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The time of ruin, the surplus prior to ruin and the deficit at ruin for the classical risk process perturbed by diffusion

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Abstract

The paper studies the joint distribution of the time of ruin, the surplus prior to ruin and the deficit at ruin for the classical risk process that is perturbed by diffusion. We prove that the expected discounted penalty satisfies an integro-differential equation of renewal type, the solution of which can be expressed as a convolution formula. The asymptotic behaviour of the expected discounted penalty as the initial capital tends to infinity is discussed.

Keywords: Deficit at ruin; Renewal equation; Time of ruin; Ruin probability; Surplus prior to ruin; Surplus process

JEL classification: G22

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1. Introduction

Consider the classical risk process that is perturbed by diffusion

$$R_t := u + ct + \sigma B_t - \sum_{i=1}^{N_t} Z_i, \quad t \geq 0, \tag{1.1}$$

where $u \geq 0$ is the initial surplus, σ a positive constant, c the positive constant premium income rate, $\{B_t, t \geq 0\}$ the standard Brownian motion, $\{N_t, t \geq 0\}$ a Poisson process with intensity $\lambda > 0$, and $\{Z_k, k \geq 1\}$ a sequence of nonnegative i.i.d. random variables. We assume that $\{B_t, t \geq 0\}$, $\{N_t, t \geq 0\}$ and $\{Z_k, k \geq 1\}$ are independent. Denote by P and μ the distribution function and the mean, respectively, of the claim sizes Z_k, with the condition that $P(0) = 0$. The safety loading $c - \lambda\mu$ is assumed to be positive to ensure that $\lim_{t\to\infty} R_t = \infty$ almost surely, provided that the process will continue even when the surplus is negative. For simplicity, P is assumed to possess a density p. The diffusion term in (1.1) contributes an additional uncertainty of the premium income or the aggregate claims to the surplus.

Let $T = \inf\{t \geq 0 : R_t < 0\}$ denote the time of ruin with the usual convention that $\inf \emptyset = \infty$. Let $w(\cdot, \cdot)$ be a nonnegative function. We are interested in the quantity

$$W_\sigma(u, \delta, w) = \mathrm{E}[\mathrm{e}^{-\delta T}\mathbf{1}(T < \infty)w(R_{T-}, |R_T|)|R_0 = u], \tag{1.2}$$

where δ is a positive constant. The values R_{T-} and $|R_T|$ are known as the surplus immediately before ruin and the deficit at ruin, respectively. According to Gerber and Shiu (1998), δ can be interpreted as a force of interest and w as some kind of penalty when ruin occurs; the function W_σ is the expectation of the discounted penalty. Let $q(\cdot, \cdot, \cdot|u)$ denote the conditional joint probability density function of $R_{T-}, |R_T|$ and T

given that $R_0 = u$, and define $q_2(\cdot|u)$ and $q_1(\cdot,\cdot|u)$ by

$$q_2(x|u) = \int_0^\infty q_1(x,y|u)\mathrm{d}y = \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} q(x,y,t|u)\mathrm{d}t\,\mathrm{d}y.$$

Then $W_\sigma(u,\delta,w)$ in (1.2) can be written as

$$\begin{aligned} W_\sigma(u,\delta,w) &= \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} w(z_1,z_2) q(z_1,z_2,t|u)\mathrm{d}t\,\mathrm{d}z_1\mathrm{d}z_2 \\ &= \int_0^\infty \int_0^\infty w(z_1,z_2) q_1(z_1,z_2|u)\mathrm{d}z_1\mathrm{d}z_2. \end{aligned} \tag{1.3}$$

For the classical model, i.e. $\sigma = 0$, the distributions of R_{T-}, $|R_T|$ and T have been investigated by many authors. Gerber and Shiu (1997) showed that

$$q(x,y,t|u) = \frac{p(x+y)}{1-P(x)} \int_0^\infty q(x,z,t|u)\mathrm{d}z,$$

and

$$q_1(x,y|u) = \frac{p(x+y)}{1-P(x)} q_2(x|u). \tag{1.4}$$

For $\delta = 0$, (1.4) was also shown by Dufresne and Gerber (1988) and Dickson and Egídio dos Reis (1994). For each case of positive, negative or zero safety loadings, Schmidli (1999) gave an explicit expression and discussed the asymptotic behaviour of $W_0(u,0,w)$ as $u \to \infty$ in the case that $w(z_1,z_2) = \mathbf{1}(z_1 > y, z_2 > x)$ for arbitrary $x, y > 0$. Gerber and Shiu (1998) considered a more general case, they derived a renewal equation for $W_0(u,\delta,w)$ and studied its asymptotic behaviour for nonnegative $w(\cdot,\cdot)$. Further study on $W_0(u,\delta,w)$ can be found in Lin and Willmot (1999, 2000).

