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Angular distributions in the decays of the triplet D_2 state of charmonium directly produced in polarized proton-antiproton collisions

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Short title: Angular distributions in the decays of the D state

Abstract. Using the helicity formulism, we calculate the combined angular distribution function of the two gamma photons and the electron in the cascade process $\bar{p}p \to {}^3D_2 \to \chi_J + \gamma_1 \to (\psi + \gamma_2) + \gamma_1 \to (e^+ + e^-) + \gamma_1 + \gamma_2$ ($J = 0, 1, 2$), when $\bar{p}$ and p are arbitrarily polarized. We also present the partially integrated angular distribution functions in six different cases. Our results show that by measuring the two-particle angular distribution of γ_1 and γ_2 and that of γ_1 and e^-, one can determine the relative magnitudes as well as the relative phases of all the helicity amplitudes in the two radiative decay processes ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$.

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1 Introduction

Recently there has been great interest in the study of charmonium spectroscopy, particularly in the energy region above the open charm $D\bar{D}$ threshold of 3.73 GeV [1, 2]. Among the higher charmonium states, the widths of the D_2 states are expected to be narrow because their decays into $D\bar{D}$ are forbidden by parity conservation and their decays into $D\bar{D}^*$ or $\bar{D}D^*$ are forbidden by energy conservation. The observation of the prominent radiative decays of these D charmonium states is a central part of the planned PANDA experiments at FAIR [3], which study charmonium spectroscopy in $\bar{p}p$ annihilation.

In a previous paper [4], it is shown that by measuring the combined angular distribution of the two photons and of the electron in the sequential decay process originating from unpolarized $\bar{p}p$ collisions, namely, $\bar{p}p \to {}^3D_2 \to \chi_J + \gamma_1 \to (\psi + \gamma_2) + \gamma_1 \to (e^+e^-) + \gamma_1 + \gamma_2$ ($J = 0, 1, 2$), one can extract the relative magnitudes as well as the cosines of the relative phases of all the angular-momentum helicity amplitudes in the radiative decay processes ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$. The sines of the relative phases can only be determined uniquely for $J = 2$. In order to get even this limited information on the relative phases of the helicity amplitudes one has to measure the simultaneous angular distribution of three particles, namely, the two gamma photons, γ_1 and γ_2, and the electron. In practice, this is probably a very difficult thing to do. By considering the sequential decay of ${}^3D_2$ produced in polarized $\bar{p}p$ collisions, one may be able to obtain more complete information on all the helicity amplitudes by measuring the angular distribution of fewer particles at a time. So in this paper we calculate the angular distribution of the final stable decay products, γ_1, γ_2 and e^-, in the above cascade process when both $\bar{p}$ and p are arbitrarily polarized. Our final model-independent expressions for the angular distribution function are valid in the $\bar{p}p$ center-of-mass frame and they are written as sums of terms involving products of the Wigner D^J functions whose arguments are the angles representing the directions of the final electron and of the two photons. The coefficients in these expansions are functions of the angular-momentum helicity amplitudes which contain all the dynamics of the individual decay processes. They are also functions of the longitudinal and the transverse components of the polarization vectors of $\bar{p}$ and p in their respective rest frames. We stress that our expressions are independent of any dynamical models and are based only on the general principles of quantum mechanics and symmetry. This is important because we can then learn about the true dynamics of the charmonium system from the decays of the charmonium states.

Once the angular distribution in polarized $\bar{p}p$ collisions is experimentally measured, our expressions will enable one to calculate the relative magnitudes as well as the relative phases of all the angular-momentum helicity amplitudes in the processes $^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$ for both $J = 1$ and $J = 2$. For the $J = 0$ case, there is only one independent helicity amplitude for each of the two radiative decay processes $^3D_2 \to \chi_0 + \gamma_1$ and $\chi_0 \to \psi + \gamma_2$ and that is fixed by our normalization. Our results on the partially integrated angular distributions where the combined angular distribution function of γ_1, γ_2 and e^- is integrated over the directions of one or two particles are quite interesting. They show that by measuring the simultaneous angular distributions of two particles, namely, (γ_1, γ_2) and (γ_1, e^-), alone at a time, we can also obtain the relative magnitudes and the relative phases of all the helicity amplitudes in the individual processes $^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$ for both $J = 1$ and $J = 2$. It is important that both $\bar{p}$ and p are polarized to get this information.

We will derive the expression for the combined angular distribution of the decay products γ_1, γ_2 and e^- in the above cascade process by means of the density matrix formalism where the density matrix elements are given in terms of the polarization vectors defined for stationary antiproton and proton. Our results are valid even when $\bar{p}$ and p have arbitrary momenta since the density matrix elements are Lorentz invariant.

The format of the rest of the paper is as follows: In Sect. 2, we define the polarization vectors and the density matrix of spin-$1/2$ particles. We then show that the density matrix elements are Lorentz invariant. We also express the transition amplitude for the process $\bar{p}p \to {}^3D_2 \to \chi_J + \gamma_1 \to \psi + \gamma_2 + \gamma_1 \to e^+ + e^- + \gamma_1 + \gamma_2$ in terms of the Wigner D^J functions and the angular-momentum helicity amplitudes in the individual sequential processes $\bar{p}p \to {}^3D_2$, $^3D_2 \to \chi_J + \gamma_1$, $\chi_J \to \psi + \gamma_2$ and $\psi \to e^- + e^+$. We then give our expression for the combined angular distribution function of the electron and of the two photons as a sum of linearly independent products of the Wigner D^J functions whose arguments are the angles defining the directions of the two photons and of the electron. The coefficients in this expansion are functions of the angular-momentum helicity amplitudes as well as the longitudinal and transverse components of the rest-frame polarization vectors of $\bar{p}$ and p. In Sect. 3, we present the results for the partially integrated angular distributions in six different cases. Again, they can all be expressed in terms of the Wigner D^J functions. We also show how the measurement of these partially integrated angular distributions will give all the information there is to get on the helicity amplitudes. Finally, in Sect. 4, we make some concluding remarks.

2 Polarization vectors, density matrices and the angular distribution function

Consider a beam of spin-$1/2$ particles consisting of an incoherent mixture of pure spin-states, denoted by i, each occurring with probability f_i. If N_i is the number of particles in the pure state $|i\rangle$ and N is the total number of particles, then

$$f_i = \frac{N_i}{N}. \tag{1}$$

In the rest frame of the beam, the polarization vector is then given by

$$\vec{P} = \sum_i f_i \hat{l}_i, \tag{2}$$

where $\hat{l}_i$ is a unit vector pointing in the direction of the filter axis that allows complete transmission of all particles in the state $|i\rangle$. The magnitude of the polarization vector ranges from zero for an unpolarized mixture to 1 for all particles in a pure state or a completely polarized beam. The density matrix for such a mixture in its rest frame can be written as

$$\rho = \frac{1}{2}(1 + \vec{P}\cdot\vec{\sigma}), \tag{3}$$

where σ_x, σ_y and σ_z are the three (2×2) Pauli matrices.

In an arbitrary Lorentz frame where either $\bar{\text{p}}$ or p has the four momentum $q^\mu = (\varepsilon, \vec{q})$, we can define a polarization four vector s^μ, which is related to its rest-frame value of $(0, \vec{P})$ by

$$s^\mu = \left(\frac{\vec{q}\cdot\vec{P}}{m}, \vec{P} + \frac{\vec{q}(\vec{q}\cdot\vec{P})}{m(\varepsilon + m)} \right), \tag{4}$$

where m is the rest mass of the particle. In the arbitrary Lorentz frame, the (4×4) density matrix can be written as

$$\rho_\pm = \frac{(\not{q} \pm m)}{2m}\frac{(1+\gamma_5 \not{s})}{2}. \tag{5}$$

In (5), the positive sign refers to the proton and the negative sign refers to the antiproton. The density matrix elements are Lorentz scalars in the following sense. Let χ_λ be a two-component Pauli spinor for either particle at rest quantized along the direction given by the unit vector $\hat{e}$. That is,

$$\vec{\sigma}\cdot\hat{e}\,\chi_\lambda = \lambda\,\chi_\lambda\,. \tag{6}$$

Let e^μ be a four-vector whose value in the rest frame of the particle is $(0,\hat{e})$. Then we write the four-momentum Dirac spinor $W_\pm(q,\lambda)$ for the proton or the antiproton in an arbitrary Lorentz frame with the properties:

$$\not{q}W_\pm(q,\lambda) = \pm m W_\pm(q,\lambda) \tag{7}$$

and

$$\gamma_5 \not{e} W_\pm(q,\lambda) = \lambda W_\pm(q,\lambda)\,. \tag{8}$$

As before, the '+' sign in (7) and (8) refers to the proton and the '−' sign refers to the antiproton and q is the four-momentum of either particle. The Lorentz invariance of the density matrix elements is now expressed by

$$\overline{W}_\pm(q,\lambda')\left(\frac{(\not{q}\pm m)}{2m}\frac{(1+\gamma_5\not{s})}{2}\right)W_\pm(q,\lambda) = \chi^\dagger_{\lambda'}\left(\frac{1+\vec{\sigma}\cdot\vec{P}}{2}\right)\chi_\lambda\,. \tag{9}$$

We now give a derivation of our formal result for the combined angular distribution of the electron and the two photons when both $\overline{\mathrm{p}}$ and p are arbitrarily polarized. We consider the cascade process, $\overline{\mathrm{p}}(\lambda_1)+\mathrm{p}(\lambda_2)\rightarrow {}^3\mathrm{D}_2(\nu)\rightarrow \chi_\mathrm{J}(\sigma)+\gamma_1(\mu)\rightarrow \psi(\rho)+\gamma_2(\kappa)+\gamma_1(\mu)\rightarrow \mathrm{e}^-(\alpha_1)$ $+\mathrm{e}^+(\alpha_2)+\gamma_2(\kappa)+\gamma_1(\mu)$ $(\mathrm{J}=0,1,2)$, in the $\overline{\mathrm{p}}\mathrm{p}$ center-of-mass frame or the ${}^3\mathrm{D}_2$ rest frame, where J is the angular momentum of the χ resonance and the Greek symbols after the particle symbols represent their helicities except for the stationary ${}^3\mathrm{D}_2$ resonance, in which case the symbol ν represents the z component of the angular momentum. We choose the z axis to be in the direction of motion of χ_J in the ${}^3\mathrm{D}_2$ rest frame. The x and y axes are arbitrary in our discussions. The experimentalists can choose them according to their convenience. The probability amplitude for the above cascade process can be written as a product of the matrix elements for the individual sequential processes. Since only the helicities of the initial and the final particles, namely, λ_1, λ_2, μ, κ, α_1 and α_2, are observed, we write the probability amplitude for the cascade process in the ${}^3\mathrm{D}_2$ rest frame as

$$\mathrm{T}^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} = \sum_{\nu}^{-2\rightarrow+2}\sum_{\rho}^{-1\rightarrow+1}\sum_{\sigma}^{-\mathrm{J}\rightarrow+\mathrm{J}} {}_\mathrm{D}\left\langle {}^3\mathrm{D}_2(\nu)\middle|\mathrm{B}\middle|\overline{\mathrm{p}}(\lambda_1),\mathrm{p}(\lambda_2)\right\rangle_\mathrm{D} \times {}_\mathrm{D}\left\langle \chi_\mathrm{J}(\sigma),\gamma_1(\mu)\middle|\mathrm{A}\middle|{}^3\mathrm{D}_2(\nu)\right\rangle_\mathrm{D}$$

$$\times\, {}_\mathrm{D}\left\langle \psi(\rho),\gamma_2(\kappa)\middle|\mathrm{E}\middle|\chi_\mathrm{J}(\sigma)\right\rangle_\mathrm{D} \times {}_\mathrm{D}\left\langle \mathrm{e}^-(\alpha_1),\mathrm{e}^+(\alpha_2)\middle|\mathrm{C}\middle|\psi(\rho)\right\rangle_\mathrm{D}\,. \tag{10}$$

