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Planar graphs with maximum degree 4 are strongly 19-edge-colorable ⋆

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Abstract

A strong edge-coloring of a graph is a proper edge-coloring such that edges at distance at most 2 receive different colors. It is known that every planar graph has a strong edge-coloring by using at most $4\Delta + 4$ colors, where Δ denotes the maximum degree of the graph. In this paper, we will show that 19 colors are enough to color a planar graph with maximum degree 4.

Key words: Strong edge-coloring, strong chromatic index, plane graph, discharging method.
AMS 2010 MSC: 05C15.

1 Introduction

A *strong k-edge-coloring* of a graph G is a proper k-edge-coloring such that edges at distance at most 2 receive different colors. The strong chromatic index of a graph G, denoted by $\chi'_s(G)$, is the minimum number of colors needed in a strong edge-coloring of G. The concept of strong edge-coloring was introduced by Fouquet and Jolivet [6]. In 1985, Erdős and Nešetřil put forward the following conjecture.

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Conjecture 1.1 (Erdős and Nešetřil [3,4]) *For every graph G,*

$$\chi'_s(G) \le \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even,} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{if } \Delta \text{ is odd.} \end{cases}$$

Andersen [1] and Horák *et al.* [7] confirmed this conjecture for the case that $\Delta = 3$.
For planar graphs with $\Delta \ge 3$, Faudree et al. proved the following theorem.

Theorem 1.1 (Faudree et al. [5]) *If G is a planar graph with $\Delta \ge 3$, then $\chi'_s(G) \le 4\Delta + 4$.*

They also constructed a class of planar graphs with $\Delta \ge 2$ such that $\chi'_s(G) = 4\Delta - 4$.

Kostochka *et al.* proved the following result and pointed out that 9 is the best possible by observing the complement of C_6.

Theorem 1.2 (Kostochka *et al.* [8]) *If G is a planar graph with $\Delta \le 3$, then $\chi'_s(G) \le 9$.*

In this paper, we focus on planar graphs with $\Delta = 4$ and obtain the following result.

Theorem 1.3 *If G is a planar graph with $\Delta = 4$, then $\chi'_s(G) \le 19$.*

In this paper, we always consider a plane drawing of a planar graph G. So we only view G as a plane graph.

The theorem will be shown by handling a special separating 4-cycle and applying the discharging method. We shall first establish some structural properties of a minimal counterexample G to the theorem. Then we shall prove that such plane graph with those properties does not exist.

2 Concepts and notation

All graphs considered in this paper are finite simple undirected plane graphs. For a plane graph G, we denote its vertex set, edge set, face set, maximum degree and minimum degree by $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$, respectively. The set of vertices adjacent to v is called the *neighborhood* of v and is denoted by $N(v)$. The number $|N(v)|$ is called the *degree* of the vertex v and is denoted by $\deg_G(v)$ (or $\deg(v)$ for short). A vertex v is called a *k-vertex* if $\deg(x) = k$; k^--vertex if $\deg(x) \le k$ and *k^+-vertex* if $\deg(x) \ge k$, respectively. Two edges are at distance 1 if they share one of their ends and they are at distance 2 if they are not at distance 1 and there exists an edge adjacent to both of them. Let $N_2(e)$ be the set of edges at distance at most 2 from the edge e. A cycle C of G is called *separating* if its interior and exterior contain each at least one vertex of G. Let $V^0(C)$ denote the set of vertices in G that lie in the interior of C. A vertex or a face in $G[V^0(C)]$ is called an *interior vertex* or *interior face* of C, respectively. For $f \in F(G)$, we use $\partial(f)$ to denote the boundary walk of f and write $f = [u_1 u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $\partial(f)$ in clockwise order. Repeated occurrences of a vertex are allowed. The *degree* of a face is the number of edge-steps in its boundary walk and is denoted by $d_G(f)$ (or $d(f)$ for short). A face f is called a *k-face* if $d(f) = k$; k^--face if $d(f) \le k$ and k^+-*face* if $d(f) \ge k$, respectively. Other concepts and notation not defined in this paper can be found in the book of Bondy and Murty [2].

3 Structural properties

Suppose Theorem 1.3 does not hold. Let G be a counterexample with the minimum number of $|V(G)|+|E(G)|$. Then the minimality of G implies that every proper subgraph G' of G admits a strong edge-coloring by using at most 19 colors and that G is connected. Let $L = \{1, 2, \ldots, 19\}$ be a color set. Suppose that σ is a strong 19-edge-coloring of a proper subgraph of G. For an edge e in G, let $c_\sigma(e)$ and $C_\sigma(e)$ denote the color assigned to e and the set of colors assigned to edges in $N_2(e)$ under the coloring σ, respectively. Then, $A_\sigma(e) = L \backslash C_\sigma(e)$ is the set of available colors for e under the coloring σ. Further, we let $S_\sigma(v)$ denote the set of colors assigned to the edges incident to v under coloring σ. In what follows, we shall keep all notation defined above.

