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*Published in:*  
Applied Mathematics and Computation

*DOI:*  
[10.1016/j.amc.2017.12.003](https://doi.org/10.1016/j.amc.2017.12.003)

Published: 01/05/2018

*Document Version:*  
Peer reviewed version

[Link to publication](#)

*Citation for published version (APA):*

Li, J., Guo, J. M., SHIU, W. C., Altındağ, B. B., & Bozkurt, D. (2018). Bounding the sum of powers of normalized Laplacian eigenvalues of a graph. *Applied Mathematics and Computation*, 324, 82-92.  
<https://doi.org/10.1016/j.amc.2017.12.003>

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# Bounding the sum of powers of normalized Laplacian eigenvalues of a graph\*

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**Abstract** Let  $G$  be a simple connected graph of order  $n$ . Its normalized Laplacian eigenvalues are  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \lambda_n = 0$ . In this paper, new bounds on  $S_\beta^*(G) = \sum_{i=1}^{n-1} \lambda_i^\beta$  ( $\beta \neq 0, 1$ ) are derived.

**AMS** classification: 05C50, 15A48.

**Keywords**: Normalized Laplacian eigenvalue, bound.

## 1 Introduction

Let  $G$  be a simple connected graph with vertex set  $V(G)$  and edge set  $E(G)$ . Its *order* is  $|V(G)|$ , denoted by  $n$ , and its *size* is  $|E(G)|$ , denoted by  $m$ . For  $v \in V(G)$ , let  $d(v)$  be the degree of  $v$ . The maximum and the minimum degrees of  $G$  are denoted by  $\Delta$  and  $\delta$ , respectively.

Let  $A(G)$  and  $D(G)$  be the adjacency matrix and the diagonal matrix of vertex degrees of  $G$ , respectively. The *Laplacian* and *normalized Laplacian* matrices of  $G$  are defined as  $L = L(G) = D(G) - A(G)$  and  $\mathcal{L} = \mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$ , respectively. Note that  $L(G)$  and  $\mathcal{L}(G)$  are symmetric positive semidefinite matrices and 0 is the smallest eigenvalue of  $L(G)$  and  $\mathcal{L}(G)$ , respectively. Let

$$\mu_1(G) \geq \dots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0 \quad \text{and} \quad \lambda_1(G) \geq \dots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0$$

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\*Partially supported by NSF of China(Nos.11471077, 61379021, 11371372); NSF of Fujian(No.2014J01020); China Postdoctoral Science Foundation(No.2014M551831); TÜBİTAK and the Office of Selçuk University Research Project (BAP); General Research Fund of Hong Kong; FRG of Hong Kong Baptist University.

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be the Laplacian eigenvalues and the normalized Laplacian eigenvalues of  $G$ , respectively. When only one graph  $G$  is under consideration, we sometimes write  $\mu_k$  and  $\lambda_k$  instead of  $\mu_k(G)$  and  $\lambda_k(G)$  for  $1 \leq k \leq n$ , respectively. For more information on the eigenvalues of  $L(G)$  and  $\mathcal{L}(G)$ , can be found in [4–6, 8, 9].

For a connected graph  $G$  of order  $n$  and a real number  $\beta \neq 0$ , Zhou [16] defined the sum of the  $\beta$ -th power of the non-zero Laplacian eigenvalues of  $G$  as

$$S_\beta(G) = \sum_{i=1}^{n-1} \mu_i^\beta. \quad (1.1)$$

For basic properties on  $S_\beta(G)$ , including various upper and lower bounds, relations between  $S_\beta(G)$  and the incidence energy, the Kirchhoff index and the Laplacian Estrada index, respectively, see [11, 16, 17].

Very recently, in full analogy with (1.1), Bozkurt *et al.* [2] defined the sum of the  $\beta$ -th power of the non-zero normalized Laplacian eigenvalues of a connected graph  $G$  of order  $n$  as

$$S_\beta^*(G) = \sum_{i=1}^{n-1} \lambda_i^\beta.$$

The cases  $\beta = 0$  and  $\beta = 1$  are trivial as  $S_0^*(G) = n - 1$  and  $S_1^*(G) = n$ , respectively. Moreover,  $2mS_{-1}(G)$  is equal to the degree Kirchhoff index  $Kf^*(G)$  of  $G$  [7]. There is an interesting relation between  $S_\beta^*(G)$  and the general Randić index,  $R_\alpha(G)$ , of  $G$  defined by

$$R_\alpha = R_\alpha(G) = \sum_{uv \in E(G)} (d(u)d(v))^\alpha,$$

where the summation is over all (unordered) edges  $uv$  in  $G$ , and  $\alpha \neq 0$  is a fixed real number [3]. Note that  $S_2^*(G)$  is equal to the trace of  $\mathcal{L}^2$ , from which it is well known that [6]

$$\sum_{i=1}^{n-1} \lambda_i^2 = n + 2R_{-1}(G).$$

For more basic properties on  $S_\beta^*(G)$ , including upper and lower bounds, see [1, 2]. And information on  $R_{-1}(G)$  and its importance to the normalized Laplacian eigenvalues, see [6, 7].

In this paper, we derive some new bounds on  $S_\beta(G)$  ( $\beta \neq 0, 1$ ). Some of those recover or improve the previous bounds in [2]. More-over, we consider the case  $\beta = -1$  which gives some new bounds on degree Kirchhoff index  $Kf^*(G)$ .

## 2 Preliminaries

For a connected graph  $G$  of order  $n$ , let

$$P = 1 + \sqrt{\frac{2R_{-1}}{n(n-1)}} \quad \text{and} \quad Q = \det(D(G)) = \prod_{u \in V(G)} d(u).$$

We begin with introducing some basic properties on the normalized Laplacian eigenvalues of a connected graph  $G$ .

