\beta + \frac{(k+1)(n-2)}{P^{\frac{\beta}{(k+1)(n-2)}}}\left(\frac{2mt}{Q}\right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} - \frac{k}{k+1}\left(\frac{2mt}{Q}\right)^{\beta/(n-1)}. \end{aligned}$$

Suppose that the equality holds. Then all above inequalities must be equalities. That is $\lambda_1^\beta = \lambda_2^\beta = \cdots = \lambda_{n-1}^\beta$, and $\lambda_1 = P$. These imply that $G \cong K_n$.

Conversely, one can easily check that the equality holds when $G \cong K_n$. This completes the proof. $\square$

Remark 3.3 *Recall that $\lambda_1 \geq \frac{\Delta+1}{\Delta}$ and $\frac{\Delta+1}{\Delta} \geq \frac{n}{n-1} \geq \left(\frac{2mt}{Q}\right)^{1/(n-1)}$, where Δ is the maximum degree of G. If we use $\lambda_1 \geq \frac{\Delta+1}{\Delta}$ in the proof of Theorem 3.10, then we find the following bound,*

$$S^*_\beta(G) \geq \left(\frac{\Delta+1}{\Delta}\right)^\beta + \frac{(k+1)(n-2)}{\left(\frac{\Delta+1}{\Delta}\right)^{\frac{\beta}{(k+1)(n-2)}}}\left(\frac{2mt}{Q}\right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} - \frac{k}{k+1}\left(\frac{2mt}{Q}\right)^{\beta/(n-1)}.$$

The equality holds if and only if $G \cong K_n$.

Moreover, when G is bipartite, the similar method can be used to drive the following bound.

Theorem 3.11 *Let G be a connected bipartite graph of order $n \geq 3$ with m edges and t spanning trees, and let β be a real number with $\beta \neq 0, 1$. Then for any real number $k \geq 0$, we have*

$$S^*_\beta(G) \geq 2^\beta + (k+1)(n-2)2^{\frac{k\beta}{(k+1)(n-1)}}\left(\frac{mt}{Q}\right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} - \frac{k}{k+1}\left(\frac{2mt}{Q}\right)^{\beta/(n-1)}. \tag{3.6}$$

Remark 3.4 *Let $k = 0$ in (3.5) and (3.6). Then we get the bounds in [2], (14) and (19), respectively.*

Similarly, we have the following bounds on $Kf^*(G)$.

Corollary 3.12 *Let G be a connected graph of order $n \geq 3$ with m edges and t spanning trees. Then for any real number $k \geq 0$, we have*

$$Kf^*(G) \geq \frac{2m}{P} + 2m(k+1)(n-2)P^{\frac{1}{(k+1)(n-2)}}\left(\frac{Q}{2mt}\right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}} - \frac{2mk}{k+1}\left(\frac{Q}{2mt}\right)^{\frac{1}{n-1}}.$$

The equality holds if and only if $G \cong K_n$.

Corollary 3.13 *Let G be a connected bipartite graph of order $n \geq 3$ with m edges and t spanning trees. Then for any real number $k \geq 0$, we have*

$$Kf^*(G) \geq m + \frac{2m(k+1)(n-2)}{2^{\frac{k}{(k+1)(n-1)}}}\left(\frac{Q}{mt}\right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}} - \frac{2mk}{k+1}\left(\frac{Q}{2mt}\right)^{\frac{1}{n-1}}.$$

Theorem 3.14 *Let G be a connected graph of order $n \geq 2$ and Randić index $R_{-1}(G)$. If $\beta < 0$ or $0 < \beta < 1$ or $\beta > 2$, then*

$$S^*_\beta(G) \geq \frac{n^{2-\beta}}{(n + 2R_{-1}(G))^{1-\beta}} \tag{3.7}$$

Equality holds in (3.7) if and only if $G \cong K_n$. If $1 < \beta < 2$, then the inequality (3.7) is reversed.

