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HO, Man Ho

Published in:
Journal of Geometry and Physics

DOI:
10.1016/j.geomphys.2016.05.016

Published: 01/09/2016

Document Version:
Peer reviewed version

THE FLAT GROTHENDIECK–RIEMANN–ROCH THEOREM WITHOUT ADIABATIC TECHNIQUES

MAN-HO HO

Abstract. In this paper we give a simplified proof of the flat Grothendieck–Riemann–Roch theorem. The proof makes use of the local family index theorem and basic computations of the Chern–Simons form. In particular, it does not involve any adiabatic limit computation of the reduced eta-invariant.

Contents

1. Introduction

In this paper we give a simplified proof of the flat Grothendieck–Riemann–Roch theorem that avoids adiabatic techniques.

1.1. **Historical background.** In this subsection we briefly review the historical background of flat K-theory, its Grothendieck–Riemann–Roch theorem and its relation to physics. For a detailed exposition, see [14, 19].

$\mathbb{R}/\mathbb{Z}$ K-theory [2] (also called flat K-theory) is a generalized cohomology theory which is defined as the cokernel of a natural homomorphism $K^*(X;\mathbb{Q}) \to K^*(X;\mathbb{Q}/\mathbb{Z}) \oplus K^*(X;\mathbb{R})$. One motivation of defining $\mathbb{R}/\mathbb{Z}$ K-theory is to prove a cohomological version of the Atiyah–Singer family

2010 *Mathematics Subject Classification.* Primary 19K56, 58J20, 19L10.

index theorem [3] (FIT) for bundles with vanishing Chern characters: if $[E] \in K(X)$ has vanishing Chern character, the Grothendieck–Riemann–Roch theorem (GRR)

$$\mathrm{ch}(\mathrm{ind}^{\mathrm{a}}(E)) = \int_{X/B} \mathrm{Todd}(X/B) \cup \mathrm{ch}(E) \tag{1.1.1}$$

implies that $\mathrm{ch}(\mathrm{ind}^{\mathrm{a}}(E)) = 0$. (Here $X \to B$ is a fibration of closed manifolds, $E \to X$ is a complex vector bundle, and all other terms are defined in the paper.) Thus one should get a refinement of GRR for bundles with vanishing Chern characters. It turns out that the flat K-group $K^{-1}(X; \mathbb{R}/\mathbb{Z})$ is the right home for such a refinement.

The first geometric model of $K^{-1}(X; \mathbb{C}/\mathbb{Z})$ is given by Karoubi [26] under the name "multiplicative K-theory". By adding Hermitian structures to elements in $K^{-1}(X; \mathbb{C}/\mathbb{Z})$, Lott gives the geometric model of $K^{-1}(X; \mathbb{R}/\mathbb{Z})$, which is denoted by $K^{-1}_{\mathrm{L}}(X)$ in this paper, and proves the FIT in K^{-1}_{L} [30] (flat FIT), which equates the flat analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}$ and the topological index $\mathrm{ind}^{\mathrm{t}}$. The flat FIT is the refinement of the FIT for bundles with vanishing Chern characters. The corresponding GRR [30, Corollary 4] (flat GRR)

$$\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E})) = \int_{X/B} \mathrm{Todd}(X/B) \cup \mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E}) \in H^{\mathrm{odd}}(B; \mathbb{R}/\mathbb{Q}), \tag{1.1.2}$$

where $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}} : K^{-1}_{\mathrm{L}}(B) \to H^{\mathrm{odd}}(B; \mathbb{R}/\mathbb{Q})$ is the flat Chern character, is the refinement of (1.1.1).

In modern language K^{-1}_{L} is the flat part of differential K-theory $\widehat{K}^0$ constructed by Hopkins–Singer [25], Bunke–Schick [12], Freed–Lott [21] and Simons–Sullivan [33] respectively, which is a generalized differential cohomology theory in the sense of Bunke–Schick [13]. In theoretical physics the motivation of differential K-theory comes from the assertion of Witten [35] that D-brane charges in string theory are described by a K-theory class of spacetime rather than by a cohomology class. Furthermore, Moore and Witten propose that Ramond–Ramond fields in type II and type I string theory, to which D-branes couple, are also classified by K-theory [31]. Freed and Hopkins propose using differential K-theory to describe Ramond–Ramond fields [20].

Freed and Lott prove a FIT in differential K-theory [21, Theorem 7.35] (dFIT), which equates the differential analytic index and the differential topological index. The GRR in differential K-theory [21, Corollary 8.26] (dGRR) is also proved by Bunke and Schick [12, Theorem 6.19] independently. The flat FIT and the flat GRR can be considered as special cases of the dFIT and dGRR respectively. See [24] for an algebraic analog of differential cohomology and the corresponding Riemann–Roch theorem. The motivation in theoretical physics for formulating and proving the dFIT, or rather its consequence for determinant line bundles dates back for proving the Green–Schwarz cancellation of local and global anomalies in type I string theory [19].

1.2. **Outline of proof and the relation to previous work.** In this subsection we first outline our proof of the flat GRR. Then we discuss the relation between our proof and the previous proofs and raise some questions.

First of all we briefly outline our proof of the flat GRR. Let $\pi : X \to B$ be a submersion with closed spin^c fibers of even relative dimension. Consider the associated submersion $\pi \times \mathrm{id} : X \times I \to B \times I$, where $I = [0, 1]$. The local FIT of the spin^c Dirac operator twisted by a $\mathbb{Z}_2$-graded Hermitian bundle $\mathscr{E} \to X \times I$ with a $\mathbb{Z}_2$-graded unitary connection $\nabla^{\mathscr{E}}$ is given by

$$d\widetilde{\eta}(\mathscr{E}) = \int_{X\times I/B\times I} \mathrm{Todd}(\nabla^{S^c(T^V(X\times I))}) \wedge \mathrm{ch}(\nabla^{\mathscr{E}}) - \mathrm{ch}(\nabla^{\ker(\mathsf{D}^{\mathscr{E}})}), \quad (1.2.1)$$

where $\widetilde{\eta}(\mathscr{E})$ is the Bismut–Cheeger eta form [8, 17]. Here we have assumed that the family of the complex vector spaces $\ker(\mathsf{D}_z^{\mathscr{E}})$ form a vector bundle $\ker(\mathsf{D}^{\mathscr{E}}) \to B \times I$. Note that (1.2.1) is an equality of differential forms on $B \times I$. By integrating (1.2.1) along the fibers of the trivial fibration $B \times I \to B$ we obtain the variational formula of the eta forms. We also prove the additivity of the eta forms. These two results enable us to prove that the flat analytic index is well defined (Proposition 3). To prove the flat GRR we choose a suitable $\mathscr{E} \to X \times I$, which is the pullback of a certain $\mathbb{Z}_2$-graded Hermitian bundle over X, in the variational formula of the eta forms. This will give us an equality of closed differential forms of odd degree on B, whose mod $\mathbb{Q}$ reduction of its de Rham class is (1.1.2) (Theorem 1). For the general case where the family of the complex vector spaces $\ker(\mathsf{D}_z^{\mathscr{E}})$ do not form a vector bundle, one can prove the corresponding results along the lines of [30, §5] (see also [21, §7]). All our arguments in the special case carry over to the general case.

One important ingredient of the previous proofs is the adiabatic limit of the reduced eta-invariant of spin (or spin^c) Dirac operator, which we briefly recall. Let B be a closed odd-dimensional spin manifold and $\pi : X \to B$ a submersion with closed spin fibers of even relative dimension. For $\varepsilon > 0$, consider the submersion metric $g_\varepsilon^{TX} = \varepsilon^{-1}\pi^* g^{TB} \oplus g^{T^V X}$ with respect to a horizontal distribution. Denote by D_ε the corresponding spin Dirac operator and by $\bar{\eta}(\mathsf{D}_\varepsilon)$ the associated reduced eta-invariant. The study of the limiting behavior, for example, of $\bar{\eta}(\mathsf{D}_\varepsilon)$ when $\varepsilon \to 0$, is called passing to adiabatic limit. It is rooted in [4] and initiated by Witten [34], who relates the adiabatic limit of the eta-invariant to the holonomy of determinant line bundle, the global anomaly. Witten's result receives rigorous proofs in [9, 10, 15, 18]. Adiabatic limit becomes an important tool in, among many other areas, local index theory (see [8, 17, 6] and the references therein).

Now we briefly outline the previous proofs of the flat GRR. As mentioned in §1.1 the flat GRR is a direct consequence of the flat FIT. One could also prove the flat GRR directly in the spirit of [30], which shares some similarities to the proof of the flat FIT and the proof of [21, Proposition 8.19].