Claim 1 $\delta(G) \ge 3$.

Proof. Suppose to the contrary that there exists a vertex v of degree at most 2. By the minimality of G, the graph $G' = G - v$ admits a strong 19-edge-coloring σ.

If $\deg(v) = 1$, then let u denote the neighbor of v. It is easy to calculate that $|C_\sigma(uv))| \le 12$ and thus $|A_\sigma(uv)| \ge 7$. Hence, we may extend the coloring σ to G by assigning an available color from $A_\sigma(uv)$ to uv.

If $\deg(v) = 2$, then let $N(v) = \{u, w\}$. One may verify that $|A_\sigma(e)| \ge 4$ for each edge $e \in \{vu, vw\}$. Therefore, we can extend σ to G by coloring vu, vw in succession.

In both cases, we always obtain a strong 19-edge-coloring of G, a contradiction. □

Claim 2 *No* 3*-vertex is incident to a* 3*-cycle.*

Proof. Suppose to the contrary that G contains a 3-vertex v incident to a 3-cycle vv_1v_2v. Let $N(v) = \{v_1, v_2, v_3\}$. By the minimality of G, the graph $G - v$ admits a strong 19-edge-coloring σ. For each edge $e \in \{vv_1, vv_2\}$, we see that $|A_\sigma(e)| \ge 5$ and $|A_\sigma(vv_3)| \ge 2$. So we can properly color vv_3, vv_1, vv_2 in succession to obtain a strong 19-edge-coloring of G, a contradiction. □

Claim 3 *The distance between two* 3*-vertices is at least* 3.

Proof. Let v and u be two 3-vertices.

Suppose the distance between u and v is 1. Let $N(v) = \{v_1, v_2, u\}$ and $N(u) = \{u_1, u_2, v\}$. By the minimality of G, the graph $G-\{v, u\}$ admits a strong 19-edge-coloring σ. We have $|A_\sigma(e)| \ge 4$ for $e \in \{vv_1, vv_2, uu_1, uu_2\}$, and $|A_\sigma(uv)| \ge 7$. So, we can color the five uncolored edges vv_1, vv_2, uu_1, uu_2, uv in succession to deduce a strong 19-edge-coloring of G, a contradiction.

Suppose the distance between u and v is 2. Let $N(v) = \{v_1, v_2, w\}$, $N(u) = \{u_1, u_2, w\}$ and $N(w) = \{u, v, w_1, w_2\}$. We consider two cases below:

Case 1: Suppose $\{u_1, u_2\} \cap \{v_1, v_2\} \neq \varnothing$. Without loss of generality, we may assume $v_1 = u_1$. Let σ be a strong 19-edge-coloring of $G' = G - \{u, v, w\}$. Now we have $|A_\sigma(e)| \geq 4$ for $e \in \{ww_1, ww_2\}$; $|A_\sigma(e)| \geq 5$ for $e \in \{uu_2, vv_2\}$; and $|A_\sigma(e)| \geq 8$ for $e \in \{uu_1, vu_1, uw, vw\}$. So we may color the uncolored edges ww_1, ww_2, uu_2, vv_2, uu_1, vu_1, uw, vw in succession.

Case 2: Suppose $\{u_1, u_2\} \cap \{v_1, v_2\} = \varnothing$. Since G is a plane graph, one of the elements of $\{v_1u_1$, v_1u_2, v_2u_1, $v_2u_2\}$ is not an edge of G. Without loss of generality, we assume $v_1u_1 \notin E(G)$. Here $vv_1 \notin N_2(uu_1)$. By the minimality of G, the graph $G' = G - \{v, w, u\} + v_1u_1$ is also a plane graph and admits a strong 19-edge-coloring φ. Assume $c_\varphi(v_1u_1) = \alpha$. Based on the coloring φ of G' we can obtain a coloring σ of $G'' = G - \{v, w, u\}$. Now we shall extend σ from G'' to G by considering the following subcases. Note that, $|A_\sigma(e)| \geq 7$ for $e \in \{uw, vw\}$, $|A_\sigma(e)| \geq 4$ for $e \in \{uu_1, uu_2, vv_1, vv_2, ww_1, ww_2\}$.