For the non-classical model, i.e. $\sigma \neq 0$, Dufresne and Gerber (1991) showed that the probability of ruin is the solution of an integro-differential equation and derived the convolution formula for it; for another derivation see Veraverbeke (1993). Gerber and Landry (1998) proved that $W_\sigma(u, \delta, w)$ satisfies a defective renewal equation with $w(z_1, z_2)$ being a univariate function of z_2 for $z_2 \geq 0$; Wang and Wu (2000) considered the function $W_\sigma(u, 0, w)$ with a univariate $w(z_1, z_2) = \mathbf{1}(z_2 \leq y)$ for an arbitrary $y > 0$, and proved that if the claim size density p is continuous on $[0, \infty)$, then $W_\sigma(u, 0, w)$, as a function of u, satisfies an integro-differential equation.

In this paper, we consider a general bivariate $w(\cdot, \cdot)$ with positive σ and δ, and prove that W_σ satisfies an integro-differential equation. An infinite series expression of the solution and its asymptotic behaviour formulae as the initial surplus tends to infinity are obtained.

2. Integro-differential equation

Consider the following integro-differential equation on the positive half line:

$$Af''(x) + Bf'(x) + D\int_0^x f(x-y)p(y)\mathrm{d}y + g(x) = Cf(x), \; x > 0, \tag{2.1}$$

where A, B, C, D are positive constants, p, as before, is a probability density function on $(0, \infty)$, and g is a nonnegative function on $[0, \infty)$. The functions p and g satisfy some mild conditions which will be specified below. Some particular forms of this type of equations frequently appear in probability theory and risk theory (see Feller, 1971; Gerber, 1979; Grandell, 1991; Embrechts *et al.*, 1997; Rolski *et al.*, 1999; Gerber and Landry, 1998; Asmussen, 2000). Our aim is to determine the exact solution of (2.1) and investigate its asymptotic behaviour as $x \to \infty$.

Throughout the paper, the Laplace transform of a function is denoted by putting a hat on the function. For a probability density p, we consider the following equation:

$$A\alpha^2 + D\hat{p}(\alpha) + B\alpha - C = 0. \tag{2.2}$$

Since $\hat{p}(\alpha)$ is defined for all nonnegative numbers α and is a decreasing convex function, equation (2.2) has a unique positive root α_1, and one negative root $-\alpha_2$, provided that $D < C$ and there exists an $\alpha_0 \in [-\infty, 0)$ such that $\lim_{\alpha \downarrow \alpha_0} \hat{p}(\alpha) = \infty$ and $\hat{p}(\alpha) < \infty$ for $\alpha > \alpha_0$. The latter condition, which is assumed throughout the paper, implies that the tail of p decreases at least exponentially fast. Moreover, if $D < B / \int_0^\infty xp(x)\mathrm{d}x$, then by the "tangent argument" used in Gerber and Shiu (1999) we have $\lim_{D-C\uparrow 0} \alpha_1 = 0$ and $\lim_{D-C\uparrow 0} \alpha_2 > 0$. These two roots α_1 and $-\alpha_2$ play a crucial role in the following. Throughout, the conditions $D < C$ and $D < B / \int_0^\infty xp(x)\mathrm{d}x$ are assumed.

For two integrable functions f_1 and f_2 defined on $[0, \infty)$, we use $f_1 * f_2$ to denote the convolution of them.