**Lemma 2.1** ([8]) *Let  $G$  be a connected graph of order  $n$ . Then  $0 \leq \lambda_i \leq 2$  for  $i = 1, 2, \dots, n$ . Moreover,  $\lambda_1 = 2$  if and only if  $G$  is bipartite;  $\lambda_1 \geq \frac{n}{n-1}$  with equality if and only if  $G \cong K_n$ .*

**Lemma 2.2** ([10]) *Let  $G$  be a connected graph of order  $n$ . Then  $\lambda_1 \geq P$ . Moreover, the equality holds if and only if  $G \cong K_n$ .*

**Lemma 2.3** ([10]) *Let  $G$  be a connected graph of order  $n > 2$ . Then*

(i)  $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1}$  if and only if  $G \cong K_n$ ;

(ii)  $\lambda_2 = \lambda_3 = \dots = \lambda_{n-1}$  if and only if  $G \cong K_n$  or  $G \cong K_{p,n-p}$ .

**Lemma 2.4** ([6]) *Let  $G$  be a connected graph of order  $n$  with  $m$  edges. Then the number of spanning trees  $t$  of  $G$  is given as*

$$t = \frac{Q}{2m} \prod_{i=1}^{n-1} \lambda_i.$$

The following result concerning the relationship between the  $\mu_i$  and  $\lambda_i$  for  $1 \leq i \leq n$ .

**Lemma 2.5** ([5, 6]) *Let  $G$  be a connected graph of order  $n \geq 3$  with maximum degree  $\Delta$  and minimum degree  $\delta$ . Then for each  $1 \leq i \leq n$ ,*

$$\frac{\mu_i}{\Delta} \leq \lambda_i \leq \frac{\mu_i}{\delta}.$$

*Equality occurs in both bounds if and only if  $G$  is a regular graph.*

**Remark 2.1** *Note that for any connected graph  $G$  of order  $n$ ,  $\mu_1 \geq \Delta + 1$  with equality if and only if  $\Delta = n - 1$  [4]. This together with Lemma 2.5 imply that  $\lambda_1 \geq \frac{\Delta+1}{\Delta}$ , the equality holds if and only if  $G \cong K_n$ .*

**Lemma 2.6** ([6]) *Let  $G$  be a connected graph of order  $n \geq 3$  with maximum degree  $\Delta$  and minimum degree  $\delta$ . Then*

$$\frac{n}{2\Delta} \leq R_{-1}(G) \leq \frac{n}{2\delta}.$$

*Equality occurs in both bounds if and only if  $G$  is a regular graph.*

**Remark 2.2** *Recall that  $P = 1 + \sqrt{\frac{2R_{-1}}{n(n-1)}}$ . This together with Lemma 2.6 imply that*

$$P = 1 + \sqrt{\frac{2R_{-1}}{n(n-1)}} \geq 1 + \sqrt{\frac{1}{(n-1)\Delta}} \geq \frac{n}{n-1} \quad \text{as } \Delta \leq n-1.$$

**Lemma 2.7** ([13]) *Let  $a_i > 0$ ,  $i = 1, 2, \dots, p$ , be  $p$  real numbers. Then*

$$p(A_p - G_p) \geq (p-1)(A_{p-1} - G_{p-1}), \quad (2.2)$$

where  $A_p = \frac{\sum_{i=1}^p a_i}{p}$  and  $G_p = \left( \prod_{i=1}^p a_i \right)^{1/p}$ .

**Lemma 2.8** ([12]) *For  $a_1, a_2, \dots, a_n \geq 0$  and  $p_1, p_2, \dots, p_n \geq 0$  such that  $\sum_{i=1}^n p_i = 1$ ,*

$$\sum_{i=1}^n p_i a_i - \prod_{i=1}^n a_i^{p_i} \geq n\lambda \left( \frac{1}{n} \sum_{i=1}^n a_i - \prod_{i=1}^n a_i^{1/n} \right), \quad (2.3)$$

where  $\lambda = \min\{p_1, p_2, \dots, p_n\}$ . Moreover, the equality holds if and only if  $a_1 = a_2 = \dots = a_n$ .

Some structural properties on graphs have exactly three distinct normalized Laplacian eigenvalues were explored in [6]. The following result concentrating the case of regular graphs.

**Lemma 2.9** ([6]) *Let  $G$  be a connected regular graph of order  $n$ . Then  $G$  has exactly three distinct normalized Laplacian eigenvalues if and only if  $G$  is strongly regular.*

### 3 Main results

In this section, more bounds on  $S_{\beta}^*(G)$  ( $\beta \neq 0, 1$ ) are deduced. Some of those recover or improve the previous bounds in [2].

Let  $G$  be a connected graph with normalized Laplacian eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \lambda_n = 0$ . For  $1 \leq k \leq n-2$ , let

$$M_k = \sum_{i=1}^k \lambda_i.$$

We begin with deriving some bounds on  $M_k$ . It is known that

$$M_k \geq \sum_{i=1}^k 1 = k \quad \text{for } 1 \leq k \leq n-2, \quad (3.4)$$

which is a consequence of Schur's Theorem [14] stating that the spectrum of any symmetric, positive definite matrix majorizes its main diagonal. The following is an improvement of (3.4).

