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The signless Laplacian spectral radius of k -connected irregular graphs [★]

Wai Chee Shiu ^a, Peng Huang ^{a,1}, Pak Kiu Sun ^a

^a*Department of Mathematics, Hong Kong Baptist University,
224 Waterloo Road, Kowloon Tong, Hong Kong, China.*

Abstract

Let G be a k -connected irregular graph of order n , size m and maximum degree Δ . Let q_1 be the signless Laplacian spectral radius of G . In this article, we prove the following lower bound on $2\Delta - q_1$:

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.$$

Moreover, we determine similar bounds for the signless Laplacian spectral radius of proper spanning subgraphs and k -edge-connected graphs.

Key words: the signless Laplacian spectrum; irregular graph; k -connected; k -edge-connected
AMS 2010 MSC: 05C50.

1 Introduction

All graphs in this paper are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph of order n and size m , where the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and $|E(G)| = m$. We denote $N_G(v)$ (or $N(v)$ for short) as the set of neighbours of v in G , and $|N_G(v)|$ as the degree of v . For $i \in \{1, 2, \dots, n\}$, let $d_i = d_G(v_i) = |N_G(v_i)|$. Moreover, the maximum and minimum degree of G are denoted by Δ and δ , respectively. A graph G is called *regular* if $\Delta = \delta$. The *local connectivity* $p(u, v)$ between distinct vertices u and v is the maximum number of pairwise internally disjoint uv -paths. A nontrivial graph G is *k -connected* if $p(u, v) \geq k$ for any two distinct vertices u and v . The *connectivity* $\kappa(G)$ of G is the maximum value of k for which G is k -connected. For two distinct vertices u and v , the *distance* between u and

[★] Supported by General Research Fund of Hong Kong; FRG, Hong Kong Baptist University
Email addresses: wcsiu@hkbu.edu.hk (Wai Chee Shiu), 13480189@life.hkbu.edu.hk
(Peng Huang), lionel@hkbu.edu.hk (Pak Kiu Sun).

¹ Corresponding author

v is the length of the shortest path between u and v . And the *diameter* of G , denoted by d , is the greatest distance between any two distinct vertices of G . The notations undefined in this article can be found in the book [1].

Let $A(G)$ be the adjacency matrix of G and $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ (or simply L and Q) are called the Laplacian matrix and the signless Laplacian matrix of G , respectively. Obviously, all the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are real numbers since these matrices are real symmetric matrices. The largest eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are denoted by $\lambda_1(G)$, $\mu_1(G)$ and $q_1(G)$ (or simply λ_1 , μ_1 and q_1) of G , respectively. They are called the *spectral radius*, the *Laplacian spectral radius* and the *signless Laplacian spectral radius*. If G is a connected graph, then Q is a nonnegative irreducible positive semi-definite matrix. Hence, by the Perron-Frobenius Theorem, the multiplicity of q_1 is one and there exists a positive eigenvector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$, which is called the *Perron vector* of Q , with $\sum_{i=1}^n x_i^2 = 1$, such that $Q\mathbf{x} = q_1\mathbf{x}$.

It is known that $q_1 \leq 2\Delta$ and equality holds if and only if G is a regular graph. We want to determine the gap between 2Δ and q_1 if G is irregular. In 2012, Ning, Li and Lu proved [9, Theorem 3.2] that

$$2\Delta - q_1 > \frac{1}{n(d - \frac{1}{4})}. \quad (1.1)$$

There are several results involving $\Delta - \lambda_1$ (see [4, 5, 8, 10–12]). Recently, Chen and Hou study the spectral radius of k -connected irregular graphs and gave a new lower bound on $\Delta - \lambda_1$ [2, Theorem 1]:

$$\Delta - \lambda_1 > \frac{(n\Delta - 2m)k^2}{(n\Delta - 2m)(n^2 - 2(n - k)) + nk^2}. \quad (1.2)$$

In this paper, we establish a similar lower bound on $2\Delta - q_1$, which will be proved in the next section. When $k \geq \sqrt{n}$, our lower bound on $2\Delta - q_1$ is better than the bound in (1.1) and with the same arguments in this paper, we can improve the bound in (1.2), which are presented in the remarks.

