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Lam, Kin; Liu, Taisheng; WONG, Wing Keung

*Published in:*  
European Journal of Operational Research

*DOI:*  
[10.1016/j.ejor.2009.07.005](https://doi.org/10.1016/j.ejor.2009.07.005)

Published: 16/05/2010

*Document Version:*  
Peer reviewed version

[Link to publication](#)

*Citation for published version (APA):*  
Lam, K., Liu, T., & WONG, W. K. (2010). A pseudo-Bayesian model in financial decision making with implications to market volatility, under- and overreaction. *European Journal of Operational Research*, 203(1), 166-175. <https://doi.org/10.1016/j.ejor.2009.07.005>

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# **A Pseudo Bayesian Model in Financial Decision Making with Implications to Market Volatility, Under- and Overreaction**

**Kin LAM**

Department of Finance & Decision Sciences  
Hong Kong Baptist University

**Taisheng LIU**

Department of Finance & Decision Sciences  
Hong Kong Baptist University

**Wing-Keung WONG**

Department of Economics  
Hong Kong Baptist University

June 15, 2009

Acknowledgement: The authors are grateful to Professor Lorenzo Peccati and anonymous referees for substantive comments that have significantly improved this manuscript. The third author would like to thank Professors Robert B. Miller and Howard E. Thompson for their continuous guidance and encouragement. The research is partially supported by the grant of number 200908 from Hong Kong Research Grants Council.

Corresponding author. Wing-Keung Wong, Department of Economics, Hong Kong Baptist University, Kowloon Tong, Hong Kong. Tel: (852)-3411-7542, Fax: (852)-3411-5580, Email: awong@hkbu.edu.hk

# **A Pseudo Bayesian Model in Financial Decision Making with Implications to Market Volatility, Under- and Overreaction**

## **Abstract**

This paper develops a model of weight assignments using a pseudo-Bayesian approach that reflects investors' behavioral biases. In this parsimonious model of investor sentiment, weights induced by investors' conservative and representative heuristics are assigned to observations of the earning shocks of stock prices. Such weight assignments enable us to provide a quantitative link between some market anomalies and investors' behavioral biases. The seriousness of an anomaly can be quantitatively assessed by investigating into its dependency on weights. New results other than the short-run underreaction and long-run overreaction can be derived and new hypotheses can be formed.

*Keywords:* Bayesian model; Representative and conservative heuristics; Underreaction; Overreaction; Stock price; Stock return

# 1. Introduction

Among many market anomalies uncovered in the last two decades, three stand out as having a long history and receiving the most substantial empirical support. They are market excess volatility, overreaction, and underreaction. Together with other market anomalies, they pose a major challenge to financial economists. To meet these challenges, advocates of behavioral biases have constructed various behavioral models to explain these anomalies. Among the behavioral biases advocated, two stand out as receiving much empirical supports from psychological literatures. These behavioral biases are investors' usage of the conservatism heuristics and the representativeness heuristics in making decisions. The most notable model in this direction is the pioneer work of Barberis, Shleifer and Vishny (1998, henceforth BSV), in which they show that underreaction in the short-run and overreaction in the long-run are a consequence of the two mentioned heuristics. However, their paper did not show that excess volatility is also a result of the conservatism and representativeness heuristics.

BSV adopt a bounded rationalism approach in which some, but not all, assumptions under the traditional rational expectations asset-pricing theory are violated. Specifically, the "consistent beliefs" made by Sargent (1993) that agents possess correct knowledge of the economic structure are assumed to be violated. In their 1998 paper, BSV assume that while earning announcements follow a random walk, investors using conservative and representative heuristics believe that the earning announcements fall into one of two regimes, a trending regime and a mean reverting regime, and transition from one regime to the other follows a Markov chain. Assuming that the investors still use a correct Bayesian methodology for decision making, BSV then deduce that such a wrong belief will lead to both short-term underreaction and long-term overreaction in the market.

This paper takes a different approach from that of BSV in modeling conservatism and representativeness. We assume that the investor knows the correct model but uses a wrong updating methodology. This approach has several advantages as follows: (1) psychological literature clearly states that the two psychological biases arise from investors' attaching wrong weights to information, rather than from their adoption of a wrong model. In this paper, the weighting of information is emphasized and it is a more accurate description of the heuristics

used by investors. (2) Since the wrong weights reflect the biases, different degree of biases can be assessed through considering a change in weights. As a result, the seriousness of an anomaly can be quantitatively assessed by investigating into its dependency on weights. (3) New results other than the short-run underreaction and long-run overreaction can be derived and new hypotheses can be formed. We will elaborate on these points below.

According to DeBondt and Thaler (1995), a good finance theory must be based on psychological evidence of how people actually behave. Psychologists observe that investors pay too much attention to extreme information and less attention to its validity when making judgments and decisions about their investments (Griffin and Tversky, 1992). When investors are overconfident about their analysis based on the past performance of stocks and underreact to recent information, thus updating their beliefs too slowly in the face of new evidence, they exhibit conservative heuristics (Edwards, 1968; Grether, 1980). On the other hand, if they are overconfident about the recent information on stocks and pay less attention to the past information on stocks or extrapolate too readily from small samples, thus leading to belief revisions that are too dramatic, they demonstrate representative heuristics (Tversky and Kahneman, 1971, 1974; Kahneman and Tversky, 1973). Most studies of conservative heuristics involve large samples, whereas most studies on representative heuristics involve smaller samples. Misunderstanding the impact of sample size on the posterior mean leads investors to make conservative revisions with large samples and radical revisions with small samples. Thus, it is obvious that behavioral biases arise from an inappropriate treatment of information, rather than from a misjudgment on the model.

One of the earliest papers addressing conservatism is Edwards (1968), who reveals that people tend to make behavioral mistakes in their decisions, although they try to employ theoretical models or methodology. He observes that investors with conservative behavior might pay little attention or even ignore the full information from an earnings announcement. They may believe that this information is mainly temporary, and thus they still cling to their prior beliefs based on past earnings. As a result, they might incorporate only partial information from recent earnings announcement in their valuation of shares. In other words, they attach too little a weight to recent information. Edwards (1968) develops a Bayesian model in which individuals tend to underweigh useful statistical evidence relative to the less useful evidence used to form

their priors. He observes that it takes two to five observations to do one observation's worth of work in inducing a subject to change his/her opinions. Grether (1980) claims that individuals who exhibit conservatism update their beliefs too slowly in the face of new evidence. Klein (1990), Mendenhall (1991), and Abarbanell and Bernard (1992) further suggest that investors tend to underreact to new information. In terms of the Bayesian rule, conservatism means that people tend to overweigh the base rate (prior) and underweigh new information. This is exactly the approach of the proposed model in this paper.

