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Chiu, Sung Nok

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Reduced Palm distribution generating function for a spatial point process

Sung Nok Chiu

Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong.

E-mail: snchiu@hkbu.edu.hk

Abstract

The generating function of a marginal distribution of the reduced Palm distribution of a spatial point process is considered. It serves as a bivariate summary function, providing more information than some other popular univariate summary functions such as the reduced second moment function and the nearest neighbour distance distribution function. Simulation confirmed that the new summary function is more informative when applied to patterns that exhibit both clustering and regularity on the same scale of observation.

Keywords: Generating function; Reduced Palm distribution; Spatial point process.

1 Introduction

In spatial point process statistics, Ripley's K-function (Ripley, 1976), also known as the reduced second moment function, is a very popular summary function. For a stationary point process with intensity λ, $\lambda K(r)$ is the mean number of points within a distance of r from the typical point. However, second order statistics, together with first order statistics such as the intensity, do not uniquely determine the distribution of a point process. The cell process (Baddeley and Silverman, 1984) is an example of a non-Poisson point process that has the same intensity and the same K-function as a stationary Poisson process; the cell process exhibits regularity and clustering on the same scale of observation simultaneously.

Naturally, third order statistics have been proposed. A straightforward generalisation of the K-function is the reduced third moment function, which is useful for directional analysis (Hanisch, 1983). Interpreting the K-function as the mean number of r-close pairs of points per unit volume, Schladitz and Baddeley (2000) generalised the K-function to the mean number of r-close triples. Møller *et al.* (1998) suggested another third order statistic to distinguish log Gaussian Cox processes from others.

Grabarnik and Chiu (2002) constructed a Pearson-like goodness-of-fit statistic $Q^2(r)$ for the intensity $\lambda_k(r)$ of points having k r-close neighbours, for $k = 0, 1, \ldots$, and showed that the Q^2-statistic is sensitive to patterns which are mixtures of regular and clustered processes. However, unlike the K-function, a large value of Q^2-statistic merely indicates deviation from the null model but provides no direct insight about its nature. Nevertheless, the functions $\lambda_k(\cdot)$, which are closely related to the reduced Palm distribution of the point process, give a fine description of the point process. This paper discusses the generating function of the sequence $\{\lambda_k(\cdot)/\lambda\}$ and shows how this generating function can be used in exploratory analysis.

2 Generating function

Suppose Φ is a stationary point process in $\mathbb{R}^d$ with intensity λ. Denote by $P_o^!$ its reduced Palm distribution (Stoyan *et al.*, 1995, p. 121) and by $\lambda_k(r)$ the intensity of points having k r-close neighbours. They are related by the following equation:

$$\lambda_k(r) = \lambda P_o^! \{\varphi : \varphi(b(o,r)) = k\} \quad \text{for } r \geq 0,$$

where $b(o,r)$ is the closed ball centred at o with radius r. Let $p_r(k) = P_o^!\{\varphi : \varphi(b(o,r)) = k\}$. Denote by $A(\cdot, r)$ the probability generating function of $\{p_r(k) : k \geq 0\}$, i.e.

$$\lambda A(s,r) = \sum_{k=0}^{\infty} s^k \lambda_k(r) \quad \text{for } r \geq 0,\ |s| < 1.$$

We call the function $A(\cdot,\cdot)$ the reduced Palm distribution generating function.

Alternatively, let $\Phi_o^!$ denote the reduced Palm version of Φ, i.e. $\Phi_o^! \sim P_o^!$. Then the function A can be defined by

$$A(s,r) = E(s^{\Phi_o^!(b(0,r))}).$$

In particular, if Φ is Poisson, then $p_r(k) = (\lambda\omega_d r^d)^k \exp(-\lambda\omega_d r^d)/k!$ and so

$$A(s,r) = \exp\{-\lambda\omega_d r^d(1-s)\}, \quad \text{for } r \geq 0,\ |s| < 1,$$

where $\omega_d = \sqrt{\pi^d}/\Gamma(1+d/2)$ is the volume of a unit ball in $\mathbb{R}^d$. This can also be obtained easily by noting that $\Phi_o^!$ has the same distribution as the original Poisson process Φ.

For a binomial point process of n points in a rectangular region W with periodic boundary condition, $p_r(k) = \binom{n-1}{k}(\omega_d r^d)^k\{\text{vol}(W) - \omega_d r^d\}^{n-1-k}/\text{vol}(W)^{n-1}$ and so

$$A(s,r) = \left\{1 - \frac{\omega_d r^d(1-s)}{\text{vol}(W)}\right\}^{n-1}, \quad \text{for } 0 \leq r \leq \{\text{vol}(W)/\omega_d\}^{1/d},\ |s| < 1,$$

where $\text{vol}(\cdot)$ denotes the volume measure.