2 Main results

Theorem 2.1 *Let G be a k -connected irregular graph ($k \geq 1$) of order n (≥ 3), size m and maximum degree Δ . We have*

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}. \quad (2.1)$$

Proof. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be the Perron vector corresponding to q_1 , thus $Q\mathbf{x} = q_1\mathbf{x}$, $x_i > 0$ and $\sum_{i=1}^n x_i^2 = 1$. Let s and u be vertices of G such that $x_s = \max_{1 \leq i \leq n} \{x_i\}$ and

$x_u = \min_{1 \leq i \leq n} \{x_i\}$. Since $\sum_{i=1}^n x_i^2 = 1$ and G is irregular, we have $x_s > \frac{1}{\sqrt{n}} > x_u$. Consider the following two cases:

Case 1: Suppose $d_s \leq \Delta - 1$. Since $Q\mathbf{x} = q_1\mathbf{x}$, we have

$$q_1x_s = d_sx_s + \sum_{j \in N(s)} x_j \leq (\Delta - 1)x_s + (\Delta - 1)x_s = 2(\Delta - 1)x_s.$$

Thus $q_1 \leq 2\Delta - 2$ as $x_s > 0$. Note that $1 \leq k \leq \delta \leq \Delta - 1 \leq n - 2$ as G is irregular. We obtain $n^2 - (\Delta - k + 2)(n - k) \geq n^2 - (n - 1 - 1 + 2)(n - k) = nk > 0$. Therefore,

$$\begin{aligned} & \frac{(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2} \\ & \leq \frac{(n\Delta - 2m)k^2}{2(n\Delta - 2m)nk + nk^2} \\ & < \frac{(n\Delta - 2m)k^2}{2(n\Delta - 2m)k^2 + nk^2} \\ & = \frac{n\Delta - 2m}{2(n\Delta - 2m) + n} \\ & < 1. \end{aligned}$$

Consequently,

$$2\Delta - q_1 \geq 2 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.$$

Case 2: Suppose $d_s = \Delta$. For the vertex u , we have

$$q_1x_u = d_u x_u + \sum_{j \in N(u)} x_j \geq d_u x_u + d_u x_u = 2d_u x_u.$$

Moreover, G is irregular implies that $q_1x_u < 2\Delta x_u$. Hence $d_u < \Delta$ since $x_u > 0$.

As G is k -connected, by Menger's Theorem, there are at least k vertex-disjoint paths between s and u . We choose k paths from them such that the sum of the lengths of these k paths is as small as possible. Let P_1, P_2, \dots, P_k be such k paths. Obviously, each of these path P_i can only contain one vertex of $N_G(s)$. Otherwise, we can find another k paths that have a smaller sum of lengths. Then there exist at least $\Delta - k$ vertices v_i , where $i \in \{1, 2, \dots, \Delta - k\}$, such that $v_i \in N_G(s)$ but $v_i \notin \bigcup_{i=1}^k V(P_i)$. As a result,

$$\sum_{i=1}^k |V(P_i)| \leq n - (\Delta - k) + 2(k - 1).$$

Following the argument of Chen [2] and Ning [9], we obtain

$$\begin{aligned}
2\Delta - q_1 &= 2\Delta - \mathbf{x}^T Q(G) \mathbf{x} \\
&= 2\Delta \sum_{i=1}^n x_i^2 - \sum_{ij \in E(G)} (x_i + x_j)^2 \\
&= 2\Delta \sum_{i=1}^n x_i^2 - \left(2 \sum_{i=1}^n d_i x_i^2 - \sum_{ij \in E(G)} (x_i - x_j)^2 \right) \\
&= 2 \sum_{i=1}^n (\Delta - d_i) x_i^2 + \sum_{ij \in E(G)} (x_i - x_j)^2 \tag{2.2}
\end{aligned}$$

$$\geq 2(n\Delta - 2m)x_u^2 + \sum_{ij \in E(G)} (x_i - x_j)^2. \tag{2.3}$$

Note that $\sum_{i=1}^k |V(P_i)| \leq n - (\Delta - k) + 2(k - 1)$. In addition, by Cauchy-Schwarz inequality, we can prove that

$$\sum_{ij \in E(P_t)} (x_i - x_j)^2 \geq \frac{1}{|V(P_t)| - 1} \left(\sum_{ij \in E(P_t)} (x_i - x_j) \right)^2$$

and

$$\sum_{t=1}^k \frac{1}{|V(P_t)| - 1} \geq \frac{k^2}{\sum_{t=1}^k (|V(P_t)| - 1)}.$$