On the representative heuristic, many experiments (see, for example, DeBondt and Thaler 1985; Lakonishok, et al., 1994; Barberis, et al., 1998) show that individuals expect key population parameters to be "represented" in any recent sequence of generated data (see Tversky and Kahneman, 1971, 1974 for a detailed discussion). Tversky and Kahneman (1971) suggest that local representativeness is a belief in the "law of small numbers," meaning that "the law of large numbers applies to small numbers as well." Investors may find that even small samples (rather than large samples) are highly representative of the populations from which they are drawn. This simply shows that investors may place excessive weights on a sample of small size and neglect distant information unjustifiably. The model proposed in this paper adopts this approach in modeling the law of small sample and in fact, the "smallness" of the sample will be taken into account in our model.

We remark that some key measures of market anomalies like market volatility, autocorrelation of market returns, and trading profit of a self-finance long-short strategy can be expressed in terms of the weights and key financial variables like risk free interest rates. The impact of the incorrect weights on the anomalous magnitudes can hence be quantitatively assessed. In doing so, we can compare the impact of conservatism and representativeness on the anomalous magnitudes. We can also study the interaction between the heuristics and the key financial variable. For example, we can show that market's excess volatility is essentially the result of the "law of small number," and under a reasonable assumption on smallness, volatility can become 28 times that of the volatility attributable to pure information, see Section 3 for further details.

Our behavioral model gives rise to a richer body of consequences than BSV. Other than demonstrating short-run underreaction and long-run overreaction, we also derive excess volatility as a consequence of the behavioral model. Furthermore, we can attribute the excess volatility to the representative heuristic and show that excess volatility is more prominent when the discount rate is small. On overreaction and underreaction, we demonstrate that there exists a magnitude effect in the under-and-overreaction phenomena. In other words, if good/bad news announcements repeatedly occur  $n$  times, the overreaction/underreaction that results increases with  $n$ . Not only we can show that the autocorrelation magnitude and the trading profit increase with  $n$ , our model actually shows that these anomalous magnitudes are a convex function of  $n$ .

The rest of the paper is organized as follows. In Section 2, we construct a pseudo-Bayesian framework to model investors' conservative and representative heuristics and develop price dynamics under this model. In Section 3, we study how the heuristics will impact on market volatility in steady state. We then outline in Section 4 the implications of our proposed model by using it to demonstrate the existence of short-run underreaction and long-run overreaction in the stock market. The trading profit resulting from the corresponding momentum/contrarian trading strategies is also derived and analyzed. In Section 5, we show that our model enables us to derive an additional result that there is a "magnitude effect" associated with the under-and overreaction in the stock market. We further show that the magnitude effect is convex in nature. Section 6 wraps up this paper with a conclusion. Some proofs are provided in the appendices.

## 2. The Model

In BSV, a representative investor observes the earnings of an asset and updates his/her belief to price the asset. It is assumed that  $N_t$ , the earnings announcement of the asset at time  $t$ , follows a random walk, i.e.,  $N_t = N_{t-1} + y_t$  where  $y_t$  is an earnings shock at time  $t$ . Using a discounting model (see, for example, Thompson and Wong, 1991,1996; Wong and Chan, 2004; Nishihara and Fukushima, 2008), the asset is priced at time  $t$  as  $P_t$  given by

$$P_t = E_t \left\{ \frac{N_{t+1}}{1+r} + \frac{N_{t+2}}{(1+r)^2} + \dots \right\} = \frac{N_t}{r} + \frac{1+r}{r} \left\{ \frac{E_t y_{t+1}}{(1+r)^1} + \frac{E_t y_{t+2}}{(1+r)^2} + \dots \right\}, \quad (1)$$

where  $r$  is the discount rate or the investor's anticipated return. In (1),  $E_t$  represents the investor's expectation at time  $t$ . In BSV, they first assume that the earnings shock  $y_t$  is independent and follows a distribution with equal chance on discrete values  $y_0$  or  $-y_0$ . They then assume the representative agent not to realize that the true process for earnings follows a random walk and thus use a wrong model to update his or her beliefs. Finally, they assume that the representative agent uses a correct Bayesian approach to update his or her beliefs. Under these assumptions, BSV deduce that both short-run underreaction and long-run overreaction exist and notice that under- and overreaction occurs as a result of the investor's mis-specification of the model and not from his or her biased updating methods.

In this paper, we propose an alternative approach by assuming that the representative investor is aware of the correct underlying model but fails to adopt a correct approach in the updating process. Brav and Heaton (2002) is the first paper to model an investor as one who misapplies Bayesian methodology. However, they model the representative heuristic only. In this paper, we modify BSV's assumptions as follows:

***Assumption 1:** The earning  $N_t$  follows a random walk model but the earnings shock  $y_t$  is independent and follows a Gaussian distribution with mean  $\mu$  and variance  $\sigma_y^2$ .*

***Assumption 2:** The representative agent knows the nature of the random walk process, except that the parameter  $\mu$  has to be estimated. In other words, the agent has to estimate  $\mu$  by employing observed data on the earnings shock  $\{y_t\}$ . For simplicity and tractability, we also assume that the representative agent knows  $\sigma_y^2$  and adopts a vague prior for  $\mu$ .*

***Assumption 3:** The agent uses a wrong statistical method to update his or her belief. It is the wrong method that reflects the investor's behavioral biases.*

Under our model,  $E_t$  incorporates investors' biases not as the result of a mis-specification of the model but as a result of an incorrect updating method. In this paper, the subjective expectation  $E_t y_{t+i}$  reflects investors' conservative and representative heuristics in estimating the



parameter  $\mu$ . The term  $E_t y_{t+i}$  can be explicitly computed after we describe the pseudo-Bayesian approach in detail in the following subsections.

## ***2.1. A quantitative behavioral model with general weights***

We first describe the correct methodology to update information on the mean level  $\mu$  of the earnings shock. We assume a vague prior for  $\mu$ , i.e.  $P_0(\mu) \propto 1$  (Matsumura, et al., 1990; Wong and Bian, 2000). Let  $y_i$  be the earnings shock observed at the end of period  $i$ ,  $i = 1, 2, \dots, t$ . The posterior distribution of  $\mu$  is given as follows:

$$P(\mu | y_1, \dots, y_t) \propto \prod_{i=1}^t L(y_{t-i+1} | \mu) \cdot 1. \quad (2)$$

Notice that equal weight is placed on each observation in  $y_1, \dots, y_t$  under the Bayesian approach (DeGroot, 1970). Consistent with the predictions of traditional efficient market, this rational expectations asset-pricing theory assumes that investors can have access both to the correct specification of the “true” economic model and to unbiased estimators of its coefficients (Friedman, 1979). Obviously, if the rational investor is endowed with an objectively correct prior and the correct likelihood function, he/she will obtain the rational expectations equilibrium and thus any structural irrationally induced financial anomaly should disappear. If investors do not recognize the effect of learning on prices to obtain equilibrium, Blume and Easley (1982) have shown that convergence of beliefs is not guaranteed within a general equilibrium learning model.