Theorem 2.1. *If* $\lim_{x\to\infty} \mathrm{e}^{-\alpha_1 x} f(x) = 0$ *and* $\lim_{x\to\infty} \mathrm{e}^{-\alpha_1 x} f'(x) = 0$, *then the solution to equation (2.1) satisfies*

$$f(x) = (f * g_1 * g_2)(x) + \mathrm{e}^{-(\alpha_1 + BA^{-1})x} f(0) + (h_1 * h_2)(x), \qquad x > 0, \tag{2.3}$$

provided that

$$\int_0^\infty \mathrm{e}^{-\alpha_1 x} g(x)\mathrm{d}x < \infty, \tag{2.4}$$

where α_1 *is the positive root of equation (2.2) and*

$$g_1(x) = \frac{D}{A}\mathrm{e}^{-(\alpha_1 + BA^{-1})x}, \qquad g_2(x) = \mathrm{e}^{\alpha_1 x} \int_x^\infty \mathrm{e}^{-\alpha_1 y} p(y)\mathrm{d}y,$$
$$h_1(x) = \frac{1}{A}\mathrm{e}^{-(\alpha_1 + BA^{-1})x}, \qquad h_2(x) = \mathrm{e}^{\alpha_1 x} \int_x^\infty \mathrm{e}^{-\alpha_1 y} g(y)\mathrm{d}y.$$

Proof. Substituting $F(x) = \mathrm{e}^{-\alpha_1 x} f(x)$ into (2.1) leads to

$$AF''(x) + (B + 2\alpha_1 A)F'(x) + D\int_0^x \mathrm{e}^{-\alpha_1 y} F(x-y)p(y)\mathrm{d}y + g(x)\mathrm{e}^{-\alpha_1 x} = D\hat{p}(\alpha_1)F(x). \tag{2.5}$$

We integrate (2.5) with respect to x from 0 to u and, after simplification, obtain

$$A(F'(u) - F'(0)) + (B + 2\alpha_1 A)(F(u) - F(0)) - D\int_0^u F(x)\left(\int_{u-x}^{\infty} \mathrm{e}^{-\alpha_1 y} p(y)\mathrm{d}y\right)\mathrm{d}x + \int_0^u g(x)\mathrm{e}^{-\alpha_1 x}\mathrm{d}x = 0. \tag{2.6}$$

By letting $u \to \infty$ we get

$$-AF'(0) - (B + 2\alpha_1 A)F(0) + \int_0^{\infty} g(x)\mathrm{e}^{-\alpha_1 x}\mathrm{d}x = 0. \tag{2.7}$$

Substituting (2.7) into (2.6) and replacing the dummy variable x by z yield

$$AF'(u) + (B + 2\alpha_1 A)F(u) - D\int_0^u F(z)\left(\int_{u-z}^{\infty} \mathrm{e}^{-\alpha_1 y} p(y)\mathrm{d}y\right)\mathrm{d}z - \int_u^{\infty} g(z)\mathrm{e}^{-\alpha_1 z}\mathrm{d}z = 0. \tag{2.8}$$

Then, equation (2.8) is multiplied by $\mathrm{e}^{(2\alpha_1 + BA^{-1})u}$, and integrated with respect to u from 0 to x. We obtain

$$A\left(\mathrm{e}^{(2\alpha_1+BA^{-1})x}F(x) - F(0)\right) - \int_0^x \mathrm{e}^{(2\alpha_1+BA^{-1})u}\left(\int_u^{\infty} g(z)\mathrm{e}^{-\alpha_1 z}\mathrm{d}z\right)\mathrm{d}u - D\int_0^x \mathrm{e}^{(2\alpha_1+BA^{-1})u}\left\{\int_0^u F(z)\left(\int_{u-z}^{\infty}\mathrm{e}^{-\alpha_1 y}p(y)\mathrm{d}y\right)\mathrm{d}z\right\}\mathrm{d}u = 0. \tag{2.9}$$

Since $f(x) = \mathrm{e}^{\alpha_1 x}F(x)$, from (2.9) we have

$$f(x) = \frac{D}{A}\int_0^x \mathrm{e}^{-(2\alpha_1+BA^{-1})(x-u)}\left\{\int_0^u \mathrm{e}^{\alpha_1(x-z)}f(z)\left(\int_{u-z}^{\infty}\mathrm{e}^{-\alpha_1 y}p(y)\mathrm{d}y\right)\mathrm{d}z\right\}\mathrm{d}u + \frac{1}{A}\mathrm{e}^{\alpha_1 x}\int_0^x\left(\mathrm{e}^{-(2\alpha_1+BA^{-1})(x-u)}\int_u^{\infty} g(y)\mathrm{e}^{-\alpha_1 y}\mathrm{d}y\right)\mathrm{d}u + \mathrm{e}^{-(\alpha_1+BA^{-1})x}f(0),$$

and the result follows.