We sum over the helicities and the spin indices of the unobserved intermediate particles in (10). The symbols B, A, E and C represent the appropriate transition operators. The subscript

D attached to the bra or the ket vector indicates that each individual matrix element is evaluated in the $^3D_2$ rest frame. In the first two matrix elements the $^3D_2$ rest frame is the same as the c.m. frame of the two particles. In the last two matrix elements $\langle \psi\gamma_2 | E | \chi_J \rangle$ and $\langle e^- e^+ | C | \psi \rangle$ this is not the case. To avoid confusion, we should clarify what we mean by the two-particle helicity states when they are not in their c.m. frame. For example, the two-particle state $|\psi(\rho), \gamma_2(\kappa)\rangle_D$ defined in the $^3D_2$ rest frame, which is not the c.m. frame of ψ and γ_2, has the following meaning. First construct the two-particle helicity state $|\psi(\rho), \gamma_2(\kappa)\rangle_{\chi_J}$ in the χ_J rest frame (which is the same as the c.m. frame of ψ and γ_2) according to the usual conventions [5] with ψ and γ_2 having equal and opposite momenta and helicities ρ and κ, respectively. Then

$$|\psi(\rho), \gamma_2(\kappa)\rangle_D = U_\Lambda(^3D_2, \chi_J)\, |\psi(\rho), \gamma_2(\kappa)\rangle_{\chi_J}, \tag{11}$$

where $U_\Lambda(A, B)$ is the unitary operator corresponding to the Lorentz transformation $\Lambda(A, B)$ which takes the system from the Lorentz frame where B is at rest to the Lorentz frame where A is at rest. It is important to clarify this point since in general ψ and γ_2 do not have definite helicities in the $^3D_2$ rest frame. A similar meaning also holds for the two-particle state $|e^-(\alpha_1), e^+(\alpha_2)\rangle_D$.

Let us now consider the matrix elements in (10) one by one. First,

$${}_D\langle ^3D_2(\nu) | B | \bar{p}(\lambda_1), p(\lambda_2)\rangle_D = \langle 2\nu | B | p(\theta,\phi); \lambda_1\lambda_2 \rangle, \tag{12}$$

where $\langle 2\nu |$ is the one-particle helicity state, or the angular-momentum state, of $^3D_2$ in its own rest frame and $p(\theta, \phi)$ is the magnitude of the c.m. momentum of $\bar{p}$, which is taken to be in the direction (θ, ϕ) in the coordinate system we have chosen. Using the usual expansion [5] of the two-particle helicity state in the c.m. frame in terms of the angular-momentum states we find [6]

$${}_D\langle ^3D_2(\nu) | B | \bar{p}(\lambda_1), p(\lambda_2)\rangle_D = \sqrt{\frac{5}{4\pi}}\, B_{\lambda_1\lambda_2} D^2_{\nu\lambda}(\phi, \theta, -\phi), \tag{13}$$

where

$$\lambda = \lambda_1 - \lambda_2, \tag{14}$$

and $B_{\lambda_1\lambda_2}$ are the angular-momentum helicity amplitudes.

Similarly, the matrix element for the process ${}^3D_2 \to \chi_J + \gamma_1$ with χ_J and γ_1 moving along the +z and –z directions, respectively, can be written as

$$ {}_D\langle \chi_J(\sigma), \gamma_1(\mu) | A | {}^3D_2(\nu) \rangle_D = \langle p_{\chi_J}(0,0); \sigma\mu | A | 2\nu \rangle $$

$$ = \sqrt{\frac{5}{4\pi}} A^J_{\sigma\mu} D^{2*}_{\nu,\sigma-\mu}(0,0,0) = \sqrt{\frac{5}{4\pi}} A^J_{\sigma\mu} \delta_{\nu,\sigma-\mu}, \tag{15} $$

where $p_{\chi_J}(0,0)$ is the magnitude of the momentum of χ_J along the z axis in the ${}^3D_2$ rest frame and $A^J_{\nu\mu}$ are the angular-momentum helicity amplitudes for this process.

Next we notice that the matrix elements for the process $\chi_J \to \psi + \gamma_2$ in the ${}^3D_2$ and the χ_J rest frames are equal. That is,

$$ {}_D\langle \psi(\rho), \gamma_2(\kappa) | E | \chi_J(\sigma) \rangle_D = {}_{\chi_J}\langle \psi(\rho), \gamma_2(\kappa) | U^\dagger_\Lambda({}^3D_2, \chi_J) E\, U_\Lambda({}^3D_2, \chi_J) | \chi_J(\sigma) \rangle_{\chi_J} $$

$$ = {}_{\chi_J}\langle \psi(\rho), \gamma_2(\kappa) | E | \chi_J(\sigma) \rangle_{\chi_J}. \tag{16} $$

In (16) we have used the fact that the transition operator E is invariant under Lorentz transformations:

$$ U^\dagger_\Lambda E\, U_\Lambda = E. \tag{17} $$

Using (16) we can now write

$$ {}_D\langle \psi(\rho), \gamma_2(\kappa) | E | \chi_J(\sigma) \rangle_D = {}_{\chi_J}\langle p'(\theta', \phi'); \rho\kappa | E | J\sigma \rangle_{\chi_J}, \tag{18} $$

where $p'(\theta', \phi')$ is the magnitude of the ψ three-momentum in the χ_J rest frame or the ψ-γ_2 c.m. frame. Moreover, in this frame, the index σ is the z-component of the total angular momentum of χ_J. Again using the expansion of the two-particle helicity state in the c.m. frame in terms of the angular-momentum states we obtain

$$ {}_D\langle \psi(\rho), \gamma_2(\kappa) | E | \chi_J(\sigma) \rangle_D = \sqrt{\frac{2J+1}{4\pi}} E^J_{\rho\kappa} D^{J*}_{\sigma,\rho-\kappa}(\phi', \theta', -\phi'), \tag{19} $$

where $E^J_{\rho\kappa}$ are the angular-momentum helicity amplitudes for the process.

For the matrix element of the final process $\psi(\rho) \to e^-(\alpha_1) + e^+(\alpha_2)$ the situation is more involved. We have

$$ {}_D\langle e^-(\alpha_1), e^+(\alpha_2) | C | \psi(\rho) \rangle_D = {}_\psi\langle e^-(\alpha_1), e^+(\alpha_2) | U^\dagger_\Lambda({}^3D_2, \psi) C U_\Lambda({}^3D_2, \chi_J) U_\Lambda(\chi_J, \psi) | \psi(\rho) \rangle_\psi $$

$$= {}_{\psi}\left\langle e^{-}(\alpha_1), e^{+}(\alpha_2)\right| U_{\Lambda}^{\dagger}({}^3D_2,\psi)\, C U_{\Lambda}({}^3D_2,\psi)\, U_{\Lambda}^{\dagger}({}^3D_2,\psi)\, U_{\Lambda}({}^3D_2,\chi_J)\, U_{\Lambda}(\chi_J,\psi) \left|\psi(\rho)\right\rangle_{\psi}$$

$$= {}_{\psi}\left\langle e^{-}(\alpha_1), e^{+}(\alpha_2)\right| C U_{\Lambda}^{\dagger}({}^3D_2,\psi)\, U_{\Lambda}({}^3D_2,\chi_J)\, U_{\Lambda}(\chi_J,\psi) \left|\psi(\rho)\right\rangle_{\psi}. \quad (20)$$

In the first equality of (20) we have made use of the fact that the single-particle state $\left|\psi(\rho)\right\rangle_D$ was also part of the two-particle helicity state in (19). It was obtained by successively performing two unitary operations corresponding to two Lorentz transformations, the first taking the ψ state from its rest frame to the χ_J rest frame and the second taking it from the χ_J rest frame to the ${}^3D_2$ rest frame. In the last equality of (20) we now make use of the fact that

$$U_{\Lambda}({}^3D_2,\chi_J)\, U_{\Lambda}(\chi_J,\psi) = U_{\Lambda}({}^3D_2,\psi)\, U_{R_W}, \quad (21)$$

where U_{R_W} is a unitary operator corresponding to a pure rotation, usually called "Wigner rotation". Using (21) and the unitarity of U_{Λ}, (20) now leads to

$${}_{D}\left\langle e^{-}(\alpha_1), e^{+}(\alpha_2)\right| C \left|\psi(\rho)\right\rangle_{D} = {}_{\psi}\left\langle e^{-}(\alpha_1), e^{+}(\alpha_2)\right| C U_{R_W} \left|\psi(\rho)\right\rangle_{\psi}$$

$$= {}_{\psi}\left\langle e^{-}(\alpha_1), e^{+}(\alpha_2)\right| U_{R_W} U_{R_W}^{\dagger} C U_{R_W} \left|\psi(\rho)\right\rangle_{\psi} = {}_{\psi}\left\langle e^{-}(\alpha_1), e^{+}(\alpha_2)\right| U_{R_W} C \left|\psi(\rho)\right\rangle_{\psi}, \quad (22)$$

since

$$U_{R_W}^{\dagger} C U_{R_W} = C. \quad (23)$$

Using the expansion of the two-particle helicity state in terms of the angular-momentum states, we can write the right-hand side of (22) as

$${}_{\psi}\left\langle e^{-}(\alpha_1), e^{+}(\alpha_2)\right| U_{R_W} C \left|\psi(\rho)\right\rangle_{\psi} = \sqrt{\frac{3}{4\pi}}\, D^{1}_{\rho\alpha}{}^{*}(R_W^{-1}\hat{e}_{\psi})\, C_{\alpha_1\alpha_2} = \sqrt{\frac{3}{4\pi}}\, C_{\alpha_1\alpha_2} D^{1}_{\rho\alpha}{}^{*}(\phi'', \theta'', -\phi''), \quad (24)$$

where

$$\alpha = \alpha_1 - \alpha_2, \quad (25)$$

$\hat{e}_{\psi}$ is a unit vector in the direction of the e^{-} three-momentum in the ψ rest frame, R_W is the (3×3) rotation matrix and $C_{\alpha_1\alpha_2}$ are the angular-momentum helicity amplitudes for the process. We should mention here that if the electron-positron pair is created by a virtual photon via $\bar{q}q \rightarrow \gamma \rightarrow e^{+}e^{-}$, the helicity zero amplitude C_{++} or C_{--} is of the order m/E when compared to the helicity 1 amplitude C_{+-} or C_{-+}. Since $E \cong M_{\psi}/2$ where M_{ψ} is the rest mass of ψ, $m/E \cong 3.3\times 10^{-4}$ and the helicity zero amplitude is relatively negligible.

