Supplement to "A New Test for Functional One-Way ANOVA with Applications to Ischemic Heart Screening"

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S.1 Effect of Discretization

We have studied the $F_{\max}$-test in the continuous setup in the paper. In practice, accessing the continuous functions $y_{ij}(t), t \in \mathcal{T}$, $j = 1, \ldots, n_i$, $i = 1, \ldots, k$, may not be always possible, and in most scenarios we have only discretized observations on them. For example, the ECG signals collected during ischemia heart screening are discretized version. When the function $\mu_i(t)$ is smooth enough and the discretization points follow some non-degenerate probability distribution function, Zhang and Chen (2007) suggested that local polynomial regression (LPR) can be applied to estimate $\mu_i(t)$. However, in general $\mu_i(t)$ might not be that smooth, and the above approach cannot be applied. In this section we study a more direct approach to approximating the $F_{\max}$-test under discretization. We show that the approximation error incurred by discretization is negligible in terms of both asymptotic level and asymptotic power. Some numerical results about the effect of the resolution on the $F_{\max}$-test are presented in Section S.2.5 of this supplementary material.

S.1.1 Discretized version of $F_{\max}$ test

Suppose we observe the random functions $y_{ij}(t)$, $j = 1, \ldots, n_i$, $i = 1, \ldots, k$, in (1) only at some discretization points $t_1, \ldots, t_M$ in the interval $\mathcal{T} = [a, b]$, such that $t_1 = a$ and $t_M = b$, and for some positive $\tau_M = O(1/M)$ as $M \to \infty$ we have

$$0 < t_{l+1} - t_l \le \tau_M, \text{ for all } l = 1, \ldots, M-1. \tag{S.1}$$

Note that the discretization might be non-uniform. Then we have k samples of random vectors: for $i = 1, \ldots, k$, the i-th sample consists of $\mathbf{y}_{ij,M} = [y_{ij}(t_1), \ldots, y_{ij}(t_M)]^T$, $j = 1, \ldots, n_i$. Let $\boldsymbol{\mu}_{i,M} = [\mu_i(t_1), \ldots, \mu_i(t_M)]^T$ denote the discretization of the ith group mean function $\mu_i(t)$, $i = 1, \ldots, k$. Given the discretized samples $\{\mathbf{y}_{ij,M}\}_{j=1}^{n_i}$, $i = 1, \ldots, k$, we have access to the sample group mean vectors $\bar{\mathbf{y}}_{i\cdot,M} = n_i^{-1} \sum_{j=1}^{n_i} \mathbf{y}_{ij,M}$, $i = 1, \ldots, k$, and the pointwise F test statistics: $F_n(t_l)$, $l = 1, \ldots, M$. Thus, parallel to the continuous case, we can test the discretized version of the null hypothesis H_0 given in (2):

$$H_{0,M} : \boldsymbol{\mu}_{1,M} = \ldots = \boldsymbol{\mu}_{k,M} \tag{S.2}$$

using the following discretized version of the $F_{\max}$ test statistic:

$$F_{\max,M} = \max_{l=1,\ldots,M} F_n(t_l). \tag{S.3}$$

For each $i = 1, \ldots, k$, based on the i-th discretized sample $\{\mathbf{y}_{ij,M}\}_{j=1}^{n_i}$, we can generate a bootstrapped sample $\{\mathbf{v}^*_{ij,M}\}_{j=1}^{n_i}$ from the estimated subject-effect vectors $\{\hat{\mathbf{v}}_{ij,M} = \mathbf{y}_{ij,M} - \bar{\mathbf{y}}_{i\cdot,M}\}_{j=1}^{n_i}$, which are estimators of the discretized subject-effect vectors $\{\mathbf{v}_{ij,M} = \mathbf{y}_{ij,M} - \boldsymbol{\mu}_{i,M}\}_{j=1}^{n_i}$. The bootstrapped $F_{\max,M}$-test statistic is then obtained as

$$F^*_{\max,M} = \max_{l=1,\ldots,M} F^*_n(t_l), \tag{S.4}$$

where $F^*_n(t_l)$ is the version of $F_n(t_l)$ using the bootstrapped k samples $\{\mathbf{v}^*_{ij,M}\}_{j=1}^{n_i},\ i = 1, \ldots, k$. Repeat the above bootstrapping process a large number of times and calculate the upper 100α-percentile of $F^*_{\max,M}$. Then we can conduct accordingly the $F_{\max,M}$-test for testing the discretized null hypothesis $H_{0,M}$ specified in (S.2).

Let $N_l(\boldsymbol{\nu}, \boldsymbol{\Gamma})$ denote the distribution of an l-vector of Gaussian entries, which has mean vector $\boldsymbol{\nu}$ and covariance matrix $\boldsymbol{\Gamma}$. Proposition S.1 studies the null distribution of $F_{\max,M}$.

Proposition S.1 *Suppose Condition A holds. Then, under the null hypothesis H_0 given in (2), we have $F_{\max,M} \xrightarrow{d} R_{0,M}$ as $n \to \infty$. Here $R_{0,M} \stackrel{d}{=} \max_{l=1,\ldots,M} \left\{(k-1)^{-1} \sum_{i=1}^{k-1} w^2_{i,M}(l)\right\}$, where $\mathbf{w}_{i,M} \equiv [w_{i,M}(1), \ldots, w_{i,M}(M)]^T, i = 1, \ldots, k-1, \stackrel{i.i.d.}{\sim} N_M(0, \boldsymbol{\Gamma}_{w,M})$ with $\boldsymbol{\Gamma}_{w,M} = \left[\gamma_w(t_p, t_q)\right]_{p,q=1,\ldots,M}$ and $w_{i,M}(l) = w_i(t_l), l = 1, \ldots, M$. In addition, for any given discretization $\{t_l\}_{l=1}^M$ of $\mathcal{T}$ satisfying condition (S.1), we may choose some $\tilde{\beta}$ such that $0 < \tilde{\beta} < \beta$ and $\left|R_0 - R_{0,M}\right| \leq \tilde{c}\tau_M^{\tilde{\beta}}$ with probability tending to 1, where R_0 is defined in (12) and the constant $\tilde{c}$ depends on the chosen $\tilde{\beta}$.*

Let $C_{\alpha,M}$ and $C^*_{\alpha,M}$ denote the upper 100α-percentiles of $R_{0,M}$ and $F^*_{\max,M}$ respectively. Proposition S.2 shows that the $F_{\max,M}$-test based on the bootstrap critical value $C^*_{\alpha,M}$ has the correct asymptotic level for testing the discretized null hypothesis (S.2). In addition, the second result in Proposition S.1 implies that $F_{\max,M}$ and $F_{\max}$ have the same limit expression R_0 under the null hypothesis. Thus the $F_{\max,M}$-test for testing (S.2) has the same asymptotic level as the $F_{\max}$-test for testing H_0.

Proposition S.2 *Under Condition A, $F^*_{\max,M} \xrightarrow{d} R_{0,M}$ and $C^*_{\alpha,M} \to C_{\alpha,M}$ as $n \to \infty$.*

Next, we study the local power of the $F_{\max,M}$-test under the local alternative H_{1n} given in (17). The asymptotic distribution of the $F_{\max,M}$ test statistic under H_1 is given in the following proposition.

Proposition S.3 *Suppose Condition A holds. Then, under the local alternative H_{1n}, as $n \to \infty$, we have $F_{\max,M} \xrightarrow{d} R_{1,M}$, where*

$$R_{1,M} \stackrel{d}{=} \max_{l=1,\ldots,M} \left\{(k-1)^{-1}\sum_{i=1}^{k-1}[w_{i,M}(l) + \delta_{i,M}(l)]^2\right\},$$

$w_{i,M}(l), l = 1,\ldots,M$, are defined in Proposition S.1 and $\delta_{i,M}(l), i = 1,\ldots,k-1$, are the $k-1$ components of $\boldsymbol{\delta}_M(l) = (\mathbf{I}_{k-1}, \mathbf{0})\,\mathbf{U}^T\mathbf{d}(t_l)/\sqrt{\gamma(t_l,t_l)}$ with $\mathbf{U}$ given in (10). Furthermore, suppose $d_i \in C^{\tilde\beta}(\mathcal{T})$, $i = 1,\ldots,k$, for a given $0 < \tilde\beta < \beta$. Then, for any given discretization $\{t_l\}_{l=1}^M$ of $\mathcal{T}$ satisfying (S.1), with probability tending to 1, we have $\left|R_1 - R_{1,M}\right| \le \tilde c \tau_M^{\tilde\beta}$, where R_1 is defined in (19) and the constant $\tilde c$ depends on the chosen $\tilde\beta$ and the Hölder modulus of $d_i(t), i = 1,\ldots,k$.

Combining the results in Propositions S.1, S.2 and S.3, as $M \to \infty$, we have that the $F_{\max,M}$-test has the same asymptotic power as the $F_{\max}$-test under the local alternative H_{1n} defined in (17). Together with Proposition 4, this implies that under the local alternative H_{1n} the power of the $F_{\max,M}$-test tends to 1, provided that $\mathbf{d}(t)$ diverges as $n \to \infty$ and both Condition A and (S.1) hold.