Proof. Let $b_1, b_2, \ldots, b_k$ be positive real numbers and let r be a real number, where $r \neq 0, \frac{1}{2}, 1$. If $r < 0$ or $r > 1$, then $\frac{2r-1}{r} > 1$. Considering the Hölder's inequality, one can arrive at

$$\sum_{i=1}^{k} b_i \geq \frac{\left(\sum_{i=1}^{k} b_i^r\right)^{\frac{2r-1}{r}}}{\left(\sum_{i=1}^{k} b_i^{2r}\right)^{\frac{r-1}{r}}}. \tag{3.8}$$

Equality holds in (3.8) if and only if $b_1 = b_2 = \cdots = b_k$, see [13].

Now we take $r = 1/\beta$ and $b_i = \lambda_i^{\beta}$, $i = 1, 2, \ldots, n-1$ in (3.8). Note that $S_1^*(G) = n$ and $S_2^*(G) = n + 2R_{-1}(G)$. Then we directly get the inequality (3.7) when $\beta < 0$ or $0 < \beta < 1$. Moreover the equality holds in (3.7) if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$. Then by Lemma 2.3, we conclude that $G \cong K_n$. Similar to the above, the proofs for the cases $\beta > 2$ and $1 < \beta < 2$ can be given taking $0 < r < \frac{1}{2}$ and $\frac{1}{2} < r < 1$, respectively. $\square$

Remark 3.5 *It is well known that many bounds for the Randić index $R_{-1}(G)$ have been established in the literature [6, 15]. Therefore the inequality (3.7) may yield some bounds for $S_\beta^*(G)$, immediately.*

Example 3.1 *From Lemma 2.6, we have that $R_{-1}(G) \leq \frac{n}{2\delta}$ with equality if and only if G is regular graph. This implies that, if $\beta < 0$ or $0 < \beta < 1$ (resp., $1 < \beta < 2$), then*

$$S_\beta^*(G) \geq (\textit{resp.}, \leq)\, n\left(1 + \frac{1}{\delta}\right)^{\beta-1}. \tag{3.9}$$

Equality holds in (3.9) if and only if $G \cong K_n$.

Example 3.2 *From Lemma 2.6, we also have that $R_{-1}(G) \geq n/2\Delta$ with equality if and only if G is regular graph. This implies that, if $\beta > 2$, then*

$$S_\beta^*(G) \geq n\left(1 + \frac{1}{\Delta}\right)^{\beta-1}. \tag{3.10}$$

Equality holds in (3.10) if and only if $G \cong K_n$.

From Theorem 3.14, we obtain the following result.

Corollary 3.15 *Let G be a connected graph of order $n \geq 2$ with m edges and Randić index $R_{-1}(G)$. Then*

$$Kf^*(G) \geq \frac{2mn^3}{(n + 2R_{-1}(G))^2}. \tag{3.11}$$

Equality holds in (3.11) if and only if $G \cong K_n$.

Theorem 3.16 *Let G be a connected bipartite graph with $n \geq 3$ vertices. If $\beta < 0$ or $0 < \beta < 1$ or $\beta > 2$, then*

$$S_\beta^*(G) \geq 2^\beta + \frac{(n-2)^{2-\beta}}{(n + 2R_{-1}(G) - 4)^{1-\beta}}. \tag{3.12}$$

Equality holds in (3.12) if and only if $G \cong K_{p,n-p}$. If $1 < \beta < 2$, then the inequality (3.12) is reversed.