Remark 2.1. Gerber and Landry (1998) obtained equation (2.3) for the special case that $g(x) = 0$.

Theorem 2.2. *If* $\lim_{x\to\infty} e^{-\alpha_1 x} f(x) = 0$ *and* $\lim_{x\to\infty} e^{-\alpha_1 x} f'(x) = 0$, *then the function* $f(x)$ *in (2.3) can be expressed as*

$$f(x) = \sum_{n=0}^{\infty} (h * g_1^{*n} * g_2^{*n})(x), \tag{2.10}$$

where g_i^{*n} *is the* n*-fold convolution of* g_i, *with the convention that* $g_i^{*0}(x) = \mathbf{1}(x \geq 0)$, *and*

$$h(x) = e^{-(\alpha_1 + BA^{-1})x} f(0) + (h_1 * h_2)(x). \tag{2.11}$$

Proof. Taking the Laplace transforms of (2.3) and (2.10), we can see that (2.10) is a solution of (2.3).

Theorem 2.3. *If* $\int_0^\infty \mathrm{e}^{\alpha_2 y} g(y)\mathrm{d}y < \infty$, $\lim_{y\to\infty} \mathrm{e}^{\alpha_2 y} g(y) = 0$, $\hat{p}'(-\alpha_2) < \infty$ *and* $B + A(\alpha_1 - \alpha_2) > 0$, *then*

$$\lim_{x\to\infty} e^{\alpha_2 x} f(x) = -\frac{Af(0)(\alpha_1 + \alpha_2) + \int_0^\infty (e^{\alpha_2 y} - e^{-\alpha_1 y}) g(y)\mathrm{d}y}{D\hat{p}'(-\alpha_2) + B - 2A\alpha_2}.$$

Proof. Multiplying (2.3) by $\mathrm{e}^{\alpha_2 x}$ yields a standard renewal equation, since $\int_0^\infty \mathrm{e}^{\alpha_2 x}(g_1 * g_2)(x)\mathrm{d}x = 1$. Using the assumed conditions we can prove that $\lim_{x\to\infty} \mathrm{e}^{\alpha_2 x} h(x) = 0$. Moreover,

$$\begin{aligned} \int_0^\infty \mathrm{e}^{\alpha_2 x} h(x)\mathrm{d}x &= \int_0^\infty \mathrm{e}^{-(\alpha_1 + BA^{-1} - \alpha_2)x} f(0)\mathrm{d}x + \int_0^\infty \mathrm{e}^{\alpha_2 x}(h_1 * h_2)(x)\mathrm{d}x \\ &= \frac{Af(0)}{B + \alpha_1 A - \alpha_2 A} + \frac{\int_0^\infty (\mathrm{e}^{\alpha_2 y} - \mathrm{e}^{-\alpha_1 y}) g(y)\mathrm{d}y}{(B + \alpha_1 A - \alpha_2 A)(\alpha_1 + \alpha_2)}. \end{aligned}$$

From the definitions of g_1 and g_2 in Theorem 2.1, we obtain for $v \leq \alpha_2$

$$\int_0^\infty \mathrm{e}^{vx}(g_1 * g_2)(x)\mathrm{d}x = \frac{D\hat{p}(-v) + A\alpha_1^2 + B\alpha_1 - C}{(\alpha_1 + v)(\alpha_1 A - vA + B)},$$

and thus

$$\int_0^\infty x\mathrm{e}^{\alpha_2 x}(g_1 * g_2)(x)\mathrm{d}x = -\widehat{g_1 * g_2}'(-\alpha_2) = -\frac{\partial}{\partial z}\left(\frac{D\hat{p}(z) + A\alpha_1^2 + B\alpha_1 - C}{(\alpha_1 - z)(\alpha_1 A + Az + B)}\right)\Bigg|_{z=-\alpha_2}$$
$$= -\frac{D\hat{p}'(-\alpha_2) + (B - 2A\alpha_2)}{(\alpha_1 + \alpha_2)(\alpha_1 A - A\alpha_2 + B)}.$$

A distribution function with density function $\mathrm{e}^{\alpha_2 x}(g_1 * g_2)(x)$ is of course spread-out, and so from the key renewal theorem (see e.g. Asmussen, 2000, p. 332, Proposition A1.1), we have

$$\lim_{x\to\infty} \mathrm{e}^{\alpha_2 x} f(x) = \frac{\int_0^\infty \mathrm{e}^{\alpha_2 x} h(x)\mathrm{d}x}{\int_0^\infty x\mathrm{e}^{\alpha_2 x}(g_1 * g_2)(x)\mathrm{d}x},$$

and the result is now obvious.