For a stationary Markov process Φ with intensity λ and Papangelou conditional intensity λ^* (Møller and Waagepetersen, 2004, p. 250), we have

$$A(s,r) = \frac{E(s^{\Phi(b(o,r))}\lambda^*(\Phi, o))}{\lambda}, \quad \text{for } r \geq 0,\ |s| < 1.$$

The primary purpose of combining the information of all $\{\lambda_k(\cdot)\}$ into the generating function $A(\cdot,\cdot)$ is to get finer information for exploratory analysis than other summary functions or the Q^2-statistic on the spatial structure of point patterns. If the aim is to test the goodness-of-fit of a certain spatial point process model, such as the stationary Poisson process, then it would be more natural to use the asymptotically χ^2-distributed Q^2-statistic (Grabarnik and Chiu, 2002),

$$Q^2(r) = (m-\mu)'\Sigma^{-1}(m-\mu),$$

where $m = (m_1, \ldots, m_q)'$, in which m_k is the number of points of Φ in a bounded $B \subset \mathbb{R}^d$ such that their numbers of r-close neighbours are exactly k, $\mu = \mathrm{vol}(B)(\lambda_1(r), \ldots, \lambda_k(r))$ and Σ^{-1} is the inverse of the covariance matrix of the vector m. The dimension q is a user-chosen parameter. The Q^2-statistic, therefore, tells us only how much, but not how, the observed intensities of points having k r-close neighbours, for $k = 0, 1, \ldots$, deviated from $\{\lambda_k(r)\}$.

3 Edge-corrected estimation

Denote by $\{x_1, x_2, \ldots\}$ a realisation of Φ observed in the sampling window W. The estimation of $\lambda A(s,r)$ can be done by estimating the intensities $\lambda_k(r)$. The simplest unbiased estimator, obtained by the border method, is the empirical intensity of points lying in the eroded window $W \ominus b(o,r)$ and having k r-close neighbours, where $\ominus$ is the Minkowski subtraction. A variant of this approach is to estimate $p_r(k)$ by the empirical proportion in the eroded window, leading to an unbiased estimator of $A(s,r)$. If W is rectangular, the periodic boundary condition may also be used.

More sophisticated edge-correction can be done by estimating the distribution function $D_k(r) = P_o^!\{\varphi : \varphi(b(o,r)) \geq k\}$ of the kth nearest neighbour distance, because

$$\lambda_k(r) = \lambda\{D_k(r) - D_{k+1}(r)\}. \tag{1}$$

An edge-corrected pointwise unbiased estimators for $\lambda D_k(r)$ was suggested by Hanisch (1984, equation (4)) and Stoyan and Stoyan (1994, equation (15.30)):

$$\widehat{\lambda D_k(r)} = \sum_i \frac{\mathbf{1}\{x_i \in W \ominus b(o, d_{k,i})\}}{\mathrm{vol}\{W \ominus b(o, d_{k,i})\}} \mathbf{1}(d_{k,i} \leq r), \quad k = 1, 2, \ldots, \tag{2}$$

where $d_{k,i}$ is the distance from x_i to its kth nearest neighbour. If n points are observed in W, $\widehat{\lambda D_k(r)} = 0$ for $k \geq n$.

Because $\lambda^2 K(r) = \sum k\lambda_k(r)$, equations (1) and (2) immediately lead to a new estimator for $\lambda^2 K(r)$:

$$\widehat{\lambda^2 K(r)} = \sum_i \sum_{j \neq i} \frac{\mathbf{1}\{x_i \in W \ominus b(o, \|x_i - x_j\|)\}}{\mathrm{vol}\{W \ominus b(o, \|x_i - x_j\|)\}} \mathbf{1}(\|x_i - x_j\| \leq r).$$

This estimator resembles the edge-corrected estimators by isotropic or translational correction (Stoyan *et al.*, 1995, pp. 134–137).