2-1: Suppose $|A_\sigma(uw)| = |A_\sigma(vw)| = 7$ and $A_\sigma(uw) = A_\sigma(vw)$. This implies that $S_\sigma(u_1) \cup S_\sigma(u_2) = S_\sigma(v_1) \cup S_\sigma(v_2)$. Since $S_\sigma(u_1) \cap S_\sigma(v_1) = \varnothing$, $S_\sigma(u_1) = S_\sigma(v_2)$ and $S_\sigma(u_2) = S_\sigma(v_1)$. Since $\alpha \notin S_\sigma(u_1) \cup S_\sigma(v_1)$, we have that $\alpha \notin S_\sigma(u_2) \cup S_\sigma(v_2)$. So we may first assign α to both vv_1 and uu_1. Let the resulting coloring be ψ. Now $|A_\psi(e)| \geq 6$ for $e \in \{uw, vw\}$, $|A_\psi(e)| \geq 3$ for $e \in \{uu_2, vv_2, ww_1, ww_2\}$. Note that if $v_2u_2 \in E(G)$, then $|A_\psi(e)| \geq 4$ for $e \in \{vv_2, uu_2\}$. If $v_2u_2 \notin E(G)$, then $uu_2 \notin N_2(vv_2)$. So we can color the remaining edges $ww_1, ww_2, vv_2, uu_2, uw, vw$ in succession to obtain a strong 19-edge-coloring of G.

2-2: Suppose $|A_\sigma(e)| \geq 8$ for some $e \in \{uw, vw\}$, say vw, or $|A_\sigma(uw)| = |A_\sigma(vw)| = 7$ and $A_\sigma(uw) \neq A_\sigma(vw)$. Now we focus on $|A_\sigma(uu_2)|$, if $u_2v_1, u_2v_2 \in E(G)$, then $|A_\sigma(uu_2)| \geq 6$. If there exists only one edge in $\{u_2v_1, u_2v_2\}$, say $u_2v_1 \in E(G)$, then $|A_\sigma(uu_2)| \geq 5$ and $uu_2 \notin N_2(vv_2)$. If $u_2v_1, u_2v_1 \notin E(G)$, then $uu_2 \notin N_2(vv_1)$ and $uu_2 \notin N_2(vv_2)$. So we may color ww_1, ww_2, vv_1, vv_2, uu_1, uu_2, uw, vw successively.

Hence we obtain a strong 19-edge-coloring for G which yields a contradiction. □

Claim 4 *There is no common vertex of two 3-cycles.*

Proof. By Claims 1 and 2, all vertices incident with these two cycles are 4-vertices. Now we consider the following two cases:

Case 1. Suppose uv is a common edge of two 3-cycles $uxvu$ and $vyuv$ counted in clockwise order. Let $N(u) = \{u_1, y, v, x\}$ and $N(v) = \{v_1, x, u, y\}$. By the minimality of G, the graph $G - \{u, v\}$ admits a strong 19-edge-coloring σ. We have that $|A_\sigma(e)| \geq 3$ for $e \in \{uu_1, vv_1\}$; $|A_\sigma(e)| \geq 6$ for $e \in \{yu, yv, xu, xv\}$ and $|A_\sigma(uv)| \geq 9$. In this case, we can color the remaining edges uu_1, vv_1, yu, yv, xu, xv, uv in succession.

Case 2. Suppose two 3-cycles $vuxv$ and $vwyv$ only have one common vertex v. By the minimality of G, the graph $G - v$ admits a strong 19-edge-coloring σ. Let $E_1 = \{vu, vx, vy, vw\}$. For each $e \in E_1$, we have that $|A_\sigma(e)| \geq 3$. If there exists an edge $e \in E_1$ such that $|A_\sigma(e)| \geq 4$, then G admits a strong 19-edge-coloring. So we assume that $|A_\sigma(e)| = 3$ for each $e \in E_1$.

2-1. Suppose each $e \in E_1$ has the same $A_\sigma(e)$. Hence $C_\sigma(vu) = C_\sigma(vx)$. We may first recolor ux. Since at least 8 colors can be used to recolor ux, we may select a color $\beta \notin A_\sigma(e)$. Now, the available colors for each $e \in E_1$ is 4. So we may always derive a strong 19-edge-coloring, a contradiction.

2-2. Suppose there is a color α such that $\alpha \in A_\sigma(e_1)$ and $\alpha \notin A_\sigma(e_2)$ for some $e_1, e_2 \in E_1$. Suppose $e_1 = uv$ and $e_2 = vx$. It suffices to first color uv with α and then color vw, vy, vx in succession. Now we obtain a 19-edge-coloring, a contradiction. For other cases, we may also obtain a contradiction by a similar argument. □

Claim 5 *No 3-vertex is incident to a 4-face.*

Proof. Suppose a 3-vertex y is incident to a 4-face $f = [vwuy]$, as depicted in Fig. 1. By Claims 1 and 3, other boundary vertices of f are 4-vertices. Let $N(v) = \{w, y, v_1, v_2\}$, $N(w) = \{u, v, w_1, w_2\}$, $N(u) = \{y, w, u_1, u_2\}$ and $N(y) = \{v, u, x\}$.