Therefore, from these two inequalities, we obtain

$$\begin{aligned}
\sum_{ij \in E(G)} (x_i - x_j)^2 &\geq \sum_{t=1}^k \sum_{ij \in E(P_t)} (x_i - x_j)^2 \\
&\geq \sum_{t=1}^k \frac{1}{|V(P_t)| - 1} \left(\sum_{ij \in E(P_t)} (x_i - x_j) \right)^2 \\
&= \sum_{t=1}^k \frac{1}{|V(P_t)| - 1} (x_s - x_u)^2 \\
&\geq \frac{k^2}{\sum_{t=1}^k (|V(P_t)| - 1)} (x_s - x_u)^2 \\
&\geq \frac{k^2}{n - \Delta + 2k - 2} (x_s - x_u)^2. \tag{2.4}
\end{aligned}$$

Combining (2.3) and (2.4), we have

$$\begin{aligned}
2\Delta - q_1 &\geq 2(n\Delta - 2m)x_u^2 + \frac{k^2}{n - \Delta + 2k - 2} (x_s - x_u)^2 \tag{2.5} \\
&= \left(2(n\Delta - 2m) + \frac{k^2}{n - \Delta + 2k - 2} \right) x_u^2 - \frac{2k^2}{n - \Delta + 2k - 2} x_s x_u \\
&\quad + \frac{k^2}{n - \Delta + 2k - 2} x_s^2.
\end{aligned}$$

Let $f(x_u) = \left(2(n\Delta - 2m) + \frac{k^2}{n - \Delta + 2k - 2} \right) x_u^2 - \frac{2k^2}{n - \Delta + 2k - 2} x_s x_u + \frac{k^2}{n - \Delta + 2k - 2} x_s^2$. If we regard $f(x_u)$

as a quadratic function, then we have

$$2\Delta - q_1 \geq \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n - \Delta + 2k - 2) + k^2} x_s^2. \quad (2.6)$$

Let

$$C = \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}$$

and consider the following two subcases:

Subcase 1: Suppose $k = 1$. We have

$$2\Delta - q_1 \geq \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n - \Delta) + 1} x_s^2 \quad (2.7)$$

and

$$C = \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n}.$$

If $x_u^2 \geq \frac{C}{2(n\Delta - 2m)}$, then from (2.5), we obtain

$$2\Delta - q_1 \geq 2(n\Delta - 2m) \cdot \frac{C}{2(n\Delta - 2m)} + \frac{k^2}{n - \Delta + 2k - 2} (x_s - x_u)^2 > C$$

since $x_s > x_u$ as G is irregular.

If $x_u^2 < \frac{C}{2(n\Delta - 2m)}$, then since $\sum_{i=1}^n x_i^2 = 1$, we have

$$x_s^2 \geq \frac{1 - x_u^2}{n - 1} > \frac{1}{n - 1} \cdot \left(1 - \frac{C}{2(n\Delta - 2m)}\right).$$

Hence from (2.7) and $n^2 - (\Delta + 1)(n - 1) = (n - \Delta)(n - 1) + 1 > (n - \Delta)(n - 1)$ we have

$$\begin{aligned} 2\Delta - q_1 &\geq \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n - \Delta) + 1} x_s^2 \\ &> \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n - \Delta) + 1} \times \frac{1}{n - 1} \\ &\quad \times \left(1 - \frac{1}{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n}\right) \\ &= \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n} \\ &\quad \times \frac{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n - 1}{2(n\Delta - 2m)(n - \Delta)(n - 1) + n - 1} \\ &> C \times 1 \\ &= C. \end{aligned}$$

Subcase 2: Suppose $k \geq 2$.