3. The joint distributions

In this section, we prove that W_σ defined by (1.2) satisfies an integro-differential equation of type (2.1), and apply theorems in the previous section to yield the corresponding results for W_σ.

Theorem 3.1. *Suppose that for fixed δ and σ, $W(u) := W_\sigma(u, \delta, w)$ is twice continuously differentiable with respect to u. Then $W(\cdot)$ satisfies*

$$\frac{1}{2}\sigma^2 W''(u) + cW'(u) + \lambda\int_0^u W(u-z)p(z)\mathrm{d}z + \lambda\int_u^\infty w(u, z-u)p(z)\mathrm{d}z$$
$$= (\lambda + \delta)W(u),\ u > 0. \qquad (3.1)$$

Proof. Let T_1 be the first epoch of the claim, and $\epsilon, t, m > 0$ such that $\epsilon < u < m$. Define $T^{\epsilon,m} = \inf\{s > 0 : u + cs + \sigma B_s \notin (\epsilon, m)\}$. Note that $W'(u)$ and $W''(u)$ are bounded on $[\epsilon, m]$ since W is twice continuously differentiable. Put $T_t^{\epsilon,m} = T^{\epsilon,m} \wedge t$ and $\tau = T_t^{\epsilon,m} \wedge T_1$. By the strong Markov property of R_t, we get

$$W(u) = \mathrm{E}[\mathrm{e}^{-\delta\tau} W(R_\tau)],$$

which can be written as

$$\begin{aligned} W(u) =& \mathrm{E}[\mathrm{e}^{-\delta T_t^{\epsilon,m}} W(R_{T_t^{\epsilon,m}}) \mathbf{1}(T_t^{\epsilon,m} < T_1)] \\ &+ \mathrm{E}[\mathrm{e}^{-\delta T_1} W(R_{T_1}) \mathbf{1}(T_t^{\epsilon,m} > T_1)] \\ &+ \mathrm{E}[\mathrm{e}^{-\delta T_1} W(R_{T_1}) \mathbf{1}(T_t^{\epsilon,m} = T_1)] \equiv I_1 + I_2 + I_3. \end{aligned}$$

Notice that $I_3 = 0$, since $P(T_t^{\epsilon,m} = T_1) = 0$, and

$$\begin{aligned} I_1 =& \int_0^t \lambda \mathrm{e}^{-\lambda s} \mathrm{E}[\mathrm{e}^{-\delta T_t^{\epsilon,m}} W(u + cT_t^{\epsilon,m} + \sigma B_{T_t^{\epsilon,m}}) \mathbf{1}(T_t^{\epsilon,m} < s)] \mathrm{d}s \\ &+ \mathrm{e}^{-\lambda t} \mathrm{E}[\mathrm{e}^{-\delta T_t^{\epsilon,m}} W(u + cT_t^{\epsilon,m} + \sigma B_{T_t^{\epsilon,m}})], \\ I_2 =& \int_0^t \lambda \mathrm{e}^{-(\lambda+\delta)s} \int_0^\infty \mathrm{E}[W(u + cs + \sigma B_s - z) \mathbf{1}(T^{\epsilon,m} > s)] p(z) \mathrm{d}z \mathrm{d}s. \end{aligned}$$