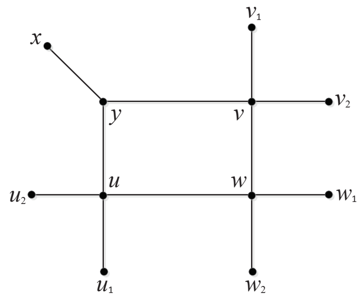

Fig. 1. A 3-vertex incident to a 4-face.

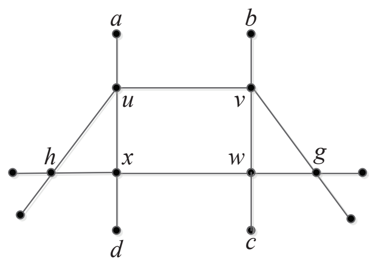

Fig. 2. A 4-face adjacent to two 3-faces.

Note that, by Claim 2, y cannot be incident to a 3-cycle. Thus $x \notin \{u_1, u_2, v_1, v_2, w\}$.

Case 1. Suppose one of the following cases occurs:

- $x \in \{w_1, w_2\}$.
- $\{w_1, w_2\} \cap \{v_1, v_2, u_1, u_2\} \neq \varnothing$.
- $u \in \{v_1, v_2\}$ or $v \in \{u_1, u_2\}$.
- $\{v_1, v_2\} \cap \{u_1, u_2\} \neq \varnothing$.

By the minimality of G, the graph $G' = G - y$ admits a strong 19-edge-coloring σ. It is easy to extend σ as a strong 19-edge-coloring of G by coloring the remaining uncolored edges xy, yu, yv in succession. Hence it is a contradiction.

Case 2. All vertices shown in Fig. 1 are distinct. Since G is a plane graph, one of the elements of $\{v_1u_1,\ v_1u_2,\ v_2u_1,\ v_2u_2\}$ is not an edge of G. Without loss of generality, we assume $v_1u_1 \notin E(G)$, then $vv_1 \notin N_2(uu_1)$. Obviously, the graph $G' = G - \{v, w, u, y\} + v_1u_1$ is also a plane graph and admits a strong 19-edge-coloring φ. Assume $c_\varphi(v_1u_1) = \alpha$. Based on the coloring φ of G' we can obtain a coloring σ of $G'' = G - \{v, w, u, y\}$. Now we shall extend σ from G'' to G. We have $|A_\sigma(e)| \geq 4$ for $e \in \{uu_1, uu_2, ww_1, ww_2, vv_1, vv_2\}$; $|A_\sigma(e)| \geq 7$ for $e \in \{xy, uw, vw\}$; $|A_\sigma(e)| \geq 10$ for $e \in \{yu, yv\}$.

2-1. Suppose there exists an edge $e_1 \in \{uw, vw\}$ such that $|A_\sigma(e_1)| \geq 8$, say vw, or $|A_\sigma(uw)| = |A_\sigma(vw)| = 7$ and $A_\sigma(uw) \neq A_\sigma(vw)$. Now, we consider the edges yu

and yv.

If there exists an edge $e_2 \in \{yu, yv\}$ such that $|A_\sigma(e_2)| \geq 11$, say yv, or $|A_\sigma(yu)| = |A_\sigma(yv)| = 10$ and $A_\sigma(yu) \neq A_\sigma(yv)$, then similar to Case 2 of the proof of Claim 3 we can color the remaining edges ww_1, ww_2, uu_1, uu_2, vv_1, vv_2, uw, vw, xy, yu, yv in succession.

If $|A_\sigma(yu)| = |A_\sigma(yv)| = 10$ and $A_\sigma(yu) = A_\sigma(yv)$, then we have that $S_\sigma(u_1) \cup S_\sigma(u_2) = S_\sigma(v_1) \cup S_\sigma(v_2)$. Since $S_\sigma(u_1) \cap S_\sigma(v_1) = \varnothing$, we get that $S_\sigma(u_1) = S_\sigma(v_2)$, $S_\sigma(u_2) = S_\sigma(v_1)$. Since $\alpha \notin S_\sigma(u_1) \cup S_\sigma(v_1)$, $\alpha \notin S_\sigma(u_2) \cup S_\sigma(v_2)$.