The universal coefficient theorem for ordinary cohomology and the divisibility of $\mathbb{R}/\mathbb{Q}$ imply that $H^{\mathrm{odd}}(B;\mathbb{R}/\mathbb{Q}) \cong \mathrm{Hom}(H_{\mathrm{odd}}(B);\mathbb{R}/\mathbb{Q})$, or equivalently the existence of a pairing $\langle,\rangle_H : H^{\mathrm{odd}}(B;\mathbb{R}/\mathbb{Q}) \times H_{\mathrm{odd}}(B) \to \mathbb{R}/\mathbb{Q}$. As (1.1.2) is an equality in $H^{\mathrm{odd}}(B;\mathbb{R}/\mathbb{Q})$, proving it is equivalent to proving

$$\left\langle \mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E})) - \int_{X/B} \mathrm{Todd}(X/B) \cup \mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E}), U \right\rangle_H \tag{1.2.2}$$

is zero in $\mathbb{R}/\mathbb{Q}$ for every $U \in H_{\mathrm{odd}}(B)$. There is a pairing analogous to $\langle,\rangle_H$ on the K-theory level, guaranteed by the universal coefficient theorem for generalized cohomology theory [36] and the divisibility of $\mathbb{R}/\mathbb{Z}$. Denote by $\langle,\rangle_K : K^{-1}_{\mathrm{L}}(B) \times K_{-1}(B) \to \mathbb{R}/\mathbb{Z}$ the pairing, where $K_\bullet$ is the topological K-homology group given by Baum–Douglas [5]. The pairings $\langle,\rangle_H$ and $\langle,\rangle_K$ are related by the flat Chern character $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}$ and the homological Chern character $\mathrm{ch}_{\mathrm{odd}} : K_{-1}(B) \to H_{\mathrm{odd}}(B;\mathbb{Q})$ in the sense that the following diagram commutes.

$$\begin{tikzcd} K^{-1}_{\mathrm{L}}(B) \times K_{-1}(B) \arrow[r, "\langle,\rangle_K"] \arrow[d, "\mathrm{ch}_{\mathbb{R}/\mathbb{Q}} \times \mathrm{ch}_{\mathrm{odd}}"'] & \mathbb{R}/\mathbb{Z} \arrow[d] \\ H^{\mathrm{odd}}(B;\mathbb{R}/\mathbb{Q}) \times H_{\mathrm{odd}}(B;\mathbb{Q}) \arrow[r, "\langle,\rangle_H"'] & \mathbb{R}/\mathbb{Q} \end{tikzcd} \tag{1.2.3}$$

The pairing $\langle,\rangle_K$ can be given by the reduced eta-invariant [30, Proposition 3] as follows. For a $\mathbb{Z}_2$-graded generator $\mathcal{E} = (E^+ \oplus E^-, h^+ \oplus h^-, \nabla^+ \oplus \nabla^-, \omega)$ of $K^{-1}_{\mathrm{L}}(B)$ (see §2.2 for the details) and a cycle $\mathcal{K} = (X, F, f)$ of $K_{-1}(B)$, which consists of a complex vector bundle $F \to X$ over a closed odd-dimensional spin^c manifold and a smooth map $f : X \to B$, define $\bar{\eta}(f^*\mathcal{E}) \in \mathbb{R}/\mathbb{Z}$ by

$$\bar{\eta}(f^*\mathcal{E}) := \bar{\eta}(\mathsf{D}^{F\otimes f^*E^+}) - \bar{\eta}(\mathsf{D}^{F\otimes f^*E^-}) - \int_X \mathrm{Todd}(\nabla^{S^c(TX)}) \wedge \mathrm{ch}(\nabla^F) \wedge f^*\omega. \tag{1.2.4}$$

Then

$$\langle [\mathcal{E}], [\mathcal{K}] \rangle_K = \bar{\eta}(f^*\mathcal{E}). \tag{1.2.5}$$

Since $\mathrm{ch}_\bullet : K_\bullet(B) \otimes \mathbb{Q} \to H_\bullet(B;\mathbb{Q})$ is an isomorphism (in particular it is surjective), it follows from the arguments in the proof of [21, Proposition 8.19] and [30, Proposition 6] that one can take U in (1.2.2) to be $\mathrm{ch}_{\mathrm{odd}}([B])$, where B is now assumed to be a closed odd-dimensional spin^c-manifold and $[B]$ is the fundamental K-homology class. By (1.2.3) and (1.2.5), proving (1.2.2) is zero in $\mathbb{R}/\mathbb{Q}$ boils down to computing the reduced eta-invariants of some spin^c Dirac operators and its adiabatic limits.

On the other hand, one can apply [7, Theorem 1.15] to prove the flat GRR, which is done in the previous version of this paper. The proof of [7, Theorem 1.15] is somewhat similar to the above proof, as it also consists of computations of the reduced eta-invariants of spin Dirac operators and their adiabatic limits.

One might ask if there is any relation between our proof of the flat GRR and the previous proofs. Since our proof of the flat GRR does not involve any adiabatic limit of the reduced eta-invariant, one might wonder whether some of the results in [9, 10, 15, 8, 17] can be proved without it. Frankly we do not have any informative answers for these questions at this moment. Perhaps a clue for these questions can be found in [1, §4], which is an interesting topic to be further investigated.

Since the flat GRR is a special case of the dGRR, one suspects that whether [7, Theorem 1.15] or even the dFIT can be proved without computing adiabatic limit of the reduced eta-invariant. Note that [7, Theorem 1.15] takes values in Cheeger–Simons differential characters [16]. Our experience shows that equality of differential characters is usually harder to prove than equality of differential forms. More precisely, the proofs of [7, Theorem 1.15], the dGRR and the dFIT depend crucially on [16, Theorem 9.2] and [17, Theorem 0.1'] (see also [30, (52)]). Thus the affirmative answer to this question depends on the previous questions.

This paper is organized as follows. In Section 2 we review the background material, including some aspects of Chern–Weil theory, the flat K-theory, the setup and the statement of the local FIT, and the definition of the flat analytic index. In Section 3 we prove the main results in this paper.

Acknowledgement

The author would like to thank the referee for raising some interesting questions and providing invaluable comments and suggestions which improve the paper in a significant way.

2. Background material

In this paper X and B are closed manifolds and I is the closed interval $[0,1]$. Given a manifold X, write $\widetilde{X} = X \times I$. Given $t \in [0,1]$, define two maps $i_{X,t} : X \to \widetilde{X}$ and $p_X : \widetilde{X} \to X$ by $i_{X,t}(x) = (x,t)$ and $p_X(x,t) = x$.

2.1. Chern character form and Chern–Simons form.

Let $E \to X$ be a complex vector bundle with a Hermitian metric h^E and a unitary connection ∇^E. The Chern character form of ∇^E is defined by

$$\mathrm{ch}(\nabla^E) = \mathrm{tr}(e^{-\frac{1}{2\pi i}(\nabla^E)^2}) \in \Omega^{\mathrm{even}}_{\mathbb{Q}}(X),$$

where $\Omega^{\mathrm{even}}_{\mathbb{Q}}(X)$ is the set of all closed even forms on X with periods in $\mathbb{Q}$.

There is a "canonical" transgression form $\mathrm{CS}(\nabla^E_1, \nabla^E_0) \in \dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$ between the Chern character forms of two connections in the sense that

$$d\,\mathrm{CS}(\nabla^E_1, \nabla^E_0) = \mathrm{ch}(\nabla^E_1) - \mathrm{ch}(\nabla^E_0). \tag{2.1.1}$$

Define $\mathrm{CS}(\nabla_1^E, \nabla_0^E)$ as follows. In the following $k \in \{0,1\}$ is fixed. Note that $p_X \circ i_{X,k} = \mathrm{id}_X$ and $i_{X,k} \circ p_X \sim \mathrm{id}_{\widetilde{X}}$. Let $\mathscr{E} \to \widetilde{X}$ be a complex vector bundle with a Hermitian metric $h^{\mathscr{E}}$ and a unitary connection $\nabla^{\mathscr{E}}$. Note that $\mathscr{E} \cong p_X^*(i_{X,k}^*\mathscr{E})$. Thus

$$E_0 := i_{X,0}^*\mathscr{E} \cong i_{X,0}^* p_X^*(i_{X,0}^*\mathscr{E}) \cong i_{X,1}^* p_X^*(i_{X,0}^*\mathscr{E}) \cong i_{X,1}^*\mathscr{E} =: E_1.$$

Write $E = E_0 \cong E_1$. Define $h^{E_k} = i_k^* h^{\mathscr{E}}$. By [27, Corollary 8.9, Chapter 1] there exists $f \in \mathrm{Aut}(E)$ such that $h^{E_0} = f^* h^{E_1}$, so we may assume that $h^{E_0} = h^{E_1}$ and denote it by h^E. Define

$$\nabla_k^E := i_k^* \nabla^{\mathscr{E}}.$$

Note that the connection ∇_k^E is compatible with h^{E_k}. The assumption $h^{E_0} = h^{E_1}$ implies that both ∇_0^E and ∇_1^E are compatible with h^E. Define

$$\mathrm{CS}(\nabla_1^E, \nabla_0^E) = \int_{\widetilde{X}/X} \mathrm{ch}(\nabla^{\mathscr{E}}) \mod \mathrm{Im}(d), \tag{2.1.2}$$

where $\widetilde{X}/X$ denotes the fiber of the fiber bundle $\widetilde{X} \to X$, and $\displaystyle\int_{\widetilde{X}/X}$ denotes integration along the fiber.