S.1.2 Proofs

Proof of Proposition S.1 By definition, the random vectors $\mathbf{v}_{ij,M} = [v_{ij}(t_1),\ldots,v_{ij}(t_M)]^T$, $j = 1,\ldots,n_i$, $i = 1,2,\ldots,k$, are i.i.d. with a zero mean vector and an $M \times M$ covariance matrix $\boldsymbol{\Gamma}_M$ whose (p,q) entry is $\mathrm{E}[v_{ij}(t_p)v_{ij}(t_q)] = \gamma(t_p,t_q)$, $p,q = 1,\ldots,M$. Note that for any finite M, taking $n \to \infty$ is exchangeable with taking maximum over the discretization points $t_1,\ldots,t_M$. Set $\mathbf{z}_{n,M} = [\mathbf{z}_n(t_1)^T,\ldots,\mathbf{z}_n(t_M)^T]^T$ and $\mathbf{z}_M = [\mathbf{z}(t_1)^T,\ldots,\mathbf{z}(t_M)^T]^T$ where $\mathbf{z}_n(t)$ is defined in (8) and $\mathbf{z}(t)$ is defined in the proof of Proposition 1. It follows from Lemma 1 that, under Condition A and as $n \to \infty$, we have

$$\mathbf{z}_{n,M} \xrightarrow{d} \mathbf{z}_M \sim N_{kM}(\mathbf{0}, \boldsymbol{\Gamma}_M \otimes \mathbf{I}_k), \quad \sqrt{n}\left\{vec(\hat{\boldsymbol{\Gamma}}_M) - vec(\boldsymbol{\Gamma}_M)\right\} \xrightarrow{d} N_{M^2}(\mathbf{0}, \mathbf{V}_M), \tag{S.5}$$

where $\otimes$ is the Kronecker product, $vec(\mathbf{A})$ denotes a column vector obtained via stacking all the column vectors of the matrix $\mathbf{A}$ one by one, and $\mathbf{V}_M$ is an $M^2 \times M^2$ matrix whose $[(k_1, l_1), (k_2, l_2)]$ entry is $\mathrm{E}\left[v_{11}(t_{k_1})v_{11}(t_{l_1})v_{11}(t_{k_2})v_{11}(t_{l_2})\right] - \gamma(t_{k_1}, t_{l_1})\gamma(t_{k_2}, t_{l_2})$, and $v_{11}(t)$ is the subject-effect function of the first subject of the first group. In addition, we have

$$\hat{\mathbf{\Gamma}}_M = \mathbf{\Gamma}_M + O_{\mathrm{UP}}\big(n^{-1/2}\big). \tag{S.6}$$

By (S.5) we have $\mathrm{SSR}_n(t_l)/(k-1) \xrightarrow{d} \mathbf{z}(t_l)^T(\mathbf{I}_k - \mathbf{b}\mathbf{b}^T)\mathbf{z}(t_l)/(k-1)$. Under the given conditions and by (S.5) and (S.6), as $n \to \infty$, we have $\mathrm{SSE}_n(t_l)/(n-k) = \hat{\gamma}(t_l, t_l) \xrightarrow{p} \gamma(t_l, t_l)$ for all $l = 1, \ldots, M$. Under the null hypothesis and by Slusky's Theorem, as $n \to \infty$, we have $F_{\max,M} = \max_{l=1,\ldots,M} \{\mathrm{SSR}_n(t_l)/(k-1)\}\{\mathrm{SSE}_n(t_l)/(n-k)\} \xrightarrow{d} R_{0,M}$ where $R_{0,M}$ is defined by $R_{0,M} = \max_{l=1,\ldots,M} \left[\big\{(k-1)\gamma(t_l, t_l)\big\}^{-1}\mathbf{z}(t_l)^T(\mathbf{I}_k - \mathbf{b}\mathbf{b}^T)\mathbf{z}(t_l)\right]$, and $\mathbf{I}_k - \mathbf{b}\mathbf{b}^T$ is the limit matrix of $\mathbf{I}_k - \mathbf{b}_n\mathbf{b}_n^T$; see (9). Note that under the null hypothesis, we have $\boldsymbol{\mu}(t_l)^T(\mathbf{I}_k - \mathbf{b}\mathbf{b}^T)\boldsymbol{\mu}(t_l) \equiv \mathbf{0}, l = 1, \ldots, M$. For $l = 1, \ldots, M$, set

$$\mathbf{w}_M(l) = (\mathbf{I}_{k-1}, \mathbf{0})\mathbf{U}^T\mathbf{z}_M(l)/\sqrt{\gamma(t_l, t_l)} = [w_{1,M}(l), \ldots, w_{k-1,M}(l)]^T, \tag{S.7}$$

where $\mathbf{U}$ comes from the singular value decomposition (10) of $\mathbf{I}_k - \mathbf{b}\mathbf{b}^T$. Then we have $R_{0,M} = \max_{l=1,\ldots,M} \big\{(k-1)^{-1}\mathbf{w}_M(l)^T\mathbf{w}_M(l)\big\} = \max_{l=1,\ldots,M} \big\{(k-1)^{-1}\sum_{i=1}^{k-1} w_{i,M}^2(l)\big\}$. Let $\mathbf{w}_{i,M} = [w_i(t_1), \ldots, w_i(t_M)]^T$, $i = 1, \ldots, k-1$. Then $\mathbf{w}_{1,M}, \ldots, \mathbf{w}_{k-1,M} \overset{i.i.d.}{\sim} N_M(\mathbf{0}, \mathbf{\Gamma}_{w,M})$ where the (p,q) entry of $\mathbf{\Gamma}_{w,M}$ is $\gamma_w(t_p, t_q), p, q = 1, \ldots, M$. This completes the proof of $F_{\max,M} \xrightarrow{d} R_{0,M}$ as $n \to \infty$.

Next, by Condition A5 on $\gamma(s,t)$ and a direct calculation by the definition of Hölder's continuity, the covariance function $\gamma_w(s,t)$ of the Gaussian process $\omega_i(t)$ defined in (23) is in $C^\beta(\mathcal{T} \times \mathcal{T})$. Thus, Kolmogorov's theorem (Koralov and Sinai, 2007, Theorem 18.19) says that, for all $i = 1, \ldots, k-1$, there exists a continuous modification $W_i(t), t \in \mathcal{T}$ of $\omega_i(t), t \in \mathcal{T}$ and an event subspace $\Omega(M)$ with $P(\Omega(M)) \to 1$ as $M \to \infty$, so that for all events in $\Omega(M)$, $W_i(t)$ is Hölder continuous with exponent $0 < \tilde{\beta} < \beta$ and Hölder's modulus $c_{\tilde{\beta}} = 2/(1-2^{-\tilde{\beta}})$. Thus, from now on we work with $\{W_i(t)\}_{i=1}^{k-1}$ instead of $\{\omega_i(t)\}_{i=1}^{k-1}$ and use the same notation for the versions of R_0 and $R_{0,M}$ based on the continuous modifications: $R_0 \overset{d}{=} \sup_{t \in \mathcal{T}} \big\{(k-1)^{-1}\sum_{i=1}^{k-1} W_i^2(t)\big\}$ and $R_{0,M} \overset{d}{=} \max_{l=1,\ldots,M} \big\{(k-1)^{-1}\sum_{i=1}^{k-1} W_{i,M}^2(l)\big\}$, where $W_{i,M}(l) = W_i(t_l), l = 1, \ldots, M; i = 1, \ldots, k-1$. Clearly $(k-1)^{-1}\sum_{i=1}^{k-1} W_i^2(t)$ is also Hölder continuous with exponent $\tilde{\beta}$ and Hölder's modulus

$\tilde{c} = c_{\tilde{\beta}}^2$. Since $\mathcal{T}$ is compact, the supremum of $\sum_{i=1}^{k-1} W_i^2(t)$ is achieved at some $t' \in [t_{l'-1}, t_{l'+1}]$. By Hölder's continuity of the process, we have

$$\left|(k-1)^{-1}\sum_{i=1}^{k-1} W_{i,M}^2(l') - \sup_{t\in[t_{l'-1},t_{l'+1}]}(k-1)^{-1}\sum_{i=1}^{k-1} W_i^2(t)\right| \le \tilde{c}\tau_M^{\tilde{\beta}}.$$

Suppose the maximum of $\sum_{i=1}^{k-1} W_{i,M}^2(l)$ is achieved at l'' instead of l', we must have

$$(k-1)^{-1}\sum_{i=1}^{k-1} W_{i,M}^2(l') \le (k-1)^{-1}\sum_{i=1}^{k-1} W_{i,M}^2(l'') \le (k-1)^{-1}\sum_{i=1}^{k-1} W_i^2(t'),$$

where the last inequality holds by definition. Thus, for all events in $\Omega(M)$, we have

$$\left|R_0 - R_{0,M}\right| \le \tilde{c}\tau_M^{\tilde{\beta}}. \tag{S.8}$$

Proof of Proposition S.2 Note that the k bootstrapped samples $\{\mathbf{v}_{ij,M}^*\}_{j=1}^{n_i}, i = 1, \ldots, k$ are i.i.d. with mean vector $\mathbf{0}$ and covariance matrix $\hat{\mathbf{\Gamma}}_M$ whose (p,q) entry is $\hat{\gamma}(t_p, t_q), p, q = 1, \ldots, M$. That is, they satisfy the discretized null hypothesis (S.2). By (S.5) and (S.6), and under Condition A, as $n \to \infty$, we have $\hat{\mathbf{\Gamma}}_M \xrightarrow{d} \mathbf{\Gamma}_M$ uniformly. Applying the same arguments as in the proof of $F_{\max,M} \xrightarrow{d} R_{0,M}$ to the k bootstrapped samples leads to the claim that $F_{\max,M}^* \xrightarrow{d} R_{0,M}$ as $n \to \infty$.