Based on the coloring σ of G'' we assign the color α to vv_1 and uu_1 to obtain a coloring of $G''' = G'' + vv_1 + uu_1$. We denote this coloring by ψ. Note that $|A_\psi(e)| \geq 3$ for $e \in \{ww_1, ww_2, vv_2, uu_2\}$; $|A_\psi(e)| \geq 6$ for $e \in \{uw, vw, xy\}$ and $|A_\psi(e)| \geq 9$ for $e \in \{yu, yv\}$. If $v_2u_2 \in E(G)$, then $|A_\psi(e)| \geq 4$ for $e \in \{vv_2, uu_2\}$. Otherwise, $v_2u_2 \notin E(G)$, then $uu_2 \notin N_2(vv_2)$. So we may first color ww_1, ww_2, vv_2, uu_2 successively. Let the resulting coloring be θ. Now $|A_\theta(e)| \geq 2$ for $e \in \{uw, vw\}$, $|A_\theta(xy)| \geq 4$ and $|A_\theta(e)| \geq 5$ for $e \in \{yu, yv\}$. We may color uw, vw, xy, yu, yv successively.

Hence we obtain a strong 19-edge-coloring for G which yields a contradiction.

2-2. If $|A_\sigma(uw)| = |A_\sigma(vw)| = 7$ and $A_\sigma(uw) = A_\sigma(vw)$, then we have that $S_\sigma(u_1) \cup S_\sigma(u_2) = S_\sigma(v_1) \cup S_\sigma(v_2)$. By the same argument as the previous case, we have $\alpha \notin S_\sigma(u_2) \cup S_\sigma(v_2)$ and obtain a contradiction. □

In addition, if G does not contain any separating 4-cycle, then G has another two structural properties shown below.

Claim 6 *Suppose G has no separating 4-cycle. Every 4-face of G is adjacent to at most one 3-face.*

Proof. Let $f = [uvwx]$ be a 4-face adjacent to two 3-faces. By Claim 4, these two 3-faces are vertex disjoint. Without loss of generality, we assume that these 3-faces are $[uxh]$ and $[vgw]$, as depicted in Fig. 2. By Claims 4, $\{a, c\} \cap \{b, d\} = \varnothing$, $a \notin \{w, g\}$, $b \notin \{x, h\}$, $g \neq h$. Since G has no separating 4-cycle, we have $a \neq c$ and $b \neq d$. Then, all vertices shown in Fig. 2 are distinct.

Since G is a plane graph, one of the elements of $\{ac, bd\}$ is not an edge of G. By symmetry, we assume $ac \notin E(G)$. By the minimality of G, the graph $G' = G - \{v, w, x, u\} + ac$ admits a strong 19-edge-coloring σ. Assume $c_\sigma(ac) = \alpha$. Now we turn back to color the graph G. Firstly, we color the edges in graph $G - \{v, w, x, u\}$ by same color as those in G'. Secondly, assign the color α to au and cw. If there exists an edge e incident to g or h with color α under σ, since $\deg_{G'}(v) = 2$ for $v \in \{g, h\}$, we can recolor e using a color different from α. Now, the number of available colors for the edges bv, dx, gv, gw, hu, hx, uv, xw, ux, vw is at least 4, 4, 7, 7, 7, 7, 8, 8, 10, 10, respectively. Color the rest edges according to the above order and obtain a strong 19-edge-coloring of G, which is a contradiction. □

Claim 7 *Suppose G has no separating 4-cycle. For two adjacent edges in a 4-face of G, if one is incident to a 3-face, then the other is incident to a 5^+-face.*

Proof. Let $[uvwx]$ be a 4-face. Suppose ux is incident with a 3-face $[uxa]$, and xw is incident with a face f. From Claim 2, we know that $\deg(x) = 4$. So ax is not an edge of f. Now we let $N(x) = \{u, w, d, a\}$. By Claim 4, f must be a 4^+-face. Next, we will show that f is a 5^+-face. Otherwise, assume that $f = [xwbd]$. For the worst case, we assume the vertices in Fig. 3 are 4-vertices. That is, $N(u) = \{v, x, a, y\}$, $N(w) = \{v, t, b, x\}$ and $N(a) = \{u, x, a_2, a_1\}$. By Claim 4, $a \neq b$, $t \notin \{a, u, d\}$ and $\{a_1, a_2\} \cap \{d, y, w\} = \varnothing$. Since G has no separating 4-cycle, we may further derive that $b \notin \{a_1, a_2\}$ and $t \notin \{y, a_1, a_2\}$. Then, all vertices shown in Fig. 3 are different.

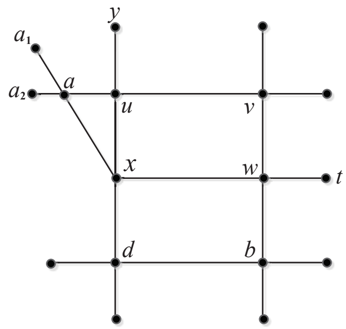

Fig. 3. A special structure of a 3-face adjacent to a 4-face.

