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Coefficients of the characteristic polynomial of the (signless, normalized) Laplacian of a graph*

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Abstract

In this paper, we give a combinatorial expression for the fifth coefficient of the (signless) Laplacian characteristic polynomial of a graph. The first five normalized Laplacian coefficients are also given.

Keywords: (Signless, normalized) Laplacian matrix; (signless, normalized) Laplacian characteristic polynomial; coefficient; graph

AMS classification: 05C50; 15A18.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E. Let $d_G(v_i)$ (or simply $d(v_i)$ or d_i) denote the degree of the vertex $v_i \in V$ $(i = 1, 2, \ldots, n)$, and $D = D(G) = diag(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. The *Laplacian matrix*, the *signless Laplacian matrix* and the *normalized Laplacian matrix* of G are defined by $L(G) = D(G) - A(G)$, $Q(G) = D(G) + A(G)$ and $\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$ (with the convention that if the degree of v is 0 then $d(v)^{-1/2} = 0$), respectively, where $A(G)$ denotes the adjacency matrix of G.

Throughout this paper, Let the *Laplacian characteristic polynomial*, the *signless Laplacian characteristic polynomial* and the *normalized Laplacian characteristic polynomial* of G be

$$\Phi(L(G)) = det(xI - L(G)) = \sum_{i=0}^{n} q_i(G)x^{n-i},$$

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$$\Phi(Q(G)) = det(xI - Q(G)) = \sum_{i=0}^{n} p_i(G)x^{n-i},$$

and

$$\Phi(\mathcal{L}(G)) = det(xI - \mathcal{L}(G)) = \sum_{i=0}^{n} r_i(G)x^{n-i},$$

respectively.

The search for isomorphism invariants has led to consideration of various algebraic properties of the (adjacency, signless Laplacian, normalized Laplacian) matrix of a graph. In particular, interest has focused on the coefficients of the characteristic polynomial of the (adjacency, signless Laplacian, normalized Laplacian) matrix.

Collatz and Sinogowitz [1] investigated the relationship between the coefficients of the characteristic polynomial of the adjacency matrix of a graph and certain subgraphs. In [5], a formula for the coefficients of the characteristic polynomial of the adjacency matrix of an arbitrary digraph was derived and it was shown that the coefficients of the polynomial of a tree count matchings.

In [6], Oliveira et al. gave the formulas for the first four coefficients of the Laplacian characteristic polynomial of a graph. In [2], Cvetković et al. gave the first three coefficients of the signless Laplacian characteristic polynomial of a graph. In [7], Wang et al. gave the fourth coefficient of the signless Laplacian characteristic polynomial of a graph. The coefficients of the (adjacency, signless Laplacian, normalized Laplacian) matrix can be used to distinguish non-isomorphic graphs in some class of graphs. By using the fifth Laplacian coefficient of trees, Lepović and Gutman [4] proved that no starlike trees are cospectral. The (signless) Laplacian coefficients were used to discuss the (signless) Laplacian spectral characterizations of two kind of graphs:∞-graphs and 3-rose graphs [8, 9]. In this paper, we give the fifth coefficient of the Laplacian characteristic polynomial and the signless Laplacian characteristic polynomial and the first five coefficients of the normalized Laplacian characteristic polynomial of a graph, respectively.

2 The Laplacian coefficient

Laplacian coefficients can be expressed in terms of subtree structures of G by the following result of Kelmans [3].

Theorem 2.1 ([3]) *Let F be a spanning forest of G with components $T_i, i = 1, 2, \cdots, k$, having n_i vertices each, and let $\gamma(F) = \Pi_{i=1}^{k} n_i$. The Laplacian coefficient q_{n-k} of a graph G is given by*

$$(-1)^{n-k} q_{n-k} = \sum_{F \in \mathscr{F}_k} \gamma(F),$$

where $\mathscr{F}_k$ is the set of all spanning forests of G with exactly k components.