Proof. Taking $r = 1/\beta$ and $b_i = \lambda_i^\beta$, $i = 2, \ldots, n-1$ in (3.8), we have

$$\sum_{i=2}^{n-1} \lambda_i^\beta \geq \frac{\left(\sum_{i=2}^{n-1} \lambda_i\right)^{2-\beta}}{\left(\sum_{i=2}^{n-1} \lambda_i^2\right)^{1-\beta}}. \tag{3.13}$$

Equality holds in (3.13) if and only if $\lambda_2 = \cdots = \lambda_{n-1}$. Then, by Eq. (3.13), Lemma 2.1, we get

$$S_\beta^*(G) \geq \lambda_1^\beta + \frac{(n-\lambda_1)^{2-\beta}}{(n+2R_{-1}(G)-\lambda_1^2)^{1-\beta}} = 2^\beta + \frac{(n-2)^{2-\beta}}{(n+2R_{-1}(G)-4)^{1-\beta}}$$

Hence the inequality (3.12) holds. Since G is bipartite, by Lemma 2.3, the equality holds in (3.12) if and only if $G \cong K_{p,n-p}$. Similar to the above, the proofs for the cases $\beta > 2$ and $1 < \beta < 2$ can be given taking $0 < r < \frac{1}{2}$ and $\frac{1}{2} < r < 1$, respectively. $\square$

As well known in graph theory, every tree is bipartite. Considering this with Theorem 3.16, we have the following example.

Example 3.3 *For a tree T of order n [6],*

$$R_{-1}(T) \leq \frac{5n+8}{18}.$$

This yields that if $\beta < 0$ or $0 < \beta < 1$ (resp., $1 < \beta < 2$), then

$$S_\beta^*(T) \geq (\mathit{resp.}, \leq)\, 2^\beta + (n-2)\left(\frac{14}{9}\right)^{\beta-1}.$$

From Theorem 3.16, we also have the following result.

Corollary 3.17 *Let G be a connected bipartite graph of order $n \geq 3$ with m edges and Randić index $R_{-1}(G)$. Then*

$$Kf^*(G) \geq m + \frac{2m(n-2)^3}{(n+2R_{-1}(G)-4)^2}. \tag{3.14}$$

Equality holds in (3.14) if and only if $G \cong K_{p,n-p}$.

The following bounds on $S_\beta^*(G)$ ($\beta \neq 0, 1$) were obtained in [2].

Lemma 3.18 ([2]) *Let G be a connected graph of order $n \geq 3$ with m edges and t spanning trees, and let β be a real number with $\beta \neq 0, 1$. Then*

$$S_\beta^*(G) \geq P^\beta + (n-2)\left(\frac{2mt}{QP}\right)^{\beta/(n-2)}, \tag{3.15}$$

with equality if and only if $G \cong K_n$.

Lemma 3.19 ([2]) *Let G be a connected bipartite graph of order $n \geq 3$ with m edges and t spanning trees, and let β be a real number with $\beta \neq 0, 1$. Then*

$$S_\beta^*(G) \geq 2^\beta + (n-2)\left(\frac{mt}{Q}\right)^{\beta/(n-2)}, \tag{3.16}$$

with equality if and only if $G \cong K_{p,n-p}$.

Bounds (3.15) and (3.16) can be improved as follows, respectively.

Theorem 3.20 *Let G be a connected graph of order $n \geq 3$ with m edges and t spanning trees, and let β be a real number with $\beta \neq 0, 1$. Then there exists a real number $\epsilon \geq 0$ such that*

$$S^*_\beta(G) \geq P^\beta + (n-2)\left(\frac{2mt}{QP}\right)^{\beta/(n-2)} + \epsilon. \tag{3.17}$$

Proof. Let $p = n-2$, $a_1 = \lambda_2^\beta$, $a_2 = \lambda_{n-1}^\beta$ and $a_i = \lambda_i^\beta$ for $i = 3, \ldots, n-2$ in (2.2). Then by Lemma 2.7, we have

$$(n-2)\left(\frac{\sum_{i=2}^{n-1}\lambda_i^\beta}{n-2} - \left(\prod_{i=2}^{n-1}\lambda_i^\beta\right)^{1/(n-2)}\right) \geq \cdots \geq 2\left(\frac{\lambda_2^\beta + \lambda_{n-1}^\beta}{2} - (\lambda_2^\beta\lambda_{n-1}^\beta)^{1/2}\right) = \left(\lambda_2^{\beta/2} - \lambda_{n-1}^{\beta/2}\right)^2.$$