By the minimality of G, the graph $G' = G - \{a, u, x\}$ admits a strong 19-edge-coloring φ. We have that $|A_\varphi(e)| \geq 4$ for $e \in \{aa_1, aa_2, uy\}$, $|A_\varphi(e)| \geq 5$ for $e \in \{uv, xd\}$, $|A_\varphi(wx)| \geq 6$, $|A_\varphi(e)| \geq 7$ for $e \in \{au, ax\}$, and $|A_\varphi(ux)| \geq 8$. We will extend φ to G properly and thus complete the proof. At this moment, we may color aa_1, aa_2, uy, uv and xd in succession. Let the resulting (partial) coloring of G be σ. Now, $|A_\sigma(wx)| \geq 1$, $|A_\sigma(e)| \geq 2$ for $e \in \{au, ax\}$ and $|A_\sigma(ux)| \geq 3$.

Suppose there exists an edge $e \in \{au, ax\}$ satisfying $|A_\sigma(e)| \geq 4$, say au. We can color the remaining edges wx, ax, ux, au in succession. So we only need to deal with $|A_\sigma(wx)| \geq 1$, $2 \leq |A_\sigma(e)| \leq 3$ for $e \in \{au, ax\}$ and $|A_\sigma(ux)| \geq 3$.

Case 1. Suppose there exists an edge $e \in \{au, ax\}$, say au, satisfying that $|A_\sigma(e)| = 3$.

If either $|A_\sigma(ux)| \geq 4$ or $|A_\sigma(ux)| = 3$ and $A_\sigma(au) \neq A_\sigma(ux)$, we can color the remaining edges wx, ax, au, ux in succession.

If $|A_\sigma(ux)| = 3$ and $A_\sigma(au) = A_\sigma(ux)$, then we see that $S_\sigma(a_1) \cup S_\sigma(a_2) \setminus \{c_\sigma(aa_1), c_\sigma(aa_2)\} = S_\sigma(d) \cup \{c_\sigma(wb), c_\sigma(wt)\} \setminus \{c_\sigma(xd)\}$. It follows immediately that $|A_\sigma(ax)| \geq 7$. In this case, we can color the remaining edges wx, au, ux, ax in succession.

Case 2. Suppose that $|A_\sigma(au)| = |A_\sigma(ax)| = 2$.

Suppose that $A_\sigma(au) = A_\sigma(ax)$. This implies that $S_\sigma(y) \cup S_\sigma(v) = S_\sigma(d) \cup S_\sigma(w)$. It is obvious that $|A_\sigma(xw)| \geq 3$ and $|A_\sigma(ux)| \geq 8$. It suffices to color the remaining uncolored edges au, ax, xw, ux in succession.

Now we assume that $A_\sigma(au) \neq A_\sigma(ax)$, i.e., $C_\sigma(au) \neq C_\sigma(ax)$. For convenience, we let $X = C_\sigma(ux)$, $Y = C_\sigma(au)$, $Z = C_\sigma(ax)$ and $K = X \cap Y \cap Z$. Then we have

$$\begin{aligned} K &= \{\sigma(uy), \sigma(uv), \sigma(vw), \sigma(dx), \sigma(aa_1), \sigma(aa_2)\} \\ X \cap Y \setminus K &= S_\sigma(y) \cup S_\sigma(v) \setminus \{\sigma(uy), \sigma(uv), \sigma(vw)\} \\ X \cap Z \setminus K &= S_\sigma(w) \cup S_\sigma(d) \setminus \{\sigma(vw), \sigma(dx)\} \\ Y \cap Z \setminus K &= S_\sigma(a_1) \cup S_\sigma(a_2) \setminus \{\sigma(aa_1), \sigma(aa_2)\} \end{aligned}$$

So $5 \le |K| \le 6$, $3 \le |X \cap Y \setminus K| \le 5$, $4 \le |X \cap Z \setminus K| \le 5$, $3 \le |Y \cap Z \setminus K| \le 6$. Moreover, $|X \cup Y \cup Z| \le 19$, $|Y| = |Z| = 17$. Thus

$$\begin{aligned} |X| &= |X \cup Y \cup Z| - |Y| - |Z| + |X \cap Y \setminus K| + |X \cap Z \setminus K| + |Y \cap Z \setminus K| + 2|K| \\ &\le 19 - 17 - 17 + 5 + 5 + 6 + 12 = 13. \end{aligned}$$