The Wigner-rotated unit vector $R_W^{-1}\hat{e}_\psi$ can be obtained in the following way. If R represents the (4×4) matrix whose spatial part gives the (3×3) matrix R_W mentioned above, then, from the definition of U_{R_W} in (21),

$$R = \Lambda^{-1}(^3\mathrm{D}_2, \psi)\Lambda(^3\mathrm{D}_2, \chi_\mathrm{J})\Lambda(\chi_\mathrm{J}, \psi), \tag{26}$$

where the Λ are the (4×4) Lorentz transformation matrices. Now we note that the electron is highly relativistic in the ψ rest frame and its four-momentum vector p_{e_ψ} can be represented to a very good approximation by

$$p_{e_\psi} = \frac{M_\psi}{2}(1, \hat{e}_\psi), \tag{27}$$

and therefore

$$R^{-1}p_{e_\psi} = \Lambda^{-1}(\chi_\mathrm{J}, \psi)\Lambda^{-1}(^3\mathrm{D}_2, \chi_\mathrm{J})\Lambda(^3\mathrm{D}_2, \psi)p_{e_\psi}$$

$$= \Lambda^{-1}(\chi_\mathrm{J}, \psi)\Lambda^{-1}(^3\mathrm{D}_2, \chi_\mathrm{J})\Lambda(^3\mathrm{D}_2, \psi)\Lambda^{-1}(^3\mathrm{D}_2, \psi)p_{e_\mathrm{D}} = \Lambda^{-1}(\chi_\mathrm{J}, \psi)\Lambda^{-1}(^3\mathrm{D}_2, \chi_\mathrm{J})p_{e_\mathrm{D}}. \tag{28}$$

In (28) the four-momentum of e^- in the $^3\mathrm{D}_2$ rest frame is given by

$$p_{e_\mathrm{D}} = E_{e_\mathrm{D}}(1, \hat{e}_\mathrm{D}), \tag{29}$$

where E_{e_D} is the relativistic energy of e^-, and $\hat{e}_\mathrm{D}$ is the unit vector in the direction of the three-momentum of e^- in the $^3\mathrm{D}_2$ rest frame. From (27) we also have

$$R^{-1}p_{e_\psi} = \frac{M_\psi}{2}(1, R_W^{-1}\hat{e}_\psi). \tag{30}$$

Combining (28)-(30) we get

$$\frac{M_\psi}{2}(1, R_W^{-1}\hat{e}_\psi) = \Lambda^{-1}(\chi_\mathrm{J}, \psi)\Lambda^{-1}(^3\mathrm{D}_2, \chi_\mathrm{J})E_{e_\mathrm{D}}(1, \hat{e}_\mathrm{D}). \tag{31}$$

The spatial part of the right-hand side of (31) gives, within a normalization factor, the Wigner-rotated unit vector $\hat{\hat{e}} = (R_W^{-1}\hat{e}_\psi)$ in terms of the angles $(\tilde{\theta}'', \tilde{\phi}'')$ which give the direction of e^- in the $^3\mathrm{D}_2$ rest frame.

We emphasize that the angles (θ', ϕ') of ψ and (θ'', ϕ'') of e^- are directions in the

χ_J and the ψ rest frames, respectively. They are not the same as the corresponding angles measured in the ${}^3D_2$ rest frame or the lab frame. However, the different reference frames are related to each other through the Lorentz transformation. The equations relating these angles are given in [7].

Using (13), (15), (19) and (24) we can now write the amplitude in (10) as

$$T^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} = \frac{5\sqrt{3(2J+1)}}{(4\pi)^2} C_{\alpha_1\alpha_2} B_{\lambda_1\lambda_2} \sum_{\rho}^{-1,0,+1} E^J_{\rho\kappa} D^{1*}_{\rho\alpha}(\phi'',\theta'',-\phi'')$$

$$\times \sum_{\sigma}^{-J\to+J} A^J_{\sigma\mu} D^{J*}_{\sigma,\rho\text{-}\kappa}(\phi',\theta',-\phi') D^2_{\sigma-\mu,\lambda}(\phi,\theta,-\phi)\,. \tag{32}$$

Because of the C and the P invariances [5], the angular-momentum helicity amplitudes in (32) are not all independent. We have

$$B_{\lambda_1\lambda_2} \overset{P}{=} -B_{-\lambda_1,-\lambda_2} \overset{C}{=} -B_{\lambda_2\lambda_1}\,,$$

$$A^J_{\sigma\mu} \overset{P}{=} (-1)^{J+1} A^J_{-\sigma,-\mu}\,,$$

$$E^J_{\rho\kappa} \overset{P}{=} (-1)^J E^J_{-\rho,-\kappa}\,,$$

and

$$C_{\alpha_1\alpha_2} \overset{P}{=} C_{-\alpha_1,-\alpha_2} \overset{C}{=} C_{\alpha_2\alpha_1}\,. \tag{33}$$

From the first equation in (33), we get

$$B_{\frac{1}{2},\frac{1}{2}} = B_{-\frac{1}{2},-\frac{1}{2}} = 0\,. \tag{34}$$

Making use of the symmetry relations of (33) we now re-label the independent angular-momentum helicity amplitudes as follows:

$$B_1 = \sqrt{2}\, B_{\frac{1}{2},-\frac{1}{2}} = -\sqrt{2}\, B_{-\frac{1}{2},\frac{1}{2}}\,,$$

$$A_\sigma = A^J_{\sigma,1} = (-1)^{J+1} A^J_{-\sigma,-1} \qquad (\sigma = -1,0,\ldots,+J)\,,$$

$$E_\rho = E^J_{\rho-1,-1} = (-1)^J E^J_{-\rho+1,1} \qquad (\rho = 0,1,\ldots,+J)\,,$$

$$C_\alpha = C_{|\alpha_1-\alpha_2|} = C_{\alpha_1\alpha_2} \qquad (\alpha = 0,1)\,. \tag{35}$$

The normalized angular distribution function for the cascade process when the initial $\bar{p}$ and p are arbitrarily polarized and the final polarizations of γ_1, γ_2, e^- and e^+ are not observed is given by

$$W(\theta,\phi;\theta',\phi';\theta'',\phi'') = N_J \sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'}^{\pm\frac{1}{2}} \sum_{\alpha_1,\alpha_2}^{\pm\frac{1}{2}} \sum_{\mu,\kappa}^{\pm 1} T_{\lambda_1\lambda_2}^{\alpha_1\alpha_2,\mu\kappa} \rho_{1_{\lambda_1\lambda_1'}} \rho_{2_{\lambda_2\lambda_2'}} T_{\lambda_1'\lambda_2'}^{\alpha_1\alpha_2,\mu\kappa*} . \quad (36)$$

The normalization constant N_J in (36) is determined by requiring that for the unpolarized case the integral of the distribution function $W(\theta,\phi;\theta',\phi';\theta'',\phi'')$ over all the directions of γ_1, γ_2 and e^- or over all the angles, $(\theta,\phi;\theta',\phi';\theta'',\phi'')$, is 1. In (36) the symbols $\rho_{1_{\lambda_1\lambda_1'}}$ and $\rho_{2_{\lambda_2\lambda_2'}}$ represent the density matrices of $\overline{p}$ and p, respectively. In the helicity basis states of the particles these matrix elements are

$$\rho_{1_{\lambda_1\lambda_1'}} = \chi_{\lambda_1}^{\dagger} \frac{1}{2}(1+\vec{P}_1\cdot\vec{\sigma})\,\chi_{\lambda_1'} \quad (37)$$

and

$$\rho_{2_{\lambda_2\lambda_2'}} = \beta_{\lambda_2}^{\dagger} \frac{1}{2}(1+\vec{P}_2\cdot\vec{\sigma})\,\beta_{\lambda_2'} . \quad (38)$$

In (37) and (38) $\vec{P}_1$ and $\vec{P}_2$ are the polarization vectors of $\overline{p}$ and p and the helicity basis states χ_{λ_1} of $\overline{p}$ and β_{λ_2} of p are defined by

$$\vec{\sigma}\cdot\hat{p}\,\chi_{\lambda_1} = \lambda_1\,\chi_{\lambda_1} \quad (39)$$

and

$$\vec{\sigma}\cdot(-\hat{p})\,\beta_{\lambda_2} = \lambda_2\,\beta_{\lambda_2} , \quad (40)$$

where λ_1 and λ_2 can take the values $+1$ or -1. In the coordinate system we defined in the beginning, we have

$$\chi_+ = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\exp(i\phi) \end{pmatrix}, \qquad \chi_- = \begin{pmatrix} -\sin(\theta/2)\exp(-i\phi) \\ \cos(\theta/2) \end{pmatrix} \quad (41)$$

and the phase of β is such that [8]

$$\beta_{\mp} = \chi_{\pm} . \quad (42)$$

Equation (37) can be rewritten as

$$\rho_{1_{\lambda_1\lambda_1'}} = \chi_{\lambda_1}^{\dagger} \frac{1}{2}\begin{pmatrix} 1+P_{1z} & P_{1x}-iP_{1y} \\ P_{1x}+iP_{1y} & 1-P_{1z} \end{pmatrix} \chi_{\lambda_1'} = \frac{1}{2}\begin{pmatrix} 1+P_{1z'} & P_{1x'}-iP_{1y'} \\ P_{1x'}+iP_{1y'} & 1-P_{1z'} \end{pmatrix}, \quad (43)$$

where the unit vectors along the new x', y' and z' axes are related to the corresponding vectors of the *xyz* coordinate system by

$$\hat{i}' = (\sin^2\phi + \cos\theta\cos^2\phi)\,\hat{i} - (\sin\phi\cos\phi - \cos\theta\sin\phi\cos\phi)\,\hat{j} - \cos\phi\sin\theta\,\hat{k} ,$$

$$\hat{j}' = (-\cos\phi\sin\phi + \cos\theta\cos\phi\sin\phi)\,\hat{i} + (\cos^2\phi + \cos\theta\sin^2\phi)\,\hat{j} - \sin\phi\sin\theta\,\hat{k}\,,$$

$$\hat{k}' = \sin\theta\cos\phi\,\hat{i} - \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\,. \tag{44}$$