Nonetheless, as evidence has mounted against this traditional Bayesian model, theories of financial anomalies have to be developed by relaxing those assumptions. One approach is to assume that investors are plagued with cognitive biases (Slovic, 1972), and they may incorrectly assign different weights to different observations. To model such behavioral biases, we assume that they place weight  $\omega_1$  on the most recent observation  $y_t$ ,  $\omega_2$  on the second most recent observation  $y_{t-1}$ , and so on, with the possibility that  $\omega_i$ 's may not equal to 1. In other words, we modify the likelihood function as

$$L(\mu) = \prod_{i=1}^t L(y_{t-i+1} | \mu)^{\omega_i} . \quad (3)$$

The posterior distribution then becomes  $P(\mu | y_1, \dots, y_t) \propto \prod_{i=1}^t L(y_{t-i+1} | \mu)^{\omega_i} \cdot 1$ . As a result, the posterior mean and posterior variance of the unknown mean can be obtained and the price dynamic under the behavioral model can be summarized in the following proposition:

**Proposition 1<sup>1</sup>** (*Price dynamic under a pseudo-Bayesian approach*).

1) Under a pseudo-Bayesian approach with a vague prior and an incorrect likelihood  $L(\mu)$  as stated in (3), for any  $k \geq 1$ , the posterior mean  $E_t y_{t+k}$  and posterior variance  $\sigma_t^2$  of  $\mu$  become

$$E_t y_{t+k} = \frac{\omega_t y_1 + \dots + \omega_t y_t}{s_t} \text{ and } \sigma_t^2 = \frac{\sigma_y^2}{s_t}, \text{ respectively where } s_t = \sum_{i=1}^t \omega_i . \quad (4)$$

2) The price at time  $t$  using the rational expectations pricing model in (1) becomes

$$P_t = \frac{\sum_{i=1}^t y_i}{r} + \frac{(1+r)}{r^2} \frac{\omega_t y_1 + \dots + \omega_t y_t}{s_t} . \quad (5)$$

## 2.2. Weight assignment schemes to reflect cognitive biases

In the model above, we incorporate general weights on observations into a simple asset-pricing setup. This allows us to examine the price formation process under a rational expectations approach with biased weights. This approach enables us to quantify investors' cognitive biases and build a quantitative relationship between anomalous asset-price phenomenon and investors' biases. We note that the idea of using different weights on evidences is not new in the finance literature. For example, Brav and Heaton (2002) consider weights given by  $\omega_1 = \dots = \omega_t = 1$ , and  $\omega_{\frac{t}{2}+1} = \dots = \omega_t = 0$ . Under this weighting scheme, investors simply ignore the distant half of

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<sup>1</sup> The proof of this proposition is trivial. We skip reporting the proof but it is available on request.

the available data. Also, it is common in the psychological literature to assume that investors calculate the posterior mean, which is a weighted average rather than a simple average as suggested by a correct Bayesian approach.

In this paper, we use a more general assumption that investors may use weights,  $\omega_1, \omega_2, \dots$ , satisfying  $0 \leq \omega_i \leq 1$  for all  $i$ . By allowing more flexibility in the choice of weights, investors' various behavioral biases can be represented quantitatively. Specifically, in (A), (B), and (C) below, we spell out three weight assignment schemes to characterize the conservative and/or representative heuristics.

(A) *Investors using a conservative heuristic assign weights as:*

$$0 \leq \omega_1 < \omega_2 < \dots < \omega_{n_0} = \omega_{n_0+1} = \dots = 1,$$

(B) *investors using a representative heuristic assign weights as:*

$$1 = \omega_1 = \omega_2 = \dots = \omega_{m_0} > \omega_{m_0+1} > \omega_{m_0+2} > \dots \geq 0, \text{ and}$$

(C) *investors using both conservative and representative heuristics assign weights as:*

$$0 \leq \omega_1 < \omega_2 < \dots < \omega_{n_0} = \omega_{n_0+1} = \dots = \omega_{m_0} = 1 > \omega_{m_0+1} > \dots \geq 0.$$

This weight assignment scheme in (A) is consistent with the psychological literature on conservative heuristics as reviewed in the introduction. Basically, people are over-conservative in that they underweigh recent information and overweigh prior information. The parameter  $n_0$  reflects the conservative heuristic that the most recent  $n_0$  observations are underweighted. If Edwards (1968) is right in that it takes two to five observations to do one observation's worth of work in inducing a subject to change his/her opinions,  $\omega_1, \omega_2, \dots, \omega_{n_0}$  can be substantially less than 1 for  $n_0 \leq 5$ . The smaller are the weights, the more conservative are the investors.

The weight assignment scheme in (B) is consistent with the representative heuristic, as reviewed in the introduction. The representative heuristic in behavioral finance is often described

as the tendency of experimental subjects to overweigh recent clusters of observations and underweigh older observations that would otherwise moderate beliefs. Heavy weights on recent data could be a reaction to concern with structural change. Whenever such change occurs, the weights placed on recent data will be very high or similarly the weight placed on the older data will be very low. The representative heuristic is characterized by a parameter  $m_0$  showing that the investor underweights the observations beyond the most recent  $m_0$  data points. Here, the parameter  $m_0$  arises from the “law of small numbers” in the mind of the investor. Because of their representative heuristic, investors have the tendency to treat a small sample size, like  $m_0$ , as large enough to represent the whole population. So, they assign weights much smaller than 1 for observations beyond the most recent  $m_0$  observations. Also, we assume that  $\sum_{i=m_0+1}^{\infty} \omega_i < \infty$  in our paper to make sure the law of small numbers holds.

Our model formulation asserts that investors are influenced by the conservative and representativeness heuristics simultaneously. In this framework, conservatism and representativeness are not mutually exclusive. The weight assignment scheme in (C) used by investors with both heuristics. When the investor is under the influence of both heuristics, the model has two parameters  $n_0$  and  $m_0$  as described above. Here, conservatism is reflected by the existence of  $n_0 > 0$  and the smallness of the sum  $\omega_1 + \dots + \omega_{n_0-1}$ , and representativeness is reflected by the existence of  $m_0 < \infty$  and the smallness of the sum  $\omega_{m_0+1} + \omega_{m_0+2} \dots$ .

Notice that a type (C) investors degenerate into a type (A) investors when  $m_0 = \infty$  and degenerate into a type (B) investors when  $n_0 = 0$ . Also, when  $m_0 = \infty$  and  $n_0 = 0$ , all weights are equal to 1 and the investor has no behavioral bias. In this sense, the third type of investors embraces all other types. Thus, it suffices to consider investors of the third type. To fully understand the price anomalies that will be introduced when incorrect weights are assigned to the likelihood function, we investigate its implication to market volatility in Section 3 and investigate its implications for under- and overreaction in Sections 4 and 5.