It follows that

$$\sum_{i=2}^{n-1}\lambda_i^\beta \geq (n-2)\left(\prod_{i=2}^{n-1}\lambda_i^\beta\right)^{1/(n-2)} + \left(\lambda_2^{\beta/2} - \lambda_{n-1}^{\beta/2}\right)^2.$$

Now let $\epsilon = \left(\lambda_2^{\beta/2} - \lambda_{n-1}^{\beta/2}\right)^2$. This together with Lemma 2.4 imply that

$$S^*_\beta(G) = \lambda_1^\beta + \sum_{i=2}^{n-1}\lambda_i^\beta \geq \lambda_1^\beta + (n-2)\left(\frac{2mt}{Q\lambda_1}\right)^{\beta/(n-2)} + \epsilon.$$

Let

$$f(x) := x^\beta + (n-2)\left(\frac{2mt}{Qx}\right)^{\beta/(n-2)}, \quad x \geq P.$$

It has been shown in [2] that $f(x)$ is an increasing function for $x \geq \left(\frac{2mt}{Q}\right)^{1/(n-1)}$ whether $\beta > 0$ or $\beta < 0$. Recall that $P \geq \left(\frac{2mt}{Q}\right)^{1/(n-1)}$. Thus we have

$$S^*_\beta(G) = f(\lambda_1) + \delta \geq f(P) + \delta = P^\beta + (n-2)\left(\frac{2mt}{QP}\right)^{\beta/(n-2)} + \epsilon.$$

This completes the proof. $\square$

In particular, recall that $\lambda_1 = 2$ when G is bipartite. Using a similar argument as above, we have

Theorem 3.21 *Let G be a connected bipartite graph of order $n \geq 3$ with m edges and t spanning trees, and let β be a real number with $\beta \neq 0, 1$. Then there exists a real number $\epsilon \geq 0$ such that*

$$S^*_\beta(G) \geq 2^\beta + (n-2)\left(\frac{mt}{Q}\right)^{\beta/(n-2)} + \epsilon. \tag{3.18}$$

Remark 3.6 *Thanks to Lemma 2.3, from the proof of Theorem 3.20, we know that $\epsilon \geq 0$ with equality if and only if $G \cong K_n$ or $G \cong K_{p,n-p}$. Hence, (3.17) and (3.18) always perform better than (3.15) and (3.16) when $G \neq K_n$ and $G \neq K_{p,n-p}$, respectively.*

Remark 3.7 *Recall that $\lambda_1 \geq \frac{\Delta+1}{\Delta} \geq \left(\frac{2mt}{Q}\right)^{1/(n-1)}$, where Δ is the maximum degree of G. If we use $\lambda_1 \geq \frac{\Delta+1}{\Delta}$ in the proof of Theorem 3.20, then we have*

$$S^*_\beta(G) \geq \left(\frac{\Delta+1}{\Delta}\right)^\beta + (n-2)\left(\frac{2\Delta mt}{(\Delta+1)Q}\right)^{\beta/(n-2)} + \epsilon.$$

Similarly, when $\beta = -1$, we then have the following bounds on $Kf^*(G)$.

Corollary 3.22 *Let G be a connected graph of order $n \geq 3$ with m edges and t spanning trees. Then there exists a real number $\epsilon \geq 0$ such that*

$$Kf^*(G) \geq \frac{2m}{P} + 2m(n-2)\left(\frac{QP}{2mt}\right)^{1/(n-2)} + \epsilon.$$

Corollary 3.23 *Let G be a connected bipartite graph of order $n \geq 3$ with m edges and t spanning trees. Then there exists a real number $\epsilon \geq 0$ such that*

$$Kf^*(G) \geq m + 2m(n-2)\left(\frac{Q}{mt}\right)^{1/(n-2)} + \epsilon.$$

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