**Lemma 3.1** *Let  $G$  be a connected graph of order  $n$ . Then  $M_k \geq \frac{nk}{n-1}$ . The equality is attained for  $G \cong K_n$ .*

**Proof.** Note that

$$\frac{M_k}{k} = \frac{\sum_{i=1}^k \lambda_i}{k} \geq \frac{\sum_{i=k+1}^{n-1} \lambda_i}{n-1-k} = \frac{n - M_k}{n-1-k} \quad \text{as } \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \quad \text{and} \quad \sum_{i=1}^{n-1} \lambda_i = n.$$

Then it follows that  $M_k \geq \frac{nk}{n-1}$ . Moreover, it is easy to check that the equality holds when  $G \cong K_n$ .  $\square$

In particular, when  $G$  is bipartite, we can find the following somewhat stronger bound.

**Lemma 3.2** *Let  $G$  be a connected bipartite graph of order  $n$ . Then  $M_k \geq k+1$ . The equality is attained for  $G \cong K_{p,n-p}$ .*

**Proof.** Note that Lemma 2.1 implies that  $\lambda_1 = 2$  when  $G$  is bipartite. Then

$$\frac{M_k - 2}{k-1} = \frac{\sum_{i=2}^k \lambda_i}{k-1} \geq \frac{\sum_{i=k+1}^{n-1} \lambda_i}{n-1-k} = \frac{n - M_k}{n-1-k} \quad \text{as } \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_{n-1} \quad \text{and} \quad \sum_{i=1}^{n-1} \lambda_i = n.$$

It follows that  $M_k \geq k+1$ . Moreover, it is easy to check that the equality holds when  $G \cong K_{p,n-p}$ .  $\square$

We now turn to derive upper bounds on  $M_k$ .

**Lemma 3.3** *Let  $G$  be a connected graph of order  $n$ . Then*

$$M_k \leq \frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}.$$

*The equality is attained for  $G \cong K_n$  or some strongly regular graphs. In particular, when  $k=1$ , the equality holds if and only if  $G \cong K_{p,n-p}$  or  $G \cong K_n$ .*

**Proof.** Note that

$$\begin{aligned} (n - M_k)^2 &= \left( \sum_{i=k+1}^{n-1} \lambda_i \right)^2 \leq (n-1-k) \left( \sum_{i=k+1}^{n-1} \lambda_i^2 \right) \\ &= (n-1-k) \left( n + 2R_{-1} - \sum_{i=1}^k \lambda_i^2 \right) \\ &\leq (n-1-k) \left( n + 2R_{-1} - \frac{M_k^2}{k} \right). \end{aligned}$$

That is,

$$\frac{n-1}{k}M_k^2 - 2nM_k + n(k+1) - 2(n-1-k)R_{-1} \leq 0.$$

It follows that

$$M_k \leq \frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}.$$

Suppose that the equality holds. Then all above inequalities must be equalities. That is  $\lambda_1 = \lambda_2 = \dots = \lambda_k$  and  $\lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_{n-1}$ , i.e.,  $G$  has exactly three (or two) distinct normalized Laplacian eigenvalues. Then by Lemma 2.9, it is easy to check that if  $G$  is  $K_n$  or a suitable strongly regular graph, then the equality is attained. In particular, when  $k = 1$ , Lemma 2.3 implies that  $G \cong K_n$  or  $G \cong K_{p,n-p}$ . This completes the proof.  $\square$

Lemma 3.3 together with Lemmas 2.6 and 2.9 imply that,

**Lemma 3.4** *Let  $G$  be a connected graph of order  $n$  with minimum degree  $\delta$ . Then*

$$M_k \leq \frac{nk + \sqrt{\frac{nk(n-1-k)(n-1-\delta)}{\delta}}}{n-1}.$$

Moreover, the equality is attained for  $G \cong K_n$  or some strongly regulars.

**Remark 3.1** *When  $k = 1$ , we then have that  $\lambda_1 \leq \frac{n + \sqrt{\frac{n(n-2)(n-1-\delta)}{\delta}}}{n-1}$ . It should be pointed out that this bound is always nontrivial for  $\delta \geq \lceil \frac{n}{2} \rceil$ , since  $\frac{n + \sqrt{\frac{n(n-2)(n-1-\delta)}{\delta}}}{n-1} \leq 2$  for  $\delta \geq \lceil \frac{n}{2} \rceil$ .*

In particular, when  $G$  is bipartite, Lemma 2.1 implies that  $\lambda_1(G) = 2$ . Hence, using the same argument as that in the proof of Lemma 3.3, we have

**Lemma 3.5** *Let  $G$  be a connected bipartite graph of order  $n$ . Then for  $1 \leq k \leq n-2$ ,*

$$M_k \leq k+1 + \frac{\sqrt{2(k-1)(n-1-k)(n-2)(R_{-1}-1)}}{n-2}.$$

The equality holds for  $1 \leq k \leq n-2$  when  $G \cong K_{p,n-p}$ .

**Remark 3.2** *When  $k = 2$ , we then have  $\lambda_2(G) \leq 1 + \frac{\sqrt{2(n-3)(n-2)(R_{-1}-1)}}{n-2}$  for any bipartite graph  $G$ .*

In what follows, we give bounds on  $S_\beta^*(G)$  ( $\beta \neq 0, 1$ ).

**Theorem 3.6** *Let  $G$  be a connected graph of order  $n \geq 2$  and  $k$  ( $1 \leq k \leq n-2$ ) be a positive integer.*

(i) *If  $0 < \beta < 1$ , then*

$$S_\beta^*(G) \leq k^{1-\beta} \left( \frac{nk}{n-1} \right)^\beta + (n-1-k) \left( \frac{n}{n-1} \right)^\beta.$$

*The equality holds if and only if  $G \cong K_n$ .*

(ii) *If  $\beta > 1$ , then*

$$S_\beta^*(G) \geq k^{1-\beta} \left( \frac{nk}{n-1} \right)^\beta + (n-1-k) \left( \frac{n}{n-1} \right)^\beta.$$

*The equality holds if and only if  $G \cong K_n$ .*

(iii) If  $\beta < 0$ , then

$$S_{\beta}^*(G) \leq k^{1-\beta} \left( \frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1} \right)^{\beta} \\ + (n-1-k)^{1-\beta} \left( \frac{n(n-1-k) - \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1} \right)^{\beta}.$$

The equality is attained for  $G \cong K_n$  or some strongly regular graphs.