To prove the Chern–Simons form defined by (2.1.2) satisfies (2.1.1), we need to invoke Stokes' theorem for integration along the fibers [22, Problem 4 (p.311)]. In general, for a smooth fiber bundle $M \to B$, where M is a manifold with boundary, with compact fibers of dimension n satisfying certain orientability assumptions, we have

$$(-1)^{k-n} \int_{\partial M/B} i^*\omega = \int_{M/B} d_M\omega - d_B \int_{M/B} \omega, \tag{2.1.3}$$

where $i : \partial M \to M$ is the inclusion map and $\omega \in \Omega^k(M)$. Applying (2.1.3) to the fiber bundle $\widetilde{X} \to X$, we have

$$\begin{aligned} d\,\mathrm{CS}(\nabla_1^E, \nabla_0^E) &= d \int_{\widetilde{X}/X} \mathrm{ch}(\nabla^{\mathscr{E}}) = \int_{\widetilde{X}/X} d\,\mathrm{ch}(\nabla^{\mathscr{E}}) + \int_{\partial\widetilde{X}/X} i^*\,\mathrm{ch}(\nabla^{\mathscr{E}}) \\ &= \mathrm{ch}(\nabla_1^E) - \mathrm{ch}(\nabla_0^E). \end{aligned}$$

Given a Hermitian bundle $E \to X$ with two unitary connections ∇_0^E and ∇_1^E, one can apply the above construction to $(p_X^*E, p_X^*h^E, \nabla^{p_X^*E})$ with

$$\nabla^{p_X^*E} := \nabla_t^E + dt \wedge \frac{\partial}{\partial t}, \tag{2.1.4}$$

where ∇_t^E is a smooth curve of unitary connections joining ∇_0^E and ∇_1^E. Note that $\mathrm{CS}(\nabla_1^E, \nabla_0^E)$ is independent of the choice of ∇_t^E [33, Proposition 1.1].

Another equivalent definition of the Chern–Simons form is given by

$$\mathrm{CS}(\nabla_1^E, \nabla_0^E) = \int_0^1 \mathrm{tr}\left(\frac{d\nabla_t^E}{dt} e^{-\frac{1}{2\pi i}(\nabla_t^E)^2}\right) dt \mod \mathrm{Im}(d). \tag{2.1.5}$$

It follows from (2.1.2) that the Chern–Simons form satisfies the following properties:

$$\mathrm{CS}(\nabla_1^E,\nabla_0^E) = -\mathrm{CS}(\nabla_0^E,\nabla_1^E), \tag{2.1.6}$$
$$\mathrm{CS}(\nabla_1^E,\nabla_0^E) = \mathrm{CS}(\nabla_1^E,\nabla_2^E)+\mathrm{CS}(\nabla_2^E,\nabla_0^E), \tag{2.1.7}$$
$$\mathrm{CS}(\nabla_1^E\oplus\nabla_1^F,\nabla_0^E\oplus\nabla_0^F) = \mathrm{CS}(\nabla_1^E,\nabla_0^E)+\mathrm{CS}(\nabla_1^F,\nabla_0^F), \tag{2.1.8}$$

where ∇_1^F,∇_0^F are unitary connections on the Hermitian bundle $F\to X$. The proofs of (2.1.6)-(2.1.8) using (2.1.5) are given in [33, Proposition 1.1, Lemma 1.4].

One can define the Chern character form and the Chern–Simons form of unitary superconnection on $\mathbb{Z}_2$-graded Hermitian bundles in the exact same way as above, except that the traces in the definitions are replaced by supertraces [32], [6, §1.4, §1.5]. Note that (2.1.1) and (2.1.6)-(2.1.8) hold for unitary superconnections.

2.2. **The flat K-theory.** In this subsection we recall the flat K-theory [30].

The flat K-group $K_{\mathrm{L}}^{-1}(X)$ is an abelian group given by generators and relations: a generator is of the form $\mathcal{E}=(E,h^E,\nabla^E,\omega)$, where $\omega\in\dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$ satisfies $\mathrm{ch}(\nabla^E)-\mathrm{rank}(E)=-d\omega$.[1] The only relation is $\mathcal{E}_1=\mathcal{E}_0$ if and only if there exists $\mathcal{G}=(G,h^G,\nabla^G,\omega_G)$ such that $E_1\oplus G\cong E_0\oplus G$ and

$$\omega_1-\omega_0=\mathrm{CS}(\nabla_0^E\oplus\nabla^G,\nabla_1^E\oplus\nabla^G).$$

Elements in $K_{\mathrm{L}}^{-1}(X)$ are required to have virtual rank zero.

A $\mathbb{Z}_2$-graded generator $\mathcal{E}$ of $K_{\mathrm{L}}^{-1}(X)$ has the form

$$\mathcal{E}=(E^+\oplus E^-,h^+\oplus h^-,\nabla^+\oplus\nabla^-,\omega), \tag{2.2.1}$$

where $E^+\oplus E^-\to X$ is a $\mathbb{Z}_2$-graded complex vector bundle with a $\mathbb{Z}_2$-graded Hermitian metric $h^+\oplus h^-$, a $\mathbb{Z}_2$-graded unitary connection $\nabla^+\oplus\nabla^-$ on $E^+\oplus E^-\to X$, and $\omega\in\dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$ satisfying

$$\mathrm{ch}(\nabla^+\oplus\nabla^-)=\mathrm{ch}(\nabla^+)-\mathrm{ch}(\nabla^-)=-d\omega.$$

Every element in $K_{\mathrm{L}}^{-1}(X)$ can be written as a $\mathbb{Z}_2$-graded generator and vice versa [30, p.286].

The flat K-group is related to other ordinary K-groups by the following exact sequence [26, §7.21], [30, (13)]

$$K^{-1}(X)\xrightarrow{r\circ\mathrm{ch}^{\mathrm{odd}}}H^{\mathrm{odd}}(X;\mathbb{R})\xrightarrow{\alpha}K_{\mathrm{L}}^{-1}(X)\xrightarrow{\beta}K(X) \tag{2.2.2}$$

where $\mathrm{ch}^{\mathrm{odd}}$ is the odd Chern character, r is induced by the inclusion of coefficients $\mathbb{Q}\hookrightarrow\mathbb{R}$, and the maps α and β are given by

$$\alpha([\omega])=(\mathbb{C}^n,h,\nabla^{\mathrm{flat}},\omega)-(\mathbb{C}^n,h,\nabla^{\mathrm{flat}},0),$$
$$\beta(\mathcal{E}-\mathcal{F})=[E]-[F],$$

[1] This differs from [30, Definition 5] by a sign.

where $\mathbb{C}^n \to X$ denotes the trivial complex vector bundle of rank n. As in the case of ordinary K-theory, there exists a unique Chern character $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}} : K_{\mathrm{L}}^{-1}(X) \to H^{\mathrm{odd}}(X;\mathbb{R}/\mathbb{Q})$, called the flat Chern character [30, Definition 9], defined as follows. For a generator $\mathcal{E} = (E, h^E, \nabla^E, \omega)$ of $K_{\mathrm{L}}^{-1}(X)$, write $N = \mathrm{rank}(E)$. The condition $\mathrm{ch}(\nabla^E) - N = -d\omega$ implies $\mathrm{ch}(E - \mathbb{C}^N) = 0 \in H^{\mathrm{even}}(X;\mathbb{Q})$. Thus there exists $k \in \mathbb{N}$ such that $kE \cong k\mathbb{C}^N$. Let ∇_0^{kE} be a unitary connection on $kE \to X$ with trivial holonomy. One can check that the odd form $\frac{1}{k}\,\mathrm{CS}(k\nabla, \nabla_0^{kE}) + \omega$ is closed. The flat Chern character $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$ is defined to be

$$\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E}) = \left[\frac{1}{k}\,\mathrm{CS}(k\nabla^E, \nabla_0^{kE}) + \omega\right] \mod \mathbb{Q}. \tag{2.2.3}$$

Note that $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$ is independent of the choices of k and ∇_0^{kE} [30, Lemma 1] and is a well defined group homomorphism [30, Proposition 1].

The flat Chern character of a $\mathbb{Z}_2$-graded generator $\mathcal{E}$ of the form (2.2.1) is defined as follows. The condition $\mathrm{ch}(\nabla^+) - \mathrm{ch}(\nabla^-) = -d\omega$ implies the existence of $k \in \mathbb{N}$ such that $kE^+ \cong kE^-$. Choose an isometric isomorphism $j : kE^+ \to kE^-$. Then $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$ is defined to be

$$\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E}) = \left[\frac{1}{k}\,\mathrm{CS}(k\nabla^+, j^*k\nabla^-) + \omega\right] \mod \mathbb{Q}, \tag{2.2.4}$$

Note that $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$ is independent of the choices of k and j [30, p.289].

2.3. **Local family index theorem.** In this subsection we recall the setup and the statement of the local FIT. We refer to [6] and the references therein for details.