So $|A_\sigma(ux)| \ge 6$. We color xw by α first, say. If $\alpha \notin A_\sigma(ax) \cup A_\sigma(au)$, then it is easy to color the remaining uncolored edges au, ax, ux in succession. Suppose $\alpha \in A_\sigma(ax) \cup A_\sigma(au)$. Without loss of generality, we may assume that $A_\sigma(ax) = \{\alpha, \beta\}$. Under our assumption, there is a color $\gamma \in A_\sigma(au) \setminus A_\sigma(ax)$. Now we may color ax and au by β and γ, respectively. Finally we may color ux. So we obtain a strong 19-edge-coloring of G, which is a contradiction. □

4 Discharging

Suppose v is a 4-vertex in G. When no 3-face is incident to v, v is called a *good* 4-vertex. When the faces incident to v are 3-face, 4-face, 5-face, 4-face in clockwise, v is called a *bad* 4-vertex. Otherwise, when v is incident to a 3-face, v is called a *weak* 4-vertex.

We assign the initial charge $w(v) = 2\deg(v) - 6$ to every vertex v, and $w(f) = d(f) - 6$ to every face f. By Euler's formula,

$$\sum_{v \in V(G)} (2\deg(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

Note that, the sum of all charges is -12, and any discharging procedure preserves the total charge of G. We redistribute the charges among the vertices and the faces in G by the following discharging rules. At the same time, we focus on the final charge $w'(x)$, $x \in V(G) \cup F(G)$.

R1: Every bad 4-vertex sends 1 to the incident 3-face.
R2: Every bad 4-vertex sends $\frac{1}{2}$ to every incident 4-face.
R3: Every weak 4-vertex sends 1 to the incident 3-face.
R4: Every weak 4-vertex sends $\frac{1}{2}$ to the incident 4-face if any.
R5: Every weak 4-vertex sends $\frac{1}{4}$ to every incident 5-face.
R6: Every good 4-vertex sends $\frac{1}{2}$ to every incident 4-face and 5-face.

Now, we are ready to prove Theorem 1.3.

Proof. Suppose, to the contrary, that G is a minimal counterexample to the theorem. We first assume G does not contain separating 4-cycle. We use the structural properties of G to show that after applying the discharging rules the charge of each vertex and face is nonnegative.

First, consider the vertices of G. By Claim 1, G does not contain 1-vertex and 2-vertex. By the discharging rules, $w(v) = w'(v) = 0$ when v is a 3-vertex; $w(v) = 2$ and $w'(v) \geq 0$ when v is a 4-vertex.

Now, we consider the final charge of a face f in terms of its degree.

Suppose f is a 3-face. We have $w(f) = -3$. By Claims 1 and 2, each vertex incident to f is a 4-vertex. By R1 and R3, f receives 1 from every incident 4-vertex. Thus $w'(f) = 0$.

Suppose f is a 4-face. We have $w(f) = -2$. By Claims 1 and 5, each vertex incident to f is a 4-vertex. By R2, R4 and R6, f receives $\frac{1}{2}$ from each incident 4-vertex. Thus $w'(f) = 0$.

Suppose f is a 5-face, we have $w(f) = -1$. By Claims 1 and 3, there is at most one 3-vertex incident to f, the others are 4-vertices. We consider the following two cases:

Case 1: Suppose all vertices incident to f are 4-vertices. By Claims 4, 6 and 7, there are at most two bad 4-vertices and at least one good 4-vertex. By R5 and R6, $w'(f) \geq -1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \geq 0$.

Case 2: Suppose there is a 3-vertex incident to f. By Claims 4, 6 and 7, there is at most one bad 4-vertex incident to f. When the number of bad 4-vertex is 1, there is at least one good 4-vertex incident to f. By R5 and R6, f receives at least $\frac{1}{2}$ from each incident good 4-vertex and $\frac{1}{4}$ from each weak 4-vertex. Thus, $w'(f) \geq -1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \geq 0$. When the number of bad 4-vertex is 0. Then, $w'(f) \geq -1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \geq 0$.

We have shown that the final charge of every vertex and face in G is nonnegative, and so is the sum of all charges. Hence, when G does not contain separating 4-cycle, a minimal counterexample to Theorem 1.3 does not exist.

When G contains a separating 4-cycle. We choose a separating 4-cycle C in G with the least sum of interior vertices and edges. Let E' be the set of edges incident with at least one interior vertex of C. Put $H = G[E'] \cup C$, where $G[E']$ is the subgraph induced by E'. For each vertex $v \in V^0(C)$, it is straightforward to see that $\deg_H(v) = \deg_G(v)$. Since G is connected, so is H. Obviously, there does not exist any separating 4-cycle in H. Let F_1 be the set of interior faces of C. According to the above discharging rules, we have $w'(x) \geq 0$, where $x \in V^0(C) \cup F_1$.