Proof of Proposition S.3 As in the proof of Proposition S.1, under Condition A, $\mathrm{SSE}_n(t_l)/(n-k) \xrightarrow{p} \gamma(t_l, t_l)$ and $\mathbf{z}_{n,M} \xrightarrow{d} \mathbf{z}_M \sim N_{kM}(\mathbf{0}, \mathbf{\Gamma}_M \otimes \mathbf{I}_k)$ as $n \to \infty$. Under H_1, we have $\mu_i(t_l) = \mu_0(t_l) + n_i^{-1/2} d_i(t_l), l = 1, \ldots, M;\ i = 1, 2, \ldots, k$. It follows that

$$\boldsymbol{\mu}(t_l) = \mu_0(t_l)\mathbf{1}_k + \mathrm{diag}(n_1^{-1/2}, \ldots, n_k^{-1/2})\mathbf{d}(t_l),\ l = 1, \ldots, M,$$

where $\boldsymbol{\mu}(t) = [\mu_1(t), \ldots, \mu_k(t)]^T$ and $\mathbf{d}(t) = [d_1(t), \ldots, d_k(t)]^T$ as defined before. Then we have $\boldsymbol{\mu}_n(t_l) = \mathrm{diag}(n_1^{1/2}, \ldots, n_k^{1/2})\boldsymbol{\mu}(t_l) = \mu_0(t_l)\mathbf{b}_n + \mathbf{d}(t_l), l = 1, \ldots, M$. Since $(\mathbf{I}_k - \mathbf{b}_n\mathbf{b}_n^T/n)\mathbf{b}_n = \mathbf{0}$, under H_1, we have $\mathrm{SSR}_n(t_l) = \left[\mathbf{z}_n(t_l) + \mathbf{d}(t_l)\right]^T(\mathbf{I}_k - \mathbf{b}_n\mathbf{b}_n^T/n)\left[\mathbf{z}_n(t_l) + \mathbf{d}(t_l)\right]$. Hence, as $n \to \infty$, we have $F_{\max,M} \xrightarrow{d} R_{1,M}$. Let $\mathbf{w}_M(l)$ be as defined in (S.7) and let $\boldsymbol{\delta}_M(l) = \boldsymbol{\delta}(t_l) = (\mathbf{I}_{k-1}, \mathbf{0})\mathbf{U}^T\mathbf{d}(t_l)/\sqrt{\gamma(t_l,t_l)} = [\delta_{1,M}, \ldots, \delta_{k-1,M}]^T$ where $\delta_{i,M}(l) = \delta_i(t_l), l = 1, \ldots, M$. Thus, $(\mathbf{I}_{k-1}, \mathbf{0})\mathbf{U}^T[\mathbf{z}(t_l) + \mathbf{d}(t_l)]/\sqrt{\gamma(t_l,t_l)} = \mathbf{w}_M(l) + \boldsymbol{\delta}_M(l)$. Therefore,

$$\begin{aligned} R_{1,M} &= \max_{l=1,\ldots,M}\left\{(k-1)^{-1}[\mathbf{w}_M(l) + \boldsymbol{\delta}_M(l)]^T[\mathbf{w}_M(l) + \boldsymbol{\delta}_M(l)]\right\} \\ &= \max_{l=1,\ldots,M}\left\{(k-1)^{-1}\sum_{i=1}^{k-1}[w_{i,M}(l) + \delta_{i,M}(l)]^2\right\}. \end{aligned}$$

Next, using the same arguments in the proof of (S.8), we can show that there exists an event space $\Omega(M)$ with $P(\Omega(M)) \to 1$ such that conditional on $\Omega(M)$, for all $i = 1, \ldots, k-1$, we have a continuous modification $W_i(t)$ of $\omega_i(t)$ defined in (23) so that $W_i(t)$ is Hölder continuous with exponent $0 < \tilde{\beta} < \beta/2$ and Hölder's modulus $c_{\tilde{\beta}} = 2/(1-2^{-\tilde{\beta}})$. We use the same notation for the versions of R_1 and $R_{1,M}$ based on the continuous modifications: $R_1 \stackrel{d}{=} \sup_{t\in\mathcal{T}} \left\{(k-1)^{-1}\sum_{i=1}^{k-1}[W_i(t)+\delta_i(t)]^2\right\}$ and $R_{1,M} \stackrel{d}{=} \max_{l=1,\ldots,M}\left\{(k-1)^{-1}\sum_{i=1}^{k-1}[W_{i,M}(l)+\delta_{i,M}(l)]^2\right\}$, where $W_{i,M}(l) = W_i(t_l), l = 1, \ldots, M$. Clearly, under the assumption on $d_i(t)$, we know $(k-1)^{-1}\sum_{i=1}^{k-1}[W_i(t)+\delta_i(t)]^2$ is also Hölder continuous with exponent $\tilde{\beta}$ and Hölder's modulus $\tilde{c} = 2(c_{\tilde{\beta}}^2 + c_{\delta}^2)$, where c_δ is the maximum of the Hölder's modulus of $d_i(t)$, $i = 1, \ldots, k$. Again, by the same arguments as those leading to (S.8), for all events in $\Omega(M)$ we have $\left|R_1 - R_{1,M}\right| \le \tilde{c}\tau_M^{\tilde{\beta}}$.

S.2 Additional Simulation Studies

Presented in the subsequent subsections are results of additional simulation studies to consider asymmetric error distributions, complicated mean function differences, non-smooth functional data, and unequal group covariance functions, and to investigate the effect of choice of the discretization resolution M.

S.2.1 Asymmetric Error Distributions

In this simulation study, we study the performance of the L^2, F, GPF and $F_{\max}$ tests when $z_{ijr}, r = 1, \ldots, q;\ j = 1, \ldots, n_i;\ i = 1, \ldots, k$, involved in the error terms of the functional data model (21) are i.i.d. following the $(\chi_4^2 - 4)/(2\sqrt{2})$ distribution, where χ_4^2 stands for a chi-squared distribution with 4 degrees of freedom. All the other settings are the same as those in Section 3 of the paper. Note that different from the symmetric distributions $N(0,1)$ and $t_4/\sqrt{2}$ used in Section 3 of the paper, the distribution of $(\chi_4^2 - 4)/(2\sqrt{2})$ is skewed. The simulation results are presented in Table S.1, from which similar conclusions as those from Tables 1 and 2 of the paper can be drawn. Comparing Table S.1 with Tables 1 and 2, we can also conclude that all of the considered tests are robust against asymmetric error distributions in terms of size control, and in the presence of asymmetric errors they lose power only slightly.

Table S.1: Empirical sizes and powers (%) of the L^2, F, GPF and $F_{\max}$ tests when the nominal level is 5%, $z_{ijr}, r=1,\ldots,q; j=1,\ldots,n_i; i=1,\ldots,k$, are i.i.d. $(\chi_4^2-4)/(2\sqrt{2})$, and $M=80$. The associated standard deviations (%) are given in parentheses.

ρ	$\mathbf{n}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$
		$\delta=0$				$\delta=0.03$				$\delta=0.06$				$\delta=0.10$				$\delta=0.13$			
	$\mathbf{n}_1$	5.02	4.70	5.10	4.68	7.20	6.60	7.20	9.70	12.32	11.58	12.36	29.48	26.22	24.90	26.76	73.32	45.82	43.98	46.32	93.68
		(0.31)	(0.30)	(0.31)	(0.30)	(0.37)	(0.35)	(0.37)	(0.42)	(0.46)	(0.45)	(0.47)	(0.64)	(0.62)	(0.61)	(0.63)	(0.63)	(0.70)	(0.70)	(0.71)	(0.34)
0.1	$\mathbf{n}_2$	4.94	4.76	4.92	4.88	8.94	8.66	8.94	15.32	19.62	19.18	19.94	59.76	55.08	54.42	55.82	97.90	81.80	81.28	82.40	99.94
		(0.31)	(0.30)	(0.31)	(0.30)	(0.40)	(0.40)	(0.40)	(0.51)	(0.56)	(0.56)	(0.57)	(0.69)	(0.70)	(0.70)	(0.70)	(0.20)	(0.55)	(0.55)	(0.54)	(0.03)
	$\mathbf{n}_3$	4.80	4.64	4.78	4.72	11.20	11.02	11.34	26.92	35.26	34.92	35.76	87.44	83.78	83.60	84.40	100	98.16	98.16	98.32	100
		(0.30)	(0.30)	(0.30)	(0.30)	(0.45)	(0.44)	(0.45)	(0.63)	(0.68)	(0.67)	(0.68)	(0.47)	(0.52)	(0.52)	(0.51)	(0)	(0.19)	(0.19)	(0.18)	(0)
		$\delta=0$				$\delta=0.05$				$\delta=0.10$				$\delta=0.15$				$\delta=0.20$			
	$\mathbf{n}_1$	5.34	5.26	5.66	5.14	6.14	5.96	6.34	6.56	10.64	10.36	11.42	17.34	21.14	20.78	22.34	41.96	34.72	34.30	36.60	71.24
		(0.32)	(0.32)	(0.33)	(0.31)	(0.34)	(0.33)	(0.34)	(0.35)	(0.44)	(0.43)	(0.45)	(0.54)	(0.58)	(0.57)	(0.59)	(0.70)	(0.67)	(0.67)	(0.68)	(0.64)
0.3	$\mathbf{n}_2$	5.16	5.18	5.52	5.14	8.28	8.26	8.62	10.90	18.90	18.92	19.74	40.86	41.36	41.26	42.72	81.86	67.90	67.80	69.96	98.12
		(0.31)	(0.31)	(0.32)	(0.31)	(0.39)	(0.39)	(0.40)	(0.44)	(0.55)	(0.55)	(0.56)	(0.70)	(0.70)	(0.70)	(0.70)	(0.55)	(0.66)	(0.66)	(0.65)	(0.19)
	$\mathbf{n}_3$	5.26	5.42	5.52	4.96	9.92	10.10	10.44	16.40	31.80	32.02	32.56	69.20	67.06	67.30	68.96	98.14	93.10	93.22	94.22	100
		(0.32)	(0.32)	(0.32)	(0.31)	(0.42)	(0.43)	(0.43)	(0.52)	(0.66)	(0.66)	(0.66)	(0.65)	(0.66)	(0.66)	(0.65)	(0.19)	(0.36)	(0.36)	(0.33)	(0)
		$\delta=0$				$\delta=0.10$				$\delta=0.20$				$\delta=0.30$				$\delta=0.40$			
	$\mathbf{n}_1$	4.52	4.64	5.22	4.30	8.18	8.44	9.38	8.62	20.28	20.58	21.94	29.06	42.60	42.94	45.36	66.32	71.12	71.36	73.56	92.36
		(0.29)	(0.30)	(0.31)	(0.29)	(0.39)	(0.39)	(0.41)	(0.40)	(0.57)	(0.57)	(0.59)	(0.64)	(0.70)	(0.70)	(0.70)	(0.67)	(0.64)	(0.64)	(0.62)	(0.38)
0.5	$\mathbf{n}_2$	4.98	5.20	5.42	5.14	11.88	12.18	12.78	15.54	40.02	40.50	42.36	64.60	79.02	79.44	81.34	96.34	97.72	97.72	98.12	99.96
		(0.31)	(0.31)	(0.32)	(0.31)	(0.46)	(0.46)	(0.47)	(0.51)	(0.69)	(0.69)	(0.70)	(0.68)	(0.58)	(0.57)	(0.55)	(0.27)	(0.21)	(0.21)	(0.19)	(0.03)
	$\mathbf{n}_3$	5.08	5.26	5.54	4.98	18.30	18.76	19.32	27.06	65.44	66.24	68.32	91.12	97.30	97.40	97.76	99.98	99.96	99.96	99.98	100
		(0.31)	(0.32)	(0.32)	(0.31)	(0.55)	(0.55)	(0.56)	(0.63)	(0.67)	(0.67)	(0.66)	(0.40)	(0.23)	(0.23)	(0.21)	(0.02)	(0.03)	(0.03)	(0.02)	(0)
		$\delta=0$				$\delta=0.20$				$\delta=0.40$				$\delta=0.60$				$\delta=0.80$			
	$\mathbf{n}_1$	4.44	4.82	5.08	4.62	13.34	14.32	14.96	13.26	47.28	49.04	50.84	52.22	85.56	86.36	87.04	91.02	98.62	98.70	99.00	99.62
		(0.29)	(0.30)	(0.31)	(0.30)	(0.48)	(0.50)	(0.50)	(0.48)	(0.71)	(0.71)	(0.71)	(0.71)	(0.50)	(0.49)	(0.48)	(0.40)	(0.16)	(0.16)	(0.14)	(0.09)
0.7	$\mathbf{n}_2$	4.62	5.00	5.20	4.88	25.50	26.46	27.30	27.32	83.32	84.10	84.96	90.40	99.62	99.68	99.70	99.94	100	100	100	100
		(0.30)	(0.31)	(0.31)	(0.30)	(0.62)	(0.62)	(0.63)	(0.63)	(0.53)	(0.52)	(0.51)	(0.42)	(0.09)	(0.08)	(0.08)	(0.03)	(0)	(0)	(0)	(0)
	$\mathbf{n}_3$	4.92	5.30	5.46	5.50	43.26	44.56	45.38	49.02	98.48	98.56	98.72	99.46	100	100	100	100	100	100	100	100
		(0.31)	(0.32)	(0.32)	(0.32)	(0.70)	(0.70)	(0.70)	(0.71)	(0.17)	(0.17)	(0.16)	(0.10)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)
		$\delta=0$				$\delta=0.30$				$\delta=0.60$				$\delta=0.90$				$\delta=1.20$			
	$\mathbf{n}_1$	4.14	4.92	4.80	4.44	12.98	14.50	14.22	10.50	52.76	55.30	55.44	42.90	92.86	93.62	93.46	86.56	99.84	99.86	99.86	99.46
		(0.28)	(0.31)	(0.30)	(0.29)	(0.48)	(0.50)	(0.49)	(0.43)	(0.71)	(0.70)	(0.70)	(0.70)	(0.36)	(0.35)	(0.35)	(0.48)	(0.06)	(0.05)	(0.05)	(0.10)
0.9	$\mathbf{n}_2$	4.72	5.14	5.12	4.98	28.08	29.82	29.78	20.94	91.88	92.34	92.46	85.36	100	100	100	99.88	100	100	100	100
		(0.30)	(0.31)	(0.31)	(0.31)	(0.64)	(0.65)	(0.65)	(0.58)	(0.39)	(0.38)	(0.37)	(0.50)	(0)	(0)	(0)	(0.05)	(0)	(0)	(0)	(0)
	$\mathbf{n}_3$	4.62	5.10	5.00	4.66	52.68	53.70	53.68	41.64	99.66	99.66	99.74	99.12	100	100	100	100	100	100	100	100
		(0.30)	(0.31)	(0.31)	(0.30)	(0.71)	(0.71)	(0.71)	(0.70)	(0.08)	(0.08)	(0.07)	(0.13)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)