Similarly, (38) can be rewritten as

$$\rho_{2_{\lambda_2\lambda_2'}} = \beta^{\dagger}_{\lambda_2}\,\frac{1}{2}\begin{pmatrix} 1+P_{2z} & P_{2x}-iP_{2y} \\ P_{2x}+iP_{2y} & 1-P_{2z} \end{pmatrix}\beta_{\lambda_2'} = \frac{1}{2}\begin{pmatrix} 1-P_{2z'} & P_{2x'}+iP_{2y'} \\ P_{2x'}-iP_{2y'} & 1+P_{2z'} \end{pmatrix}. \tag{45}$$

In (43) and (45), $P_{1z'}$ and $-P_{2z'}$ are the longitudinal components (components along the momenta of the respective particles) and the x' and y' components are the transverse components of the polarization vectors. Note that the angular distribution function $W(\theta,\phi;\theta',\phi';\theta'',\phi'')$ is now given in terms of the density matrix elements defined for stationary proton and antiproton. But (36) is of course valid in the $\bar{p}p$ c.m. frame, where $\bar{p}$ and p are moving with relativistic velocities, since the density matrix elements are Lorentz invariant as expressed by (9).

Substituting (32) into (36) and performing the various sums will give us a useful expression for the angular distribution function $W(\theta,\phi;\theta',\phi';\theta'',\phi'')$ in terms of the Wigner D^J functions. Before we do the sums we make use of the Clebsch-Gordan series relation for the D^J functions, namely,

$$D^{j_1}_{m_1m_2} D^{j_2}_{m_1'm_2'} = \sum_{J=|j_1-j_2|}^{j_1+j_2} \langle j_1 j_2 m_1 m_1' \,|\, J, m_1+m_1' \rangle \langle j_1 j_2 m_2 m_2' \,|\, J, m_2+m_2' \rangle\, D^{J}_{m_1+m_1', m_2+m_2'} \tag{46}$$

and the relation

$$D^{j*}_{m_1m_2} = (-1)^{m_1-m_2} D^{j}_{-m_1-m_2}\,. \tag{47}$$

After a very long algebra, we then obtain the following expression for the normalized angular distribution function:

$$W(\theta,\phi;\theta',\phi';\theta'',\phi'') = \frac{1}{4(4\pi)^3}\sum_{L_3}^{0,2}\gamma_{L_3}\sum_{L_1}^{0\to 4}\sum_{L_2}^{0\to 2J}\sum_{d}^{0\to d_m}\sum_{d'}^{0\to d_m'}$$

$$\times\, \varepsilon^{L_3L_2}_{d'}\,\alpha^{L_1L_2}_{d}\sum_{M(L_1)}\Big[\beta^{L_1}_{M}\,Y^{L_1L_2L_3}_{dd'M} + (-1)^{L_1}\beta^{L_1*}_{M}\,Y^{L_1L_2L_3*}_{dd'M}\Big], \tag{48}$$

where

$d_m = \mathrm{Min}(L_1, L_2, 3)$,

$d_m' = \mathrm{Min}(L_3, L_2, J)$,

$$M(L_1) = \begin{cases} 0 & \text{when } L_1 < 2 \\ 0,2 & \text{when } L_1 \ge 2 \end{cases}, \tag{49}$$

and we have used the following normalizations for the angular-momentum helicity amplitudes B_1, C_α, A_σ and E_ρ defined in (35):

$$\left|B_1\right|^2 = \left|C_0\right|^2 + \left|C_1\right|^2 = \sum_{\sigma}^{-1\to J}\left|A_\sigma\right|^2 = \sum_{\rho}^{0\to J}\left|E_\rho\right|^2 = 1 . \tag{50}$$

The angle-dependent function $Y^{L_1L_2L_3}_{dd'M}$ in (48) is defined by

$$Y^{L_1L_2L_3}_{dd'M} = D^{L_3 *}_{d',0}\, D^{L_2 *}_{d,d'}\, D^{L_1}_{d,M} + (-1)^{L_2+L_1}\, D^{L_3 *}_{d',0}\, D^{L_2 *}_{-d,d'}\, D^{L_1}_{-d,M} . \tag{51}$$

The arguments of the Wigner functions D^{L_1}, D^{L_2} and D^{L_3} in (51) are $(\phi,\theta,-\phi)$ the direction of $\overline{p}$ with respect to χ_J in the $^3D_2$ rest frame, $(\phi',\theta',-\phi')$ the direction of ψ in the χ_J rest frame, and $(\phi'',\theta'',-\phi'')$ the direction of the e^- momentum in the ψ rest frame, respectively. The coefficients γ_{L_3}, $\varepsilon^{L_3L_2}_{d'}$ and $\alpha^{L_1L_2}_{d}$, which are independent of the angles in (48), are defined as follows:

$$\gamma_{L_3} = -\sqrt{3}\sum_{\alpha}^{0,1}(-1)^\alpha \langle 11;\alpha-\alpha|L_3\,0\rangle \left|C_\alpha\right|^2 , \tag{52}$$

$$\varepsilon^{L_3L_2}_{d'} = (-1)^J\sqrt{3(2J+1)}\left(1-\frac{\delta_{d'0}}{2}\right)\sum_{s'(d')}\left[E_{\frac{s'+d'}{2}}\, E^*_{\frac{s'-d'}{2}} + (-1)^{L_2}\, E^*_{\frac{s'+d'}{2}}\, E_{\frac{s'-d'}{2}}\right]$$
$$\times\left\langle JJ;\frac{s'+d'}{2},-\frac{s'-d'}{2}\middle|L_2\,d'\right\rangle\left\langle 11;\frac{s'+d'-2}{2},-\frac{s'-d'-2}{2}\middle|L_3\,d'\right\rangle,$$
$$s'(d') = d', d'+2, \ldots, 2J-d' , \tag{53}$$

$$\alpha^{L_1L_2}_{d} = (-1)^{J+1}\sqrt{5(2J+1)}\left(1-\frac{\delta_{d0}}{2}\right)\sum_{s(d)}\left[A_{\frac{s+d}{2}}\, A^*_{\frac{s-d}{2}} + (-1)^{L_1+L_2}\, A^*_{\frac{s+d}{2}}\, A_{\frac{s-d}{2}}\right]$$
$$\times\left\langle JJ;\frac{s+d}{2},-\frac{s-d}{2}\middle|L_2\,d\right\rangle\left\langle 22;\frac{s+d-2}{2},-\frac{s-d-2}{2}\middle|L_1\,d\right\rangle,$$
$$s(d) = -(2J-d), -(2J-d)+2, \ldots, +(2J-d) . \tag{54}$$

In (48) the components of the polarization vectors are contained in the coefficients $\beta^{L_1}_{M}$ defined as follows:

$$\beta^{L_1}_{0} = -\frac{\sqrt{5}}{2}\langle 22;1-1|L_1\,0\rangle \left|B_1\right|^2\left[(P_+ + P_{1z'} + P_{2z'}) + (-1)^{L_1}(P_+ - P_{1z'} - P_{2z'})\right],$$

$$\beta^{L_1}_{2} = \frac{\sqrt{5}}{2}\langle 22;11|L_1\,2\rangle \left|B_1\right|^2\left[1+(-1)^{L_1}\right](P_B - iP_C),$$

$$\beta^{L_1}_{-2} = \beta^{L_1 *}_{2} , \tag{55}$$

where

$$P_+ = 1 + P_{1z'} P_{2z'} \,,$$

$$P_B = P_{1x'} P_{2x'} - P_{1y'} P_{2y'} \,,$$

$$P_C = P_{1x'} P_{2y'} + P_{1y'} P_{2x'} \,. \tag{56}$$

Using the normalization conditions in (50), the explicit expressions for the nonzero coefficients $\beta_{\rm M}^{\rm L_1}$, $\gamma_{\rm L_3}$, $\varepsilon_{\rm d'}^{\rm L_3 L_2}$ and $\alpha_{\rm d}^{\rm L_1 L_2}$ in (48) are given in the following; we have

$$\beta_0^0 = P_+ \,,$$

$$\beta_0^1 = \frac{1}{\sqrt{2}} (P_{1z'} + P_{2z'}) \,,$$

$$\beta_0^2 = -\sqrt{\frac{5}{14}} P_+ \,,$$

$$\beta_0^3 = -\sqrt{2}\, (P_{1z'} + P_{2z'}) \,,$$

$$\beta_0^4 = -\sqrt{\frac{8}{7}} P_+ \,,$$

$$\beta_2^2 = -\sqrt{\frac{15}{7}}\, (P_B - iP_C) \,,$$

$$\beta_2^4 = \sqrt{\frac{20}{7}}\, (P_B - iP_C) \,, \tag{57}$$

$$\gamma_0 = 1 \,,$$

$$\gamma_2 = \frac{1}{\sqrt{2}} \left(\left| \mathrm{C}_1 \right|^2 - 2 \left| \mathrm{C}_0 \right|^2 \right) \cong \frac{1}{\sqrt{2}} \,. \tag{58}$$