Let $\pi : X \to B$ be a submersion with closed spin^c fibers of even relative dimension. Denote by $T^VX \to X$ its vertical tangent bundle. Put a metric g^{T^VX} on $T^VX \to X$. Given a horizontal distribution $T^HX \to X$ and a Riemannian metric g^{TB} on $TB \to B$, we can define a metric on $TX \to X$ by $g^{TX} := g^{T^VX} \oplus \pi^* g^{TB}$. If ∇^{TX} is the corresponding Levi-Civita connection, then $\nabla^{T^VX} := P \circ \nabla^{TX} \circ P$ is a connection on $T^VX \to X$, where $P : TX \to T^VX$ is the orthogonal projection. Denote by $S^c(T^VX) \to X$ the $\mathbb{Z}_2$-graded spin^c bundle and by $L^VX \to X$ the associated characteristic Hermitian line bundle with a unitary connection ∇^{L^VX}. Note that the connection ∇^{T^VX} lifts uniquely to the local spinor bundle and preserves its grading and the isomorphism $S^c(T^VX) \cong S(T^VX) \otimes L^VX$ exists globally [28, p.397]. The connection $\nabla^{S^c(T^VX)}$ on $S^c(T^VX) \to X$, defined by

$$\nabla^{S^c(T^VX)} := \nabla^{T^VX} \otimes \nabla^{L^VX},$$

preserves the grading of $S^c(T^VX) \to X$. The Todd form $\mathrm{Todd}(\nabla^{S^c(T^VX)})$ of $S^c(T^VX) \to X$ is defined to be

$$\mathrm{Todd}(\nabla^{S^c(T^VX)}) = \widehat{A}(\nabla^{T^VX}) \wedge e^{\frac{1}{2}c_1(\nabla^{L^VX})}.$$

Define an infinite-rank bundle $\pi_*E \to B$ whose fiber over $z \in B$ is given by

$$(\pi_*E)_z := \Gamma(X_z, (S^c(T^VX) \otimes E)_z).$$

Since $S^c(T^VX) \otimes E \to X$ is $\mathbb{Z}_2$-graded whose even and odd part are given by

$$(S^c(T^VX) \otimes E)^{\pm} = S^c(T^VX)^{\pm} \otimes E, \tag{2.3.1}$$

it follows that the bundle $\pi_*E \to B$ is also $\mathbb{Z}_2$-graded whose even and odd part are given by

$$(\pi_*E)^{\pm}_z = \Gamma(X_z, (S^c(T^VX)^{\pm} \otimes E)_z) \tag{2.3.2}$$

for each $z \in B$. The space of sections of $\pi_*E \to B$ is defined to be

$$\Gamma(B, \pi_*E) := \Gamma(X, S^c(T^VX) \otimes E). \tag{2.3.3}$$

Note that $\pi_*E \to B$ admits an L^2-metric and a $\mathbb{Z}_2$-graded unitary connection ∇^{π_*E} [6, Proposition 9.13].

The spin^c Dirac operator $\mathsf{D}^E : \Gamma(X, S^c(T^VX) \otimes E) \to \Gamma(X, S^c(T^VX) \otimes E)$ is an odd operator given by

$$\mathsf{D}^E = \sum_k c(e^k) \nabla^{S^c(T^VX)\otimes E}_{e_k}, \tag{2.3.4}$$

where c is the Clifford multiplication, $\nabla^{S^c(T^VX)\otimes E} := \nabla^{S^c(T^VX)} \otimes \nabla^E$, $\{e_k\}$ is a local orthonormal frame for $T^VX \to X$ and $\{e^k\}$ its dual frame for $(T^VX)^* \to X$. By (2.3.3), D^E can be regarded as an odd operator on $\pi_*E \to B$. Assume that the family of complex vector spaces $\ker(\mathsf{D}^E_z)$ has locally constant dimension for $z \in B$. Then $\ker(\mathsf{D}^E_z)$ form a finite-rank $\mathbb{Z}_2$-graded complex vector bundle over B, denoted by $\ker(\mathsf{D}^E) \to B$ and is called the index bundle of $E \to X$. The analytic index $\mathrm{ind}^{\mathrm{a}}(E)$ of $E \to X$ is defined by $\mathrm{ind}^{\mathrm{a}}(E) = [\ker(\mathsf{D}^E)] \in K(B)$, and is a ring homomorphism $\mathrm{ind}^{\mathrm{a}} : K(X) \to K(B)$.

Write $\mathbb{E} = (E, h^E, \nabla^E)$. The Bismut superconnection $\mathbb{B}(\mathbb{E})$ on $\pi_*E \to B$ is defined to be

$$\mathbb{B}(\mathbb{E}) = \mathsf{D}^E + \nabla^{\pi_*E} - \frac{c(T)}{4},$$

where T is the curvature 2-form of the fiber bundle $X \to B$. For each $z \in B$, denote by $P^z_0 : (\pi_*E)_z \to \ker(\mathsf{D}^E)_z$ the orthogonal projection. Then P_0 is a family of smoothing operators. Note that $\nabla^{\ker(\mathsf{D}^E)} := P_0\mathbb{B}(\mathbb{E})_{[1]}P_0$ is a $\mathbb{Z}_2$-graded unitary connection on $\ker(\mathsf{D}^E) \to B$ [6, Lemma 9.18]. The rescaled Bismut superconnection $\mathbb{B}(\mathbb{E})_t$ is defined to be

$$\mathbb{B}(\mathbb{E})_t = \sqrt{t}\mathsf{D}^E + \nabla^{\pi_*E} - \frac{c(T)}{4\sqrt{t}}.$$

By [6, Theorem 10.32], we have

$$\lim_{t\to 0}\mathrm{ch}(\mathbb{B}(\mathbb{E})_t) = \int_{X/B}\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\mathrm{ch}(\nabla^E), \tag{2.3.5}$$

$$\lim_{t\to\infty}\mathrm{ch}(\mathbb{B}(\mathbb{E})_t) = \mathrm{ch}(\nabla^{\ker(\mathsf{D}^E)}). \tag{2.3.6}$$

Note that

$$\frac{d\,\mathrm{ch}(\mathbb{B}(\mathbb{E})_t)}{dt} = -d\,\mathrm{str}\left(\frac{d\mathbb{B}(\mathbb{E})_t}{dt}e^{-\frac{1}{2\pi i}(\mathbb{B}(\mathbb{E})_t)^2}\right) \tag{2.3.7}$$

and the integral $\int_0^\infty \mathrm{str}\left(\frac{d\mathbb{B}(\mathbb{E})_t}{dt}e^{-\frac{1}{2\pi i}(\mathbb{B}(\mathbb{E})_t)^2}\right)dt$ converges [6, Theorem 10.32]. The eta form [8, 17] of $\mathbb{E}$ is defined to be

$$\widetilde{\eta}(\mathbb{E}) := \int_0^\infty \mathrm{str}\left(\frac{d\mathbb{B}(\mathbb{E})_t}{dt}e^{-\frac{1}{2\pi i}(\mathbb{B}(\mathbb{E})_t)^2}\right)dt. \tag{2.3.8}$$

The local FIT [6, Theorem 10.32] states that

$$d\widetilde{\eta}(\mathbb{E}) = \int_{X/B}\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\mathrm{ch}(\nabla^E) - \mathrm{ch}(\nabla^{\ker(\mathsf{D}^E)}). \tag{2.3.9}$$

which follows from (2.3.5)–(2.3.8).

Remark 1. We use the slightly unconventional symbol $\mathbb{B}(\mathbb{E})$ and $\widetilde{\eta}(\mathbb{E})$ to emphasize the dependence of the Bismut superconnection and the eta form on $\mathbb{E}$. Of course they also depend on other data: the metrics g^{T^VX} and g^{L^VX}, the horizontal distribution T^HX and the unitary connection ∇^{L^VX}. Henceforth we choose and fix these data. Because of the definition of K_{L}^{-1} we are only interested in the deformation of the unitary connection on $E\to X$.

2.4. **The flat analytic index.** In this subsection we recall the definition of the flat analytic index [30, Definition 13]. Given a $\mathbb{Z}_2$-graded generator $\mathcal{E}$ of $K_{\mathrm{L}}^{-1}(X)$, its flat analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E})\in K_{\mathrm{L}}^{-1}(B)$ is, roughly speaking, given by the analytic index of the $\mathbb{Z}_2$-graded data of $\mathcal{E}$ and a pushforward of the form ω. We refer to the construction of the analytic index in §2.3, and indicate the changes as follows.