$\mathbf{n}_1=[20,30,30], \mathbf{n}_2=[40,30,70], \mathbf{n}_3=[80,70,100]$.

S.2.2 Complicated Mean Function Differences

In this simulation study, we consider two different ways in specifying the group mean function differences $\mu_i(t)-\mu_1(t), i=2,\ldots,k$. We set $\mu_1(t)$ the same as in Section 3 of the paper, but with

$$\mu_i(t)=\mu_1(t)+\delta f(t),\ i=2,3, \tag{S.9}$$

where $f(t)$ is defined in two ways: (1) $1 - 2|t - 0.5|$, (2) the density function of $N(0, 1/8)$. Note that in the first case $f(t)$ is not differentiable at $t = 0.5$, and in the second case $f(t)$ has a tall local spike at $t = 0$.

The simulation results are presented in Tables S.2 and S.3. The results are generally similar to those presented in Table 1 of the paper, except that in Tables S.3 $F_{\max}$ test is more powerful than L^2, F and GPF tests even when $\rho = 0.9$. The possible reason is that under Case (2) the mean function difference has a tall local spike which favors $F_{\max}$. Comparing Tables S.2 and S.3 with Table 1, we can see that overall the considered tests still have good size control when the mean function differences are more complicated. And, in terms of power, all the considered tests are less powerful under Cases (1) and (2), and $F_{\max}$ test is less sensitive to spiked mean function difference like in Case (2) than the other competitors.

Table S.2: Empirical sizes and powers (%) of the L^2, F, GPF and $F_{\max}$ tests when the nominal level is 5%, $z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k$, are i.i.d. $N(0,1)$, $M = 80$, and $f(t) = 1 - 2|t - 0.5|$ in (S.9). The associated standard deviations (%) are given in parentheses.

ρ	$\mathbf{n}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$
		$\delta=0$				$\delta=0.03$				$\delta=0.06$				$\delta=0.10$				$\delta=0.13$			
	$\mathbf{n}_1$	6.02	5.60	6.00	5.32	6.02	5.58	6.08	6.08	7.92	7.48	7.96	9.66	14.28	13.34	14.48	24.06	19.72	18.50	19.76	38.30
		(0.34)	(0.33)	(0.34)	(0.32)	(0.34)	(0.32)	(0.34)	(0.34)	(0.38)	(0.37)	(0.38)	(0.42)	(0.49)	(0.48)	(0.50)	(0.60)	(0.56)	(0.55)	(0.56)	(0.69)
0.1	$\mathbf{n}_2$	5.64	5.38	5.64	5.32	6.28	6.10	6.32	7.28	11.38	11.04	11.50	18.72	24.66	24.18	24.86	50.52	41.08	40.20	41.68	76.58
		(0.33)	(0.32)	(0.33)	(0.32)	(0.34)	(0.34)	(0.34)	(0.37)	(0.45)	(0.44)	(0.45)	(0.55)	(0.61)	(0.61)	(0.61)	(0.71)	(0.70)	(0.69)	(0.70)	(0.60)
	$\mathbf{n}_3$	4.84	4.74	4.92	4.52	7.72	7.64	7.86	9.92	15.02	14.80	15.18	30.16	40.02	39.64	40.76	78.24	65.46	65.14	66.38	96.36
		(0.30)	(0.30)	(0.31)	(0.29)	(0.38)	(0.38)	(0.38)	(0.42)	(0.51)	(0.50)	(0.51)	(0.65)	(0.69)	(0.69)	(0.69)	(0.58)	(0.67)	(0.67)	(0.67)	(0.26)
		$\delta=0$				$\delta=0.05$				$\delta=0.10$				$\delta=0.15$				$\delta=0.20$			
	$\mathbf{n}_1$	5.66	5.54	5.84	5.00	6.00	5.86	6.42	6.08	7.54	7.34	8.02	7.58	11.66	11.28	12.14	13.30	16.42	16.08	17.32	22.14
		(0.33)	(0.32)	(0.33)	(0.31)	(0.34)	(0.33)	(0.35)	(0.34)	(0.37)	(0.37)	(0.38)	(0.37)	(0.45)	(0.45)	(0.46)	(0.48)	(0.52)	(0.52)	(0.54)	(0.59)
0.3	$\mathbf{n}_2$	5.00	5.00	5.24	4.86	6.38	6.38	6.66	6.18	10.22	10.22	10.70	12.16	17.92	17.86	18.68	25.62	30.90	30.94	32.50	48.10
		(0.31)	(0.31)	(0.32)	(0.30)	(0.35)	(0.35)	(0.35)	(0.34)	(0.43)	(0.43)	(0.44)	(0.46)	(0.54)	(0.54)	(0.55)	(0.62)	(0.65)	(0.65)	(0.66)	(0.71)
	$\mathbf{n}_3$	5.22	5.26	5.36	5.24	7.20	7.28	7.56	7.88	13.90	14.04	14.50	20.36	29.86	30.02	31.40	48.08	52.38	52.76	54.46	77.50
		(0.31)	(0.32)	(0.32)	(0.32)	(0.37)	(0.37)	(0.37)	(0.38)	(0.49)	(0.49)	(0.50)	(0.57)	(0.65)	(0.65)	(0.66)	(0.71)	(0.71)	(0.71)	(0.70)	(0.59)
		$\delta=0$				$\delta=0.10$				$\delta=0.20$				$\delta=0.30$				$\delta=0.40$			
	$\mathbf{n}_1$	5.60	5.72	6.06	5.54	5.52	5.68	6.20	5.80	10.76	11.02	11.78	10.80	19.82	20.06	21.38	19.68	33.18	33.66	35.18	36.24
		(0.33)	(0.33)	(0.34)	(0.32)	(0.32)	(0.33)	(0.34)	(0.33)	(0.44)	(0.44)	(0.46)	(0.44)	(0.56)	(0.57)	(0.58)	(0.56)	(0.67)	(0.67)	(0.68)	(0.68)
0.5	$\mathbf{n}_2$	4.96	5.22	5.36	5.40	8.20	8.38	9.00	7.68	18.42	18.70	19.68	18.64	39.26	39.92	41.58	44.62	65.90	66.54	68.64	75.42
		(0.31)	(0.31)	(0.32)	(0.32)	(0.39)	(0.39)	(0.40)	(0.38)	(0.55)	(0.55)	(0.56)	(0.55)	(0.69)	(0.69)	(0.70)	(0.70)	(0.67)	(0.67)	(0.66)	(0.61)
	$\mathbf{n}_3$	5.10	5.38	5.66	5.04	9.18	9.64	10.28	10.00	30.50	31.16	32.28	35.36	64.30	64.90	66.92	73.66	90.50	90.84	91.76	96.02
		(0.31)	(0.32)	(0.33)	(0.31)	(0.41)	(0.42)	(0.43)	(0.42)	(0.65)	(0.66)	(0.66)	(0.68)	(0.68)	(0.68)	(0.67)	(0.62)	(0.41)	(0.41)	(0.39)	(0.28)
		$\delta=0$				$\delta=0.20$				$\delta=0.40$				$\delta=0.60$				$\delta=0.80$			
	$\mathbf{n}_1$	5.02	5.54	5.72	5.06	8.08	8.84	9.32	7.68	21.08	22.28	22.92	17.30	44.02	45.34	46.60	36.42	72.48	73.90	74.76	64.28
		(0.31)	(0.32)	(0.33)	(0.31)	(0.39)	(0.40)	(0.41)	(0.38)	(0.58)	(0.59)	(0.59)	(0.53)	(0.70)	(0.70)	(0.71)	(0.68)	(0.63)	(0.62)	(0.61)	(0.68)
0.7	$\mathbf{n}_2$	4.74	5.06	5.26	5.16	11.66	12.40	12.74	9.94	43.24	44.92	45.92	36.34	82.00	83.12	83.86	75.42	97.76	97.98	98.14	95.72
		(0.30)	(0.31)	(0.32)	(0.31)	(0.45)	(0.47)	(0.47)	(0.42)	(0.70)	(0.70)	(0.70)	(0.68)	(0.54)	(0.53)	(0.52)	(0.61)	(0.21)	(0.20)	(0.19)	(0.29)
	$\mathbf{n}_3$	5.20	5.78	5.64	5.16	18.72	19.62	20.24	16.34	69.40	70.34	71.02	60.72	97.96	98.12	98.20	96.12	100	100	100	99.90
		(0.31)	(0.33)	(0.33)	(0.31)	(0.55)	(0.56)	(0.57)	(0.52)	(0.65)	(0.65)	(0.64)	(0.69)	(0.20)	(0.19)	(0.19)	(0.27)	(0)	(0)	(0)	(0.04)
		$\delta=0$				$\delta=0.30$				$\delta=0.60$				$\delta=0.90$				$\delta=1.20$			
	$\mathbf{n}_1$	4.64	5.42	5.58	5.40	8.28	9.30	9.26	6.50	21.42	23.84	23.58	14.94	53.04	55.50	54.90	33.22	82.72	84.20	83.98	59.88
		(0.30)	(0.32)	(0.32)	(0.32)	(0.39)	(0.41)	(0.41)	(0.35)	(0.58)	(0.60)	(0.60)	(0.50)	(0.71)	(0.70)	(0.70)	(0.67)	(0.53)	(0.52)	(0.52)	(0.69)
0.9	$\mathbf{n}_2$	4.38	5.18	4.90	5.28	13.28	14.18	13.84	10.00	51.20	52.88	52.54	32.90	91.24	91.78	91.52	72.06	99.78	99.78	99.78	95.40
		(0.29)	(0.31)	(0.31)	(0.32)	(0.48)	(0.49)	(0.49)	(0.42)	(0.71)	(0.71)	(0.71)	(0.66)	(0.40)	(0.39)	(0.39)	(0.63)	(0.07)	(0.07)	(0.07)	(0.30)
	$\mathbf{n}_3$	4.86	5.10	5.24	5.20	21.84	22.80	22.82	15.00	80.44	81.02	81.14	58.32	99.66	99.70	99.64	94.90	100	100	100	99.96
		(0.30)	(0.31)	(0.32)	(0.31)	(0.58)	(0.59)	(0.59)	(0.51)	(0.56)	(0.55)	(0.55)	(0.70)	(0.08)	(0.08)	(0.08)	(0.31)	(0)	(0)	(0)	(0.03)