$\mathrm{J} = 0$:

$$\varepsilon_0^{00} = 1 \,,$$

$$\varepsilon_0^{20} = \frac{1}{\sqrt{2}} \,,$$

$$\alpha_0^{00} = 1 \,,$$

$$\alpha_0^{20} = -\sqrt{\frac{5}{14}} \,,$$

$$\alpha_0^{40} = -\sqrt{\frac{8}{7}} \,. \tag{59}$$

$J = 1$:

$$\varepsilon_0^{00} = 1,$$

$$\varepsilon_0^{02} = -\sqrt{2}\left(|E_0|^2 - \frac{1}{2}|E_1|^2 \right),$$

$$\varepsilon_0^{20} = \frac{1}{\sqrt{2}}\left(|E_0|^2 - 2|E_1|^2 \right),$$

$$\varepsilon_1^{21} = -3i\,\mathrm{Im}\left(E_1E_0^*\right) ,$$

$$\varepsilon_0^{22} = -1,$$

$$\varepsilon_1^{22} = -3\,\mathrm{Re}\left(E_1E_0^*\right) , \qquad (60)$$

$$\alpha_0^{00} = 1,$$

$$\alpha_0^{02} = \frac{1}{\sqrt{2}}\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2 \right),$$

$$\alpha_0^{11} = \sqrt{3}\,|A_{-1}|^2,$$

$$\alpha_1^{11} = \sqrt{6}\left[\mathrm{Re}\left(A_0A_{-1}^*\right) + \sqrt{\frac{3}{2}}\,\mathrm{Re}\left(A_1A_0^*\right) \right],$$

$$\alpha_1^{12} = -\sqrt{6}\,i\left[\mathrm{Im}\left(A_0A_{-1}^*\right) - \sqrt{\frac{3}{2}}\,\mathrm{Im}\left(A_1A_0^*\right) \right],$$

$$\alpha_0^{20} = \sqrt{\frac{10}{7}}\left(|A_{-1}|^2 - \frac{1}{2}|A_0|^2 - |A_1|^2 \right),$$

$$\alpha_1^{21} = -\sqrt{\frac{90}{7}}\,i\left[\mathrm{Im}\left(A_0A_{-1}^*\right) + \frac{1}{\sqrt{6}}\mathrm{Im}\left(A_1A_0^*\right) \right],$$

$$\alpha_0^{22} = \sqrt{\frac{5}{7}}\left(|A_{-1}|^2 + |A_0|^2 - |A_1|^2 \right),$$

$$\alpha_1^{22} = \sqrt{\frac{90}{7}}\left[\mathrm{Re}\left(A_0A_{-1}^*\right) - \frac{1}{\sqrt{6}}\mathrm{Re}\left(A_1A_0^*\right) \right],$$

$$\alpha_2^{22} = 2\sqrt{\frac{30}{7}}\,\mathrm{Re}\left(A_1A_{-1}^*\right),$$

$$\alpha_0^{31} = \frac{\sqrt{3}}{2}\left|A_{-1}\right|^2,$$

$$\alpha_1^{31} = 3\left[\ \mathrm{Re}\left(A_0 A_{-1}^*\right) - \sqrt{\frac{2}{3}}\,\mathrm{Re}\left(A_1 A_0^*\right)\ \right],$$

$$\alpha_1^{32} = -3i\left[\ \mathrm{Im}\left(A_0 A_{-1}^*\right) + \sqrt{\frac{2}{3}}\,\mathrm{Im}\left(A_1 A_0^*\right)\ \right],$$

$$\alpha_2^{32} = -\sqrt{30}\,i\,\mathrm{Im}\left(A_1 A_{-1}^*\right),$$

$$\alpha_0^{40} = \frac{1}{\sqrt{14}}\left(\ \left|A_{-1}\right|^2 - 4\left|A_0\right|^2 + 6\left|A_1\right|^2\ \right),$$

$$\alpha_1^{41} = -\sqrt{\frac{15}{7}}\,i\left[\ \mathrm{Im}\left(A_0 A_{-1}^*\right) - \sqrt{6}\,\mathrm{Im}\left(A_1 A_0^*\right)\ \right],$$

$$\alpha_0^{42} = \frac{1}{2\sqrt{7}}\left(\ \left|A_{-1}\right|^2 + 8\left|A_0\right|^2 + 6\left|A_1\right|^2\ \right),$$

$$\alpha_1^{42} = \sqrt{\frac{15}{7}}\left[\ \mathrm{Re}\left(A_0 A_{-1}^*\right) + \sqrt{6}\,\mathrm{Re}\left(A_1 A_0^*\right)\ \right],$$

$$\alpha_2^{42} = \sqrt{\frac{90}{7}}\,\mathrm{Re}\left(A_1 A_{-1}^*\right). \tag{61}$$

$J = 2$:

$$\varepsilon_0^{00} = 1,$$

$$\varepsilon_0^{02} = -\sqrt{\frac{10}{7}}\left(\ \left|E_0\right|^2 + \frac{1}{2}\left|E_1\right|^2 - \left|E_2\right|^2\ \right),$$

$$\varepsilon_0^{04} = 3\sqrt{\frac{2}{7}}\left(\ \left|E_0\right|^2 - \frac{2}{3}\left|E_1\right|^2 + \frac{1}{6}\left|E_2\right|^2\ \right),$$

$$\varepsilon_0^{20} = \frac{1}{\sqrt{2}}\left(\ \left|E_0\right|^2 - 2\left|E_1\right|^2 + \left|E_2\right|^2\ \right),$$

$$\varepsilon_1^{21} = -3i\left[\ \mathrm{Im}\left(E_1 E_0^*\right) - \sqrt{\frac{2}{3}}\,\mathrm{Im}\left(E_2 E_1^*\right)\ \right],$$

$$\varepsilon_0^{22} = -\sqrt{\frac{5}{7}}\left(\ \left|E_0\right|^2 - \left|E_1\right|^2 - \left|E_2\right|^2\ \right),$$

$$\varepsilon_1^{22} = -\sqrt{\frac{15}{7}}\left[\ \mathrm{Re}\left(E_1 E_0^*\right) - \sqrt{6}\,\mathrm{Re}\left(E_2 E_1^*\right)\ \right],$$

$$\varepsilon_2^{22} = 2\sqrt{\frac{30}{7}}\,\mathrm{Re}\left(E_2 E_0^*\right)\ ,$$

$$\varepsilon_1^{23} = \sqrt{6}\, i \left[\operatorname{Im}\left(E_1 E_0^*\right) + \sqrt{\frac{3}{2}} \operatorname{Im}\left(E_2 E_1^*\right) \right],$$

$$\varepsilon_2^{23} = \sqrt{30}\, i \operatorname{Im}\left(E_2 E_0^*\right) \ ,$$

$$\varepsilon_0^{24} = \frac{3}{\sqrt{7}} \left(\left|E_0\right|^2 + \frac{4}{3}\left|E_1\right|^2 + \frac{1}{6}\left|E_2\right|^2 \right),$$

$$\varepsilon_1^{24} = \sqrt{\frac{90}{7}} \left[\operatorname{Re}\left(E_1 E_0^*\right) + \frac{1}{\sqrt{6}} \operatorname{Re}\left(E_2 E_1^*\right) \right],$$

$$\varepsilon_2^{24} = \sqrt{\frac{90}{7}} \operatorname{Re}\left(E_2 E_0^*\right) \ , \qquad (62)$$

$$\alpha_0^{00} = 1 \,,$$

$$\alpha_0^{02} = -\sqrt{\frac{5}{14}} \left(\left|A_{-1}\right|^2 + 2\left|A_0\right|^2 + \left|A_1\right|^2 - 2\left|A_2\right|^2 \right),$$

$$\alpha_0^{04} = -2\sqrt{\frac{2}{7}} \left(\left|A_{-1}\right|^2 - \frac{3}{2}\left|A_0\right|^2 + \left|A_1\right|^2 - \frac{1}{4}\left|A_2\right|^2 \right),$$

$$\alpha_0^{11} = \left|A_{-1}\right|^2 + \left|A_2\right|^2 ,$$

$$\alpha_1^{11} = \sqrt{6} \left[\operatorname{Re}\left(A_0 A_{-1}^*\right) + \sqrt{\frac{3}{2}} \operatorname{Re}\left(A_1 A_0^*\right) + \operatorname{Re}\left(A_2 A_1^*\right) \right],$$

$$\alpha_1^{12} = -\sqrt{\frac{10}{7}}\, i \left[\operatorname{Im}\left(A_0 A_{-1}^*\right) - \sqrt{\frac{3}{2}} \operatorname{Im}\left(A_1 A_0^*\right) - 3\operatorname{Im}\left(A_2 A_1^*\right) \right],$$

$$\alpha_0^{13} = -2\left(\left|A_{-1}\right|^2 - \frac{1}{4}\left|A_2\right|^2 \right),$$

$$\alpha_1^{13} = -2\left[\operatorname{Re}\left(A_0 A_{-1}^*\right) + \sqrt{\frac{3}{2}} \operatorname{Re}\left(A_1 A_0^*\right) - \frac{3}{2}\operatorname{Re}\left(A_2 A_1^*\right) \right],$$

$$\alpha_1^{14} = 2\sqrt{\frac{15}{7}}\, i \left[\operatorname{Im}\left(A_0 A_{-1}^*\right) - \sqrt{\frac{3}{2}} \operatorname{Im}\left(A_1 A_0^*\right) + \frac{1}{2}\operatorname{Im}\left(A_2 A_1^*\right) \right],$$

$$\alpha_0^{20} = \sqrt{\frac{10}{7}} \left(\left|A_{-1}\right|^2 - \frac{1}{2}\left|A_0\right|^2 - \left|A_1\right|^2 - \frac{1}{2}\left|A_2\right|^2 \right),$$

$$\alpha_1^{21} = -3\sqrt{\frac{10}{7}}\, i \left[\operatorname{Im}\left(A_0 A_{-1}^*\right) + \frac{1}{\sqrt{6}} \operatorname{Im}\left(A_1 A_0^*\right) - \frac{1}{3}\operatorname{Im}\left(A_2 A_1^*\right) \right],$$

$$\alpha_0^{22} = -\frac{5}{7}\left(|A_{-1}|^2 - |A_0|^2 - |A_1|^2 + |A_2|^2 \right),$$

$$\alpha_1^{22} = \frac{5\sqrt{6}}{7}\left[\mathrm{Re}\left(A_0 A_{-1}^*\right) - \frac{1}{\sqrt{6}}\mathrm{Re}\left(A_1 A_0^*\right) + \mathrm{Re}\left(A_2 A_1^*\right) \right],$$

$$\alpha_2^{22} = \frac{10\sqrt{6}}{7}\left[\mathrm{Re}\left(A_1 A_{-1}^*\right) + \mathrm{Re}\left(A_2 A_0^*\right) \right],$$

$$\alpha_1^{23} = 2\sqrt{\frac{15}{7}}\, i\left[\mathrm{Im}\left(A_0 A_{-1}^*\right) + \frac{1}{\sqrt{6}}\mathrm{Im}\left(A_1 A_0^*\right) + \frac{1}{2}\mathrm{Im}\left(A_2 A_1^*\right) \right],$$

$$\alpha_2^{23} = 5\sqrt{\frac{6}{7}}\, i\,\mathrm{Im}\left(A_2 A_0^*\right),$$

$$\alpha_0^{24} = -\frac{4\sqrt{5}}{7}\left(|A_{-1}|^2 + \frac{3}{4}|A_0|^2 - |A_1|^2 + \frac{1}{8}|A_2|^2 \right),$$