Let $\pi: X\to B$ be a submersion with closed spin^c fibers of even relative dimension, and $\mathcal{E}$ a $\mathbb{Z}_2$-graded generator of $K_{\mathrm{L}}^{-1}(X)$ of the form (2.2.1). As in §2.3, the spin^c bundle $S^c(T^VX)\to X$ of $T^VX\to X$ is $\mathbb{Z}_2$-graded. Since $E^+\oplus E^-\to X$ is also $\mathbb{Z}_2$-graded, the even and the odd part of $S^c(T^VX)\widehat{\otimes}E\to X$ become

$$(S^c(T^VX)\widehat{\otimes}E)^\pm = S^c(T^VX)^+\otimes E^\pm\oplus S^c(T^VX)^-\otimes E^\mp.$$

It follows from (2.3.2) that the even and the odd part of $\pi_*E\to B$ has a similar decomposition, so the same is true for $\ker(\mathsf{D}^E)\to B$, that is,

$$\ker(\mathsf{D}^E)^\pm = \ker(\mathsf{D}^{E^+})^\pm\oplus\ker(\mathsf{D}^{E^-})^\mp.$$

The L^2-metric and the Bismut superconnection $\mathbb{B}(\mathbb{E})$ on $\pi_*E \to B$ are defined accordingly. The $\mathbb{Z}_2$-graded unitary connection on $\ker(\mathsf{D}^E) \to B$ given by

$$\nabla^{\ker(\mathsf{D}^E)} = \nabla^{\ker(\mathsf{D}^E)^+} \oplus \nabla^{\ker(\mathsf{D}^E)^-},$$

where

$$\nabla^{\ker(\mathsf{D}^E)^\pm} := \nabla^{\ker(\mathsf{D}^{E^+})^\pm} \oplus \nabla^{\ker(\mathsf{D}^{E^-})^\mp},$$

is a direct sum of connections. In this case the local FIT takes the form

$$\mathrm{ch}(\nabla^{\ker(\mathsf{D}^E)}) = \int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^VX)}) \wedge \mathrm{ch}(\nabla^+ \oplus \nabla^-) - d\widetilde{\eta}(\mathbb{E}^+ \oplus \mathbb{E}^-), \tag{2.4.1}$$

where $\mathbb{E}^+ \oplus \mathbb{E}^- = (E^+ \oplus E^-, h^+ \oplus h^-, \nabla^+ \oplus \nabla^-)$.

The flat analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}} : K^{-1}_{\mathrm{L}}(X) \to K^{-1}_{\mathrm{L}}(B)$ of a $\mathbb{Z}_2$-graded generator $\mathcal{E}$ of the form (2.2.1) is defined by

$$\begin{aligned} &\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}) \\ &:= \bigg(\ker(\mathsf{D}^E), h^{\ker(\mathsf{D}^E)}, \nabla^{\ker(\mathsf{D}^E)}, \int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^VX)}) \wedge \omega + \widetilde{\eta}(\mathbb{E}^+ \oplus \mathbb{E}^-)\bigg). \end{aligned} \tag{2.4.2}$$

It follows from (2.3.9) that $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}) \in K^{-1}_{\mathrm{L}}(B)$ is a $\mathbb{Z}_2$-graded generator.

3. Main results

In this section we will prove the main results of this paper.

3.1. **Some properties of the eta form.**

In this subsection we provide proofs of the additivity and the variational formula of the eta forms.

Although $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}$ is defined using the spin^c Dirac operator, we prove the additivity of the eta forms in a slightly more general setting. Instead of working on the twisted spin^c bundle $S^c(T^VX) \otimes E \to X$ we work on Clifford modules. Before we state and prove the result we briefly recall the Clifford modules in our setup. We refer to [6, §10.2, §10.3] for the details.

Let $\pi : X \to B$ be a submersion with closed fibers of even relative dimension. Put a Riemannian metric g^{TB} on $TB \to B$ and a metric g^{T^VX} on the vertical bundle $T^VX \to X$. Recall from [6, p.322] that a Clifford module along the fibers of $\pi : X \to B$ is given by $\mathscr{E} = (E, h^E, \nabla^E)$, where $E \to X$ is a $\mathbb{Z}_2$-graded complex vector bundle, h^E a $\mathbb{Z}_2$-graded Hermitian metric and ∇^E a $\mathbb{Z}_2$-graded unitary connection, with a skew-adjoint action

$$c : \mathrm{C}\ell((T^VX)^*) \to \mathrm{End}(E),$$

where $(T^VX)^* \to X$ denotes the dual bundle of $T^VX \to X$ and $\mathrm{C}\ell((T^VX)^*) \to X$ is the Clifford bundle of $(T^VX)^* \to X$, such that

$$[\nabla^E_V, c(\alpha)] = c(\nabla^{(T^VX)^*}_V \alpha)$$

for $V \in \Gamma(X, TX)$ and $\alpha \in \Gamma(X, (T^VX)^*)$. One can define a Dirac operator $\mathsf{D}^{\mathscr{E}} : \Gamma(X, \mathscr{E}) \to \Gamma(X, \mathscr{E})$ for a Clifford module $\mathscr{E}$ along the fibers in a way similar to (2.3.4).

Here we recall the definition of the Bismut superconnection $\mathbb{B}(\mathscr{E})$ associated to $\mathscr{E}$ [6, Proposition 10.15]. Given a Clifford module $\mathscr{E} = (E, h^E, \nabla^E)$ along the fibers of $X \to B$, define a complex vector bundle $\widetilde{E} := \pi^*\Lambda(T^*B) \otimes E$ over X and equip it with the Hermitian metric $\pi^* g^{T^*B} \otimes h^E$. Consider the Clifford algebra bundle $\mathrm{C}\ell_0(T^*X) \to X$, where $\mathrm{C}\ell_0(T^*X)$ denotes $\mathrm{C}\ell(T^*X)$ equipped with the degenerate metric $g_0^{T^*X}$, with $g_\varepsilon^{T^*X} := g^{(T^VX)^*} \oplus \varepsilon\pi^* g^{T^*B}$. Since $T^*X \cong (T^VX)^* \oplus \pi^*T^*B$, it follows that $\mathrm{C}\ell_0(T^*X) \cong \pi^*\Lambda(T^*B) \otimes \mathrm{C}\ell((T^VX)^*)$. Define a Clifford multiplication $m_0 : \mathrm{C}\ell_0(T^*X) \to \mathrm{End}(\widetilde{E})$ by $m_0(\alpha) = \alpha \wedge \cdot$ if $\alpha \in \Gamma(X, \pi^*T^*B)$ and $m_0(\alpha) = c(\alpha)$ if $\alpha \in \Gamma(X, (T^VX)^*)$. Define a connection $\nabla^{\widetilde{E}}$ on $\widetilde{E} \to X$ by

$$\nabla^{\widetilde{E}} = \pi^*\nabla^{T^*B} \otimes \nabla^E + \frac{1}{2} m_0(\omega), \tag{3.1.1}$$

where $\omega \in \Omega^1(X, \Lambda^2(T^*X))$ is characterized by [6, Proposition 10.6]. By [6, Proposition 10.10] $\nabla^{\widetilde{E}}$ is a Clifford connection. Thus $\widetilde{\mathscr{E}} := (\widetilde{E}, \pi^*g^{T^*B} \otimes h^E, \nabla^{\widetilde{E}})$ is a Clifford module over the Clifford algebra bundle $\mathrm{C}\ell_0(T^*X) \to X$. Note that $\Omega(B, \pi_*E)$ is defined to be $\Gamma(X, \widetilde{E})$. The Bismut superconnection $\mathbb{B}(\mathscr{E}) : \Omega(B, \pi_*E) \to \Omega(B, \pi_*E)$ is defined as a Dirac operator $\mathbb{B}(\mathscr{E}) : \Gamma(X, \widetilde{E}) \to \Gamma(X, \widetilde{E})$ by the formula

$$\mathbb{B}(\mathscr{E}) = \sum_k m_0(e^k)\nabla^{\widetilde{E}}_{e_k}, \tag{3.1.2}$$

where $\{e_k\}$ is a local orthonormal frame for $TX \to X$ and $\{e^k\}$ its dual frame for $T^*X \to X$.

Proposition 1. Let $\pi : X \to B$ be a submersion with closed fibers of even relative dimension, $\mathscr{E} = (E, h^E, \nabla^E)$ a Clifford module over X along the fibers of $X \to B$ and $\mathsf{D}^{\mathscr{E}}$ the Dirac operator associated to $\mathscr{E}$. Let $\widetilde{\eta}(\mathscr{E})$ be the eta form of the Bismut superconnection $\mathbb{B}(\mathscr{E})$. If $\mathscr{F} = (F, h^F, \nabla^F)$ is another Clifford module over X along the fibers of $X \to B$, then

$$\widetilde{\eta}(\mathscr{E} \oplus \mathscr{F}) = \widetilde{\eta}(\mathscr{E}) + \widetilde{\eta}(\mathscr{F})$$

up to exact forms.