### 3. Model's Implications for Excess Volatility

#### 3.1. Market volatility under the behavioral model

In this section, we study the magnitude of market volatility under our behavioral model in which we define market volatility as the variance of the 1-period return for the asset. Under our behavior model, this volatility is time varying and will reach a steady state when time goes to infinity. To calculate the time-varying volatility with mis-specified weights  $\omega_i$ 's, we recall that the asset price  $P_t$  measured in a log-scale follows a stochastic process given by (5) in which the earnings  $y_i$ 's are i.i.d.  $N(\mu, \sigma^2)$  random variables as stated in Assumption 1. Similarly, the price,  $P_{t+1}$ , at  $t+1$  can be expressed as

$$P_{t+1} = \frac{N_{t+1}}{r} + \frac{1+r}{r^2} \left( \frac{\omega_{t+1}y_1 + \dots + \omega_1y_{t+1}}{s_{t+1}} \right). \quad (6)$$

Thus, the 1-period return,  $R_{t,t+1} = P_{t+1} - P_t$ , from time  $t$  to time  $t+1$  is given by

$$R_{t,t+1} = \frac{1+r}{r^2} \left[ \left( \frac{\omega_{t+1}}{s_{t+1}} - \frac{\omega_t}{s_t} \right) y_1 + \dots + \left( \frac{\omega_2}{s_{t+1}} - \frac{\omega_1}{s_t} \right) y_t \right] + \left( \frac{1}{r} + \frac{1+r}{r^2} \frac{\omega_1}{s_{t+1}} \right) y_{t+1}. \quad (7)$$

Hence, its variance can be expressed as

$$\text{Var}(R_{t,t+1}) = \left( \frac{1+r}{r^2} \right)^2 \left[ \left( \frac{\omega_{t+1}}{s_{t+1}} - \frac{\omega_t}{s_t} \right)^2 + \dots + \left( \frac{\omega_2}{s_{t+1}} - \frac{\omega_1}{s_t} \right)^2 \right] \sigma_y^2 + \left( \frac{1}{r} + \frac{1+r}{r^2} \frac{\omega_1}{s_{t+1}} \right)^2 \sigma_y^2. \quad (8)$$

#### 3.2. Market volatility at steady state

Notice that when  $t$  is small, the investor is still learning about the economic structure. The learning process becomes complete when  $t$  gets large. Since we want to distinguish whether the excess volatility is contributed by learning or by behavioral biases, we study the steady state when  $t$  tends to infinity. When  $s_t \rightarrow \infty$  as  $t \rightarrow \infty$ , one can easily show that

$\lim_{t \rightarrow \infty} \text{Var}(R_{t,t+1}) = \frac{1}{r^2} \sigma_y^2$  which can be regarded as the basic volatility due to information

uncertainty. It is intuitive that information uncertainty is a determinant of price volatility because the information process  $N_t$  is a random walk, and hence, past information will have a permanent price effect. Thus, the most recent earnings shock  $y_t$ , with a variance equal to  $\sigma_y^2$ , will induce volatility in prices. In addition to information uncertainty, volatility may arise because of investors' uncertainty about the values of the valuation-relevant parameters. However, under the condition that  $s_t \rightarrow \infty$  when  $t \rightarrow \infty$ , uncertainty due to parameter estimation will vanish when the system reaches a steady state.

However, the situation is different if there are substantial behavioral biases, characterized by the condition that  $\lim_{t \rightarrow \infty} s_t = s_\infty < \infty$ . Under this condition, true parameter values will never become known to market participants, and hence the condition  $s_\infty < \infty$  introduces another kind of uncertainty to the price process. As a result of the cognitive bias, volatility can also arise from the uncertainty in the valuation-relevant parameter estimation. Proposition 2 below shows how market volatility will be affected by the presence of behavioral biases.

**Proposition 2.** *If we assume that behavioral biases are severe, i.e.,  $\lim_{t \rightarrow \infty} s_t = s_\infty < \infty$ , the market*

*volatility is given by* 
$$\left[ \frac{1}{r^2} + 2 \left( \frac{1}{r} \right) \left( \frac{1+r}{r^2} \right) \frac{s_1}{s_\infty} \right] \sigma_y^2 + \frac{(1+r)^2}{r^4} \left[ \frac{1}{s_\infty^2} \right] A_\infty \sigma_y^2$$

*where* 
$$A_\infty = \omega_1^2 + \sum_{i=1}^{\infty} (\omega_{i+1} - \omega_i)^2 .$$

The proof of Proposition 2 is in the Appendix. In Proposition 2, market volatility is decomposed into two parts: the volatility due to information uncertainty and the volatility arising from behavioral biases. Specifically, the volatility due to information uncertainty at steady state is given by

$$\text{Var}_\infty(\text{Information uncertainty}) = \frac{1}{r^2} \sigma_y^2, \quad (9)$$

and the steady-state volatility attributable to behavioral biases is given by

$$\text{Var}_\infty(\text{behavioral biases}) = \left[ 2 \left( \frac{1}{r} \right) \left( \frac{1+r}{r^2} \right) \left( \frac{s_1}{s_\infty} \right) + \left( \frac{1+r}{r^2} \right)^2 \left( \frac{1}{s_\infty^2} \right) A_\infty \right] \sigma_y^2. \quad (10)$$

It is interesting to compare the volatilities arising from these two different sources by computing their ratio. Dividing (10) by (9), the volatilities ratio is equal to

$$\frac{\text{Var}_\infty(\text{behavioral biases})}{\text{Var}_\infty(\text{information uncertainty})} = 2 \left( \frac{1+r}{r} \right) \left( \frac{s_1}{s_\infty} \right) + \left( \frac{1}{2} \right) \left( \frac{1+r}{r} \right)^2 \left( \frac{1}{s_\infty^2} \right) A_\infty. \quad (11)$$

Expression (11) shows that the ratio depends on the mis-specified weights  $\omega_i$ 's. To have a better idea of the magnitude of this ratio, we consider the following special case:

$$\omega_1 = \dots = \omega_{p_0} = 1 > \omega_{p_0+1} = \dots = 0. \quad \text{Expression (11) then reduces to } 2 \left( \frac{1}{p_0 r} \right) + 2 \left( \frac{1}{p_0 r} \right)^2.$$

Obviously, from this ratio, the percentage of excess volatility increases as  $p_0$  decreases. Since  $p_0$  represents a bias coming from the “law of small numbers,” for this kind of bias, the small sample can be as small as 15 or 30, representing a horizon of 3.75 to 7.5 years of data, considering that the time period from  $t$  to  $t+1$  is a quarter of year. Furthermore, if we assume a quarterly discount rate  $r = 2\%$ , the volatility ratio equals 8.83 when  $p_0 = 30$ . This shows that the volatility attributable to cognitive bias can be much larger than the volatility attributable to information uncertainty. In fact, if the “law of small numbers” operates on an even smaller scale, say,  $p_0 = 15$ , then the volatility ratio can be as large as 28.8.