Applying the Itô formula to the second term of I_1 yields

$$\begin{aligned} W(u) =& \mathrm{e}^{-\lambda t} W(u) - \delta \mathrm{e}^{-\lambda t} \mathrm{E}\left(\int_0^{T_t^{\epsilon,m}} \mathrm{e}^{-\delta s} W(R_s) \mathrm{d}s \right) \\ &+ \mathrm{e}^{-\lambda t} \mathrm{E}\left(\int_0^{T_t^{\epsilon,m}} \mathrm{e}^{-\delta s} [cW'(u + cs + \sigma B_s) + \frac{1}{2}\sigma^2 W''(u + cs + \sigma B_s)] \mathrm{d}s \right) \\ &+ \lambda \int_0^t \mathrm{e}^{-\lambda s} \mathrm{E}[\mathrm{e}^{-\delta T_t^{\epsilon,m}} W(u + cT_t^{\epsilon,m} + \sigma B_{T_t^{\epsilon,m}}) \mathbf{1}(T_t^{\epsilon,m} < s)] \mathrm{d}s \\ &+ \lambda \int_0^t \mathrm{e}^{-\lambda s - \delta s} \int_0^\infty \mathrm{E}[W(u + cs + \sigma B_s - z) \mathbf{1}(T_t^{\epsilon,m} > s)] p(z) \mathrm{d}z \mathrm{d}s. \end{aligned} \tag{3.2}$$

Subtracting $\mathrm{e}^{-\lambda t}W(u)$ from each side of equation (3.2), dividing by t and then letting $t \to 0$, we find that equation (3.1) holds in (ϵ, m). Hence, it holds in $(0,\infty)$ since ϵ and m are arbitrary.

Remark 3.1. Our proof is similar to but not the same as Wang and Wu (2000). The condition that $W(u)$ is twice continuously differentiable does not seem particularly restrictive. Tsai and Willmot (2002, p. 54) also imposed implicitly the same condition on a similar function. In fact, using the technique in Wang and Wu (2000), we can prove that the condition holds if w is a nonnegative bounded continuous function and p is continuous. In particular, when $\delta = 0$ and $w(z_1, z_2) = \mathbf{1}(z_2 \geq y)$ for an arbitrary $y > 0$, (3.1) reduces to a result of Wang and Wu (2000). If we set $w(z_1, z_2) = w_0\mathbf{1}(z_2 = 0)$, where w_0 is a constant, then (3.1) reduces to Gerber and Landry (1998, equation (6)). If we set $\delta = 0$ and $w \equiv 1$ in (3.1), we can retrieve Dufresne and Gerber (1991, equation (2.1)). Schmidli (1999, equation (1)) can also be obtained from our (3.1) by taking $\delta = 0, \sigma = 0$ and $w(z_1, z_2) = \mathbf{1}(z_1 > y, z_2 > x)$ for arbitrary $x, y > 0$. When $\sigma = 0$, (3.1) reduces to Gerber and Shiu (1998, equation (2.16)).

Let β_1 and $-\beta_2$, respectively, be the positive and negative roots of

$$\frac{1}{2}\sigma^2\beta^2 + \lambda\hat{p}(\beta) + c\beta - \lambda - \delta = 0, \tag{3.3}$$

and assume that the nonnegative function $w(\cdot,\cdot)$ satisfies

$$\int_0^\infty \int_0^\infty \mathrm{e}^{\beta_2 z_1} w(z_1, z_2) p(z_1 + z_2) \mathrm{d}z_1 \mathrm{d}z_2 < \infty. \tag{3.4}$$

Theorem 3.2. *For fixed δ, fixed σ and W as in Theorem 3.1, if $\lim_{u\to\infty} \mathrm{e}^{-\alpha_1 u}W(u) = 0$ and $\lim_{u\to\infty} \mathrm{e}^{-\alpha_1 u}W'(u) = 0$, then the solution to equation (3.1) can be expressed as*

$$W(u, \delta, w) = \sum_{n=0}^{\infty} (H * G_1^{*n} * G_2^{*n})(u), \tag{3.5}$$

where

$$H(u) = e^{-(\beta_1+2c\sigma^{-2})u}w(0,0) + (H_1 * H_2)(u)$$

with

$$H_1(u) = 2\sigma^{-2}e^{-(\beta_1+2c\sigma^{-2})u},$$
$$H_2(u) = \lambda e^{\beta_1 u}\int_u^\infty e^{-\beta_1 y}\int_y^\infty w(y, z-y)p(z)\mathrm{d}z\,\mathrm{d}y,$$
$$G_1(u) = 2\lambda\sigma^{-2}e^{-(\beta_1+2c\sigma^{-2})u},$$
$$G_2(u) = e^{\beta_1 u}\int_u^\infty e^{-\beta_1 y}p(y)\mathrm{d}y.$$

Proof. Apply Theorem 2.2 to equation (3.1).