Proof. First of all we claim that $\mathbb{B}(\mathscr{E} \oplus \mathscr{F}) = \mathbb{B}(\mathscr{E}) \oplus \mathbb{B}(\mathscr{F})$. By (3.1.2) it suffices to prove that

$$\nabla^{\widetilde{E \oplus F}} = \nabla^{\widetilde{E}} \oplus \nabla^{\widetilde{F}}.$$

To see this, let $\beta\otimes(\alpha_1\oplus\alpha_2)\in\Gamma(X,\pi^*\Lambda(T^*B)\otimes(E\oplus F))$. By (3.1.1) we have

$$\begin{aligned}
&\nabla^{\widetilde{E\oplus F}}(\beta\otimes(\alpha_1\oplus\alpha_2))\\
&=(\pi^*\nabla^{TB}\otimes\nabla^{E\oplus F})(\beta\otimes(\alpha_1\oplus\alpha_2))+\frac{1}{2}m_0(\omega)(\beta\otimes(\alpha_1\oplus\alpha_2))\\
&=\pi^*\nabla^{TB}\beta\otimes(\alpha_1\oplus\alpha_2)+\beta\otimes(\nabla^E\alpha_1\oplus\nabla^F\alpha_2)+\frac{1}{2}m_0(\omega)\beta\otimes(\alpha_1\oplus\alpha_2)\\
&=\left(\pi^*\nabla^{TB}\beta\otimes\alpha_1+\beta\otimes\nabla^E\alpha_1+\frac{1}{2}m_0(\omega)\beta\otimes\alpha_1\right)\\
&\qquad\oplus\left(\pi^*\nabla^{TB}\beta\otimes\alpha_2+\beta\otimes\nabla^F\alpha_2+\frac{1}{2}m_0(\omega)\beta\otimes\alpha_2\right)\\
&=\nabla^{\widetilde{E}}(\beta\otimes\alpha_1)\oplus\nabla^{\widetilde{F}}(\beta\otimes\alpha_2)=(\nabla^{\widetilde{E}}\oplus\nabla^{\widetilde{F}})(\beta\otimes(\alpha_1\oplus\alpha_2)).
\end{aligned}$$

The additivity of the Bismut superconnections holds for the rescaled Bismut superconnection; i.e.,

$$\mathbb{B}(\mathscr{E}\oplus\mathscr{F})_t=\mathbb{B}(\mathscr{E})_t\oplus\mathbb{B}(\mathscr{F})_t.$$

Consider the Chern–Simons form

$$\mathrm{CS}(\mathbb{B}(\mathscr{E})_T,\mathbb{B}(\mathscr{E})_t)=\int_t^T\mathrm{str}\left(\frac{d\mathbb{B}(\mathscr{E})_s}{ds}e^{-\frac{1}{2\pi i}(\mathbb{B}(\mathscr{E})_s)^2}\right)ds,$$

where $0<t<T$ are fixed. Properties (2.1.6)-(2.1.8) extend to this case. Therefore

$$\begin{aligned}
\mathrm{CS}(\mathbb{B}(\mathscr{E}\oplus\mathscr{F})_T,\mathbb{B}(\mathscr{E}\oplus\mathscr{F})_t)&=\mathrm{CS}(\mathbb{B}(\mathscr{E})_T\oplus\mathbb{B}(\mathscr{F})_T,\mathbb{B}(\mathscr{E})_t\oplus\mathbb{B}(\mathscr{F})_t)\\
&=\mathrm{CS}(\mathbb{B}(\mathscr{E})_T,\mathbb{B}(\mathscr{E})_t)+\mathrm{CS}(\mathbb{B}(\mathscr{F})_T,\mathbb{B}(\mathscr{F})_t).
\end{aligned}$$

By letting $T\to\infty$ and $t\to 0$ in above, the convergence of all the integrals involved [6, Theorem 10.32] shows that $\widetilde{\eta}(\mathscr{E}\oplus\mathscr{F})=\widetilde{\eta}(\mathscr{E})+\widetilde{\eta}(\mathscr{F})$ up to exact forms. □

Let $\mathbb{E}=(E,h^E,\nabla^E)$ and $\mathbb{F}=(F,h^F,\nabla^F)$ be Hermitian bundles with unitary connections. By applying Proposition 1 to the twisted spin^c bundle $S^c(T^VX)\otimes(E\oplus F)\to X$ where the fibers of $\pi:X\to B$ are assumed to be spin^c, we have

$$\widetilde{\eta}(\mathbb{E}\oplus\mathbb{F})=\widetilde{\eta}(\mathbb{E})+\widetilde{\eta}(\mathbb{F})\tag{3.1.3}$$

up to exact forms.

Some remarks for Proposition 2.

Remark 2. Let $p:M\to X$ be a smooth fiber bundle with compact fibers. By [11, Chapter 1], we have

$$\int_{M/X}p^*\alpha\wedge\beta=\alpha\wedge\left(\int_{M/X}\beta\right),\tag{3.1.4}$$

for all $\alpha\in\Omega(X)$ and $\beta\in\Omega(M)$.

If $q : X \to B$ is another smooth fiber bundle with compact fibers, then $q \circ p : M \to B$ is a smooth fiber bundle with compact fibers. It is straightforward to check (or see [22, Problem 3 (p.311)]) that

$$\int_{M/B} = \int_{X/B} \circ \int_{M/X} . \tag{3.1.5}$$

Proposition 2. Let $\pi : X \to B$ be a submersion with closed spin^c fibers of even relative dimension. Write $\mathbb{E}_k = (E, h^E, \nabla^E_k)$, where $k \in \{0, 1\}$, as in §2.1. Then

$$\widetilde{\eta}(\mathbb{E}_1)-\widetilde{\eta}(\mathbb{E}_0) = \int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^V X)}) \wedge \mathrm{CS}(\nabla^E_1, \nabla^E_0) - \mathrm{CS}(\nabla_1^{\ker(\mathsf{D}^E)}, \nabla_0^{\ker(\mathsf{D}^E)}) \tag{3.1.6}$$

up to exact forms.

Proposition 2 is a special case of the variational formula of the equivariant eta forms [29, Theorem 1.7], where the geometric data on $\pi : X \to B$ and the connection on $E \to X$ are deformed.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{p_X} & X \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{B} & \xrightarrow[p_B]{} & B \end{array}$$

The geometric data on $\widetilde{\pi} : \widetilde{X} \to \widetilde{B}$ is obtained by pulling back the geometric data on $\pi : X \to B$. Then the local FIT (2.3.9) for $\widetilde{\mathbb{E}} = (\widetilde{E}, h^{\widetilde{E}}, \nabla^{\widetilde{E}})$ gives

$$d\widetilde{\eta}(\widetilde{\mathbb{E}}) = \int_{\widetilde{X}/\widetilde{B}} \mathrm{Todd}(\nabla^{S^c(T^V \widetilde{X})}) \wedge \mathrm{ch}(\nabla^{\widetilde{E}}) - \mathrm{ch}(\nabla^{\ker(\mathsf{D}^{\widetilde{E}})}). \tag{3.1.7}$$

Consider $\ker(\mathsf{D}^{\widetilde{E}}) \to \widetilde{B}$. By the same reason as in §2.1 we have $i_0^* \ker(\mathsf{D}^{\widetilde{E}}) \cong i_1^* \ker(\mathsf{D}^{\widetilde{E}})$. Moreover, for $k \in \{0, 1\}$ the connection defining the spin^c Dirac operator $\mathsf{D}^{\widetilde{E}|_{X\times\{k\}}}$ on $(S^c(T^V \widetilde{X}) \otimes \widetilde{E})|_{X\times\{k\}} \to X \times \{k\}$ is $\nabla^{S^c(T^V \widetilde{X})} \otimes \nabla_k$. Thus $\ker(\mathsf{D}^{E_k}) \cong i_k^* \ker(\mathsf{D}^{\widetilde{E}})$, so $\ker(\mathsf{D}^{E_0}) \cong \ker(\mathsf{D}^{E_1})$, and is therefore denoted by $\ker(\mathsf{D}^E)$. Write $\nabla_k^{\ker(\mathsf{D}^E)}$ for the unitary connection on $\ker(\mathsf{D}^E) \to B$ induced by ∇_k. Denote by $i : \partial\widetilde{B} \to \widetilde{B}$ the inclusion map. By (2.1.3), we have

$$\widetilde{\eta}(\mathbb{E}_1) - \widetilde{\eta}(\mathbb{E}_0) = \int_{\partial\widetilde{B}/B} i^*\widetilde{\eta}(\widetilde{\mathbb{E}}) = \int_{\widetilde{B}/B} d_{\widetilde{B}}\widetilde{\eta}(\widetilde{\mathbb{E}}) - d_B \int_{\widetilde{B}/B} \widetilde{\eta}(\widetilde{\mathbb{E}}).$$

By modding out exact forms, it follows from (3.1.7) that

$$\begin{aligned}\widetilde{\eta}(\mathbb{E}_1)-\widetilde{\eta}(\mathbb{E}_0)&=\int_{\widetilde{B}/B}d_{\widetilde{B}}\widetilde{\eta}(\widetilde{\mathbb{E}})\\&=\int_{\widetilde{B}/B}\left(\int_{\widetilde{X}/\widetilde{B}}\mathrm{Todd}(\nabla^{S^c(T^V\widetilde{X})})\wedge\mathrm{ch}(\nabla^{\widetilde{E}})-\mathrm{ch}(\nabla^{\ker(\mathsf{D}^{\widetilde{E}})})\right)\\&=\int_{\widetilde{B}/B}\int_{\widetilde{X}/\widetilde{B}}p_X^*\,\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\mathrm{ch}(\nabla^{\widetilde{E}})-\int_{\widetilde{B}/B}\mathrm{ch}(\nabla^{\ker(\mathsf{D}^{\widetilde{E}})}).\end{aligned}$$