$$\alpha_1^{24} = -\frac{30}{7}\left[\mathrm{Re}\left(A_0 A_{-1}^*\right) - \frac{1}{\sqrt{6}}\mathrm{Re}\left(A_1 A_0^*\right) - \frac{1}{6}\mathrm{Re}\left(A_2 A_1^*\right) \right],$$

$$\alpha_2^{24} = -\frac{20\sqrt{2}}{7}\left[\mathrm{Re}\left(A_1 A_{-1}^*\right) - \frac{3}{4}\mathrm{Re}\left(A_2 A_0^*\right) \right],$$

$$\alpha_0^{31} = \frac{1}{2}\left(|A_{-1}|^2 - 4|A_2|^2 \right),$$

$$\alpha_1^{31} = 3\left[\mathrm{Re}\left(A_0 A_{-1}^*\right) - \sqrt{\frac{2}{3}}\mathrm{Re}\left(A_1 A_0^*\right) - \frac{2}{3}\mathrm{Re}\left(A_2 A_1^*\right) \right],$$

$$\alpha_1^{32} = -\sqrt{\frac{15}{7}}\, i\left[\mathrm{Im}\left(A_0 A_{-1}^*\right) + \sqrt{\frac{2}{3}}\mathrm{Im}\left(A_1 A_0^*\right) + 2\,\mathrm{Im}\left(A_2 A_1^*\right) \right],$$

$$\alpha_2^{32} = -5\sqrt{\frac{6}{7}}\, i\,\mathrm{Im}\left(A_1 A_{-1}^*\right),$$

$$\alpha_0^{33} = -\left(|A_{-1}|^2 + |A_2|^2 \right),$$

$$\alpha_1^{33} = -\sqrt{6}\left[\mathrm{Re}\left(A_0 A_{-1}^*\right) - \sqrt{\frac{2}{3}}\mathrm{Re}\left(A_1 A_0^*\right) + \mathrm{Re}\left(A_2 A_1^*\right) \right],$$

$$\alpha_3^{33} = 5\,\mathrm{Re}\left(A_2 A_{-1}^*\right),$$

$$\alpha_1^{34} = 3\sqrt{\frac{10}{7}}\, i\left[\mathrm{Im}\left(A_0 A_{-1}^*\right) + \sqrt{\frac{2}{3}}\mathrm{Im}\left(A_1 A_0^*\right) - \frac{1}{3}\mathrm{Im}\left(A_2 A_1^*\right) \right],$$

$$\alpha_2^{34} = 10\sqrt{\frac{2}{7}}\, i\, \mathrm{Im}\left(A_1 A_{-1}^*\right),$$

$$\alpha_3^{34} = 5 i\, \mathrm{Im}\left(A_2 A_{-1}^*\right),$$

$$\alpha_0^{40} = \frac{1}{\sqrt{14}}\left(|A_{-1}|^2 - 4|A_0|^2 + 6|A_1|^2 - 4|A_2|^2 \right),$$

$$\alpha_1^{41} = -\sqrt{\frac{15}{7}}\, i \left[\mathrm{Im}\left(A_0 A_{-1}^*\right) - \sqrt{6}\, \mathrm{Im}\left(A_1 A_0^*\right) + 2\, \mathrm{Im}\left(A_2 A_1^*\right) \right],$$

$$\alpha_0^{42} = -\frac{\sqrt{5}}{14}\left(|A_{-1}|^2 - 8|A_0|^2 + 6|A_1|^2 + 8|A_2|^2 \right),$$

$$\alpha_1^{42} = \frac{5}{7}\left[\mathrm{Re}\left(A_0 A_{-1}^*\right) + \sqrt{6}\, \mathrm{Re}\left(A_1 A_0^*\right) - 6\, \mathrm{Re}\left(A_2 A_1^*\right) \right],$$

$$\alpha_2^{42} = \frac{15\sqrt{2}}{7}\left[\mathrm{Re}\left(A_1 A_{-1}^*\right) - \frac{4}{3}\mathrm{Re}\left(A_2 A_0^*\right) \right],$$

$$\alpha_1^{43} = \sqrt{\frac{10}{7}}\, i \left[\mathrm{Im}\left(A_0 A_{-1}^*\right) - \sqrt{6}\, \mathrm{Im}\left(A_1 A_0^*\right) - 3\, \mathrm{Im}\left(A_2 A_1^*\right) \right],$$

$$\alpha_2^{43} = -10\sqrt{\frac{2}{7}}\, i\, \mathrm{Im}\left(A_2 A_0^*\right),$$

$$\alpha_3^{43} = -5 i\, \mathrm{Im}\left(A_2 A_{-1}^*\right),$$

$$\alpha_0^{44} = -\frac{2}{7}\left(|A_{-1}|^2 + 6|A_0|^2 + 6|A_1|^2 + |A_2|^2 \right),$$

$$\alpha_1^{44} = -\frac{5\sqrt{6}}{7}\left[\mathrm{Re}\left(A_0 A_{-1}^*\right) + \sqrt{6}\, \mathrm{Re}\left(A_1 A_0^*\right) + \mathrm{Re}\left(A_2 A_1^*\right) \right],$$

$$\alpha_2^{44} = -\frac{10\sqrt{6}}{7}\left[\mathrm{Re}\left(A_1 A_{-1}^*\right) + \mathrm{Re}\left(A_2 A_0^*\right) \right],$$

$$\alpha_3^{44} = -5\, \mathrm{Re}\left(A_2 A_{-1}^*\right). \qquad (63)$$

Equation (48) together with the expressions for the non-vanishing coefficients in (57)-(63) give the angular distribution of the two gamma photons γ_1 and γ_2 and of the electron as a function of the angles (θ,ϕ), (θ',ϕ') and (θ'',ϕ''). Equation (48) looks complicated only because it gives the combined angular distribution of three particles. Nevertheless, it is useful. Since the result is expressed as a sum of products of the orthogonal Wigner D^J functions, we can obtain the values for the coefficients $\beta_M^{L_1}$, γ_{L_3}, $\varepsilon_{d'}^{L_3L_2}$ and $\alpha_d^{L_1L_2}$ of the $Y_{dd'M}^{L_1L_2L_3}$ angular function from

$$\frac{1}{(2L_1+1)(2L_2+1)(2L_3+1)}\gamma_{L_3}\,\alpha_d^{L_1L_2}\,\varepsilon_{d'}^{L_3L_2}\left[\beta_M^{L_1}+(-1)^{L_1}\beta_M^{L_1*}\right]$$

$$\times\left[1+(-1)^{L_1+L_2}\delta_{d0}\right]\left[1+(-1)^{L_1+L_2}\delta_{d'0}\,\delta_{M0}\right]$$

$$=\int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\left[Y_{dd'M}^{L_1L_2L_3}+Y_{dd'M}^{L_1L_2L_3*}\right]d\Omega\,d\Omega'\,d\Omega''. \tag{64}$$

In calculating (64), we made use of the orthogonality relation:

$$\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} D_{mm'}^{j*}(\alpha,\beta,\gamma)\, D_{\mu\mu'}^{j'}(\alpha,\beta,\gamma)\sin\beta\, d\beta = \frac{8\pi^2}{(2j+1)}\delta_{m\mu}\,\delta_{m'\mu'}\,\delta_{jj'}\,. \tag{65}$$

When we have sufficient experimental data for the angular distribution function $W(\theta,\phi;\theta',\phi';\theta'',\phi'')$, the integral on the right side of (64) can then be determined numerically for all possible allowed values of L_1, L_2, L_3, d, d' and M. A close examination of the expressions for the coefficients $\beta_M^{L_1}$, γ_{L_3}, $\varepsilon_{d'}^{L_3L_2}$ and $\alpha_d^{L_1L_2}$ in (57)-(63) shows that this will enable us to determine the relative magnitudes as well as the relative phases of all the angular-momentum helicity amplitudes A_σ and E_ρ in the radiative decay processes $^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$, respectively, for both the $J=1$ and the $J=2$ cases, when one or both of the incoming particles, $\bar{p}$ and p, are polarized. For the $J=0$ case, there is only one independent helicity amplitude (A_0 or E_0) for each of the radiative decay process and that is fixed by our normalization. Moreover, we can always get the relative magnitude of the two independent helicity amplitudes C_α in the final decay process $\psi \to e^+e^-$ for all values of J. It should be noted that the coefficients $\beta_M^{L_1}$ are functions of the longitudinal ($P_{z'}$) and the transverse ($P_{x'}, P_{y'}$) components of the polarization vectors of $\bar{p}$ and p. If the polarization vectors $\vec{P}_1$ and $\vec{P}_2$ go to zero, then $\beta_M^{L_1}=0$ when M is nonzero or when L_1 is odd, and we will recover the results of the unpolarized $\bar{p}p$ collisions given in [4].

3 Partially integrated angular distributions

The partially integrated angular distributions obtained from (48) will look a lot simpler and we will gain greater insight from them. We will find that we can also get complete information on all the helicity amplitudes A_σ and E_ρ by considering these partially integrated angular distributions only. We now consider six different cases of partially integrated angular distributions. In deriving these results, we frequently make use of (65) and the following property of the D^J functions:

$$\int_0^{2\pi} d\phi \int_0^{\pi} D^{L*}_{MM'}(\phi,\theta,-\phi)\sin\theta\, d\theta = \int_0^{2\pi} d\phi \int_0^{\pi} D^{L}_{MM'}(\phi,\theta,-\phi)\sin\theta\, d\theta$$

$$= 2\pi\, \delta_{M-M',0} \int_0^{\pi} d^L_{MM'}(\theta)\sin\theta\, d\theta = 2\pi\, k_{LM}\,, \qquad (66)$$

where

$$k_{LM} = \int_0^{\pi} d^L_{MM}(\theta)\sin\theta\, d\theta\,. \qquad (67)$$

We will express the final results in terms of the orthogonal spherical harmonics by making use of the relation:

$$D^L_{M0} = \sqrt{\frac{4\pi}{2L+1}}\, Y^*_{LM}\,. \qquad (68)$$

Case 1: We will integrate over (θ',ϕ') and (θ'',ϕ''). Only the angular distribution of the first gamma photon γ_1 is measured. We obtain

$$\tilde{W}(\theta,\phi) = \int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\, d\Omega'\, d\Omega''$$

$$= \frac{1}{\sqrt{4\pi}}\Big\{\beta^0_0\, Y_{00}(\theta) + \frac{1}{\sqrt{5}}\,\alpha^{20}_0\,\beta^2_0\, Y_{20}(\theta) + \frac{1}{\sqrt{5}}\,\alpha^{20}_0\,\mathrm{Re}\big[\beta^2_2\, Y^*_{22}(\theta,\phi)\big]$$

$$+ \frac{1}{3}\,\alpha^{40}_0\,\beta^4_0\, Y_{40}(\theta) + \frac{1}{3}\,\alpha^{40}_0\,\mathrm{Re}\big[\beta^4_2\, Y^*_{42}(\theta,\phi)\big]\Big\}, \qquad (69)$$

where the angles (θ,ϕ) represent the direction of $\bar{p}$ measured from the z axis, which is taken to be the direction of the momentum of χ_J. This angle is the same as that of γ_1 measured in the ${}^3D_2$ rest frame with the z axis taken to be the direction of the proton. The x and y axes are arbitrary. With the normalization condition $|A_{-1}|^2 + |A_0|^2 + |A_1|^2 = 1$ and the expressions for α^{20}_0 and α^{40}_0 in (61), equation (69) allows us to determine the magnitudes of all the A helicity amplitudes for the $J=1$ case since there are three equations and three unknowns. For the $J=2$ case, this is not possible since there are only three equations but four unknowns. In fact, this is the same situation as the unpolarized $\bar{p}p$ collisions in [4].