Other possible edge-corrected estimators for D_k include the Kaplan–Meier type estimators suggested by Baddeley and Gill (1997):

$$\widehat{D_k(r)} = 1 - \prod_{s \le r} \left[1 - \frac{\sum_i \mathbf{1}(d_{k,i} = s)\mathbf{1}\{x_i \in W \ominus b(o,s)\}}{\sum_i \mathbf{1}(d_{k,i} \ge s)\mathbf{1}\{x_i \in W \ominus b(o,s)\}} \right].$$

4 Simulation

Patterns exhibiting clustering and regularity, respectively, will have their A-functions lying below and above the A-function for a completely spatially random pattern. However, K and D_1 are also very successful in discriminating regularity and clustering from complete spatial randomness. Nevertheless, Grabarnik and Chiu (2002) showed that K and D_1 may not be so successful if the empirical patterns exhibit both regularity and clustering on the same scale of observation.

Here we consider such a mixed process: a fixed number n_1 of points are placed in the unit square according to conditional Poisson cluster process (Diggle, 1979, p. 93), with periodic boundary condition, where the n_1 points are assigned to one of n_c clusters uniformly and the distances to the centres of the corresponding clusters is uniformly distributed on $(0, r_{cl})$, and they are followed by $n - n_1$ points according to the simple sequential inhibition process (Diggle *et al.*, 1976, p. 663) with minimum distance r_{SSI}.

For ease of implementation, let us estimate $p_r(k)$ by the corresponding empirical proportion in the eroded window and all other summary functions were also estimated by the border method. Using the same edge-correction for all summary functions provides a fair comparison. The purpose of this simulation study is to establish the advantages of $A(\cdot, \cdot)$ and so we do not compare different edge-correction methods. Interested readers may find such comparison studies for the estimation of K and D_1 in e.g. Stoyan (2006) and Tscheschel and Chiu (2007).

Figure 1 shows the means and the upper and lower envelopes of the estimates of the L-function, where $L(r) = \sqrt{K(r)/\pi}$, the nearest neighbour distance distribution function $D = D_1$, the empty space function $F(r) = \Pr\{\varphi : \varphi(b(o,r)) \ge 1\}$, the J-function (van Lieshout and Baddeley, 1996), where $J(r) = \{1 - D(r)\}/\{1 - F(r)\}$ for $F(r) > 1$, and cross-sections of the A-function obtained from 10 simulated independent realisations of such a process with $n = 100$, $n_1 = 45$, $n_c = 20$, $r_{cl} = 0.025$ and $r_{SSI} = 0.05$. The corresponding theoretical means and the upper and lower envelopes from 99 independent simulated realisations of the binomial process with the same number of points are also shown. We can see that the empirical means of the L, D, F and J of the mixed process exceeded the corresponding envelopes of the binomial process around r_{cl}, revealing the clustering of the mixed process, but they all failed to offer strong evidence for the regularity. However, cross-sections of the mean of the estimates of the A-function show that A could capture both clustering and regularity, respectively, around r_{cl} and r_{SSI}.

Fixing the total number n and reducing the number n_1 of points of the cluster process, we will get less clustered and more regular patterns. Figure 2 shows the results for such patterns with $n_1 = 36$. The number of clusters is also reduced to $n_c = 16$, so that the mean number of points per cluster remains the same as in the previous simulation. We can see that the empirical mean of L still revealed the clustering only, whilst that of D and J showed evidence not for clustering but for regularity; the empirical mean of F was lying between the envelopes obtained from 99 realisations of the binomial process. On the other hand, cross-sections of the mean of the empirical A-functions again captured both clustering and regularity.

From these two examples, we can see that the advantage of the A-function over the other popular univariate summary functions. One reason is that, instead of a curve depending on r, the A-function is a surface depending on the distance r as well as the argument of the generating function s, which is a parameter controlling the weight given to each $p_r(k)$; the smaller the absolute value of s, the faster the weight s^k decays, leading to the weighting scheme that the larger the k, the smaller the weight assigned to $p_r(k)$. Thus, there is no recommended value for s; for a fixed s, $A(s, \cdot)$ describes the behaviour of a weighted sum of $p_r(k)$, whereas for a fixed r, $A(\cdot, r)$ provides a summary of $p_r(k)$, under different weighting scheme, for detecting possible interaction between points with inter-point distance not greater than r. We simply use contour plots or cross-sections to see whether there is evidence for clustering or regularity. The plot of the differences between the mean of the empirical A from the same ten realisations as in Figure 1 and the lower and upper envelopes of 99 independent realisations of the binomial process in Figure 3 shows clearly that the A-function is able to reveal the clustering around $r = r_{cl}$ and the regularity around $r = r_{SSI}$.