**Proof.** (i) By power mean inequality with  $0 < \beta < 1$ , we then have

$$\left( \frac{\sum_{i=1}^k \lambda_i^{\beta}}{k} \right)^{1/\beta} \leq \frac{M_k}{k}, \quad \text{i.e.,} \quad \sum_{i=1}^k \lambda_i^{\beta} \leq k^{1-\beta} M_k^{\beta},$$

with equality holds if and only if  $\lambda_1 = \lambda_2 = \dots = \lambda_k$ .

Similarly,

$$\sum_{i=k+1}^{n-1} \lambda_i^{\beta} \leq (n-1-k)^{1-\beta} (n - M_k)^{\beta} \quad \text{as} \quad M_k + \sum_{i=k+1}^{n-1} \lambda_i = n,$$

with equality holds if and only if  $\lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_{n-1}$ .

Therefore,

$$S_{\beta}^*(G) = \sum_{i=1}^{n-1} \lambda_i^{\beta} = \sum_{i=1}^k \lambda_i^{\beta} + \sum_{i=k+1}^{n-1} \lambda_i^{\beta} \\ \leq k^{1-\beta} M_k^{\beta} + (n-1-k)^{1-\beta} (n - M_k)^{\beta}.$$

Let

$$f(x) := k^{1-\beta} x^{\beta} + (n-1-k)^{1-\beta} (n-x)^{\beta}, \quad x \geq \frac{nk}{n-1}.$$

Note that

$$f'(x) = \beta \left[ \left( \frac{x}{k} \right)^{\beta-1} - \left( \frac{n-x}{n-1-k} \right)^{\beta-1} \right] \leq 0 \quad \text{as} \quad 0 < \beta < 1, \quad x \geq \frac{nk}{n-1}.$$

Then  $f(x)$  is a decreasing function on  $x \geq \frac{nk}{n-1}$ . Moreover, note that  $M_k \geq \frac{nk}{n-1}$  from Lemma 3.1. Hence we have

$$S_{\beta}^*(G) = f(M_k) \leq f\left(\frac{nk}{n-1}\right) = k^{1-\beta} \left(\frac{nk}{n-1}\right)^{\beta} + (n-1-k) \left(\frac{n}{n-1}\right)^{\beta}.$$

Suppose that the equality holds. Then all above inequalities must be equalities. That is  $\lambda_1 = \lambda_2 = \dots = \lambda_k$ ,  $\lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_{n-1}$  and  $M_k = \frac{nk}{n-1}$ . These imply that

$$\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = \frac{n}{n-1} \quad \text{as} \quad \sum_{i=1}^{n-1} \lambda_i = n.$$

Hence, Lemma 2.3 implies that  $G \cong K_n$ .

Conversely, one can easily check that the equality holds when  $G \cong K_n$ .

(ii) When  $\beta > 1$ , using power mean inequality, form (i), we then have

$$S_{\beta}^*(G) \geq k^{1-\beta} M_k^{\beta} + (n-1-k)^{1-\beta} (n - M_k)^{\beta}.$$

This together with the fact that  $f(x)$  is a increasing function on  $x \geq \frac{nk}{n-1}$  as  $\beta > 1$ . Hence

$$S_{\beta}^*(G) = f(M_k) \geq f\left(\frac{nk}{n-1}\right) = k^{1-\beta} \left(\frac{nk}{n-1}\right)^{\beta} + (n-1-k) \left(\frac{n}{n-1}\right)^{\beta}.$$

Similarly, the equality holds if and only if  $G \cong K_n$ .

(iii) Recall that  $f(x)$  is a increasing function on  $x \geq \frac{nk}{n-1}$  as  $\beta < 0$ . From Lemmas 3.1 and 3.3, we then have

$$\frac{nk}{n-1} \leq M_k \leq \frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}$$

Hence,

$$\begin{aligned} S_{\beta}^*(G) &\leq f\left(\frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}\right) \\ &= k^{1-\beta} \left(\frac{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}\right)^{\beta} \\ &\quad + (n-1-k)^{1-\beta} \left(\frac{n(n-1-k) - \sqrt{(n-1-k)k[2(n-1)R_{-1} - n]}}{n-1}\right)^{\beta}. \end{aligned}$$

Moreover, by Lemma 2.9, it is easy to check that the equality is attained for  $G \cong K_n$  or some strongly regular graphs.  $\square$

Recall that  $\lambda_1 = 2$  when  $G$  is bipartite. Combining Lemmas 3.2 and 3.5, the same approach can be used to find the following bounds for bipartite graphs.