$\mathbf{n}_1 = [20, 30, 30], \mathbf{n}_2 = [40, 30, 70], \mathbf{n}_3 = [80, 70, 100]$.

Table S.3: Empirical sizes and powers (%) of the L^2, F, GPF and $F_{\max}$ tests when the nominal level is 5%, $z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k$, are i.i.d. $N(0, 1)$, $M = 80$, and $f(t)$ in model (S.9) is the density function of $N(0, 1/8)$. The associated standard deviations (%) are given in parentheses.

ρ	$\mathbf{n}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$
		$\delta = 0$				$\delta = 0.03$				$\delta = 0.06$				$\delta = 0.10$				$\delta = 0.13$			
	$\mathbf{n}_1$	5.76	5.38	5.74	5.20	5.36	5.04	5.38	7.76	6.94	6.46	7.04	21.80	9.40	8.64	9.76	59.48	13.46	12.44	13.90	84.82
		(0.33)	(0.32)	(0.33)	(0.31)	(0.32)	(0.31)	(0.32)	(0.38)	(0.36)	(0.35)	(0.36)	(0.58)	(0.41)	(0.40)	(0.42)	(0.69)	(0.48)	(0.47)	(0.49)	(0.51)
0.1	$\mathbf{n}_2$	5.22	5.02	5.28	4.90	5.74	5.44	5.82	13.54	7.70	7.44	7.96	46.98	15.90	15.58	16.60	94.02	27.74	26.60	29.36	99.94
		(0.31)	(0.31)	(0.32)	(0.31)	(0.33)	(0.32)	(0.33)	(0.48)	(0.38)	(0.37)	(0.38)	(0.71)	(0.52)	(0.51)	(0.53)	(0.34)	(0.63)	(0.62)	(0.64)	(0.03)
	$\mathbf{n}_3$	5.08	5.04	5.08	4.58	6.36	6.26	6.46	20.40	10.30	10.12	10.60	74.60	27.74	27.30	29.44	99.96	60.38	59.68	64.62	100
		(0.31)	(0.31)	(0.31)	(0.30)	(0.35)	(0.34)	(0.35)	(0.57)	(0.43)	(0.43)	(0.44)	(0.62)	(0.63)	(0.63)	(0.64)	(0.03)	(0.69)	(0.69)	(0.68)	(0)
		$\delta = 0$				$\delta = 0.05$				$\delta = 0.10$				$\delta = 0.15$				$\delta = 0.20$			
	$\mathbf{n}_1$	5.44	5.34	5.80	4.82	5.36	5.26	5.66	5.92	6.60	6.46	7.24	13.34	8.24	8.02	9.04	30.56	11.38	11.14	12.50	55.16
		(0.32)	(0.32)	(0.33)	(0.30)	(0.32)	(0.32)	(0.33)	(0.33)	(0.35)	(0.35)	(0.37)	(0.48)	(0.39)	(0.38)	(0.41)	(0.65)	(0.45)	(0.44)	(0.47)	(0.70)
0.3	$\mathbf{n}_2$	5.66	5.66	5.92	5.88	5.68	5.64	5.94	8.68	8.00	8.02	8.78	29.58	12.16	12.14	13.32	67.46	21.74	21.78	24.52	94.28
		(0.33)	(0.33)	(0.33)	(0.33)	(0.33)	(0.33)	(0.33)	(0.40)	(0.38)	(0.38)	(0.40)	(0.65)	(0.46)	(0.46)	(0.48)	(0.66)	(0.58)	(0.58)	(0.61)	(0.33)
	$\mathbf{n}_3$	5.22	5.28	5.42	5.08	6.00	6.12	6.32	12.50	9.60	9.74	10.26	53.24	20.90	21.08	23.22	93.62	41.50	41.82	46.68	99.94
		(0.31)	(0.32)	(0.32)	(0.31)	(0.34)	(0.34)	(0.34)	(0.47)	(0.42)	(0.42)	(0.43)	(0.71)	(0.58)	(0.58)	(0.60)	(0.35)	(0.70)	(0.70)	(0.71)	(0.03)
		$\delta = 0$				$\delta = 0.10$				$\delta = 0.20$				$\delta = 0.30$				$\delta = 0.40$			
	$\mathbf{n}_1$	5.82	5.98	6.52	5.34	5.78	5.90	6.42	6.40	8.72	8.88	9.72	18.14	13.22	13.50	15.06	47.08	23.82	24.36	27.92	80.68
		(0.33)	(0.34)	(0.35)	(0.32)	(0.33)	(0.33)	(0.35)	(0.35)	(0.40)	(0.40)	(0.42)	(0.55)	(0.48)	(0.48)	(0.51)	(0.71)	(0.60)	(0.61)	(0.63)	(0.56)
0.5	$\mathbf{n}_2$	5.40	5.54	5.54	5.04	6.84	7.00	7.74	11.18	12.82	13.48	14.76	46.16	30.02	30.58	34.28	89.84	58.38	59.54	64.92	99.68
		(0.32)	(0.32)	(0.32)	(0.31)	(0.36)	(0.36)	(0.38)	(0.45)	(0.47)	(0.48)	(0.50)	(0.71)	(0.65)	(0.65)	(0.67)	(0.43)	(0.70)	(0.69)	(0.67)	(0.08)
	$\mathbf{n}_3$	4.88	5.28	5.26	4.98	8.24	8.64	9.26	19.00	21.08	21.84	23.66	78.66	56.56	57.60	62.88	99.64	93.12	93.70	95.94	100
		(0.30)	(0.32)	(0.32)	(0.31)	(0.39)	(0.40)	(0.41)	(0.55)	(0.58)	(0.58)	(0.60)	(0.58)	(0.70)	(0.70)	(0.68)	(0.08)	(0.36)	(0.34)	(0.28)	(0)
		$\delta = 0$				$\delta = 0.20$				$\delta = 0.40$				$\delta = 0.60$				$\delta = 0.80$			
	$\mathbf{n}_1$	4.80	5.22	5.42	4.82	7.24	7.76	8.38	8.96	15.32	16.54	17.74	30.06	35.96	37.72	39.44	71.96	65.04	66.84	69.00	96.56
		(0.30)	(0.31)	(0.32)	(0.30)	(0.37)	(0.38)	(0.39)	(0.40)	(0.51)	(0.53)	(0.54)	(0.65)	(0.68)	(0.69)	(0.69)	(0.64)	(0.67)	(0.67)	(0.65)	(0.26)
0.7	$\mathbf{n}_2$	5.26	5.74	5.78	5.16	8.80	9.48	9.76	13.88	33.70	35.12	36.88	70.32	76.58	77.78	79.98	98.78	97.92	98.22	98.34	100
		(0.32)	(0.33)	(0.33)	(0.31)	(0.40)	(0.41)	(0.42)	(0.49)	(0.67)	(0.68)	(0.68)	(0.65)	(0.60)	(0.59)	(0.57)	(0.16)	(0.20)	(0.19)	(0.18)	(0)
	$\mathbf{n}_3$	4.72	5.06	5.00	5.38	15.04	15.72	16.52	29.10	60.88	62.44	65.10	94.98	97.72	98.04	98.42	100	100	100	100	100
		(0.30)	(0.31)	(0.31)	(0.32)	(0.51)	(0.51)	(0.53)	(0.64)	(0.69)	(0.68)	(0.67)	(0.31)	(0.21)	(0.20)	(0.18)	(0)	(0)	(0)	(0)	(0)
		$\delta = 0$				$\delta = 0.30$				$\delta = 0.60$				$\delta = 0.90$				$\delta = 1.20$			
	$\mathbf{n}_1$	4.66	5.44	5.24	5.26	6.68	7.68	7.94	7.52	16.60	18.66	18.48	21.34	41.40	44.28	43.90	57.00	74.52	76.96	76.40	89.86
		(0.30)	(0.32)	(0.32)	(0.32)	(0.35)	(0.38)	(0.38)	(0.37)	(0.53)	(0.55)	(0.55)	(0.58)	(0.70)	(0.70)	(0.70)	(0.70)	(0.62)	(0.60)	(0.60)	(0.43)
0.9	$\mathbf{n}_2$	4.38	5.04	4.82	4.96	10.66	11.56	11.54	12.16	40.58	42.28	42.34	55.28	84.84	85.90	85.58	95.66	99.26	99.34	99.34	100
		(0.29)	(0.31)	(0.30)	(0.31)	(0.44)	(0.45)	(0.45)	(0.46)	(0.69)	(0.70)	(0.70)	(0.70)	(0.51)	(0.49)	(0.50)	(0.29)	(0.12)	(0.11)	(0.11)	(0)
	$\mathbf{n}_3$	4.36	4.66	4.94	5.42	16.76	17.78	17.66	20.76	70.48	71.58	71.82	87.40	99.12	99.20	99.26	99.98	100	100	100	100
		(0.29)	(0.30)	(0.31)	(0.32)	(0.53)	(0.54)	(0.54)	(0.57)	(0.65)	(0.64)	(0.64)	(0.47)	(0.13)	(0.13)	(0.12)	(0.02)	(0)	(0)	(0)	(0)