By (2.1.2), the last term of the right-hand side is equal to $\mathrm{CS}(\nabla_1^{\ker(\mathsf{D}^E)},\nabla_0^{\ker(\mathsf{D}^E)})$. Then

$$\begin{aligned}&\widetilde{\eta}(\mathbb{E}_1)-\widetilde{\eta}(\mathbb{E}_0)\\&=\int_{\widetilde{B}/B}\int_{\widetilde{X}/\widetilde{B}}p_X^*\,\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\mathrm{ch}(\nabla^{\widetilde{E}})-\mathrm{CS}(\nabla_1^{\ker(\mathsf{D}^E)},\nabla_0^{\ker(\mathsf{D}^E)})\\&=\int_{\widetilde{X}/B}p_X^*\,\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\mathrm{ch}(\nabla^{\widetilde{E}})-\mathrm{CS}(\nabla_1^{\ker(\mathsf{D}^E)},\nabla_0^{\ker(\mathsf{D}^E)})\\&=\int_{X/B}\int_{\widetilde{X}/X}p_X^*\,\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\mathrm{ch}(\nabla^{\widetilde{E}})-\mathrm{CS}(\nabla_1^{\ker(\mathsf{D}^E)},\nabla_0^{\ker(\mathsf{D}^E)})\\&=\int_{X/B}\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\left(\int_{\widetilde{X}/X}\mathrm{ch}(\nabla^{\widetilde{E}})\right)-\mathrm{CS}(\nabla_1^{\ker(\mathsf{D}^E)},\nabla_0^{\ker(\mathsf{D}^E)})\\&=\int_{X/B}\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\mathrm{CS}(\nabla_1^E,\nabla_0^E)-\mathrm{CS}(\nabla_1^{\ker(\mathsf{D}^E)},\nabla_0^{\ker(\mathsf{D}^E)}),\end{aligned}$$

up to exact forms, where the second and the third equalities follow from (3.1.5), the fourth equality follows from (3.1.4) and the last equality follows from (2.1.2). □

We call (3.1.6) the variational formula of the eta forms of the pair $(\mathbb{E}_1,\mathbb{E}_0)$.

3.2. **The flat GRR.** In this subsection we prove that the flat analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}:K_{\mathrm{L}}^{-1}(X)\to K_{\mathrm{L}}^{-1}(B)$ is well defined and the flat GRR. The proof of Proposition 3 is the essentially the same as [23, Proposition 3].

Proposition 3. Let $\pi:X\to B$ be a submersion with closed spin^c fibers of even relative dimension. The flat analytic index

$$\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}:K_{\mathrm{L}}^{-1}(X)\to K_{\mathrm{L}}^{-1}(B)$$

is well defined.

Proof. For $k=0,1$, let $\mathcal{E}_k=(E_k^+\oplus E_k^-,h_k^+\oplus h_k^-,\nabla_k^+\oplus\nabla_k^-,\omega_k)$ be $\mathbb{Z}_2$-graded generators of $K_{\mathrm{L}}^{-1}(X)$ such that the classes of $\mathcal{E}_0$ and $\mathcal{E}_1$ are equal in $K_{\mathrm{L}}^{-1}(X)$. For notational clarity we write E_k for $E_k^+\oplus E_k^-$ and similarly for other $\mathbb{Z}_2$-graded objects. By the definition of K_{L}^{-1}, there exists a $\mathbb{Z}_2$-graded generator $\mathcal{G}=(G,h^G,\nabla^G,\omega^G)$ of $K_{\mathrm{L}}^{-1}(X)$ such that

$$E_1\oplus G\cong E_0\oplus G,\tag{3.2.1}$$

and

$$\omega_1 - \omega_0 = \mathrm{CS}(\nabla_0 \oplus \nabla^G, \nabla_1 \oplus \nabla^G). \tag{3.2.2}$$

Since the analytic index is additive, (3.2.1) implies

$$\begin{aligned} \mathrm{ind}^{\mathrm{a}}(E_1) \oplus \mathrm{ind}^{\mathrm{a}}(G) &\cong \mathrm{ind}^{\mathrm{a}}(E_1 \oplus G) \\ &\cong \mathrm{ind}^{\mathrm{a}}(E_0 \oplus G) \cong \mathrm{ind}^{\mathrm{a}}(E_0) \oplus \mathrm{ind}^{\mathrm{a}}(G). \end{aligned} \tag{3.2.3}$$

Since the diagram

$$\begin{tikzcd} K^{-1}(X) \arrow[r, "\beta"] \arrow[d, "\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}"'] & K(X) \arrow[d, "\mathrm{ind}^{\mathrm{a}}"] \\ K^{-1}(B) \arrow[r, "\beta"'] & K(B) \end{tikzcd}$$

commutes, it follows that

$$\beta(\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}_1) - \mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}_0)) = \mathrm{ind}^{\mathrm{a}}(E_1) - \mathrm{ind}^{\mathrm{a}}(E_0) = 0.$$

It follows from the exact sequence (2.2.2) that there exists $[\omega] \in H^{\mathrm{odd}}(B; \mathbb{R})$ such that

$$\alpha([\omega]) = \mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}_1) - \mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}_0). \tag{3.2.4}$$

It suffices to prove that ω is an exact form. By the definition of α and (2.4.2), (3.2.4) implies

$$\begin{aligned} &\left(\ker(\mathsf{D}^{E_1}), h^{\ker(\mathsf{D}^{E_1})}, \nabla^{\ker(\mathsf{D}^{E_1})}, \int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^V X)}) \wedge \omega_1 + \widetilde{\eta}(\mathbb{E}_1)\right) \\ &\qquad - \left(\ker(\mathsf{D}^{E_0}), h^{\ker(\mathsf{D}^{E_0})}, \nabla^{\ker(\mathsf{D}^{E_0})}, \int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^V X)}) \wedge \omega_0 + \widetilde{\eta}(\mathbb{E}_0)\right) \\ &= (\mathbb{C}^n, h, \nabla^{\mathrm{flat}}, \omega) - (\mathbb{C}^n, h, \nabla^{\mathrm{flat}}, 0). \end{aligned}$$

It follows from the definition of K^{-1}_{L} that

$$\begin{aligned} \omega = {} & \mathrm{CS}(\nabla^{\ker(\mathsf{D}^{E_1})} \oplus \nabla^{\mathrm{flat}}, \nabla^{\ker(\mathsf{D}^{E_0})} \oplus \nabla^{\mathrm{flat}}) + \widetilde{\eta}(\mathbb{E}_1) - \widetilde{\eta}(\mathbb{E}_0) \\ & + \int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^V X)}) \wedge (\omega_1 - \omega_0) \end{aligned} \tag{3.2.5}$$

up to exact forms. By the additivity of the eta forms (3.1.3), we have

$$\widetilde{\eta}(\mathbb{E}_1) - \widetilde{\eta}(\mathbb{E}_0) = \widetilde{\eta}(\mathbb{E}_1 \oplus \mathbb{G}) - \widetilde{\eta}(\mathbb{E}_0 \oplus \mathbb{G}).$$

Together with (2.1.8) the right-hand side of (3.2.5) becomes

$$\begin{aligned} &\mathrm{CS}(\nabla^{\ker(\mathsf{D}^{E_1})} \oplus \nabla^{\ker(\mathsf{D}^{G})} \oplus \nabla^{\mathrm{flat}}, \nabla^{\ker(\mathsf{D}^{E_0})} \oplus \nabla^{\ker(\mathsf{D}^{G})} \oplus \nabla^{\mathrm{flat}}) \\ &\quad + \int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^V X)}) \wedge \mathrm{CS}(\nabla_0 \oplus \nabla^G, \nabla_1 \oplus \nabla^G) + \widetilde{\eta}(\mathbb{E}_1 \oplus \mathbb{G}) - \widetilde{\eta}(\mathbb{E}_0 \oplus \mathbb{G}). \end{aligned}$$

By (3.2.1) and (2.1.8), it becomes

$$\begin{aligned}&\mathrm{CS}(\nabla^{\ker(\mathsf{D}^{E_1})}\oplus\nabla^{\ker(\mathsf{D}^{G})},\nabla^{\ker(\mathsf{D}^{E_0})}\oplus\nabla^{\ker(\mathsf{D}^{G})})+\widetilde{\eta}(\mathbb{E}_1\oplus\mathbb{G})-\widetilde{\eta}(\mathbb{E}_0\oplus\mathbb{G})\\&+\int_{X/B}\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\mathrm{CS}(\nabla_0\oplus\nabla^G,\nabla_1\oplus\nabla^G).\end{aligned}$$

Because of (3.2.3), we can apply the variational formula for the eta forms (Proposition 2) to the pair $(\mathbb{E}_0\oplus\mathbb{G},\mathbb{E}_1\oplus\mathbb{G})$, which shows that the above form is exact. From (3.2.5) we see that ω is exact, so $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}_1)=\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}_0)$. Therefore the flat analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}$ is well defined. □

We are now ready to prove the flat GRR.