**Theorem 3.7** *Let  $G$  be a connected bipartite graph of order  $n \geq 2$  and  $k$  ( $1 \leq k \leq n-2$ ) be a positive integer.*

(i) *If  $0 < \beta < 1$ , then*

$$S_{\beta}^*(G) \leq k^{1-\beta} (k+1)^{\beta} + (n-1-k).$$

*The equality holds if and only if  $G \cong K_{p,n-p}$ .*

(ii) *If  $\beta > 1$ , then*

$$S_{\beta}^*(G) \geq k^{1-\beta} (k+1)^{\beta} + (n-1-k).$$

*The equality holds if and only if  $G \cong K_{p,n-p}$ .*

(iii) *If  $\beta < 0$ , then*

$$\begin{aligned} S_{\beta}^*(G) &\leq k^{1-\beta} \left(k+1 + \frac{\sqrt{2(k-1)(n-1-k)(n-2)(R_{-1}-1)}}{n-2}\right)^{\beta} \\ &\quad + (n-1-k)^{1-\beta} \left(n-k-1 - \frac{\sqrt{2(k-1)(n-1-k)(n-2)(R_{-1}-1)}}{n-2}\right)^{\beta}. \end{aligned}$$

*The equality is attained for  $G \cong K_{p,n-p}$ .*

Note that bounds on the degree Kirchhoff index  $Kf^*(G)$  can be easily derived by bounds on  $S_{-1}^*(G)$ . Then we have



**Corollary 3.8** Let  $G$  be a connected graph of order  $n$  with  $m$  edges. Then for positive integer  $k$  ( $1 \leq k \leq n-2$ ), we have

$$Kf^*(G) \leq \frac{2m(n-1)k^2}{nk + \sqrt{(n-1-k)k[2(n-1)R_{-1}-n]}} + \frac{2m(n-1)(n-1-k)^2}{n(n-1-k) - \sqrt{(n-1-k)k[2(n-1)R_{-1}-n]}}.$$

**Corollary 3.9** Let  $G$  be a connected bipartite graph of order  $n$  with  $m$  edges. Then for positive integer  $k$  ( $1 \leq k \leq n-2$ ), we have

$$Kf^*(G) \leq \frac{2m(n-2)k^2}{(n-2)(k+1) + \sqrt{2(k-1)(n-1-k)(n-2)(R_{-1}-1)}} + \frac{2m(n-2)(n-1-k)^2}{(n-2)(n-1-k) - \sqrt{2(k-1)(n-1-k)(n-2)(R_{-1}-1)}}.$$

**Theorem 3.10** Let  $G$  be a connected graph of order  $n \geq 3$  with  $m$  edges and  $t$  spanning trees, and let  $\beta$  be a real number with  $\beta \neq 0, 1$ . Then for any real number  $k \geq 0$ , we have

$$S_{\beta}^*(G) \geq P^{\beta} + \frac{(k+1)(n-2)}{P^{\frac{\beta}{(k+1)(n-2)}}} \left( \frac{2mt}{Q} \right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} - \frac{k}{k+1} \left( \frac{2mt}{Q} \right)^{\beta/(n-1)}. \quad (3.5)$$

Moreover, the equality holds if and only if  $G \cong K_n$ .

**Proof.** Putting in (2.3) with  $a_i = \lambda_i^{\beta}$  for  $i = 1, 2, \dots, n-1$ ,  $p_1 = \frac{k}{(k+1)(n-1)}$  and  $p_i = \frac{(k+1)n-(2k+1)}{(k+1)(n-1)(n-2)}$  for  $i = 2, \dots, n-1$ , where  $k \geq 0$  is a real number. Then by Lemma 2.8, we have

$$\begin{aligned} & \frac{k\lambda_1^{\beta}}{(k+1)(n-1)} + \frac{(k+1)n-(2k+1)}{(k+1)(n-1)(n-2)} \sum_{i=2}^{n-1} \lambda_i^{\beta} - \lambda_1^{\frac{k\beta}{(k+1)(n-1)}} \prod_{i=2}^{n-1} \lambda_i^{\frac{(k+1)n-(2k+1)}{(k+1)(n-1)(n-2)}\beta} \\ & \geq \frac{k}{(k+1)(n-1)} \sum_{i=1}^{n-1} \lambda_i^{\beta} - \frac{k}{k+1} \prod_{i=1}^{n-1} \lambda_i^{\beta/(n-1)}. \end{aligned}$$

It follows that

$$S_{\beta}^*(G) \geq \lambda_1^{\beta} + \frac{(k+1)(n-2)}{\lambda_1^{\frac{\beta}{(k+1)(n-2)}}} \left( \frac{2mt}{Q} \right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} - \frac{k}{k+1} \left( \frac{2mt}{Q} \right)^{\beta/(n-1)}.$$

Let

$$f(x) := x^{\beta} + \frac{(k+1)(n-2)}{x^{\frac{\beta}{(k+1)(n-2)}}} \left( \frac{2mt}{Q} \right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta}.$$

Then solving

$$f'(x) = \beta \left[ x^{\beta-1} - \left( \frac{2mt}{Q} \right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} x^{\frac{-\beta}{(k+1)(n-2)}-1} \right] \geq 0,$$

we can see that  $f(x)$  is an increasing function for  $x \geq \left( \frac{2mt}{Q} \right)^{1/(n-1)}$  whether  $\beta > 0$  or  $\beta < 0$ . Note that by Lemmas 2.2 and 2.4, and Remark 2.2, we have

$$\lambda_1 \geq P \geq \frac{n}{n-1} = \frac{\sum_{i=1}^{n-1} \lambda_i}{n-1} \geq \left( \prod_{i=1}^{n-1} \lambda_i \right)^{1/(n-1)} = \left( \frac{2mt}{Q} \right)^{1/(n-1)}.$$

Then, by Lemma 2.2, we have

$$\begin{aligned}
S_{\beta}^*(G) &= f(\lambda_1) - \frac{k}{k+1} \left( \frac{2mt}{Q} \right)^{\beta/(n-1)} \\
&\geq f(P) - \frac{k}{k+1} \left( \frac{2mt}{Q} \right)^{\beta/(n-1)} \\
&= P^{\beta} + \frac{(k+1)(n-2)}{P^{\frac{\beta}{(k+1)(n-2)}}} \left( \frac{2mt}{Q} \right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} - \frac{k}{k+1} \left( \frac{2mt}{Q} \right)^{\beta/(n-1)}.
\end{aligned}$$

Suppose that the equality holds. Then all above inequalities must be equalities. That is  $\lambda_1^{\beta} = \lambda_2^{\beta} = \dots = \lambda_{n-1}^{\beta}$ , and  $\lambda_1 = P$ . These imply that  $G \cong K_n$ .