$\mathbf{n}_1 = [20, 30, 30], \mathbf{n}_2 = [40, 30, 70], \mathbf{n}_3 = [80, 70, 100]$.

S.2.3 Non-smooth Functional Data

In this simulation study, we consider the case when the observed functional data are non-smooth. To this end, we modified the data generating model (21) described in Section 3 of the paper by adding an additional Brownian motion process term so that we have

$$y_{ij}(t) = \mu_i(t) + v_{ij}(t) + \epsilon_{ij}(t), j = 1, \ldots, n_i; i = 1, \ldots, k, \tag{S.10}$$

where $\epsilon_{ij}(t), j = 1, \ldots, n_i; i = 1, \ldots, k$, are i.i.d. Brownian motion processes. All the other settings are the same as those described in Section 3 of the paper. The results of this simulation study presented in Table S.4 are similar to those presented in Table 1 of the paper. From Table S.4 and Table 1, we can also observe that when the functional data are non-smooth the considered tests still have good size control, and they all lose power to some extent. It is understandable that the considered tests all suffer from some loss of power because of the non-smooth Brownian motion term. Fortunately the $F_{\max}$ test still preserves its advantages over the other competitors in this difficult case.

Table S.4: Empirical sizes and powers (%) of the L^2, F, GPF and $F_{\max}$ tests when the nominal level is 5%, $z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k$, are i.i.d. $N(0,1)$ and $M = 80$ in the non-smooth functional data model (S.10). The associated standard deviations (%) are given in parentheses.

ρ	$\mathbf{n}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$
		$\delta = 0$				$\delta = 0.06$				$\delta = 0.12$				$\delta = 0.20$				$\delta = 0.26$			
	$\mathbf{n}_1$	5.30	5.10	5.52	4.68	7.52	7.24	7.24	6.36	13.68	13.30	12.22	11.34	31.08	30.34	25.68	29.76	50.20	49.38	42.28	53.66
		(0.32)	(0.31)	(0.32)	(0.30)	(0.37)	(0.37)	(0.37)	(0.35)	(0.49)	(0.48)	(0.46)	(0.45)	(0.65)	(0.65)	(0.62)	(0.65)	(0.71)	(0.71)	(0.70)	(0.71)
0.1	$\mathbf{n}_2$	5.72	5.64	5.86	5.06	9.62	9.50	8.82	8.74	24.02	23.82	19.86	23.00	58.40	58.10	48.96	66.20	82.88	82.62	74.74	92.08
		(0.33)	(0.33)	(0.33)	(0.31)	(0.42)	(0.41)	(0.40)	(0.40)	(0.60)	(0.60)	(0.56)	(0.60)	(0.70)	(0.70)	(0.71)	(0.67)	(0.53)	(0.54)	(0.61)	(0.38)
	$\mathbf{n}_3$	4.92	4.92	5.22	5.08	12.14	12.20	10.86	10.26	36.96	37.04	30.64	41.02	84.64	84.68	76.14	93.84	97.56	97.56	94.38	99.70
		(0.31)	(0.31)	(0.31)	(0.31)	(0.46)	(0.46)	(0.44)	(0.43)	(0.68)	(0.68)	(0.65)	(0.70)	(0.51)	(0.51)	(0.60)	(0.34)	(0.22)	(0.22)	(0.33)	(0.08)
		$\delta = 0$				$\delta = 0.10$				$\delta = 0.20$				$\delta = 0.30$				$\delta = 0.40$			
	$\mathbf{n}_1$	5.18	5.06	5.66	4.74	7.80	7.74	7.80	6.90	19.18	18.92	16.86	21.20	42.98	42.50	37.76	53.26	68.88	68.32	62.10	84.72
		(0.31)	(0.31)	(0.33)	(0.30)	(0.38)	(0.38)	(0.38)	(0.36)	(0.56)	(0.55)	(0.53)	(0.58)	(0.70)	(0.70)	(0.69)	(0.71)	(0.65)	(0.66)	(0.69)	(0.51)
0.3	$\mathbf{n}_2$	4.88	4.90	5.38	5.14	12.52	12.52	11.36	12.54	40.00	39.96	35.20	51.12	77.42	77.58	70.86	92.30	96.02	96.00	93.58	99.74
		(0.30)	(0.31)	(0.32)	(0.31)	(0.47)	(0.47)	(0.45)	(0.47)	(0.69)	(0.69)	(0.68)	(0.71)	(0.59)	(0.59)	(0.64)	(0.38)	(0.28)	(0.28)	(0.35)	(0.07)
	$\mathbf{n}_3$	5.18	5.22	5.08	4.68	18.40	18.68	15.78	20.60	63.64	63.88	56.96	82.14	96.04	96.10	93.22	99.84	99.90	99.90	99.78	100
		(0.31)	(0.31)	(0.31)	(0.30)	(0.55)	(0.55)	(0.52)	(0.57)	(0.68)	(0.68)	(0.70)	(0.54)	(0.28)	(0.27)	(0.36)	(0.06)	(0.04)	(0.04)	(0.07)	(0)
		$\delta = 0$				$\delta = 0.10$				$\delta = 0.20$				$\delta = 0.30$				$\delta = 0.40$			
	$\mathbf{n}_1$	4.88	4.96	5.76	4.62	7.78	7.88	7.88	7.06	15.30	15.52	15.00	15.16	30.78	31.08	29.54	34.58	52.10	52.34	49.16	63.28
		(0.30)	(0.31)	(0.33)	(0.30)	(0.38)	(0.38)	(0.38)	(0.36)	(0.51)	(0.51)	(0.51)	(0.51)	(0.65)	(0.65)	(0.65)	(0.67)	(0.71)	(0.71)	(0.71)	(0.68)
0.5	$\mathbf{n}_2$	5.06	5.22	5.28	4.34	10.34	10.62	10.58	10.24	28.58	29.06	26.96	33.46	59.48	60.20	56.20	74.68	86.90	87.10	84.10	96.74
		(0.31)	(0.31)	(0.32)	(0.29)	(0.43)	(0.44)	(0.44)	(0.43)	(0.64)	(0.64)	(0.63)	(0.67)	(0.69)	(0.69)	(0.70)	(0.62)	(0.48)	(0.47)	(0.52)	(0.25)
	$\mathbf{n}_3$	5.30	5.52	5.52	5.06	13.84	14.18	13.16	14.30	47.72	48.54	44.90	60.58	86.00	86.40	83.46	96.58	98.84	98.96	98.40	99.98
		(0.32)	(0.32)	(0.32)	(0.31)	(0.49)	(0.49)	(0.48)	(0.50)	(0.71)	(0.71)	(0.70)	(0.69)	(0.49)	(0.48)	(0.53)	(0.26)	(0.15)	(0.14)	(0.18)	(0.02)
		$\delta = 0$				$\delta = 0.20$				$\delta = 0.40$				$\delta = 0.60$				$\delta = 0.80$			
	$\mathbf{n}_1$	5.32	5.78	5.94	5.08	11.68	12.44	12.40	9.98	36.70	37.94	37.02	35.64	75.16	76.12	74.20	77.36	95.68	95.98	95.40	97.36
		(0.32)	(0.33)	(0.33)	(0.31)	(0.45)	(0.47)	(0.47)	(0.42)	(0.68)	(0.69)	(0.68)	(0.68)	(0.61)	(0.60)	(0.62)	(0.59)	(0.29)	(0.28)	(0.30)	(0.23)
0.7	$\mathbf{n}_2$	4.58	4.82	5.04	4.52	19.92	20.92	20.10	18.84	71.88	73.04	70.94	74.96	98.28	98.44	98.20	99.22	99.98	100	99.98	100
		(0.30)	(0.30)	(0.31)	(0.29)	(0.56)	(0.58)	(0.57)	(0.55)	(0.64)	(0.63)	(0.64)	(0.61)	(0.18)	(0.18)	(0.19)	(0.12)	(0.02)	(0)	(0.02)	(0)
	$\mathbf{n}_3$	5.84	6.06	6.12	6.08	34.42	35.80	34.18	34.36	93.20	93.76	92.66	96.80	100	100	100	100	100	100	100	100
		(0.33)	(0.34)	(0.34)	(0.34)	(0.67)	(0.68)	(0.67)	(0.67)	(0.36)	(0.34)	(0.37)	(0.25)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)
		$\delta = 0$				$\delta = 0.30$				$\delta = 0.60$				$\delta = 0.90$				$\delta = 1.20$			
	$\mathbf{n}_1$	4.54	5.32	5.46	4.66	13.46	15.02	14.34	10.48	49.02	51.24	50.16	37.72	88.80	90.18	89.16	79.26	99.42	99.56	99.40	98.24
		(0.29)	(0.32)	(0.32)	(0.30)	(0.48)	(0.51)	(0.50)	(0.43)	(0.71)	(0.71)	(0.71)	(0.69)	(0.45)	(0.42)	(0.44)	(0.57)	(0.11)	(0.09)	(0.11)	(0.19)
0.9	$\mathbf{n}_2$	5.22	5.64	5.70	5.54	27.46	28.94	27.90	19.98	87.14	87.94	87.04	78.16	99.94	99.94	99.92	99.68	100	100	100	100
		(0.31)	(0.33)	(0.33)	(0.32)	(0.63)	(0.64)	(0.63)	(0.57)	(0.47)	(0.46)	(0.48)	(0.58)	(0.03)	(0.03)	(0.04)	(0.08)	(0)	(0)	(0)	(0)
	$\mathbf{n}_3$	4.50	4.78	4.94	5.16	46.64	47.76	46.70	34.44	99.12	99.16	99.08	97.38	100	100	100	100	100	100	100	100
		(0.29)	(0.30)	(0.31)	(0.31)	(0.71)	(0.71)	(0.71)	(0.67)	(0.13)	(0.13)	(0.14)	(0.23)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)