Case 2: We will integrate over (θ,ϕ) and (θ'',ϕ''). Only the angular distribution of the second gamma photon γ_2 is measured. We get

$$\tilde{W}(\theta',\phi') = \int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\, d\Omega\, d\Omega''$$

$$= \frac{1}{8\sqrt{\pi}} \sum_{L_1}^{0\to 4} \sum_{L_2}^{0,2\to 2J} \sum_{M(L_1)}^{0\to \mathrm{Min}(L_1,L_2,3)} \alpha_M^{L_1 L_2}\, \varepsilon_0^{0 L_2}\, k_{L_1 M} \sqrt{\frac{1}{2L_2+1}}$$

$$\times\left[\beta_M^{L_1}\, Y_{L_2 M}(\theta',\phi') + (-1)^{L_1} \beta_M^{L_1 *}\, Y_{L_2 M}^{*}(\theta',\phi')\right]. \qquad (70)$$

Here, (θ',ϕ') are the angles between ${}^3D_2$ and γ_2 in the χ_J rest frame. We will write the results separately for the $J=1$ and $J=2$ cases in terms of the helicity amplitudes A_σ and E_ρ and the coefficients $\beta_M^{L_1}$, which are functions of the polarization vectors as follows:

$J=1$:

$$\tilde{W}(\theta',\phi') = \frac{1}{\sqrt{4\pi}}\left\{\beta_0^0\, Y_{00}(\theta') - \frac{1}{\sqrt{5}}\left(|E_0|^2 - \frac{1}{2}|E_1|^2\right)\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2\right)\beta_0^0\, Y_{20}(\theta')\right.$$

$$\left. - \frac{2}{5}\sqrt{\frac{7}{3}}\left(|E_0|^2 - \frac{1}{2}|E_1|^2\right)\mathrm{Re}\left(A_1 A_{-1}^*\right)\mathrm{Re}\left[\beta_2^2\, Y_{22}(\theta',\phi')\right]\right\}. \qquad (71)$$

$J=2$:

$$\tilde{W}(\theta',\phi') = \frac{1}{\sqrt{4\pi}}\left\{\beta_0^0\, Y_{00}(\theta')\right.$$

$$+ \frac{\sqrt{5}}{7}\left(|E_0|^2 + \frac{1}{2}|E_1|^2 - |E_2|^2\right)\left(|A_{-1}|^2 + 2|A_0|^2 + |A_1|^2 - 2|A_2|^2\right)\beta_0^0\, Y_{20}(\theta')$$

$$- \frac{2}{\sqrt{21}}\left(|E_0|^2 + \frac{1}{2}|E_1|^2 - |E_2|^2\right)\left[\mathrm{Re}\left(A_1 A_{-1}^*\right) + 2\,\mathrm{Re}\left(A_2 A_0^*\right)\right]\mathrm{Re}\left[\beta_2^2\, Y_{22}(\theta',\phi')\right]$$

$$- \frac{4}{7}\left(|E_0|^2 - \frac{2}{3}|E_1|^2 + \frac{1}{6}|E_2|^2\right)\left(|A_{-1}|^2 - \frac{3}{2}|A_0|^2 + |A_1|^2 - \frac{1}{4}|A_2|^2\right)\beta_0^0\, Y_{40}(\theta')$$

$$\left. - \frac{4}{3\sqrt{7}}\left(|E_0|^2 - \frac{2}{3}|E_1|^2 + \frac{1}{6}|E_2|^2\right)\left[\mathrm{Re}\left(A_1 A_{-1}^*\right) - \frac{3}{2}\mathrm{Re}\left(A_2 A_0^*\right)\right]\mathrm{Re}\left[\beta_2^2\, Y_{42}(\theta',\phi')\right]\right\}. \qquad (72)$$

From the measurement of the angular distribution of γ_2 alone we get the following information on the helicity amplitudes in the $J=1$ and $J=2$ cases:

$J=1$:

From the measurement of γ_1 alone we already know $|A_{-1}|$, $|A_0|$ and $|A_1|$. So the coefficient of Y_{20} in (71) now gives $|E_0|^2 - \frac{1}{2}|E_1|^2$. With the normalization condition this will now fix both $|E_0|$ and $|E_1|$. The coefficient of $\mathrm{Re}\left[\beta_2^2 Y_{22}\right]$ in (71) then gives $\mathrm{Re}\left(A_1 A_{-1}^*\right)$. So we also get the cosine of the relative phase between A_{-1} and A_1. It should be noted that the parameter β_2^2 defined by (57) will vanish if there is no polarization in the p and $\bar{p}$ beams, and we will not get any information on $\mathrm{Re}\left(A_1 A_{-1}^*\right)$. So the polarization is crucial for extracting this information from the single-particle angular distribution of γ_2.

$J = 2$:

From the angular distribution of e^- (case 3), as we will see later, we can determine $|E_1|^2$ and express $|E_2|^2$ linearly in terms of $|E_0|^2$. Now dividing the coefficient of $\beta_0^0 Y_{20}$ by that of $\beta_0^0 Y_{40}$ in (72) gives us another relation among $|A_{-1}|^2$, $|A_0|^2$, $|A_1|^2$ and $|A_2|^2$. From this and the result for $J = 2$ in case 1 we can uniquely obtain the relative magnitudes of all the A helicity amplitudes and hence the relative magnitudes of all the E helicity amplitudes. Once these magnitudes are known, we can then determine $\mathrm{Re}\left(A_1 A_{-1}^*\right)$ and $\mathrm{Re}\left(A_2 A_0^*\right)$ from the coefficients of $\mathrm{Re}\left[\beta_2^2 Y_{22}\right]$ and $\mathrm{Re}\left[\beta_2^2 Y_{42}\right]$.

Case 3: We will integrate over (θ,ϕ) and (θ',ϕ'). Only the angular distribution of the electron is measured. We have

$$\tilde{W}(\theta'',\phi'') = \int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\, d\Omega\, d\Omega'$$

$$= \frac{1}{64\sqrt{\pi}} \sum_{L_3}^{0,2} \gamma_{L_3} \sum_{L_1}^{0\to 4} \sum_{L_2}^{0\to 2J} \sum_{M(L_1)}^{0\to \mathrm{Min}(L_1,L_2,L_3,J)} \alpha_M^{L_1 L_2}\, \varepsilon_M^{L_3 L_2}\, k_{L_2 M}\, k_{L_1 M} \sqrt{\frac{1}{2L_3+1}}$$

$$\times (1+\delta_{M0})^2 \left[\beta_M^{L_1}\, Y_{L_3 M}(\theta'',\phi'') + (-1)^{L_1} \beta_M^{L_1 *}\, Y_{L_3 M}^*(\theta'',\phi'')\right], \tag{73}$$

where (θ'',ϕ'') are the angles between the directions of the momenta of e^- and χ_J in the ψ rest frame. Again we give the results for the $J = 1$ and $J = 2$ cases separately in terms of the helicity amplitudes A_σ and E_ρ and the coefficients $\beta_M^{L_1}$:

$J = 1$:

$$\tilde{W}(\theta'',\phi'') = \frac{1}{\sqrt{4\pi}} \left[\beta_0^0 Y_{00}(\theta'') + \frac{1}{\sqrt{20}} \left(|E_0|^2 - 2|E_1|^2\right) \beta_0^0 Y_{20}(\theta'')\right]. \tag{74}$$

$J=2$:

$$\tilde{W}(\theta'',\phi'') = \frac{1}{\sqrt{4\pi}}\left[\beta_0^0 Y_{00}(\theta'') + \frac{1}{\sqrt{20}}\left(|E_0|^2 - 2|E_1|^2 + |E_2|^2\right)\beta_0^0 Y_{20}(\theta'')\right]$$

$$+\frac{1}{48}\sqrt{\frac{7}{2\pi}}\left\{\frac{1}{5}\left[2\,\mathrm{Re}\left(A_1A_{-1}^*\right)+7\,\mathrm{Re}\left(A_2A_0^*\right)\right]\mathrm{Re}\left(E_2E_0^*\right)\right.$$

$$\left.+\,\mathrm{Im}\left(A_2A_0^*\right)\mathrm{Im}\left(E_2E_0^*\right)\right\}\mathrm{Re}\left[\beta_2^2 Y_{22}(\theta'',\phi'')\right]. \qquad (75)$$

In calculating (74) and (75) we have neglected $|C_0|^2$ when compared to $|C_1|^2$ for the reason mentioned earlier. From the measurement of the angular distribution of the electron alone we get the following information on the helicity amplitudes for the $J=1$ and $J=2$ cases.

$J=1$:

The coefficient of $\beta_0^0 Y_{20}$ in (74), with the normalization condition, will fix both $|E_0|$ and $|E_1|$.

$J=2$:

From the coefficient of $\beta_0^0 Y_{20}$ in (75) we get $|E_0|^2 - 2|E_1|^2 + |E_2|^2$. This information together with the normalization condition, $|E_0|^2 + |E_1|^2 + |E_2|^2 = 1$, enables us to determine $|E_1|^2$ and express $|E_2|^2$ linearly in terms of $|E_0|^2$.

So from cases 1-3, we see that we can obtain the relative magnitudes of all the helicity amplitudes in the processes ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$ by measuring the single-particle angular distributions of γ_1, γ_2 and e^-. We can also obtain the cosines of some of the relative phases of the helicity amplitudes in the process ${}^3D_2 \to \chi_J + \gamma_1$. In order to get complete information on the relative phases of the helicity amplitudes we need to measure the simultaneous angular distributions of two particles as we will see in cases 4-6.