In addition, from Proposition 2, one could easily obtain the following three interesting observations about excess volatility:

**Observation 1.** *Excess volatility is a decreasing function of the discount rate or investor's anticipated return  $r$ .*

**Observation 2.** *Conservative heuristics will reduce excess volatility.*

**Observation 3.** *Representative heuristics will increase excess volatility.*

## 4. Model's Implications for Under- and Overreaction

### 4.1. Measures of Under- and Overreaction

Overreaction refers to the predictability of good (bad) future returns from bad (good) past performance, while underreaction refers to the predictability of good (bad) future returns from good (bad) past performance (DeBondt and Thaler, 1985; Lakonishok, et al., 1994). In the context of a single asset, under- and overreaction could be demonstrated either through return autocorrelations or through the abnormal return under an event approach (BSV). In this paper, both approaches are employed to illustrate the under-and-overreaction phenomena documented by psychologists. Section 4.1.1 adopts a correlation approach while Section 4.1.2 deals with the same concept using an event approach.

#### 4.1.1. Under- and overreaction in terms of correlation coefficients

Consider the  $k$ -period return  $R_{t,t+k} = P_{t+k} - P_t$  from time  $t$  to time  $t+k$  and the  $k$ -period return  $R_{t-k,t} = P_t - P_{t-k}$  from time  $t-k$  to time  $t$ . The correlation coefficient between these two returns can be interpreted as the lag-one autocorrelation of the  $k$ -period return. Since underreaction is associated with positive autocorrelation and overreaction is associated with negative autocorrelation, we define short-term underreaction and long-term overreaction as follows:

*(I). Prices of a single asset exhibit a short-term underreaction (long-term overreaction) in terms of return correlation if the  $k$ -period return has a positive (negative) lag-one autocorrelation for sufficiently small (large)  $k$ .*

We note that the above definition of under- and overreaction is consistent with the mean reversion phenomena reported by Fama and French (1988), who show that long-holding-period returns are significantly negatively serially correlated. Because of the existence of such negative serial correlation, a large percentage of the variance of large-horizon returns is predictable from past returns. This phenomenon is called mean reversion in the finance literature. Notice that the



short-term underreaction and long-term overreaction as defined above are by no means mutually exclusive. In other words,  $K_1$  and  $K_2$  can exist simultaneously so that the  $k$ -period returns are positively autocorrelated for  $k < K_1$  and are negatively autocorrelated if  $k > K_2$ .

#### 4.1.2. Under- and overreaction under an event approach

In this section, we consider an alternative way to measure under- and overreaction using the event approach by BSV. The market is said to have underreacted when the average return on the company's stock in a period following an announcement of good news is higher than that in a period following an announcement of bad news. However, when pieces of news come in continuing strings, the opposite phenomenon may occur. In other words, the average return following a series of good news announcements turns out to be lower than that following a series of bad news announcements. This is described as the long-term overreaction phenomenon documented in psychology. To quantify such under- and overreaction in the sense used by BSV, we note that the earnings shock,  $y_t$ , provides a measure of how good or bad the earning is. Since the earnings shock  $y_t$  follows a  $N(\mu, \sigma^2)$  distribution, a piece of good (bad) news can be viewed as one in which the earnings shock is larger (smaller) than  $\mu + s\sigma_y$  ( $\mu - s\sigma_y$ ) where  $s > 0$  is a pre-specified constant. Now consider the difference in average returns after string of good or bad news denoted by  $U_t(s, j)$  as follows:

$$U_t(s, j) = E\left\{R_{t+1} \mid y_t > \mu + s \cdot \sigma_y, \dots, y_{t-j+1} > \mu + s \cdot \sigma_y\right\} - E\left\{R_{t+1} \mid y_t < \mu - s \cdot \sigma_y, \dots, y_{t-j+1} < \mu - s \cdot \sigma_y\right\}. \quad (12)$$

In (12),  $j$  represents the time length of the string of good or bad news,  $s > 0$  represents the intensity of the news content, and the quantity  $U_t(s, j)$  represents the expected profit of a momentum trading strategy that dictates buying when there is string of good news and selling when there is string of bad news. On the other hand, if one adopts a contrarian trading strategy of selling when there is string of good news and buying when there is string of bad news, the expected profit of such a contrarian trading strategy is represented by  $-U_t(s, j)$ . If  $U_t(s, j)$  is positive, the momentum trading strategy is profitable resulting from a market underreaction. Otherwise, the contrarian trading strategy is profitable, signifying the existence of an

overreaction. A formal definition of short-term underreaction and long-term overreaction can now be given as follows:

**(II).** *Prices exhibit a short-term underreaction (long-term overreaction) if  $U_t(s, j) > (<)0$  for sufficiently small (large)  $j$ .*

Notice that the short-term underreaction and long-term overreaction as defined above are by no means mutually exclusive. Just like the existence of constants  $K_1$  and  $K_2$  in the previous section, constants  $J_1$  and  $J_2$  can co-exist, so that  $U_t(s, j) > 0$  for  $j < J_1$  and  $U_t(s, j) < 0$  for  $j \geq J_2$ .

#### **4.2. Under- and overreaction in the presence of behavioral biases**

In this section, we assume that the representative investor possesses both conservative and representative heuristics and assigns weights to data as described by (C) in Section 2. We will show in Proposition 3 that asset prices will exhibit underreaction in the short run and overreaction in the long run, where under- and overreaction is measured by return autocorrelations. Proposition 3 is important because it shows that our behavioral model implies that returns are predictable, a well-documented market anomaly in the finance literature. In Proposition 3, predictability results even after the system has reached steady state and hence it does not arise only from the investors' learning process. We then demonstrate short-term underreaction and long-term overreaction phenomena using an event approach in Proposition 4.

**Proposition 3.**<sup>2</sup> *If investors possess both conservative and representative heuristics, then prices exhibit short-term underreaction and long-term overreaction in terms of return autocorrelations (see I in Section 4.1.1). Specifically, there exist positive integers  $K_1$  and  $K_2$  such that for*

$$\text{sufficiently large } t, \text{ we have } \begin{cases} \text{Corr}(R_{t-k,t}, R_{t,t+k}) > 0 & \text{for } k \leq K_1, \\ \text{Corr}(R_{t-k,t}, R_{t,t+k}) < 0 & \text{for } k > K_2. \end{cases}$$

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<sup>2</sup> As Proposition 3 is a straightforward consequence of the weight assignments, we skip its proof in our paper.