Let $A^k = (a_{ij}^{(k)})$, where $A = (a_{ij})$ is the adjacency matrix of a connected graph G. It is a well-known fact in graph theory that $a_{ij}^{(k)}$ is the number of walks from v_i to v_j with length k of G. By using Theorem 2.1, it is easy to see that $q_0(G) = 1$, $q_1(G) = -2m$, where m is the number of edges of G. Furthermore, Oliveira et al. [6] gave the third and the fourth Laplacian coefficients, as follows:

Theorem 2.2 *[6] Let G be a simple graph with n vertices and m edges and let $d = (d_1, d_2, \cdots, d_n)$ be its degree sequence. Then*

$$q_2(G) = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2;$$

$$q_3(G) = \frac{1}{3}[-4m^3 + 6m^2 + 3(m-1)\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 + tr(A^3)],$$

where A is the adjacency matrix of G.

Note that $tr(A^3) = 6n_3(G)$, where $n_3(G)$ is the number of C_3, a cycle with length 3, in G. Thus from Theorem 2.2, we also have

$$q_3(G) = \frac{1}{3}[-4m^3 + 6m^2 + (3m-1)\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 + 6n_3(G)]. \tag{2.1}$$

We define an H_2-spanning forest in G as a spanning graph with only one component isomorphic to P_3 and $(n-3)$ components isomorphic to K_1. Just as Oliveira et al. pointed that in order to find the third Laplacian coefficient, all one needs to do is to count the number of H_2-spanning forests in G. On the other hand, determining $q_3(G)$ is not that easy [6]. In order to obtain the fifth Laplacian coefficient, we need the following:

Lemma 2.3 *[10] Let $B = (b_{ij})$ be a matrix with characteristic polynomial*

$$\Phi(B) = det(xI - B) = x^n + \sum_{i=1}^{n} a_i x^{n-i}.$$

Let $s_k = tr(B^k)$. Then the coefficients of $\Phi(B)$ satisfy the following:

$$a_1 = -s_1 \quad and \quad ka_k = -s_k - a_1 s_{k-1} - a_2 s_{k-2} - \cdots - a_{k-1}s_1, \; (k = 2, 3, \cdots, n).$$

Lemma 2.4 *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ symmetric real matrices and $C = diag(c_1, c_2, \ldots, c_n)$ be the real diagonal matrix. Then we have*

$$tr(ACB) = tr(CAB).$$

Proof. By direct computation, it is easy to see that

$$tr(ACB) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}c_k b_{ki} = \sum_{i=1}^{n} c_i \sum_{k=1}^{n} a_{ki}b_{ik}$$

and

$$tr(CAB) = \sum_{i=1}^{n} c_i \sum_{k=1}^{n} a_{ik}b_{ki}.$$

Note that $A = A^T$ and $B = B^T$. The result follows. □

In the following, we will give a simpler proof of Theorem 2.2. Furthermore, we will give the fifth Laplacian coefficient.

Theorem 2.5 *Let G be a simple graph with n vertices and m edges and let $d = (d_1, d_2, \cdots, d_n)$ be its degree sequence. Then the fifth Laplacian coefficient*

$$q_4(G) = -\frac{1}{4}\sum_{i=1}^{n} d_i^4 + (\frac{2m}{3} - 1)\sum_{i=1}^{n} d_i^3 + \frac{1}{8}(\sum_{i=1}^{n} d_i^2)^2 - \frac{1}{2}(2m^2 - 5m + 1)\sum_{i=1}^{n} d_i^2 - \sum_{v_iv_j \in E(G)} d_i d_j$$
$$+ 2\sum_{i=1}^{n} d_i t_3(v_i) - 2n_4(G) - 4mn_3(G) + \frac{2}{3}m^4 - 2m^3 + \frac{1}{2}m^2 + \frac{1}{2}m, \qquad (2.2)$$

where $t_3(v_i)$ denotes the number of C_3 of G through the vertex v_i and $n_4(G)$ is the number of C_4, a cycle with length 4, in G.