If $x_u^2 \geq \frac{C}{2(n\Delta-2m)}$, then the result can be obtained by using a similar argument of Subcase 1.

Since $d_u \geq k$. We can choose $k-1$ vertices from $N_G(u)$, denoted by u_1, u_2, \dots, u_{k-1} , such that $s \notin \{u_1, u_2, \dots, u_{k-1}\}$. If $\sum_{t=1}^{k-1} x_{u_t}^2 > C \cdot (1 + \frac{k-1}{2(n\Delta-2m)})$, then by (2.3) and the method to derive (2.6), we obtain

$$\begin{aligned}
2\Delta - q_1 &\geq 2(n\Delta - 2m)x_u^2 + \sum_{t=1}^{k-1} (x_{u_t} - x_u)^2 \\
&= \sum_{t=1}^{k-1} \left(\frac{2(n\Delta - 2m)}{k-1} x_{u_t}^2 + (x_{u_t} - x_u)^2 \right) \\
&\geq \sum_{t=1}^{k-1} \frac{2(n\Delta - 2m)}{2(n\Delta - 2m) + k - 1} x_{u_t}^2 \\
&= \frac{2(n\Delta - 2m)}{2(n\Delta - 2m) + k - 1} \cdot \sum_{t=1}^{k-1} x_{u_t}^2 \\
&> \frac{2(n\Delta - 2m)}{2(n\Delta - 2m) + k - 1} \cdot \frac{2(n\Delta - 2m) + k - 1}{2(n\Delta - 2m)} \cdot C \\
&= C.
\end{aligned}$$

It remains to show that our result is valid when $x_u^2 < \frac{C}{2(n\Delta-2m)}$ and $\sum_{t=1}^{k-1} x_{u_t}^2 \leq C \cdot (1 + \frac{k-1}{2(n\Delta-2m)})$. Using $\sum_{i=1}^n x_i^2 = 1$ again, we have

$$x_s^2 \geq \frac{1}{n-k} \left(1 - x_u^2 - \sum_{t=1}^{k-1} x_{u_t}^2 \right) > \frac{1}{n-k} \left(1 - \frac{2(n\Delta - 2m) + k}{2(n\Delta - 2m)} \cdot C \right).$$

Therefore, from (2.6),

$$\begin{aligned}
2\Delta - q_1 &> \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n - \Delta + 2k - 2) + k^2} \cdot \frac{1}{n-k} \cdot \left(1 - \frac{2(n\Delta - 2m) + k}{2(n\Delta - 2m)} \cdot C \right) \\
&= \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2} \\
&= C.
\end{aligned}$$

This completes the proof.

Remark 1 Note that the bound in (2.1) increases when $n\Delta - 2m$ increases. It is obvious that $n\Delta - 2m \geq 1$ as G is irregular. Hence from Theorem 2.1 we can easily find that

$$2\Delta - q_1 > \frac{2k^2}{2(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.$$

Let

$$f(k) = \frac{2k^2}{2(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.$$

Then

$$\begin{aligned} f(k) &\geq \frac{2k^2}{2(n^2 - 3(n - k)) + nk^2} \\ &= \frac{1}{n/2 + (n^2 - 3n)/k^2 + 3/k}. \end{aligned}$$

And $\frac{1}{n/2 + (n^2 - 3n)/k^2 + 3/k}$ is also an increasing function on k . Thus when $k \geq \sqrt{n}$, we have

$$f(k) \geq \frac{1}{(3n)/2 + 3/\sqrt{n} - 3} > \frac{1}{(7n)/4} \geq \frac{1}{n(d - 1/4)}.$$

The second inequality holds as $n \geq 3$ and the third holds since G is irregular, which implies $d \geq 2$. Therefore, when $k \geq \sqrt{n}$, the bound in (2.1) is better than the bound in (1.1).

Remark 2 In the proof of Theorem 2.1, we use the fact that there are at least $\Delta - k$ vertices of G that do not belong to the subgraph $H = G[\cup_{t=1}^k V(P_t)]$. The lower bound on $2\Delta - q_1$ can be improved when the vertices outside the subgraph H are increased. In fact, one can show that if there are l vertices outside the subgraph H , then

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (l + 2)(n - k)) + nk^2}. \quad (2.8)$$

The bound in (2.1) is a particular case ($l = \Delta - k$) of the bound in (2.8).