$\mathbf{n}_1 = [20, 30, 30]$, $\mathbf{n}_2 = [40, 30, 70]$, $\mathbf{n}_3 = [80, 70, 100]$.

S.2.4 Unequal Covariance Functions

In this simulation study, we consider the heteroscedastic case where different groups have different covariance functions. For simplicity, we only consider two groups with group sample sizes $\mathbf{n}_1 = [80, 80]$ and $\mathbf{n}_2 = [50, 100]$. The first group is generated using model (21) of the paper, and the second group is generated using the following modified model

$$y_{2j}(t) = \mu_2(t) + 2v_{2j}(t), j = 1, \ldots, n_2, \tag{S.11}$$

with all the other settings being the same as in the paper. In this case, we have $\gamma_2(s,t) = 4\gamma_1(s,t)$, where γ_i denotes the covariance function of the ith group. Note that the $F_{\max}$-test based on the bootstrap procedure described in Section 2.2 relies on the assumption that all the groups have a common covariance function. However, it can be easily modified to take into account possibly unequal group covariance functions via replacing the bootstrap procedure described in Section 2.2 by that described by Zhang (2013) (p. 329), see also Zhang et al. (2010) and Paparoditis and Sapatinas (2016). Namely, the bootstrap k samples in (15) are replaced by k within-group bootstrap samples drawn from the estimated subject-effect functions (14). We denote the $F_{\max}$-test using the new bootstrap procedure as $F_{\max,c}$.

The simulation results are presented in Table S.5. From Table S.5, we can see that when the two groups have the same group sample size (i.e., when $\mathbf{n} = \mathbf{n}_1$) the effect of unequal covariance functions is relatively minor. And, when the two groups have very different group sample sizes (i.e., when $\mathbf{n} = \mathbf{n}_2$) the L^2, F, GPF and $F_{\max}$ tests are all rather conservative and become much less powerful. Note that the $F_{\max,c}$-test based on the modified bootstrap procedure works well in both cases and is robust against group covariance function heteroscedasticity. Extension of $F_{\max}$ test to functional samples with unequal group covariance functions is interesting and may deserve a further study.

Table S.5: Empirical sizes and powers (%) of the L^2, F, GPF, $F_{\max}$ and $F_{\max,c}$ tests when the nominal level is 5%, $z_{ijr}, r = 1, \ldots, q; j = 1, \ldots, n_i; i = 1, \ldots, k$, are i.i.d. $N(0, 1)$ and $M = 80$ for the heteroscedastic covariance function case (S.11). The associated standard deviations (%) are given in parentheses.

ρ	$\mathbf{n}$	L^2	F	GPF	$F_{\max}$	$F_{\max,c}$	L^2	F	GPF	$F_{\max}$	$F_{\max,c}$	L^2	F	GPF	$F_{\max}$	$F_{\max,c}$
				$\delta = 0$					$\delta = 0.13$					$\delta = 0.20$		
	$\mathbf{n}_1$	5.14	5.00	5.08	5.32	5.18	20.42	20.14	20.56	56.84	56.24	44.90	44.24	45.24	93.06	92.88
0.1		(0.31)	(0.31)	(0.31)	(0.32)	(0.31)	(0.57)	(0.57)	(0.57)	(0.70)	(0.70)	(0.70)	(0.70)	(0.70)	(0.36)	(0.36)
	$\mathbf{n}_2$	1.54	1.42	1.50	1.32	4.62	10.40	10.12	10.42	37.68	57.48	25.94	25.34	26.44	83.88	93.76
		(0.17)	(0.17)	(0.17)	(0.16)	(0.30)	(0.43)	(0.43)	(0.43)	(0.69)	(0.70)	(0.62)	(0.62)	(0.62)	(0.52)	(0.34)
				$\delta = 0$					$\delta = 0.26$					$\delta = 0.40$		
	$\mathbf{n}_1$	4.72	4.68	4.92	4.98	4.62	26.44	26.40	27.54	54.98	54.26	56.48	56.36	58.08	92.86	92.50
0.3		(0.30)	(0.30)	(0.31)	(0.31)	(0.30)	(0.62)	(0.62)	(0.63)	(0.70)	(0.70)	(0.70)	(0.70)	(0.70)	(0.36)	(0.37)
	$\mathbf{n}_2$	1.50	1.48	1.62	1.10	4.74	12.70	12.58	13.38	33.24	55.48	34.16	34.06	36.08	82.36	93.82
		(0.17)	(0.17)	(0.18)	(0.15)	(0.30)	(0.47)	(0.47)	(0.48)	(0.67)	(0.70)	(0.67)	(0.67)	(0.68)	(0.54)	(0.34)
				$\delta = 0$					$\delta = 0.42$					$\delta = 0.65$		
	$\mathbf{n}_1$	4.64	4.86	5.06	4.94	4.72	35.80	36.16	38.00	56.58	55.34	74.08	74.50	76.36	93.60	93.24
0.5		(0.30)	(0.30)	(0.31)	(0.31)	(0.30)	(0.68)	(0.68)	(0.69)	(0.70)	(0.70)	(0.62)	(0.62)	(0.60)	(0.35)	(0.36)
	$\mathbf{n}_2$	1.26	1.30	1.48	0.98	4.92	17.28	17.66	18.82	31.62	56.66	53.46	54.08	56.02	84.16	94.62
		(0.16)	(0.16)	(0.17)	(0.14)	(0.31)	(0.53)	(0.54)	(0.55)	(0.66)	(0.70)	(0.71)	(0.70)	(0.70)	(0.52)	(0.32)
				$\delta = 0$					$\delta = 0.63$					$\delta = 0.95$		
	$\mathbf{n}_1$	4.78	5.10	5.20	5.06	4.66	49.36	50.62	51.42	54.28	52.50	86.82	87.54	88.06	91.74	91.22
0.7		(0.30)	(0.31)	(0.31)	(0.31)	(0.30)	(0.71)	(0.71)	(0.71)	(0.70)	(0.71)	(0.48)	(0.47)	(0.46)	(0.39)	(0.40)
	$\mathbf{n}_2$	0.58	0.72	0.68	0.58	4.40	22.84	24.26	24.86	27.08	54.02	67.32	68.56	69.40	76.16	92.56
		(0.11)	(0.12)	(0.12)	(0.11)	(0.29)	(0.59)	(0.61)	(0.61)	(0.63)	(0.70)	(0.66)	(0.66)	(0.65)	(0.60)	(0.37)
				$\delta = 0$					$\delta = 1.00$					$\delta = 1.60$		
	$\mathbf{n}_1$	4.44	4.84	5.14	5.36	4.82	63.48	65.18	65.30	53.52	51.64	98.40	98.50	98.54	95.62	94.96
0.9		(0.29)	(0.30)	(0.31)	(0.32)	(0.30)	(0.68)	(0.67)	(0.67)	(0.71)	(0.71)	(0.18)	(0.17)	(0.17)	(0.29)	(0.31)
	$\mathbf{n}_2$	0.12	0.12	0.16	0.48	4.18	29.22	30.62	30.56	21.68	53.26	89.68	90.72	90.70	78.44	95.44
		(0.05)	(0.05)	(0.06)	(0.10)	(0.28)	(0.64)	(0.65)	(0.65)	(0.58)	(0.71)	(0.43)	(0.41)	(0.41)	(0.58)	(0.30)

$\mathbf{n}_1 = [80, 80], \mathbf{n}_2 = [50, 100]$.