Obviously, $\deg_H(v) \geq 2$ for the vertex $v \in V(C)$. Now, we consider the existence of cut vertex in G, and get the following claim.

Claim 8 *G does not contain any cut vertex.*

Proof. Assume v is a cut vertex in G. Here $G - v = G_1 \cup G_2$, where G_1 and G_2 are subgraphs of G. Let $G_1' = G[V(G_1) \cup \{v\}]$ and $G_2' = G[V(G_2) \cup \{v\}]$ be the induced subgraphs of G, respectively. By the minimality property of G, there are strong 19-edge-colorings σ_1 and σ_2 of G_1' and G_2', respectively. By symmetry, we have to deal with the following two cases:

Case 1: Suppose v has only one neighbor in G_1. Without loss of generality, we may assume that the set of colors assigned by σ_1 to edges incident to v or its neighbor in G_1 is $\{1, 2, 3, 4\}$, and the set of colors assigned by σ_2 to edges incident to v or its neighbors in G_2 is $\{5, 6, \ldots, 16\}$.

Case 2: Suppose v has exactly two neighbors in G_1. Without loss of generality, we may assume that the set of colors assigned by σ_1 to edges incident to v or its neighbors in G_1 is $\{1, 2, \ldots, 8\}$, and the set of colors assigned by σ_2 to edges incident to v or its neighbors in G_2 is $\{9, 10, \ldots, 16\}$.

In the above two cases, we can easily obtain a strong 19-edge-coloring of G, which is a contradiction. □

By Claim 8, there are at least two 3^+-vertices in $V(C)$.

In the following, we show that after applying the discharging rules the sum of the charges of H is greater than -12. We shall consider the discharging rules among vertices $v \in V(C)$ and faces $f \in F(H) \setminus F_1$. Now we define the discharging rule on vertices in $V(C)$ as follows:

R7: Every vertex $v \in V(C)$ sends 1 to the incident 3-face, $\frac{1}{2}$ to the incident 4-face or 5-face.

By Claims 1, 2 and 5 each vertex in $V^0(C)$ incident to 4^--face must be a 4-vertex. According to the above discharging rules, each face $f \in F(H) \setminus F_1$ has final charge $w'(f) \geq 0$. Next, we focus on the final charge of the vertex $v \in V(C)$.

When the degree of v is 2 in H, the initial charge $w(v) = -2$. By R7, v sends at most 1 to the incident inner face and $\frac{1}{2}$ to the outer 4-face. Now, the final charge $w'(v) \geq -\frac{7}{2}$.

When the degree of v is 3 in H, the initial charge $w(v) = 0$. By Claim 4, there is at most one 3-face incident with v. By R7, v sends at most 1 to the incident 3-face and $\frac{1}{2}$ to the incident inner 4-face or 5-face, $\frac{1}{2}$ to the outer 4-face. Now, the final charge $w'(v) \geq -2$.

When the degree of v is 4 in H, the initial charge $w(v) = 2$. By Claim 4, each vertex in $V(C)$ has at most one incident 3-face. By R7, v gives 1 to the incident 3-face and $\frac{1}{2}$ to the incident internal 4-face or 5-face, $\frac{1}{2}$ to the external 4-face, then the final charge $w'(v) \geq -\frac{1}{2}$.

In summary, we have $\sum_{v \in V(C)} w'(v) \geq -\frac{7}{2} \times 2 + -2 \times 2 \geq -11$. So $\sum_{x \in V(H) \cup F(H)} w'(x) \geq -11$. This is a contradiction. Hence, a minimal counterexample to Theorem 1.3 does not exist. ■

5 Concluding remarks

Theorem 1.3 affirms that every planar graph G with $\Delta = 4$ has $\chi'_s(G) \leq 19$. Faudree *et al.* [5] constructed planar graphs G with $\Delta = 4$ and $\chi'_s(G) = 12$, so the upper bound 19 still has room for improvement. In the following, we give two examples of planar graphs G with $\Delta = 4$ and $\chi'_s(G) = 13$ (see Fig. 4).

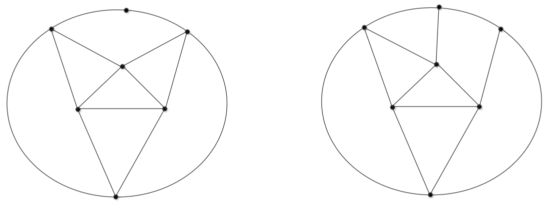

Fig. 4. Two planar graphs G with $\Delta = 4$ and $\chi'_s(G) = 13$.

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