Conversely, one can easily check that the equality holds when  $G \cong K_n$ . This completes the proof.  $\square$

**Remark 3.3** Recall that  $\lambda_1 \geq \frac{\Delta+1}{\Delta}$  and  $\frac{\Delta+1}{\Delta} \geq \frac{n}{n-1} \geq \left( \frac{2mt}{Q} \right)^{1/(n-1)}$ , where  $\Delta$  is the maximum degree of  $G$ . If we use  $\lambda_1 \geq \frac{\Delta+1}{\Delta}$  in the proof of Theorem 3.10, then we find the following bound,

$$S_{\beta}^*(G) \geq \left( \frac{\Delta+1}{\Delta} \right)^{\beta} + \frac{(k+1)(n-2)}{\left( \frac{\Delta+1}{\Delta} \right)^{\frac{\beta}{(k+1)(n-2)}}} \left( \frac{2mt}{Q} \right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} - \frac{k}{k+1} \left( \frac{2mt}{Q} \right)^{\beta/(n-1)}.$$

The equality holds if and only if  $G \cong K_n$ .

Moreover, when  $G$  is bipartite, the similar method can be used to drive the following bound.

**Theorem 3.11** Let  $G$  be a connected bipartite graph of order  $n \geq 3$  with  $m$  edges and  $t$  spanning trees, and let  $\beta$  be a real number with  $\beta \neq 0, 1$ . Then for any real number  $k \geq 0$ , we have

$$S_{\beta}^*(G) \geq 2^{\beta} + (k+1)(n-2)2^{\frac{k\beta}{(k+1)(n-1)}} \left( \frac{mt}{Q} \right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}\beta} - \frac{k}{k+1} \left( \frac{2mt}{Q} \right)^{\beta/(n-1)}. \quad (3.6)$$

**Remark 3.4** Let  $k = 0$  in (3.5) and (3.6). Then we get the bounds in [2], (14) and (19), respectively.

Similarly, we have the following bounds on  $Kf^*(G)$ .

**Corollary 3.12** Let  $G$  be a connected graph of order  $n \geq 3$  with  $m$  edges and  $t$  spanning trees. Then for any real number  $k \geq 0$ , we have

$$Kf^*(G) \geq \frac{2m}{P} + 2m(k+1)(n-2)P^{\frac{1}{(k+1)(n-2)}} \left( \frac{Q}{2mt} \right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}} - \frac{2mk}{k+1} \left( \frac{Q}{2mt} \right)^{\frac{1}{n-1}}.$$

The equality holds if and only if  $G \cong K_n$ .

**Corollary 3.13** Let  $G$  be a connected bipartite graph of order  $n \geq 3$  with  $m$  edges and  $t$  spanning trees. Then for any real number  $k \geq 0$ , we have

$$Kf^*(G) \geq m + \frac{2m(k+1)(n-2)}{2^{\frac{k}{(k+1)(n-1)}}} \left( \frac{Q}{mt} \right)^{\frac{(k+1)n-(2k+1)}{(k+1)(n-2)(n-1)}} - \frac{2mk}{k+1} \left( \frac{Q}{2mt} \right)^{\frac{1}{n-1}}.$$

**Theorem 3.14** Let  $G$  be a connected graph of order  $n \geq 2$  and Randić index  $R_{-1}(G)$ . If  $\beta < 0$  or  $0 < \beta < 1$  or  $\beta > 2$ , then

$$S_{\beta}^*(G) \geq \frac{n^{2-\beta}}{(n+2R_{-1}(G))^{1-\beta}} \quad (3.7)$$

Equality holds in (3.7) if and only if  $G \cong K_n$ . If  $1 < \beta < 2$ , then the inequality (3.7) is reversed.

**Proof.** Let  $b_1, b_2, \dots, b_k$  be positive real numbers and let  $r$  be a real number, where  $r \neq 0, \frac{1}{2}, 1$ . If  $r < 0$  or  $r > 1$ , then  $\frac{2r-1}{r} > 1$ . Considering the Hölder's inequality, one can arrive at

$$\sum_{i=1}^k b_i \geq \frac{\left(\sum_{i=1}^k b_i^r\right)^{\frac{2r-1}{r}}}{\left(\sum_{i=1}^k b_i^{2r}\right)^{\frac{r-1}{r}}}. \quad (3.8)$$

Equality holds in (3.8) if and only if  $b_1 = b_2 = \dots = b_k$ , see [13].