Remark 3.2. Equation (3.5) generalizes Dufresne and Gerber (1991, equation (3.4)) and the classical Beekman's convolution formula, which is also referred to as Pollaczeck–Khinchine formula (e.g. Asmussen, 2000, pp. 61-62).

Theorem 3.3. *If* $\hat{p}'(-\beta_2) < \infty$, $c + \frac{1}{2}\sigma^2(\beta_1 - \beta_2) > 0$ *and* $\lim_{y\to\infty} e^{\beta_2 y}\int_y^\infty w(y, z - y)p(z)\mathrm{d}z = 0$, *then*

$$\lim_{u\to\infty} e^{\beta_2 u}W_\sigma(u,\delta,w) = \frac{-\frac{1}{2}\sigma^2 w(0,0)(\beta_1+\beta_2) - \lambda\int_0^\infty\int_0^\infty(e^{\beta_2 z_1} - e^{-\beta_1 z_1})w(z_1,z_2)p(z_1+z_2)\mathrm{d}z_2\mathrm{d}z_1}{\lambda\hat{p}'(-\beta_2) + c - \sigma^2\beta_2}. \quad (3.6)$$

Proof. Notice that if $u = 0$, ruin takes place immediately (Gerber and Landry, 1998), and hence $W_\sigma(0,\delta,w) = w(0,0)$. The result follows from Theorems 2.3 and 3.1.

Theorem 3.4. *Suppose that* $\hat{p}'(-\beta_2) < \infty$ *and* $c + \frac{1}{2}\sigma^2(\beta_1 - \beta_2) > 0$. *If* $\sigma \neq 0$ *then for* $x, y > 0$ *or if* $\sigma = 0$ *then for* $x, y \geq 0$,

$$\lim_{u\to\infty} e^{\beta_2 u}q_1(x,y|u) = \frac{-\lambda(e^{\beta_2 x} - e^{-\beta_1 x})p(x+y)}{\lambda\hat{p}'(-\beta_2) + c - \sigma^2\beta_2}.$$

Proof. It follows from equation (1.3) and Theorem 3.3.

Remark 3.3. For $\sigma = 0$, (3.6) reduces to Gerber and Shiu (1998, equation (4.10)). If $w(z_1, z_2) = \mathbf{1}(z_1 \le x, z_2 \le y)$ for arbitrary $x, y > 0$, then

$$\lim_{u\to\infty} \mathrm{e}^{\beta_2 u}\mathrm{E}[\mathrm{e}^{-\delta T}\mathbf{1}(T<\infty)\mathbf{1}(R_{T-}\le x, |R_T|\le y)]$$
$$= \frac{-\frac{1}{2}\sigma^2(\beta_1+\beta_2) - \lambda\int_0^x (\mathrm{e}^{\beta_2 z} - \mathrm{e}^{-\beta_1 z})\{P(z+y)-P(z)\}\mathrm{d}z}{\lambda\hat{p}'(-\beta_2)+c-\sigma^2\beta_2}.$$

For $w(z_1, z_2) \equiv 1$, (3.6) becomes

$$\lim_{u\to\infty} \mathrm{e}^{\beta_2 u}\mathrm{E}[\mathrm{e}^{-\delta T}\mathbf{1}(T<\infty)]$$
$$= \frac{-\frac{1}{2}\sigma^2(\beta_1+\beta_2) - \lambda\int_0^\infty (\mathrm{e}^{\beta_2 x} - \mathrm{e}^{-\beta_1 x})\{(1-P(x)\}\mathrm{d}x}{\lambda\hat{p}'(-\beta_2)+c-\sigma^2\beta_2}.$$

In particular, for $\delta = 0$, and hence $\beta_1 = 0$, we have

$$\lim_{u\to\infty} \mathrm{e}^{\beta_2 u}\mathrm{P}(T<\infty|R_0=u) = \frac{-c+\lambda\mu}{\lambda\hat{p}'(-\beta_2)+c-\sigma^2\beta_2}, \tag{3.7}$$

where μ is the expected claim size. When $\sigma = 0$, (3.7) reduces to the classical Cramér–Lundberg asymptotic formula (e.g. Asmussen, 2000, p. 71, Theorem 5.3; Embrechts *et al.*, 1997, Theorem 1.2.2; Grandell, 1991, p. 7, (III)).

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