Case 4: We will integrate over the angles (θ'',ϕ''), the direction of the final electron. The combined angular distribution of the two photons γ_1 and γ_2 is measured. We get

$$\tilde{\tilde{W}}(\theta,\phi;\theta',\phi') = \int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta''\,d\theta''\,d\phi''$$

$$=\frac{1}{64\pi^2}\sum_{L_1}^{0\to 4}\ \sum_{L_2}^{0,2\to 2J}\ \sum_{d}^{0\to \mathrm{Min}(L_1,L_2,3)} \alpha_d^{L_1L_2}\,\varepsilon_0^{0L_2}\sum_{M(L_1)}^{-2\to 2}(1+\delta_{M0})$$

$$\times\left[\beta_M^{L_1}\,D_{d0}^{L_2*}(\phi',\theta',-\phi')\,D_{dM}^{L_1}(\phi,\theta,-\phi)+(-1)^{L_1}\beta_M^{L_1*}\,D_{d0}^{L_2}(\phi',\theta',-\phi')\,D_{dM}^{L_1*}(\phi,\theta,-\phi)\right], \qquad (76)$$

where $M(L_1)$ defined in (49) also takes the value -2 when $L_1 \geq 2$.

Since the explicit expressions for the partially integrated angular distributions of two particles are rather long, we only give the results in terms of the sums of the coefficients defined in (57)-(63). In (76), however, we can obtain the coefficients of the D^J angular functions from

$$\frac{1}{4(2L_1+1)(2L_2+1)}\,\alpha_d^{L_1L_2}\,\varepsilon_0^{0L_2}\,[1+\delta_{M0}][1+\delta_{d0}\delta_{M0}]\left[\beta_M^{L_1}+(-1)^{L_1}\beta_M^{L_1*}\right]$$

$$=\int\tilde{\tilde{W}}(\theta,\phi;\theta',\phi')\left[D_{d0}^{L_2*}\,D_{dM}^{L_1}+D_{d0}^{L_2}\,D_{dM}^{L_1*}\right]d\Omega\,d\Omega'. \qquad (77)$$

A close examination of the expressions for $\alpha_d^{L_1L_2}$ and $\varepsilon_0^{0L_2}$ in (60)-(63) shows that (77) enables us to obtain the sines and the cosines of the relative phases of all the A helicity amplitudes for both the $J=1$ and $J=2$ cases. It also enables us to determine the relative magnitudes of all the A and the E helicity amplitudes for $J=1$ and $J=2$. Only the relative phases among the E helicity amplitudes remain undetermined. We can get these only by measuring the simultaneous angular distribution of γ_1 and of e^- as we will see in the following case.

Case 5: Here we will integrate over (θ',ϕ') or the direction of γ_2. We will measure the combined angular distribution of γ_1 and e^-. We have

$$\tilde{\tilde{W}}(\theta,\phi;\theta'',\phi'')=\int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta'\,d\theta'\,d\phi'$$

$$=\frac{1}{256\pi^2}\sum_{L_3}^{0,2}\gamma_{L_3}\sum_{L_1}^{0\to4}\sum_{L_2}^{0\to2J}\sum_{d}^{0\to\mathrm{Min}(L_1,L_2,L_3,J)}\alpha_d^{L_1L_2}\,\varepsilon_d^{L_3L_2}\,k_{L_2d}\,(1+\delta_{d0})\sum_{M(L_1)}^{-2\to2}(1+\delta_{M0})$$

$$\times\left[\beta_M^{L_1}\,D_{d0}^{L_3*}(\phi'',\theta'',-\phi'')\,D_{dM}^{L_1}(\phi,\theta,-\phi)+(-1)^{L_1}\beta_M^{L_1*}\,D_{d0}^{L_3}(\phi'',\theta'',-\phi'')\,D_{dM}^{L_1*}(\phi,\theta,-\phi)\right]. \qquad (78)$$

When we have enough experimental data to perform the required numerical integrations, the coefficients of the D^J angular functions for the different possible values of L_1, L_2, L_3, d and M in (78) can be obtained from

$$\frac{1}{16(2L_1+1)(2L_3+1)}\,\gamma_{L_3}\,\alpha_d^{L_1L_2}\,\varepsilon_d^{L_3L_2}\,k_{L_2d}\,[1+\delta_{M0}][1+\delta_{d0}][1+\delta_{d0}\delta_{M0}]\left[\beta_M^{L_1}+(-1)^{L_1}\beta_M^{L_1*}\right]$$

$$=\int\tilde{\tilde{W}}(\theta,\phi;\theta'',\phi'')\left[D_{d0}^{L_3}\,D_{dM}^{L_1*}+D_{d0}^{L_3*}\,D_{dM}^{L_1}\right]d\Omega\,d\Omega''. \qquad (79)$$

From the previous case we already know the values for all $\alpha_d^{L_1L_2}$. Now a close examination of the expressions for the coefficients in (79) shows that we will get complete information on the relative phases of all the E helicity amplitudes for both the $J=1$ and $J=2$ cases. In addition, the relative magnitudes of the A and the E helicity amplitudes can be determined for the $J=1$ case. For the $J=2$ case, these relative magnitudes have to be determined from the single-particle angular distributions separately or from the simultaneous angular distribution of the two photons in case 4.

Case 6: This time we will integrate over the angles (θ,ϕ) to obtain the combined angular distribution of γ_2 and e^- alone. We obtain

$$\tilde{\tilde{W}}(\theta',\phi';\theta'',\phi'') = \int W(\theta,\phi;\theta',\phi';\theta'',\phi'') \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{256\pi^2} \sum_{L_3}^{0,2} \gamma_{L_3} \sum_{L_1}^{0\to 4} \sum_{L_2}^{0\to 2J} \sum_{d}^{0\to \mathrm{Min}(L_2,L_3,J)} \varepsilon_d^{L_3 L_2} \sum_{M(L_1)}^{-2\to 2} \alpha_M^{L_1 L_2}\, k_{L_1 M}\, (1+\delta_{M0})^2$$

$$\times \left[\beta_M^{L_1} D_{d0}^{L_3 *}(\phi'',\theta'',-\phi'') D_{Md}^{L_2 *}(\phi',\theta',-\phi') + (-1)^{L_1} \beta_M^{L_1 *} D_{d0}^{L_3}(\phi'',\theta'',-\phi'') D_{Md}^{L_2}(\phi',\theta',-\phi') \right]. \quad (80)$$

Using the orthogonality of the D^J functions, the coefficients for all possible values of L_1, L_2, L_3, d and M in (80) can be obtained from

$$\frac{1}{16(2L_2+1)(2L_3+1)} \gamma_{L_3}\, \varepsilon_d^{L_3 L_2}\, \alpha_M^{L_1 L_2}\, k_{L_1 M} \left[1+\delta_{M0}\right]^2 \left[1+\delta_{d0}\delta_{M0}\right] \left[\beta_M^{L_1} + (-1)^{L_1} \beta_M^{L_1 *} \right]$$

$$= \int \tilde{\tilde{W}}(\theta',\phi';\theta'',\phi'') \left[D_{d0}^{L_3} D_{Md}^{L_2} + D_{d0}^{L_3 *} D_{Md}^{L_2 *} \right] d\Omega'\, d\Omega''. \quad (81)$$

From (81), we can determine the relative magnitudes as well as the relative phases of all the E helicity amplitudes only for the $J=2$ case by measuring this simultaneous angular distribution of γ_2 and e^-. For the $J=1$ case, we can only determine the cosine of the relative phase between E_1 and E_0 as well as the relative magnitudes of these two helicity amplitudes. In addition, the relative magnitudes of C_0 and C_1 can also be determined for both the $J=1$ and the $J=2$ cases.

4 Concluding remarks

We have derived a model-independent expression for the combined angular distribution of the final electron and the two gamma photons in the cascade process, $\bar{p}p \to {}^3D_2 \to \chi_J + \gamma_1 \to \psi + \gamma_2 + \gamma_1 \to e^+ + e^- + \gamma_2 + \gamma_1 \quad (J=0,1,2)$, when $\bar{p}$ and p are arbitrarily polarized. Our expression is based only on the general principles of quantum mechanics and the symmetry of the problem. We have also derived the partially integrated angular distribution functions which give the angular distributions of γ_1, γ_2 and e^- alone and of (γ_1,γ_2), (γ_1,e^-) and (γ_2,e^-). Once these angular distributions are experimentally measured, our expressions can be used to extract all the independent helicity amplitudes in the radiative decays ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$ for all values of J. In fact, the analysis of the angular correlations in the final decay products will serve to verify the value of J for the intermediate χ state in the cascade process. The experimentally determined values of the helicity amplitudes can then be compared with the predictions of various dynamical models. The great advantage of studying polarized $\bar{p}p$ collisions is that one can get a lot of information on the helicity amplitudes from the measurement of the simultaneous angular distributions of two particles. For example we can get complete information on all the helicity amplitudes in the process ${}^3D_2 \to \chi_J + \gamma_1$ $(J = 1, 2)$ from the simultaneous angular distribution of γ_1 and γ_2 and also in the process $\chi_J \to \psi + \gamma_2$ $(J = 1, 2)$ from the simultaneous angular distribution of γ_1 and e^-, when both $\bar{p}$ and p are polarized with both transverse and longitudinal polarization vector components in their respective frames. In the case of unpolarized $\bar{p}p$ collisions [4] one has to measure the simultaneous angular distribution of all the three particles γ_1, γ_2 and e^- in order to obtain nearly the same information on the A helicity amplitudes in the process ${}^3D_2 \to \chi_2 + \gamma_1$ and the E helicity amplitudes in the process $\chi_2 \to \psi + \gamma_2$. We still could not get complete information on the sines of the relative phases for the $J = 1$ case. Polarizations of both $\bar{p}$ and p are necessary to get this information.

For the $J = 0$ case, there is only one independent helicity amplitude in each of the processes ${}^3D_2 \to \chi_0 + \gamma_1$ and $\chi_0 \to \psi + \gamma_2$ and that is fixed by normalization. In addition, the relative magnitudes of the two independent helicity amplitudes C_0 and C_1 in $\psi \to e^+ + e^-$ can also be obtained.

We should also emphasize that the angular distributions alone will not give the absolute strengths of the helicity amplitudes. We get the magnitudes of all the helicity amplitudes only with the arbitrary normalization convention of (50). In order to get the true absolute values which are physically significant one has to measure the branching ratios of each of the above processes and the parent particle's lifetime or decay width. The measurement of the angular distributions alone will only give the relative magnitudes and the relative phases of the helicity amplitudes in each radiative decay process.

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