Tables 1–3 show the estimated powers of the summary functions for detecting clustering and regularity in the mixed process against the binomial process with the same number of points under various parameter values for $n = 25$, 50 and 100. The presence of regularity (clustering, respectively) is detected if the estimated A, F or J does not lie entirely below (above) the upper (lower) envelope of 99 independent realisations of the binomial process, or the estimated K or D does not lie entirely above (below) the lower (upper) envelope. The powers are estimated by 100 simulated realisations of the mixed process. We can see from these tables that if n is small, none of these summary functions can detect the presence of both clustering and regularity in individual realisations, but when n is large, the A function is more able to detect the presence of both in individual realisations. Moreover, the A-function is not worse than any one of K, D, F and J in their ability to capture clustering. For pure regularity, the J-function is the most sensitive one but its superiority in detecting regularity is lost when a pattern exhibits both clustering and regularity. However, for pure clustered patterns with only a small number of clusters and small cluster radii, the A-function occasionally gave false alarm for the regularity. The reason may be that in such patterns these dense clusters sometimes may be so far apart that the number of long inter-point distances is excessive, which is a feature of regularity.

In conclusion, the reduced Palm distribution generating function A provides more information on patterns exhibiting both regularity and clustering than K, D, F or J alone does.

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Table 1: The estimated powers (in percentages) of the summary functions for detecting clustering and regularity in the mixed process, n=25.

		Regular	Cluster	Both			Regular	Cluster	Both
$n_1 = 0$	A	5	0	0	$n_1 = 0$	A	7	0	0
$n_c = —$	K	5	0	0	$n_c = —$	K	7	0	0
$r_{cl} = —$	D	4	0	0	$r_{cl} = —$	D	7	0	0
$r_{SSI} = 0.08$	F	2	0	0	$r_{SSI} = 0.1$	F	4	0	0
	J	19	0	0		J	37	0	0
$n_1 = 0$	A	12	0	0	$n_1 = 10$	A	7	46	3
$n_c = —$	K	12	0	0	$n_c = 2$	K	4	38	0
$r_{cl} = —$	D	12	0	0	$r_{cl} = 0.06$	D	4	6	0
$r_{SSI} = 0.12$	F	5	0	0	$r_{SSI} = 0.12$	F	0	12	0
	J	36	0	0		J	0	5	0
$n_1 = 10$	A	3	16	0	$n_1 = 10$	A	0	89	0
$n_c = 2$	K	1	14	0	$n_c = 5$	K	0	86	0
$r_{cl} = 0.08$	D	1	2	0	$r_{cl} = 0.02$	D	0	53	0
$r_{SSI} = 0.1$	F	0	1	0	$r_{SSI} = 0.06$	F	0	16	0
	J	0	3	0		J	0	54	0
$n_1 = 10$	A	0	83	0	$n_1 = 10$	A	0	85	0
$n_c = 8$	K	0	75	0	$n_c = 8$	K	0	82	0
$r_{cl} = 0.01$	D	0	73	0	$r_{cl} = 0.01$	D	0	71	0
$r_{SSI} = 0.03$	F	0	14	0	$r_{SSI} = 0.06$	F	0	28	0
	J	0	79	0		J	0	79	0
$n_1 = 15$	A	1	100	1	$n_1 = 15$	A	0	57	0
$n_c = 3$	K	0	100	0	$n_c = 5$	K	0	44	0
$r_{cl} = 0.04$	D	0	93	0	$r_{cl} = 0.05$	D	0	40	0
$r_{SSI} = 0.01$	F	0	70	0	$r_{SSI} = 0.1$	F	0	18	0
	J	0	98	0		J	0	41	0
$n_1 = 15$	A	0	21	0	$n_1 = 15$	A	0	98	0
$n_c = 5$	K	0	17	0	$n_c = 10$	K	0	91	0
$r_{cl} = 0.08$	D	0	5	0	$r_{cl} = 0.02$	D	0	94	0
$r_{SSI} = 0.1$	F	0	1	0	$r_{SSI} = 0.03$	F	0	34	0
	J	0	6	0		J	0	95	0
$n_1 = 15$	A	0	98	0	$n_1 = 25$	A	0	95	0
$n_c = 10$	K	0	96	0	$n_c = 10$	K	0	78	0
$r_{cl} = 0.02$	D	0	85	0	$r_{cl} = 0.08$	D	0	68	0
$r_{SSI} = 0.06$	F	0	17	0	$r_{SSI} = —$	F	0	41	0
	J	0	88	0		J	0	73	0
$n_1 = 25$	A	0	79	0	$n_1 = 25$	A	0	71	0
$n_c = 10$	K	0	33	0	$n_c = 15$	K	0	41	0
$r_{cl} = 0.09$	D	0	58	0	$r_{cl} = 0.08$	D	0	40	0
$r_{SSI} = —$	F	0	29	0	$r_{SSI} = —$	F	0	14	0
	J	0	60	0		J	0	39	0

Table 2: The estimated powers (in percentages) of the summary functions for detecting clustering and regularity in the mixed process, n=50.