Remark 3 Using the same arguments, we can prove that

$$\Delta - \lambda_1 > \frac{(n\Delta - 2m)k^2}{(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}$$

which improves the bound in (1.2). Similar to (2.8), if there are l vertices outside the subgraph H , then

$$\Delta - \lambda_1 > \frac{(n\Delta - 2m)k^2}{(n\Delta - 2m)(n^2 - (l + 2)(n - k)) + nk^2}.$$

To consider the signless Laplacian spectral radius of a subgraph, we begin with the following lemma.

Lemma 2.2 [7, Theorem 2.1] *Let G be a graph on n vertices and m edges and let e be an edge of G . Let q_1, q_2, \dots, q_n ($q_1 \geq q_2 \geq \dots \geq q_n$) and s_1, s_2, \dots, s_n ($s_1 \geq s_2 \geq \dots \geq s_n$) be the signless Laplacian spectra of G and of $G - e$, respectively. Then*

$$0 \leq s_n \leq q_n \leq \dots \leq s_2 \leq q_2 \leq s_1 \leq q_1.$$

Let G be an irregular connected graph of order n and maximum degree Δ . We cannot always find a Δ -regular graph G' such that G is a proper spanning subgraph of G' . In fact, if $n\Delta$ is an odd number, then the Δ -regular graph G' cannot be found as $n\Delta = 2m \equiv 0$

(mod 2) for G' . Thus if G is a proper spanning subgraph of Δ -regular graph G' , then we can prove the following theorem.

Theorem 2.3 *Let G be a proper spanning subgraph of a Δ -regular k -connected graph G' of order n . If $k \geq 2$, then*

$$2\Delta - q_1(G) > \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta+2k-4) + (n+1)(k-1)^2}.$$

Proof. Lemma 2.2 implies that the signless Laplacian spectral radius cannot be increased by deleting an edge. Thus we may assume that $G = G' - e$ for some edge e of G' . Let $e = uv$, we have $d_G(u) = d_G(v) = \Delta - 1$ and $d_G(w) = \Delta$ for other vertices w . Since G is connected when $k \geq 2$, there exist a Perron vector, $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ with $x_i > 0$, corresponding to $q_1(G)$. Let s be a vertex of G such that $x_s = \max_{1 \leq i \leq n} \{x_i\}$. Hence

$$q_1(G)x_s = d_G(s)x_s + \sum_{j \in N_G(s)} x_j \leq d_G(s)x_s + d_G(s)x_s = 2d_G(s)x_s$$

which means that $q_1(G) \leq 2d_G(s)$. Moreover, using the fact that $\lambda_1(G) > \frac{2|E(G)|}{n} = \frac{n\Delta-2}{n}$ (see [6]) together with $2 \leq k \leq n-1$, we have

$$q_1(G) \geq 2\lambda_1(G) > 2\frac{2|E(G)|}{n} = 2\frac{n\Delta-2}{n} = 2\Delta - \frac{4}{n} \geq 2\Delta - \frac{4}{3} > 2\Delta - 2.$$

Combining this with $q_1(G) \leq 2d_G(s)$, we obtain $d_G(s) > \Delta - 1$. Therefore, $d_G(s) = \Delta$ and so $s \neq u$ and $s \neq v$.

In addition, formula (2.2) gives

$$\begin{aligned} 2\Delta - q_1(G) &= 2\sum_{i=1}^n (\Delta - d_G(v_i))x_i^2 + \sum_{ij \in E(G)} (x_i - x_j)^2 \\ &= 2(x_u^2 + x_v^2) + \sum_{ij \in E(G)} (x_i - x_j)^2. \end{aligned} \quad (2.9)$$

With the same argument as Chen in [2], we know that

$$\sum_{ij \in E(G)} (x_i - x_j)^2 \geq \frac{(k-1)^2}{n-\Delta+2k-4} (x_s - x_u)^2.$$