S.2.5 Effect of Discretization Resolution

In this simulation study, we study the effect of discretization in the implementation of the proposed $F_{\max}$-test. To this end, we redo the simulation studies which produce Table 1 when $\rho = 0.5$ in the paper and Table S.4 when $\rho = 0.1$ for different discretization resolutions $M = 150, 300, 500$, and 1000. The simulation results are presented in Tables S.6 and S.7, respectively. Compared with the associated results in Tables 1 in the paper and Table S.4, we can see the results are comparable for different discretization resolutions. This indicates that implementation of the

Table S.6: Empirical sizes and powers (%) of the L^2, F, GPF and $F_{\max}$ when $\rho = 0.5$ under the same simulation settings that produce Table 1 of the paper, but with $M = 150, 300, 500,$ and 1000.

M	$\mathbf{n}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$
		$\delta = 0$				$\delta = 0.20$				$\delta = 0.40$			
	$\mathbf{n}_1$	5.60	5.70	6.04	5.52	20.06	20.42	21.50	29.36	69.90	70.32	72.64	92.40
		(0.33)	(0.33)	(0.34)	(0.32)	(0.57)	(0.57)	(0.58)	(0.64)	(0.65)	(0.65)	(0.63)	(0.37)
150	$\mathbf{n}_2$	4.76	4.92	5.02	4.66	39.14	39.72	41.22	63.90	97.42	97.50	97.84	100
		(0.30)	(0.31)	(0.31)	(0.30)	(0.69)	(0.69)	(0.70)	(0.68)	(0.22)	(0.22)	(0.21)	(0)
	$\mathbf{n}_3$	5.30	5.52	5.54	5.04	64.54	65.08	66.34	90.46	99.96	99.96	99.98	100
		(0.32)	(0.32)	(0.32)	(0.31)	(0.68)	(0.67)	(0.67)	(0.42)	(0.03)	(0.03)	(0.02)	(0)
		$\delta = 0$				$\delta = 0.20$				$\delta = 0.40$			
	$\mathbf{n}_1$	5.46	5.52	5.96	5.46	19.58	19.90	21.28	28.84	68.50	68.80	70.82	92.26
		(0.32)	(0.32)	(0.33)	(0.32)	(0.56)	(0.56)	(0.58)	(0.64)	(0.66)	(0.66)	(0.64)	(0.38)
300	$\mathbf{n}_2$	5.36	5.58	5.84	5.46	37.66	38.18	39.92	63.34	97.24	97.36	97.82	99.98
		(0.32)	(0.32)	(0.33)	(0.32)	(0.69)	(0.69)	(0.69)	(0.68)	(0.23)	(0.23)	(0.21)	(0.02)
	$\mathbf{n}_3$	5.06	5.20	5.46	5.18	64.82	65.72	67.24	90.98	99.98	99.98	100	100
		(0.31)	(0.31)	(0.32)	(0.31)	(0.68)	(0.67)	(0.66)	(0.41)	(0.02)	(0.02)	(0)	(0)
		$\delta = 0$				$\delta = 0.20$				$\delta = 0.40$			
	$\mathbf{n}_1$	5.26	5.42	5.98	5.30	19.42	19.84	21.12	28.26	69.08	69.44	71.28	92.96
		(0.32)	(0.32)	(0.34)	(0.32)	(0.56)	(0.56)	(0.58)	(0.64)	(0.65)	(0.65)	(0.64)	(0.36)
500	$\mathbf{n}_2$	5.26	5.40	5.62	4.96	39.12	39.82	41.12	64.10	97.56	97.60	98.02	100
		(0.32)	(0.32)	(0.33)	(0.31)	(0.69)	(0.69)	(0.70)	(0.68)	(0.22)	(0.22)	(0.20)	(0)
	$\mathbf{n}_3$	4.72	4.96	5.06	4.78	64.22	65.14	66.76	91.10	99.98	99.98	99.98	100
		(0.30)	(0.31)	(0.31)	(0.30)	(0.68)	(0.67)	(0.67)	(0.40)	(0.02)	(0.02)	(0.02)	(0)
		$\delta = 0$				$\delta = 0.20$				$\delta = 0.40$			
	$\mathbf{n}_1$	5.30	5.46	5.92	5.04	18.54	18.94	20.30	28.74	68.22	68.50	70.62	92.48
		(0.32)	(0.32)	(0.33)	(0.31)	(0.55)	(0.55)	(0.57)	(0.64)	(0.66)	(0.66)	(0.64)	(0.37)
1000	$\mathbf{n}_2$	5.02	5.24	5.36	5.02	39.66	40.16	41.34	64.72	97.56	97.84	98.06	99.98
		(0.31)	(0.32)	(0.32)	(0.31)	(0.69)	(0.69)	(0.70)	(0.68)	(0.22)	(0.21)	(0.20)	(0.02)
	$\mathbf{n}_3$	5.32	5.58	5.78	4.90	64.28	64.96	66.46	91.36	99.98	99.98	100	100
		(0.32)	(0.32)	(0.33)	(0.31)	(0.68)	(0.67)	(0.67)	(0.40)	(0.02)	(0.02)	(0)	(0)

$\mathbf{n}_1 = [20, 30, 30], \mathbf{n}_2 = [40, 30, 70], \mathbf{n}_3 = [80, 70, 100]$.

tests by discretization with $M = 80 \sim 150$ may be generally sufficient. See also Section 4.5.6 of Zhang (2013) for the effect of the resolution number M on an F-type test for functional data.

Table S.7: Empirical sizes and powers (%) of the L^2, F, GPF and $F_{\max}$ when $\rho = 0.1$ under the same simulation settings that produce Table S.4, but with $M = 150, 300, 500, 1000$.

M	$\mathbf{n}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$	L^2	F	GPF	$F_{\max}$
		$\delta = 0$				$\delta = 0.12$				$\delta = 0.26$			
	$\mathbf{n}_1$	5.44	5.28	5.22	4.52	14.50	14.08	12.84	11.88	48.14	47.38	39.36	51.98
		(0.32)	(0.32)	(0.31)	(0.29)	(0.50)	(0.49)	(0.47)	(0.46)	(0.71)	(0.71)	(0.69)	(0.71)
150	$\mathbf{n}_2$	4.54	4.42	4.84	4.36	22.08	21.92	18.54	21.48	82.34	82.24	73.16	91.18
		(0.29)	(0.29)	(0.30)	(0.29)	(0.59)	(0.59)	(0.55)	(0.58)	(0.54)	(0.54)	(0.63)	(0.40)
	$\mathbf{n}_3$	5.16	5.16	5.18	5.10	38.98	39.02	32.20	42.10	97.24	97.24	93.98	99.64
		(0.31)	(0.31)	(0.31)	(0.31)	(0.69)	(0.69)	(0.66)	(0.70)	(0.23)	(0.23)	(0.34)	(0.08)
		$\delta = 0$				$\delta = 0.12$				$\delta = 0.26$			
	$\mathbf{n}_1$	5.38	5.26	5.50	5.18	13.18	12.74	11.80	11.32	47.32	46.52	39.74	51.56
		(0.32)	(0.32)	(0.32)	(0.31)	(0.48)	(0.47)	(0.46)	(0.45)	(0.71)	(0.71)	(0.69)	(0.71)
300	$\mathbf{n}_2$	5.24	5.14	5.28	4.82	23.10	22.94	19.46	22.44	82.18	81.86	73.20	90.84
		(0.32)	(0.31)	(0.32)	(0.30)	(0.60)	(0.59)	(0.56)	(0.59)	(0.54)	(0.55)	(0.63)	(0.41)
	$\mathbf{n}_3$	5.60	5.66	5.78	5.56	37.52	37.62	30.56	41.06	97.82	97.82	94.44	99.72
		(0.33)	(0.33)	(0.33)	(0.32)	(0.68)	(0.69)	(0.65)	(0.70)	(0.21)	(0.21)	(0.32)	(0.07)
		$\delta = 0$				$\delta = 0.12$				$\delta = 0.26$			
	$\mathbf{n}_1$	6.06	5.88	6.40	5.84	13.78	13.38	12.34	10.92	46.78	46.06	39.08	51.76
		(0.34)	(0.33)	(0.35)	(0.33)	(0.49)	(0.48)	(0.47)	(0.44)	(0.71)	(0.70)	(0.69)	(0.71)
500	$\mathbf{n}_2$	6.08	6.00	6.08	5.60	23.26	23.12	19.54	22.10	82.10	81.86	73.28	91.40
		(0.34)	(0.34)	(0.34)	(0.33)	(0.60)	(0.60)	(0.56)	(0.59)	(0.54)	(0.55)	(0.63)	(0.40)
	$\mathbf{n}_3$	4.70	4.74	5.22	4.72	37.30	37.32	30.46	40.50	97.20	97.20	93.80	99.76
		(0.30)	(0.30)	(0.31)	(0.30)	(0.68)	(0.68)	(0.65)	(0.69)	(0.23)	(0.23)	(0.34)	(0.07)
		$\delta = 0$				$\delta = 0.12$				$\delta = 0.26$			
	$\mathbf{n}_1$	5.18	5.10	5.36	4.76	13.74	13.36	12.26	11.26	46.92	46.14	39.24	50.70
		(0.31)	(0.31)	(0.32)	(0.30)	(0.49)	(0.48)	(0.46)	(0.45)	(0.71)	(0.71)	(0.69)	(0.71)
1000	$\mathbf{n}_2$	5.06	5.02	5.48	5.04	23.72	23.46	20.06	22.36	81.64	81.52	72.56	90.38
		(0.31)	(0.31)	(0.32)	(0.31)	(0.60)	(0.60)	(0.57)	(0.59)	(0.55)	(0.55)	(0.63)	(0.42)
	$\mathbf{n}_3$	5.22	5.24	5.40	5.26	38.08	38.16	30.86	40.62	97.50	97.50	94.14	99.68
		(0.31)	(0.32)	(0.32)	(0.32)	(0.69)	(0.69)	(0.65)	(0.69)	(0.22)	(0.22)	(0.33)	(0.08)

$\mathbf{n}_1 = [20, 30, 30], \mathbf{n}_2 = [40, 30, 70], \mathbf{n}_3 = [80, 70, 100]$.

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