Proof. Let $s_k = tr(L^k(G))$ in Lemma 2.3. It is easy to see that $s_1 = 2m$. From Lemma 2.3, we have $q_1(G) = -2m$. By direct calculation, we have

$$(D - A)^2 = D^2 - DA - AD + A^2; \qquad (2.3)$$
$$(D - A)^3 = D^3 - D^2A - DAD - AD^2 + DA^2 + ADA + A^2D - A^3; \qquad (2.4)$$
$$(D - A)^4 = D^4 - D^3A - D^2AD - DAD^2 + D^2A^2 + DA^2D + DADA - DA^3$$
$$- AD^3 + AD^2A + ADAD - ADA^2 + A^2D^2 - A^2DA - A^3D + A^4. \qquad (2.5)$$

Since

$$tr(D^2) = \sum_{i=1}^{n} d_i^2, tr(DA) = tr(AD) = 0, tr(A^2) = \sum_{i=1}^{n} d_i = 2m,$$

from Eq. (2.3), we have $s_2 = \sum_{i=1}^{n} d_i^2 + 2m$. From Lemma 2.3, we have

$$q_2(G) = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2.$$

From Lemma 2.4 and by simple computation, we have

$$tr(D^3) = \sum_{i=1}^{n} d_i^3,$$
$$tr(D^2A) = tr(DAD) = tr(AD^2) = 0,$$
$$tr(DA^2) = tr(A^2D) = tr(ADA) = \sum_{i=1}^{n} d_i^2,$$
$$tr(A^3) = 6n_3(G).$$

From Eq. (2.4), we have $s_3 = \sum_{i=1}^{n} d_i^3 + 3\sum_{i=1}^{n} d_i^2 - 6n_3(G)$. From Lemma 2.3, Eq. (2.1) holds.

Similarly, we have

$$\begin{aligned}
&tr(D^4) = \sum_{i=1}^{n} d_i^4,\\
&tr(D^3A) = tr(D^2AD) = tr(DAD^2) = tr(AD^3) = 0,\\
&tr(D^2A^2) = tr(DA^2D) = tr(AD^2A) = tr(A^2D^2) = \sum_{i=1}^{n} d_i^3,\\
&tr(DADA) = tr(ADAD) = 2\sum_{v_iv_j\in E(G)} d_id_j,\\
&tr(DA^3) = tr(ADA^2) = tr(A^2DA) = tr(A^3D) = 2\sum_{v_i\in V(G)} d_it_3(v_i),\\
&tr(A^4) = 8n_4(G) + 2\sum_{i=1}^{n} d_i^2 - 2m.
\end{aligned}$$

From Eq. (2.5), we have

$$s_4 = \sum_{i=1}^{n} d_i^4 + 4\sum_{i=1}^{n} d_i^3 + 4\sum_{v_iv_j\in E(G)} d_id_j - 8\sum_{v_i\in V(G)} d_it_3(v_i) + 8n_4(G) + 2\sum_{i=1}^{n} d_i^2 - 2m.$$

From Lemma 2.3, we find that Eq. (2.2) holds. □

3 The signless Laplacian coefficient

A TU-subgraph of a graph G is a spanning subgraph whose components are trees or odd-unicyclic graphs. If H is a TU-subgraph of G consisting of c unicyclic graphs and trees $T_1, T_2, \cdots, T_s$, then we define the quantity $W(H) = 4^c\prod_{i=1}^{s}(1+|E(T_i)|)$ as the weight of H. In [2], Cvetkovic et al. expressed the signless Laplacian coefficients by the weights of TU-graphs of G, and they proved the following result.

Theorem 3.1 ([2]) *Let $\aleph_i$ be the set of all TU-subgraph of G with i edges. Then, for each $1 \le i \le n$,*

$$p_i(G) = \sum_{H\in\aleph_i} (-1)^i W(H).$$

By using the above result, Cvetković et al. and Wang et al. gave the first three and the fourth signless Laplacian coefficients of a graph, respectively. Their results are as follows.

Theorem 3.2 *[2] Let G be a simple graph with n vertices and m edges and let $d = (d_1, d_2, \cdots, d_n)$ be its degree sequence. Then*

$$p_0(G) = 1,\ p_1(G) = -2m,\ p_2(G) = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2;$$

Theorem 3.3 *[7] Let G be a simple graph with n vertices and m edges and let $d = (d_1, d_2, \cdots, d_n)$ be its degree sequence. Then*

$$p_3(G) = \frac{1}{3}[-4m^3 + 6m^2 + 3(m-1)\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 6n_3(G)].$$

In the following, we will give a simpler proof of Theorems 3.2 and 3.3. Furthermore, we will give the fifth signless Laplacian coefficient.