Furthermore, the correlation coefficients above is non-trivial for sufficiently large  $t$ , i.e. the limiting correlation coefficients for  $t \rightarrow \infty$  is non-zero.

This proposition reveals the short-term underreaction phenomenon that the correlation of stock returns are positively correlated (DeBondt and Thaler, 1985) and long-term overreaction phenomenon that the correlation of stock returns are negative correlated (Fama and French, 1996) by the conservative and representative heuristics.

As explained in Section 4.1.2, under- and overreaction can also be treated using an event approach in which under- and overreaction is measured by the expected momentum profit  $U_i(s, j)$  defined in (12). We will show in Proposition 4 below that, for a representative investor of type (C), an investor's  $U_i(s, j)$  is positive when  $j$  is small and is negative when  $j$  is large. In other words, momentum trading is profitable on a short run of good or bad news but contrarian trading is profitable on a long-run of good or bad news.

**Proposition 4.** *If investors possess both conservative and representative heuristics modeling by weight assignment scheme (C), prices exhibit short-term underreaction and long-term overreaction using an event approach (see II in Section 4.1.2). Specifically, there exist integers*

$J_1$  and  $J_2$  such that for given  $s > 0$  and for large  $t$ , we have 
$$\begin{cases} U_i(s, j) > 0 & \text{for } j \leq J_1, \\ U_i(s, j) < 0 & \text{for } j > J_2. \end{cases}$$

Furthermore, the expected momentum trading profit  $U_i(s, j)$  is non-trivial when  $t$  tends to infinity, i.e., the limiting trading profit is non-zero for  $t \rightarrow \infty$ .

The proof of Proposition 4 is in the appendix. From the proof of Proposition 4, we can see that the representative heuristic contributes to the contrarian profit, while the conservative heuristic contributes to the momentum profit, and Proposition 4 links investors' irrational cognitive biases to financial anomalies of overreaction and underreaction. It shows that overreaction occurs after long-run periods of good or bad performance, whereas underreaction happens after short-run periods of good or bad performance. In addition to demonstrating the existence of overreaction in the long run, Proposition 4 also provides good insights into how the

contrarian/momentum profits arise. The representative heuristic has to overpower the conservative heuristic for a contrarian profit to surface. The long-run assumption is necessary for a contrarian profit because under a long-run situation the representativeness bias will become noticeable.

Another interesting observation is that both momentum/contrarian profits are sensitive to the discount rate  $r$ . The smaller the discount rate, the larger the momentum/contrarian profits. This is because when  $r$  is small, future cash flows become important, and a mis-estimation of future cash flows will intensify the over- or underreaction phenomena.

## **5. Model's Implications for the Magnitude Effect**

### ***5.1. Existence of a magnitude effect***

In this section, we will provide theoretical support for the magnitude effect of the under-and-overreaction hypotheses. In an early paper on the overreaction hypothesis, DeBondt and Thaler (1985) write “Specifically, two hypotheses are suggested: (1) extreme movements in stock prices will be followed by subsequent price movements in the opposite direction. (2) The more extreme the initial price movement, the greater will be the subsequent adjustment.” While (1) is usually referred to as the overreaction hypothesis, and (2) states that overreaction does have a magnitude effect. In this paper, we show that not only can our quantitative behavioral model provide theoretical support to (1) but it can also be used to demonstrate that the resulting under- and overreaction does entail a magnitude effect as specified in (2). Recall that the definition of  $U_i(s, j)$  in (12) stands for the expected profit of the momentum trading strategy. Observe that both parameters  $s$  and  $j$  represent an “event magnitude.” For the parameter  $s$ , the larger is  $s$ , the more extreme is the earnings shock, and the more extreme is the event under study. On the other hand, the parameter  $j$  represents another dimension of “event magnitude.” If  $j$  is large, the event consists of a bigger clustering of good or bad news and the event becomes more

extreme as  $j$  gets larger. Thus, the “magnitude effect” associated with the under- or overreaction may have two meanings:

- (1) the momentum (contrarian) profit  $U_t(s, j)$  ( $-U_t(s, j)$ ) increases as  $s$  increases,
- (2) the momentum (contrarian) profit  $U_t(s, j)$  ( $-U_t(s, j)$ ) increases as  $j$  increases.

We demonstrate in Propositions 5 and 6 that both magnitude effects exist.

**Proposition 5** *(a magnitude effect in  $s$ ). If investors possess both conservative and representative heuristics, both the long-term overreaction and the short-term underreaction established in Proposition 4 will exhibit a magnitude effect in  $s$ . Specifically, there exist integers  $J_1$  and  $J_2 > 0$  such that*

- (a) *for sufficiently small  $t$ , and for  $j \leq J_1$ , the momentum profit  $U_t(s, j)$  is positive and is monotonically increasing with  $s$ ; and*
- (b) *for sufficiently large  $t$ , and for  $j \geq J_2$ , the contrarian profit  $-U_t(s, j)$  is positive and is monotonically increasing with  $s$ .*

**Proposition 6** *(a magnitude effect in  $j$ ).*

- (1) *When  $j$  is sufficiently large, the contrarian profit based on  $j$  consecutive good or bad news increases as  $j$  increases.*
- (2) *When  $j$  is sufficiently small, the momentum profit based on  $j$  consecutive good or bad news increases as  $j$  decreases.*

The proof of Proposition 5 is in the appendix. The proof of Propositions 6 is similar to that of Proposition 5 and is skipped in this paper. We note that the conditions in Propositions 5 are equivalently to the conditions in which both the momentum and contrarian strategies of trading on string of good or bad news will have a profit that will increase with the magnitude of the impact of the news.

## 5.2. Convexity in the magnitude effect

In Section 5.1, we demonstrate that there is a magnitude effect in the under-and-overreaction phenomena, in the sense that momentum/contrarian trading profit increases with the magnitude of the earnings shock. In this section, we go one step further to show in Proposition 7 that when  $s$  is used as a magnitude measure, the magnitude effect is convex in nature. For example, when magnitude doubles, the momentum/contrarian trading profit is more than doubled.

**Proposition 7.** *If investors possess both conservative and representative heuristics, the momentum/contrarian trading profit  $U_t(s, j)$  is a convex function in  $s$ .*

The proof of Proposition 7 is in the Appendix.

## 6. Concluding Remarks

We posit that some investors possess conservative and/or representative heuristics that lead them to underweigh recent observations and/or underweigh past observations in the earnings shocks of stock prices. We introduce a quantitative pseudo-Bayesian approach to model such investors' behavior. Compared with other behavioral models in which investors possess either conservative heuristics at one time or representative heuristics at another time but not both, our specification captures the essential feature of having a parsimonious model that allows investors to possess conservative or representative heuristics at the same time.