Hence, similar to the proof of (2.6), we have

$$\begin{aligned} 2\Delta - q_1(G) &\geq 2(x_u^2 + x_v^2) + \frac{(k-1)^2}{n-\Delta+2k-4} (x_s - x_u)^2 \\ &> 2x_u^2 + \frac{(k-1)^2}{n-\Delta+2k-4} (x_s - x_u)^2 \\ &\geq \frac{2(k-1)^2}{2(n-\Delta+2k-4) + (k-1)^2} x_s^2. \end{aligned} \quad (2.10)$$

Define

$$C' = \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta+2k-4) + (n+1)(k-1)^2}.$$

If $x_u^2 + x_v^2 > \frac{C'}{2}$, then from (2.9),

$$2\Delta - q_1(G) = 2(x_u^2 + x_v^2) + \sum_{ij \in E(G)} (x_i - x_j)^2 > 2\frac{C'}{2} + \sum_{ij \in E(G)} (x_i - x_j)^2 \geq C'.$$

Since $d_G(u) = \Delta - 1$, it is possible to choose $\Delta - 2$ vertices $\{u_1, u_2, \dots, u_{\Delta-2}\}$ from $N_G(u)$ such that $s \notin \{u_1, u_2, \dots, u_{\Delta-2}\}$. Hence if $\sum_{t=1}^{\Delta-2} x_{u_t}^2 \geq \frac{\Delta}{2}C'$, then by (2.9) again, we have

$$\begin{aligned} 2\Delta - q_1(G) &> 2x_u^2 + \sum_{t=1}^{\Delta-2} (x_{u_t} - x_u)^2 \\ &= \sum_{t=1}^{\Delta-2} \left(\frac{2}{\Delta-2} x_u^2 + (x_{u_t} - x_u)^2 \right) \\ &\geq \sum_{t=1}^{\Delta-2} \frac{\frac{2}{\Delta-2}}{1 + \frac{2}{\Delta-2}} x_{u_t}^2 \\ &\geq \frac{2}{\Delta} \frac{\Delta}{2} C' \\ &= C'. \end{aligned}$$

The remaining case is $x_u^2 + x_v^2 \leq \frac{C'}{2}$ and $\sum_{t=1}^{\Delta-2} x_{u_t}^2 < \frac{\Delta}{2}C'$. Obviously,

$$x_s^2 \geq \frac{1 - x_u^2 - x_v^2 - \sum_{t=1}^{\Delta-2} x_{u_t}^2}{n - \Delta} > \frac{1}{n - \Delta} \left(1 - \frac{C'}{2} - \frac{\Delta}{2} C' \right) = \frac{1}{n - \Delta} \left(1 - \frac{\Delta + 1}{2} C' \right),$$

and from (2.10) we obtain

$$\begin{aligned} 2\Delta - q_1(G) &> \frac{2(k-1)^2}{2(n-\Delta+2k-4) + (k-1)^2} \times \frac{1}{n-\Delta} \times \left(1 - \frac{\Delta+1}{2} C' \right) \\ &= \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta+2k-4) + (n+1)(k-1)^2} \\ &= C'. \end{aligned}$$

This completes the proof.

Remark 4 It can be shown that Theorem 2.3 holds for any proper subgraph no matter whether the proper subgraph is spanning or not. Indeed, if G is a proper subgraph but not a spanning subgraph of G' , then we can construct a new graph G'' by adding some isolated vertices such that the order of G'' is the same as the order of G' . For G'' , we know that $q_1(G'') = q_1(G)$. And obviously, G'' is a proper spanning subgraph of G' . Hence Theorem 2.3 holds for any proper subgraph G .

The most critical condition in estimating the value of $\sum_{ij \in E(G)} (x_i - x_j)^2$ in the proof of Theorem 2.1 and Theorem 2.3 is that we can use each edge of G at most one time. As a

consequence, if a graph G is k -edge-connected, then this condition is satisfied because we can find pairwise edge-disjoint paths by Menger's Theorem. Therefore we have

Theorem 2.4 *Let G be a k -edge-connected irregular graph ($k \geq 1$) of order n (≥ 3), size m and maximum degree Δ . Then we have*

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.$$

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