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DISCUSSIONS to

“The Time Value of Ruin in a Sparre Andersen Model,”

Hans U. Gerber and Elias S. W. Shiu, April 2005

CHUANCUN YIN* AND SUNG NOK CHIU**

The ruin problem for a renewal risk model is often difficult to solve; Professors Gerber and Shiu have elegantly extended the results for the classical model in Gerber and Shiu (1998) to a Sparre Andersen risk model with Generalized Erlang interclaim times. The purpose of this discussion is to present explicit formulas for $\phi(u)$ and $f(x, y|u)$ for the special case in which $\hat{p}(\xi)$ is a rational function.

Gerber and Shiu express $\hat{\phi}$ by (eq. 7.2)

$$\hat{\phi}(\xi) = \frac{\hat{\omega}(\xi) - q(\xi)}{\hat{\gamma}(\xi) - \hat{p}(\xi)}$$

It is showed by Gerber and Shiu in Section 4 that, in the right half of the complex plane, the function in the denominator of (1) has n zeros $\rho_1, \rho_2, \dots, \rho_n$. For simplicity, we assume that they are distinct. By (9.2) and (11.1), we can write

$$\hat{\phi}(\xi) = \frac{\widehat{S\omega}(\xi)}{1 - \widehat{Sp}(\xi)}, \quad (1)$$

where the operator S is defined by

$$S = \frac{\lambda_1 \cdots \lambda_n}{c^n} \prod_{j=1}^n T_{\rho_j}.$$

Moreover, by (9.10), $(Sp)(y)$ can be written as a linear function of $T_{\rho_j}p(y)$'s:

$$(Sp)(y) = g(y) = \frac{\prod_{i=1}^n \lambda_i}{c^n} \left[\sum_{j=1}^n \left(\prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) T_{\rho_j} p \right] (y).$$

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It follows that the denominator of (1) is a rational function if and only if $\hat{p}(\xi)$ is a rational function. Let $\theta_1, \dots, \theta_k$ be the distinct roots with negative real parts of the equation $\hat{g}(\xi) = 1$ and n_1, \dots, n_k be their multiplicities, respectively. From equation (11.4) we can see that the θ_k 's are exactly the zeros with negative real parts of equation (4.2). By the principle of partial fractions we may write

$$\frac{1}{1 - \hat{g}(\xi)} = P_1 \left(\frac{1}{\xi - \theta_1} \right) + \dots + P_k \left(\frac{1}{\xi - \theta_k} \right) + 1,$$

where

$$P_j(\alpha) = c_{j,n_j} \alpha^{n_j} + c_{j,n_j-1} \alpha^{n_j-1} + \dots + c_{j,1} \alpha, \quad j = 1, 2, \dots, k.$$

Denote by

$$Q_j(u) = c_{j,n_j} \frac{1}{(n_j - 1)!} u^{n_j-1} e^{\theta_j(\delta)u} + c_{j,n_j-1} \frac{1}{(n_j - 2)!} u^{n_j-2} e^{\theta_j(\delta)u} + \dots + c_{j,1} e^{\theta_j(\delta)u},$$

we have

$$\hat{Q}_j(\xi) = P_j \left(\frac{1}{\xi - \theta_j} \right), \quad j = 1, 2, \dots, k.$$

Hence (1) can be written as

$$\hat{\phi}(\xi) = \sum_{j=1}^k \left(\hat{Q}_j(\xi) \cdot \hat{S}\omega(\xi) \right) + \widehat{S}\omega(\xi).$$

Inverting the Laplace transform yields

$$\phi(u) = \sum_{j=1}^k \int_0^u Q_j(u-x) \cdot (S\omega)(x) dx + (S\omega)(u). \quad (2)$$

In particular, when $n_1 = \dots = n_k = 1$, we have

$$\phi(u) = \sum_{j=1}^k A_j \int_0^u e^{\theta_j(u-x)} \cdot (S\omega)(x) dx + (S\omega)(u), \quad (3)$$

where

$$A_j = -\frac{1}{\hat{g}'(\theta_j)}, \quad j = 1, 2, \dots, k.$$

Moreover, the discounted joint probability density function $f(x, y|u)$ defined by (2.1) can be obtained from (2) or (3). For simplicity, we assume that all roots with negative real parts of $\hat{g}(\xi) = 1$ are simple, that is, $n_1 = \dots = n_k = 1$. It follows from (9.11) that

$$(S\omega)(u) = \int_0^\infty \int_0^\infty w(x, y) 1(x \geq u) f(x - u, y + u|0) dx dy,$$

from which we obtain

$$\int_0^u e^{\theta_j(u-z)} (S\omega)(z) dz = \int_0^\infty \int_0^\infty w(x, y) \left(\int_0^{u \wedge x} e^{\theta_j(u-z)} f(x - z, y + z|0) dz \right) dx dy.$$

Consequently,

$$\begin{aligned} \phi(u) &= \int_0^\infty \int_0^\infty w(x, y) \sum_{j=1}^k A_j \left(\int_0^{u \wedge x} e^{\theta_j(u-z)} f(x - z, y + z|0) dz \right) dx dy \\ &= \int_0^\infty \int_0^\infty w(x, y) 1(x \geq u) f(x - u, y + u|0) dx dy. \end{aligned} \quad (4)$$

Recall that $\phi(u) = \int_0^\infty \int_0^\infty w(x, y) f(x, y|u) dx dy$; by comparing this with (4) we find that

$$\begin{aligned} f(x, y|u) &= \sum_{j=1}^k A_j \left(\int_0^{u \wedge x} e^{\theta_j(u-z)} f(x - z, y + z|0) dz \right) \\ &\quad + 1(x \geq u) f(x - u, y + u|0) \\ &= \frac{\prod_{l=1}^n \lambda_l}{c^n} p(x + y) \sum_{j=1}^k A_j \left[e^{\theta_j u} \sum_{i=1}^n e^{-\rho_i x} \frac{(1 - e^{-(u \wedge x)(\theta_j - \rho_i)})}{\theta_j - \rho_i} \prod_{k=1, k \neq i}^n \frac{1}{\rho_k - \rho_i} \right] \\ &\quad + 1(x \geq u) f(x - u, y + u|0), \end{aligned}$$

with (see eq. 8.3)

$$f(x, y|0) = \frac{\prod_{l=1}^n \lambda_l}{c^n} p(x + y) \sum_{j=1}^n \left[e^{-\rho_j x} \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right].$$

In particular, we get the marginal discounted probability density function

$$\begin{aligned} f_{U(T-)}(x|u) &= \frac{\prod_{l=1}^n \lambda_l}{c^n} \bar{P}(x) \sum_{j=1}^k A_j \left[e^{\theta_j u} \sum_{i=1}^n e^{-\rho_i x} \frac{(1 - e^{-(u \wedge x)(\theta_j - \rho_i)})}{\theta_j - \rho_i} \prod_{k=1, k \neq i}^n \frac{1}{\rho_k - \rho_i} \right] \\ &\quad + 1(x \geq u) \frac{\prod_{l=1}^n \lambda_l}{c^n} \bar{P}(x) \sum_{j=1}^n \left[e^{-\rho_j(x-u)} \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right]. \end{aligned}$$

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