This paper develops a model of weight assignments using a pseudo-Bayesian approach to reflect investors' behavioral biases. In this parsimonious model of investors' sentiment, weights induced by investors' conservative and representative heuristics are assigned to observations of the earnings shocks of stock prices. Our behavioral model provides a quantitative link between some market anomalies and investors' behavioral biases. While learning may contribute to market anomalies, anomalies still exist even after the learning process has been completed. In particular, we can deduce the following: (1) Excess market volatility will result from investors' biased heuristics. The representative heuristic, rather than the conservative heuristic, contributes

to excess volatility in the market. Excess volatility is more prominent when the discount rate is small. (2) Through a misapplication of Bayes' rule, investors' behavioral biases lead to short-term underreaction and long-term overreaction in the markets. The more conservative/representative the heuristic, the larger is the magnitude of the return auto-correlation. Further analysis shows that the representative heuristic contributes to the contrarian trading profit and the conservative heuristic contributes to the momentum profit. The smaller the discount rate, the larger the contrarian/momentum profit. (3) Investors' behavioral biases induce a magnitude effect in the under-and-overreaction phenomena documented in psychology, i.e., the more severe the earning shock, the larger the market autocorrelation and the larger the momentum/contrarian trading profit. (4) The magnitude effect described in (3) is convex in nature. This paper studies behaviors of investors with conservative and/or representative heuristics. Further study could include studying the behaviors of investors who assign weights according to different criteria without following a particular and rigid rule.<sup>3</sup>

At last, we note that, recently, Fong et al. (2005), Wong et al. (2006), and Sriboonchita et al. (2009) incorporate stochastic dominance, utility maximization and mean-variance approaches to obtain another resolution to explain the under-and-overreaction phenomena. Further research could include incorporating our approach and their approach to find a better explanation of the under-and-overreaction phenomena. In addition, we note that our approach could be benefited from the fuzzy logic theory in the development of the heuristic model. One could call our approach to be intelligent model by using fuzzy logic approach to focus on behavioral biases incorporated in it.<sup>4</sup> Further research could also incorporate fuzzy logic approach to our approach. In addition, we note that Lam et al. (2008) have developed new test statistics to test the magnitudes of the under-and-overreaction phenomena in the international markets and confirm that there exist the magnitudes effects in the under-and-overreaction phenomena.

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<sup>3</sup> We would like to show our appreciation to the anonymous referee for this suggestion.

<sup>4</sup> We would like to show our appreciation to the anonymous referee for telling us this information.

## Appendix

### Proof of Proposition 2:

As  $t \rightarrow \infty$ ,  $s_{t+i} \rightarrow s_\infty$  for all positive  $i$ . From (8) and the definition of  $s_t$ , we have

$$\begin{aligned} \text{Var}_\infty &= \left(\frac{1+r}{r^2}\right)^2 \left[ \left(\frac{\omega_{t+1}}{s_\infty} - \frac{\omega_t}{s_\infty}\right)^2 + \dots + \left(\frac{\omega_2}{s_\infty} - \frac{\omega_1}{s_\infty}\right)^2 \right] \sigma_y^2 + \left[ \frac{1}{r^2} + \frac{2\omega_1}{rs_\infty} \left(\frac{1+r}{r^2}\right) + \frac{\omega_1^2}{s_\infty^2} \left(\frac{1+r}{r^2}\right)^2 \right] \sigma_y^2 \\ &= \frac{1}{r^2} \sigma_y^2 + \frac{2s_1}{rs_\infty} \left(\frac{1+r}{r^2}\right) \sigma_y^2 + \left(\frac{1+r}{r^2 s_\infty}\right)^2 \left[ \omega_1^2 + (\omega_2 - \omega_1)^2 + (\omega_3 - \omega_2)^2 + \dots \right] \sigma_y^2. \quad \square \end{aligned}$$

### Proof of Proposition 4:

Before we prove Proposition 4, we state Lemma 1 as follows:

**Lemma 1.**  $U_t(s, j) = 2\sigma_y \frac{1+r}{r^2} [\Delta(t, j)] D(s)$ , where  $\Delta(t, j) = \frac{s_{j+1} - s_1}{s_{t+1}} - \frac{s_j}{s_t}$ ,

$D(s) = E(Z|Z > s)$ , and  $Z$  is a standard normal random variable with mean zero and unit standard deviation.

**Proof of Lemma 1:** Denoting the return from period  $t$  to  $t+1$  by  $R_{t+1} = R_{t,t+1}$ , from (7) we have

$$R_{t+1} = \frac{1+r}{r^2} \frac{1}{s_t s_{t+1}} \left[ (s_t \omega_{t+1} - s_{t+1} \omega_t) y_1 + \dots + (s_t \omega_2 - s_{t+1} \omega_1) y_t + \left( s_t \omega_1 + \frac{r}{1+r} s_t s_{t+1} \right) y_{t+1} \right].$$

Let  $\frac{y_t - \mu}{\sigma_y} = z_t$  and  $z_t$ 's are  $N(0,1)$  variables.  $R_{t+1}$  can now be expressed in terms of  $z_t$ 's as

follows:

$$R_{t+1} = \frac{1+r}{r^2} \frac{\sigma_y}{s_t s_{t+1}} \left[ (s_t \omega_{t+1} - s_{t+1} \omega_t) z_1 + \dots + (s_t \omega_2 - s_{t+1} \omega_1) z_t + \left( s_t \omega_1 + \frac{r}{1+r} s_t s_{t+1} \right) z_{t+1} + K_t \right]$$



where  $K_t = \frac{\mu}{\sigma_y} \left[ (s_t \omega_{t+1} - s_{t+1} \omega_t) + \dots + (s_t \omega_2 - s_{t+1} \omega_1) + \left( s_t \omega_1 + \frac{r}{1+r} s_t s_{t+1} \right) \right] = \frac{\mu}{\sigma_y} \left[ \frac{r}{r+1} \right] s_t s_{t+1}$ .