Theorem 3.4 *Let G be a simple graph with n vertices and m edges and let $d = (d_1, d_2, \cdots, d_n)$ be its degree sequence. Then the fifth signless Laplacian coefficient*

$$p_4(G) = -\frac{1}{4}\sum_{i=1}^{n} d_i^4 + (\frac{2m}{3} - 1)\sum_{i=1}^{n} d_i^3 + \frac{1}{8}(\sum_{i=1}^{n} d_i^2)^2 - \frac{1}{2}(2m^2 - 5m + 1)\sum_{i=1}^{n} d_i^2 - \sum_{v_iv_j \in E(G)} d_i d_j$$
$$- 2\sum_{i=1}^{n} d_i t_3(v_i) - 2n_4(G) + 4mn_3(G) + \frac{2}{3}m^4 - 2m^3 + \frac{1}{2}m^2 + \frac{1}{2}m. \qquad (3.6)$$

Proof. Let $s_k = tr(Q^k(G))$ in Lemma 2.3. By direct calculation, we have

$$(D + A)^2 = D^2 + DA + AD + A^2;$$
$$(D + A)^3 = D^3 + D^2A + DAD + DA^2 + AD^2 + ADA + A^2D + A^3;$$
$$(D + A)^4 = D^4 + D^3A + D^2AD + DAD^2 + D^2A^2 + DADA + DA^2D + DA^3$$
$$+ AD^3 + AD^2A + ADAD + ADA^2 + A^2D^2 + A^2DA + A^3D + A^4.$$

From the proof of Theorem 2.5, we have $s_1 = 2m$, $s_2 = \sum_{i=1}^{n} d_i^2 + 2m$, $s_3 = \sum_{i=1}^{n} d_i^3 + 3\sum_{i=1}^{n} d_i^2 + 6n_3(G)$, $s_4 = \sum_{i=1}^{n} d_i^4 + 4\sum_{i=1}^{n} d_i^3 + 4\sum_{v_iv_j \in E(G)} d_i d_j + 8\sum_{v_i \in V(G)} d_i t_3(v_i) + 8n_4(G) + 2\sum_{i=1}^{n} d_i^2 - 2m$.

From Lemma 2.3, we have $p_1 = -2m$, $p_2(G) = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2$, $p_3(G) = \frac{1}{3}[-4m^3 + 6m^2 + 3(m - 1)\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 6n_3(G)]$ and Eq. (3.6).

4 The normalized Laplacian coefficient

Suppose that $Z = v_1v_2\cdots v_lv_1$ is a cycle of G. Let $d_G(Z) = \Pi_{i=1}^{l} d(v_i)$ or simply $d_Z = \Pi_{i=1}^{l} d(v_i)$ if G is clear from the context. In the following, we will give the first five normalized Laplacian coefficients.

Theorem 4.1 *Let G be a simple graph with n vertices and m edges and let $d = (d_1, d_2, \cdots, d_n)$ be its degree sequence. Then the first five normalized Laplacian coefficients are as follows*

$$r_0(G) = 1,$$
$$r_1(G) = -n,$$
$$r_2(G) = \frac{n(n-1)}{2} - \sum_{v_iv_j \in E(G)} \frac{1}{d_i d_j},$$
$$r_3(G) = \frac{n(n-1)(2-n)}{6} + (n-2)\sum_{v_iv_j \in E(G)} \frac{1}{d_i d_j} + 2\sum_{Z \in \Theta_3} \frac{1}{d_Z},$$
$$r_4(G) = \frac{n(n-1)(n-2)(n-3)}{24} - \frac{(n-2)(n-3)}{2}\sum_{v_iv_j \in E(G)} \frac{1}{d_i d_j} - (2n-6)\sum_{Z \in \Theta_3} \frac{1}{d_Z}$$
$$- \frac{1}{2}\sum_{i=1}^{n} \frac{1}{d_i^2}(\sum_{v_iv_j \in E(G)} \frac{1}{d_j})^2 - 2\sum_{Z \in \Theta_4} \frac{1}{d_Z} + \frac{1}{2}\sum_{v_iv_j \in E(G)} \frac{1}{d_i^2 d_j^2} + \frac{1}{2}(\sum_{v_iv_j \in E(G)} \frac{1}{d_i d_j})^2, \qquad (4.7)$$

where Θ_3 and Θ_4 are the sets of 3-cycles and 4-cycles in G, respectively.