Now we take  $r = 1/\beta$  and  $b_i = \lambda_i^\beta$ ,  $i = 1, 2, \dots, n-1$  in (3.8). Note that  $S_1^*(G) = n$  and  $S_2^*(G) = n + 2R_{-1}(G)$ . Then we directly get the inequality (3.7) when  $\beta < 0$  or  $0 < \beta < 1$ . Moreover the equality holds in (3.7) if and only if  $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1}$ . Then by Lemma 2.3, we conclude that  $G \cong K_n$ . Similar to the above, the proofs for the cases  $\beta > 2$  and  $1 < \beta < 2$  can be given taking  $0 < r < \frac{1}{2}$  and  $\frac{1}{2} < r < 1$ , respectively.  $\square$

**Remark 3.5** *It is well known that many bounds for the Randić index  $R_{-1}(G)$  have been established in the literature [6, 15]. Therefore the inequality (3.7) may yield some bounds for  $S_\beta^*(G)$ , immediately.*

**Example 3.1** *From Lemma 2.6, we have that  $R_{-1}(G) \leq \frac{n}{2\delta}$  with equality if and only if  $G$  is regular graph. This implies that, if  $\beta < 0$  or  $0 < \beta < 1$  (resp.,  $1 < \beta < 2$ ), then*

$$S_\beta^*(G) \geq (\text{resp., } \leq) n \left(1 + \frac{1}{\delta}\right)^{\beta-1}. \quad (3.9)$$

*Equality holds in (3.9) if and only if  $G \cong K_n$ .*

**Example 3.2** *From Lemma 2.6, we also have that  $R_{-1}(G) \geq n/2\Delta$  with equality if and only if  $G$  is regular graph. This implies that, if  $\beta > 2$ , then*

$$S_\beta^*(G) \geq n \left(1 + \frac{1}{\Delta}\right)^{\beta-1}. \quad (3.10)$$

*Equality holds in (3.10) if and only if  $G \cong K_n$ .*

From Theorem 3.14, we obtain the following result.

**Corollary 3.15** *Let  $G$  be a connected graph of order  $n \geq 2$  with  $m$  edges and Randić index  $R_{-1}(G)$ . Then*

$$Kf^*(G) \geq \frac{2mn^3}{(n + 2R_{-1}(G))^2}. \quad (3.11)$$

*Equality holds in (3.11) if and only if  $G \cong K_n$ .*

**Theorem 3.16** *Let  $G$  be a connected bipartite graph with  $n \geq 3$  vertices. If  $\beta < 0$  or  $0 < \beta < 1$  or  $\beta > 2$ , then*

$$S_\beta^*(G) \geq 2^\beta + \frac{(n-2)^{2-\beta}}{(n + 2R_{-1}(G) - 4)^{1-\beta}}. \quad (3.12)$$

*Equality holds in (3.12) if and only if  $G \cong K_{p,n-p}$ . If  $1 < \beta < 2$ , then the inequality (3.12) is reversed.*

**Proof.** Taking  $r = 1/\beta$  and  $b_i = \lambda_i^\beta$ ,  $i = 2, \dots, n-1$  in (3.8), we have

$$\sum_{i=2}^{n-1} \lambda_i^\beta \geq \frac{\left(\sum_{i=2}^{n-1} \lambda_i\right)^{2-\beta}}{\left(\sum_{i=2}^{n-1} \lambda_i^2\right)^{1-\beta}}. \quad (3.13)$$

Equality holds in (3.13) if and only if  $\lambda_2 = \dots = \lambda_{n-1}$ . Then, by Eq. (3.13), Lemma 2.1, we get

$$S_\beta^*(G) \geq \lambda_1^\beta + \frac{(n - \lambda_1)^{2-\beta}}{(n + 2R_{-1}(G) - \lambda_1^2)^{1-\beta}} = 2^\beta + \frac{(n - 2)^{2-\beta}}{(n + 2R_{-1}(G) - 4)^{1-\beta}}$$

Hence the inequality (3.12) holds. Since  $G$  is bipartite, by Lemma 2.3, the equality holds in (3.12) if and only if  $G \cong K_{p,n-p}$ . Similar to the above, the proofs for the cases  $\beta > 2$  and  $1 < \beta < 2$  can be given taking  $0 < r < \frac{1}{2}$  and  $\frac{1}{2} < r < 1$ , respectively.  $\square$

As well known in graph theory, every tree is bipartite. Considering this with Theorem 3.16, we have the following example.

**Example 3.3** For a tree  $T$  of order  $n$  [6],

$$R_{-1}(T) \leq \frac{5n + 8}{18}.$$

This yields that if  $\beta < 0$  or  $0 < \beta < 1$  (resp.,  $1 < \beta < 2$ ), then

$$S_\beta^*(T) \geq (\text{resp.}, \leq) 2^\beta + (n - 2) \left(\frac{14}{9}\right)^{\beta-1}.$$

From Theorem 3.16, we also have the following result.

**Corollary 3.17** Let  $G$  be a connected bipartite graph of order  $n \geq 3$  with  $m$  edges and Randić index  $R_{-1}(G)$ . Then

$$Kf^*(G) \geq m + \frac{2m(n-2)^3}{(n + 2R_{-1}(G) - 4)^2}. \quad (3.14)$$

Equality holds in (3.14) if and only if  $G \cong K_{p,n-p}$ .

The following bounds on  $S_\beta^*(G)$  ( $\beta \neq 0, 1$ ) were obtained in [2].

**Lemma 3.18** ([2]) Let  $G$  be a connected graph of order  $n \geq 3$  with  $m$  edges and  $t$  spanning trees, and let  $\beta$  be a real number with  $\beta \neq 0, 1$ . Then

$$S_\beta^*(G) \geq P^\beta + (n - 2) \left(\frac{2mt}{QP}\right)^{\beta/(n-2)}, \quad (3.15)$$

with equality if and only if  $G \cong K_n$ .

**Lemma 3.19** ([2]) Let  $G$  be a connected bipartite graph of order  $n \geq 3$  with  $m$  edges and  $t$  spanning trees, and let  $\beta$  be a real number with  $\beta \neq 0, 1$ . Then

$$S_\beta^*(G) \geq 2^\beta + (n - 2) \left(\frac{mt}{Q}\right)^{\beta/(n-2)}, \quad (3.16)$$

with equality if and only if  $G \cong K_{p,n-p}$ .