Hence, we have

$$\begin{aligned}
& E \left[ R_{t+1} \mid y_t > \mu + s\sigma_y, \dots, y_{t-j+1} > \mu + s\sigma_y \right] = E \left[ R_{t+1} \mid z_t > s, \dots, z_{t-j+1} > s \right] \\
& = \frac{1+r}{r^2} \frac{\sigma_y}{s_t s_{t+1}} \left[ 0 + \dots + 0 + E \left( (s_t \omega_{j+1} - s_{t+1} \omega_j) z_{t-j+1} \mid z_{t-j+1} > s \right) + \dots + E \left( (s_t \omega_2 - s_{t+1} \omega_1) z_t \mid z_t > s \right) + K_t \right] \\
& = \frac{1+r}{r^2} \frac{\sigma_y}{s_t s_{t+1}} \left[ K_t + (s_t \omega_{j+1} - s_{t+1} \omega_j) E \left( z_{t-j+1} \mid z_{t-j+1} > s \right) + \dots + (s_t \omega_2 - s_{t+1} \omega_1) E \left( z_t \mid z_t > s \right) \right] \\
& = \frac{1+r}{r^2} \frac{\sigma_y}{s_t s_{t+1}} \left[ K_t + \left( s_t (\omega_2 + \dots + \omega_{j+1}) - s_{t+1} (\omega_1 + \dots + \omega_j) \right) D(s) \right] = \frac{1+r}{r^2} \sigma_y \left[ \frac{K_t}{s_t s_{t+1}} + \Delta(t, j) D(s) \right]
\end{aligned}$$

where  $\Delta(t, j) = \frac{s_{j+1} - s_1}{s_{t+1}} - \frac{s_j}{s_t}$ . Similarly, from the symmetry of  $Z$ , one can show that

$$E \left( R_{t+1} \mid y_t < \mu - s\sigma_y, \dots, y_{t-j+1} < \mu - s\sigma_y \right) = \frac{1+r}{r^2} \sigma_y \left[ \frac{K_t}{s_t s_{t+1}} - \Delta(t, j) D(s) \right].$$

By subtraction, we have  $U(s, j) = 2\sigma_y \frac{1+r}{r^2} (\Delta(t, j)) D(s)$ . This completes the proof of the lemma.

Now we come back to the proof of Proposition 4. Since  $D(s) > 0$  for  $s > 0$ , the sign of  $U_t(s, j)$

depends on the sign of  $\Delta(t, j) = \frac{s_{j+1} - s_1}{s_{t+1}} - \frac{s_j}{s_t}$ . As  $t$  tends to infinity,

$$\Delta(t, j) \rightarrow \frac{s_{j+1} - s_1 - s_j}{s_\infty} = \frac{\omega_{j+1} - \omega_1}{s_\infty}.$$

In addition, since the conservative heuristic guarantees that  $\omega_1 < \omega_2 < \dots < \omega_{n_0}$ ,  $\Delta(t, j)$  is positive for small values of  $j$ . On the other hand, the representative heuristic imposes decreasing

weights when  $j$  is large. Also, under the assumption that  $\sum_{i=1}^{\infty} \omega_i = s_{\infty} < \infty$ ,  $\omega_j \rightarrow 0$  as  $j \rightarrow \infty$ .

As a consequence, if  $j$  is large enough,  $\omega_{j+1} < \omega_j$  and the sign of  $\Delta(t, j)$  becomes negative.

This completes the proof of Proposition 4.  $\square$

### Proof of Proposition 5:

Let  $\varphi$  and  $\Phi$  are the pdf and cdf of  $Z$ , a standard normal random variable with mean zero and unit standard deviation, respectively. Before we prove proposition 5, we first prove the Property that  $\varphi(a) - a + a\Phi(a) > 0$  as follows:

$$\text{Since } E(Z - a \mid Z \geq a) = \frac{\int_a^{\infty} x\varphi(x)dx - a \int_a^{\infty} \varphi(x)dx}{1 - \Phi(a)} = \frac{\varphi(a) - a + a\Phi(a)}{1 - \Phi(a)}, \text{ we have}$$

$\varphi(a) - a + a\Phi(a) > 0$  because the conditional expectation is obviously positive.

Now, we come back to prove proposition 5. From Lemma 1, we have

$$U_i(s, j) = 2\sigma_y \left(\frac{1+r}{r^2}\right) \Delta(t, j) D(s). \text{ Hence, } \frac{d}{ds} U_i(s, j) = 2\sigma_y \frac{1+r}{r^2} \Delta(t, j) D'(s).$$

To demonstrate a magnitude effect in the under-or-overreaction phenomenon (depending on the sign of  $\Delta(t, j)$ ), it suffices to show that  $D'(s) > 0$ . The proof that  $D(s)$  is monotonically increasing in  $s$  is shown as follows:

$$\text{Conditional on } Z > s, Z \text{ has a p.d.f } \frac{\varphi(x)}{1 - \Phi(s)} \text{ for } x > s. \text{ Hence, } D(s) = E(Z \mid Z > s) = \frac{\int_s^{\infty} x\varphi(x)dx}{1 - \Phi(s)}.$$

Differentiating the above, we have

$$D'(s) = \frac{(-s\varphi(s))(1-\Phi(s)) - \varphi(s)^2}{(1-\Phi(s))^2} = \frac{\varphi(s)[s\Phi(s) + \varphi(s) - s]}{(1-\Phi(s))^2}.$$

This expression is positive because  $s\Phi(s) + \varphi(s) - s > 0$ . This completes the proof of Proposition

5. □

### Proof of Proposition 7:

It suffices to show that  $D(s)$  is a convex function in  $s$ . To prove the convexity of  $D(s)$  is equivalent to prove  $D''(a) > 0$ .

Since  $D'(a) = \frac{(-a\varphi(a))(1-\Phi(a)) - \varphi(a)^2}{(1-\Phi(a))^2}$ , we have

$$\begin{aligned} D''(a) &= \frac{(1-\Phi(a))^2 ((2\varphi(a))(-a\varphi(a)) - \varphi(a)^2 (2)(1-\Phi(a))(-\varphi(a)))}{(1-\Phi(a))^4} \\ &\quad - \frac{((1-\Phi(a))(a(-a\varphi(a)) + \varphi(a)) - a\varphi(a)(-\varphi(a)))}{(1-\Phi(a))^2} \\ &= \frac{-2a\varphi(a)^2 (1-\Phi(a))^2 + 2\varphi(a)^3 (1-\Phi(a))}{(1-\Phi(a))^4} - \frac{(1-\Phi(a))(\varphi(a)(1-a^2) + a\varphi(a)^2)}{(1-\Phi(a))^2} \\ &= \frac{-2a\varphi(a)^2}{(1-\Phi(a))^2} + \frac{2\varphi(a)^3}{(1-\Phi(a))^3} - \frac{(1-a^2)\varphi(a)}{1-\Phi(a)} - \frac{a\varphi(a)^2}{(1-\Phi(a))^2} \\ &= \frac{2\varphi(a)^2}{(1-\Phi(a))^3} \left[ -a(1-\Phi(a)) + \varphi(a) \right] - \left( \frac{(1-a^2)\varphi(a)(1-\Phi(a)) + a\varphi(a)^2}{(1-\Phi(a))^2} \right). \end{aligned}$$

By the result in Proposition 5, the first term is positive for all  $a$ . For the second term, when  $a > 1$ , it is positive because  $1-a$  is negative. Thus, it follows that  $D''(a) > 0$  when  $a > 1$ . □

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