Theorem 1. Let $\pi: X\to B$ be a submersion with closed spin^c fibers of even relative dimension. The following diagram commutes.

$$\begin{array}{ccc}K^{-1}_{\mathrm{L}}(X) & \xrightarrow{\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}} & H^{\mathrm{odd}}(X;\mathbb{R}/\mathbb{Q})\\ {\scriptstyle \mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}}\big\downarrow & & \big\downarrow{\scriptstyle \int_{X/B}\mathrm{Todd}(X/B)\cup(\cdot)}\\ K^{-1}_{\mathrm{L}}(B) & \xrightarrow[\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}]{} & H^{\mathrm{odd}}(B;\mathbb{R}/\mathbb{Q})\end{array}$$

i.e., for a $\mathbb{Z}_2$-graded generator $\mathcal{E}$ of $K^{-1}_{\mathrm{L}}(X)$ of the form (2.2.1), we have

$$\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}))=\int_{X/B}\mathrm{Todd}(X/B)\cup\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E}).\tag{3.2.6}$$

Proof. By (2.2.4) and (2.4.2), $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}))$ is given by the mod $\mathbb{Q}$ reduction of the de Rham class of

$$\frac{1}{\ell}\,\mathrm{CS}(\ell\nabla^{\ker(\mathsf{D}^E)^+},j_1^*\ell\nabla^{\ker(\mathsf{D}^E)^-})+\int_{X/B}\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\omega+\widetilde{\eta}(\mathbb{E}),\tag{3.2.7}$$

where $\ell\in\mathbb{N}$ and $j_1:\ell\ker(\mathsf{D}^E)^+\to\ell\ker(\mathsf{D}^E)^-$ is an isometric isomorphism. Similarly $\int_{X/B}\mathrm{Todd}(X/B)\cup\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$ is given by the mod $\mathbb{Q}$ reduction of the de Rham class of

$$\frac{1}{k}\int_{X/B}\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\mathrm{CS}(k\nabla^+,j_2^*k\nabla^-)+\int_{X/B}\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\omega,\tag{3.2.8}$$

where $k\in\mathbb{N}$ and $j_2:kE^+\to kE^-$ is an isometric isomorphism. Consider the difference between (3.2.7) and (3.2.8), which is given by

$$\begin{aligned}h:=&\frac{1}{\ell}\,\mathrm{CS}(\ell\nabla^{\ker(\mathsf{D}^E)^+},j_1^*\ell\nabla^{\ker(\mathsf{D}^E)^-})+\widetilde{\eta}(\mathbb{E})\\&-\frac{1}{k}\int_{X/B}\mathrm{Todd}(\nabla^{S^c(T^VX)})\wedge\mathrm{CS}(k\nabla^+,j_2^*k\nabla^-).\end{aligned}\tag{3.2.9}$$

Thus to prove (3.2.6) it suffices to prove $h=0$ up to forms with periods in $\mathbb{Q}$. Let m be the least common multiple of k and ℓ. Then there exist unique

$d_1, d_2 \in \mathbb{N}$ such that $m = \ell d_1$ and $m = k d_2$. Since $j_2 : kE^+ \to kE^-$ is an isometric isomorphism, the same is true for

$$d_2 j_2 : \overbrace{kE^+ \oplus \cdots \oplus kE^+}^{d_2} \to \overbrace{kE^- \oplus \cdots \oplus kE^-}^{d_2},$$

where $d_2 j_2 := \overbrace{j_2 \oplus \cdots \oplus j_2}^{d_2}$. Moreover, $d_2(k\nabla^+) = m\nabla^+$ and $(d_2 j_2)^*(d_2 k \nabla^+) = (d_2 j_2)^* m\nabla^-$ are unitary connections on $mE^+ \to X$. Note that

$$\begin{aligned}\frac{1}{k}\,\mathrm{CS}(k\nabla^+, j_2^* k\nabla^-) = \frac{1}{m}\,\mathrm{CS}(m\nabla^+, (d_2 j_2)^* m\nabla^-) + \frac{1}{k}\,\mathrm{CS}(k\nabla^+, j_2^* k\nabla^-)\\ - \frac{1}{m}\,\mathrm{CS}(m\nabla^+, (d_2 j_2)^* m\nabla^-).\end{aligned} \tag{3.2.10}$$

By (2.1.6) and (2.1.7), the last two terms of the right-hand side of (3.2.10) equal

$$\begin{aligned}&\frac{1}{k}\,\mathrm{CS}(k\nabla^+, j_2^* k\nabla^-) - \frac{1}{m}\,\mathrm{CS}(m\nabla^+, (d_2 j_2)^* m\nabla^-)\\ &= \frac{1}{d_2 k}\,\mathrm{CS}(d_2 k\nabla^+, (d_2 j_2)^*(d_2 k\nabla^-)) - \frac{1}{m}\,\mathrm{CS}(m\nabla^+, (d_2 j_2)^* m\nabla^-)\\ &= \frac{1}{m}\Big(\mathrm{CS}(m\nabla^+, (d_2 j_2)^*(m\nabla^-)) - \mathrm{CS}(m\nabla^+, (d_2 j_2)^* m\nabla^-)\Big)\\ &= \frac{1}{m}\Big(\mathrm{CS}(m\nabla^+, (d_2 j_2)^*(m\nabla^-)) + \mathrm{CS}((d_2 j_2)^* m\nabla^-, m\nabla^+)\Big)\\ &= \frac{1}{m}\,\mathrm{CS}(m\nabla^+, m\nabla^+) = 0\end{aligned}$$

up to exact forms.[2] Thus

$$\begin{aligned}&\frac{1}{k}\int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^V X)}) \wedge \mathrm{CS}(k\nabla^+, j_2^* k\nabla^-)\\ &\qquad = \frac{1}{m}\int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^V X)}) \wedge \mathrm{CS}(m\nabla^+, (d_2 j_2)^* m\nabla^-)\end{aligned}$$

up to exact forms. By the same argument, the first term of the right-hand side of (3.2.9) becomes

$$\frac{1}{\ell}\,\mathrm{CS}(\ell\nabla^{\ker(\mathsf{D}^E)^+}, j_1^* \ell\nabla^{\ker(\mathsf{D}^E)^-}) = \frac{1}{m}\,\mathrm{CS}(m\nabla^{\ker(\mathsf{D}^E)^+}, (d_1 j_1)^* m\nabla^{\ker(\mathsf{D}^E)^-})$$

[2]The argument is similar to the proof of [30, Lemma 1], which says that the definition of $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}$ is, in particular, independent of the choice of $k \in \mathbb{N}$ (see (2.2.3)).

up to exact forms. Since exact forms have zero periods, it follows that (3.2.9) becomes

$$h = \frac{1}{m}\,\mathrm{CS}(m\nabla^{\ker(\mathsf{D}^E)^+}, (d_1 j_1)^* m\nabla^{\ker(\mathsf{D}^E)^-}) + \widetilde{\eta}(\mathbb{E}) \\ - \frac{1}{m}\int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^V X)}) \wedge \mathrm{CS}(m\nabla^+, (d_2 j_2)^* m\nabla^-).$$

Since $\widetilde{\eta}(m\mathbb{E}) = m\widetilde{\eta}(\mathbb{E})$ by (3.1.3), it follows that

$$h = \frac{1}{m}\bigg(\mathrm{CS}(m\nabla^{\ker(\mathsf{D}^E)^+}, (d_1 j_1)^* m\nabla^{\ker(\mathsf{D}^E)^-}) + \widetilde{\eta}(m\mathbb{E}) \\ - \int_{X/B} \mathrm{Todd}(\nabla^{S^c(T^V X)}) \wedge \mathrm{CS}(m\nabla^+, (d_2 j_2)^* m\nabla^-)\bigg).$$

A priori the variational formula of the eta forms (Proposition 2) cannot be applied to the pair $(m\mathbb{E}^+, (d_2 j_2)^* m\mathbb{E}^-)$ since the isometric isomorphisms $d_1 j_1$ and $d_2 j_2$ are not related in general. However, as remarked in [30, p.289], the flat Chern character $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}$ of $\mathbb{Z}_2$-graded generator is independent of the choice of the isometric isomorphism involved. Thus, without loss of generality, we can assume $d_1 j_1$ is induced by $d_2 j_2$, so Proposition 2 can be applied. Thus $h = 0$ up to exact forms, and therefore (3.2.6) holds.

We have proved (3.2.6) under the assumption that the kernel bundle $\ker(\mathsf{D}^E) \to B$ exists. Without this assumption one can proceed as in [30, §5] and [21, §7.12] to prove (3.2.6). □

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Department of Mathematics, Hong Kong Baptist University
E-mail address: homanho@math.hkbu.edu.hk