Bounds (3.15) and (3.16) can be improved as follows, respectively.

**Theorem 3.20** *Let  $G$  be a connected graph of order  $n \geq 3$  with  $m$  edges and  $t$  spanning trees, and let  $\beta$  be a real number with  $\beta \neq 0, 1$ . Then there exists a real number  $\epsilon \geq 0$  such that*

$$S_{\beta}^*(G) \geq P^{\beta} + (n-2) \left( \frac{2mt}{QP} \right)^{\beta/(n-2)} + \epsilon. \quad (3.17)$$

**Proof.** Let  $p = n-2$ ,  $a_1 = \lambda_2^{\beta}$ ,  $a_2 = \lambda_{n-1}^{\beta}$  and  $a_i = \lambda_i^{\beta}$  for  $i = 3, \dots, n-2$  in (2.2). Then by Lemma 2.7, we have

$$(n-2) \left( \frac{\sum_{i=2}^{n-1} \lambda_i^{\beta}}{n-2} - \left( \prod_{i=2}^{n-1} \lambda_i^{\beta} \right)^{1/(n-2)} \right) \geq \dots \geq 2 \left( \frac{\lambda_2^{\beta} + \lambda_{n-1}^{\beta}}{2} - (\lambda_2^{\beta} \lambda_{n-1}^{\beta})^{1/2} \right) = (\lambda_2^{\beta/2} - \lambda_{n-1}^{\beta/2})^2.$$

It follows that

$$\sum_{i=2}^{n-1} \lambda_i^{\beta} \geq (n-2) \left( \prod_{i=2}^{n-1} \lambda_i^{\beta} \right)^{1/(n-2)} + (\lambda_2^{\beta/2} - \lambda_{n-1}^{\beta/2})^2.$$

Now let  $\epsilon = (\lambda_2^{\beta/2} - \lambda_{n-1}^{\beta/2})^2$ . This together with Lemma 2.4 imply that

$$S_{\beta}^*(G) = \lambda_1^{\beta} + \sum_{i=2}^{n-1} \lambda_i^{\beta} \geq \lambda_1^{\beta} + (n-2) \left( \frac{2mt}{Q\lambda_1} \right)^{\beta/(n-2)} + \epsilon.$$

Let

$$f(x) := x^{\beta} + (n-2) \left( \frac{2mt}{Qx} \right)^{\beta/(n-2)}, \quad x \geq P.$$

It has been shown in [2] that  $f(x)$  is an increasing function for  $x \geq \left( \frac{2mt}{Q} \right)^{1/(n-1)}$  whether  $\beta > 0$  or  $\beta < 0$ . Recall that  $P \geq \left( \frac{2mt}{Q} \right)^{1/(n-1)}$ . Thus we have

$$S_{\beta}^*(G) = f(\lambda_1) + \delta \geq f(P) + \delta = P^{\beta} + (n-2) \left( \frac{2mt}{QP} \right)^{\beta/(n-2)} + \epsilon.$$

This completes the proof.  $\square$

In particular, recall that  $\lambda_1 = 2$  when  $G$  is bipartite. Using a similar argument as above, we have

**Theorem 3.21** *Let  $G$  be a connected bipartite graph of order  $n \geq 3$  with  $m$  edges and  $t$  spanning trees, and let  $\beta$  be a real number with  $\beta \neq 0, 1$ . Then there exists a real number  $\epsilon \geq 0$  such that*

$$S_{\beta}^*(G) \geq 2^{\beta} + (n-2) \left( \frac{mt}{Q} \right)^{\beta/(n-2)} + \epsilon. \quad (3.18)$$

**Remark 3.6** *Thanks to Lemma 2.3, from the proof of Theorem 3.20, we know that  $\epsilon \geq 0$  with equality if and only if  $G \cong K_n$  or  $G \cong K_{p,n-p}$ . Hence, (3.17) and (3.18) always perform better than (3.15) and (3.16) when  $G \neq K_n$  and  $G \neq K_{p,n-p}$ , respectively.*

**Remark 3.7** *Recall that  $\lambda_1 \geq \frac{\Delta+1}{\Delta} \geq \left( \frac{2mt}{Q} \right)^{1/(n-1)}$ , where  $\Delta$  is the maximum degree of  $G$ . If we use  $\lambda_1 \geq \frac{\Delta+1}{\Delta}$  in the proof of Theorem 3.20, then we have*

$$S_{\beta}^*(G) \geq \left( \frac{\Delta+1}{\Delta} \right)^{\beta} + (n-2) \left( \frac{2\Delta mt}{(\Delta+1)Q} \right)^{\beta/(n-2)} + \epsilon.$$

Similarly, when  $\beta = -1$ , we then have the following bounds on  $Kf^*(G)$ .

**Corollary 3.22** *Let  $G$  be a connected graph of order  $n \geq 3$  with  $m$  edges and  $t$  spanning trees. Then there exists a real number  $\epsilon \geq 0$  such that*

$$Kf^*(G) \geq \frac{2m}{P} + 2m(n-2) \left( \frac{QP}{2mt} \right)^{1/(n-2)} + \epsilon.$$

**Corollary 3.23** *Let  $G$  be a connected bipartite graph of order  $n \geq 3$  with  $m$  edges and  $t$  spanning trees. Then there exists a real number  $\epsilon \geq 0$  such that*

$$Kf^*(G) \geq m + 2m(n-2) \left( \frac{Q}{mt} \right)^{1/(n-2)} + \epsilon.$$

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