Proof. It is obvious that $r_0 = 1$. Let $s_k = tr(\mathcal{L}^k(G))$ in Lemma 2.3. By direct calculation, we have

$$(D^{-1/2}LD^{-1/2})^2 = D^{-1/2}(D - 2A + AD^{-1}A)D^{-1/2}; \tag{4.8}$$

$$(D^{-1/2}LD^{-1/2})^3 = D^{-1/2}(D - 3A + 3AD^{-1}A - AD^{-1}AD^{-1}A)D^{-1/2}; \tag{4.9}$$

$$\begin{aligned}(D^{-1/2}L(G)D^{-1/2})^4 = D^{-1/2}(&D - 4A + 6AD^{-1}A - 4AD^{-1}AD^{-1}A \\ &+ AD^{-1}AD^{-1}AD^{-1}A)D^{-1/2}.\end{aligned} \tag{4.10}$$

Since

$$\begin{aligned}&tr(D^{-1/2}DD^{-1/2}) = n,\\ &tr(D^{-1/2}AD^{-1/2}) = 0,\\ &tr(D^{-1/2}AD^{-1}AD^{-1/2}) = 2\sum_{v_iv_j\in E(G)}\frac{1}{d_id_j},\end{aligned}$$

from Eq. (4.8), we have

$$s_1 = n, s_2 = n + 2\sum_{v_iv_j\in E(G)}\frac{1}{d_id_j}.$$

From Lemma 2.3, we have

$$r_1 = -n, r_2(G) = \frac{n(n-1)}{2} - \sum_{v_iv_j\in E(G)}\frac{1}{d_id_j}.$$

Since

$$tr(D^{-1/2}AD^{-1}AD^{-1}AD^{-1/2}) = \sum_{i=1}^{n}\frac{1}{d_i}\sum_{j=1}^{n}(\sum_{k=1}^{n}\frac{a_{ik}a_{kj}}{d_k})\frac{a_{ji}}{d_j} = 6\sum_{Z\in\Theta_3}\frac{1}{d_Z},$$

from Eq. (4.9), we have

$$s_3 = n + 6\sum_{v_iv_j\in E(G)}\frac{1}{d_id_j} - 6\sum_{Z\in\Theta_3}\frac{1}{d_Z}.$$

From Lemma 2.3, we have

$$r_3(G) = \frac{n(n-1)(2-n)}{6} + (n-2)\sum_{v_iv_j\in E(G)}\frac{1}{d_id_j} + 2\sum_{Z\in\Theta_3}\frac{1}{d_Z}.$$

Since

$$\begin{aligned}&tr(D^{-1/2}AD^{-1}AD^{-1}AD^{-1}AD^{-1/2})\\ &= \sum_{i=1}^{n}\frac{1}{d_i}\sum_{j=1}^{n}\frac{1}{d_j}(\sum_{k=1}^{n}\frac{a_{ik}a_{kj}}{d_k})^2\\ &= 2\sum_{i=1}^{n}\frac{1}{d_i^2}(\sum_{v_iv_j\in E(G)}\frac{1}{d_j})^2 + 8\sum_{Z\in\Theta_4}\frac{1}{d_Z} - 2\sum_{v_iv_j\in E(G)}\frac{1}{d_i^2d_j^2},\end{aligned}$$

from Eq. (4.10), we have

$$s_4 = n + 12\sum_{v_iv_j\in E(G)}\frac{1}{d_id_j} - 24\sum_{Z\in\Theta_3}\frac{1}{d_Z} + 2\sum_{i=1}^{n}\frac{1}{d_i^2}(\sum_{v_iv_j\in E(G)}\frac{1}{d_j})^2 + 8\sum_{Z\in\Theta_4}\frac{1}{d_Z} - 2\sum_{v_iv_j\in E(G)}\frac{1}{d_i^2d_j^2}.$$

From Lemma 2.3, we have Eq. (4.7) holds. □

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