		Regular	Cluster	Both			Regular	Cluster	Both
$n_1 = 0$	A	1	1	0	$n_1 = 0$	A	57	0	0
$n_c = $ —	K	1	0	0	$n_c = $ —	K	56	0	0
$r_{cl} = $ —	D	1	0	0	$r_{cl} = $ —	D	55	0	0
$r_{SSI} = 0.03$	F	0	0	0	$r_{SSI} = 0.05$	F	12	0	0
	J	5	0	0		J	68	0	0
$n_1 = 0$	A	100	0	0	$n_1 = 10$	A	26	96	25
$n_c = $ —	K	100	0	0	$n_c = 2$	K	3	96	3
$r_{cl} = $ —	D	100	0	0	$r_{cl} = 0.02$	D	6	20	1
$r_{SSI} = 0.07$	F	31	0	0	$r_{SSI} = 0.05$	F	0	16	0
	J	100	0	0		J	2	21	1
$n_1 = 10$	A	57	29	18	$n_1 = 10$	A	9	29	2
$n_c = 2$	K	10	30	3	$n_c = 5$	K	3	29	1
$r_{cl} = 0.04$	D	30	0	0	$r_{cl} = 0.02$	D	1	1	0
$r_{SSI} = 0.07$	F	1	3	0	$r_{SSI} = 0.05$	F	0	1	0
	J	16	0	0		J	2	0	0
$n_1 = 10$	A	1	47	1	$n_1 = 10$	A	7	5	0
$n_c = 8$	K	0	44	0	$n_c = 8$	K	2	5	0
$r_{cl} = 0.01$	D	0	24	0	$r_{cl} = 0.02$	D	3	0	0
$r_{SSI} = 0.03$	F	0	7	0	$r_{SSI} = 0.05$	F	0	0	0
	J	0	24	0		J	0	1	0
$n_1 = 30$	A	15	100	15	$n_1 = 30$	A	13	100	13
$n_c = 5$	K	0	100	0	$n_c = 5$	K	1	100	1
$r_{cl} = 0.01$	D	1	100	1	$r_{cl} = 0.03$	D	2	100	2
$r_{SSI} = 0.03$	F	0	100	0	$r_{SSI} = 0.06$	F	0	100	0
	J	0	100	0		J	0	100	0
$n_1 = 30$	A	4	100	4	$n_1 = 30$	A	3	100	3
$n_c = 10$	K	0	100	0	$n_c = 20$	K	0	100	0
$r_{cl} = 0.01$	D	0	100	0	$r_{cl} = 0.01$	D	0	100	0
$r_{SSI} = 0.03$	F	0	100	0	$r_{SSI} = 0.03$	F	0	96	0
	J	0	100	0		J	0	100	0
$n_1 = 30$	A	4	69	4	$n_1 = 50$	A	9	100	9
$n_c = 20$	K	0	65	0	$n_c = 5$	K	0	100	0
$r_{cl} = 0.03$	D	0	42	0	$r_{cl} = 0.02$	D	0	100	0
$r_{SSI} = 0.06$	F	0	16	0	$r_{SSI} = $ —	F	0	100	0
	J	0	41	0		J	0	100	0
$n_1 = 50$	A	0	100	0	$n_1 = 50$	A	0	97	0
$n_c = 25$	K	0	100	0	$n_c = 40$	K	0	91	0
$r_{cl} = 0.04$	D	0	100	0	$r_{cl} = 0.04$	D	0	92	0
$r_{SSI} = $ —	F	0	99	0	$r_{SSI} = $ —	F	0	71	0
	J	0	100	0		J	0	95	0

Table 3: The estimated powers (in percentages) of the summary functions for detecting clustering and regularity in the mixed process, n=100.

		Regular	Cluster	Both			Regular	Cluster	Both
$n_1 = 25$	A	23	100	23	$n_1 = 25$	A	100	84	84
$n_c = 5$	K	1	100	1	$n_c = 5$	K	2	85	2
$r_{cl} = 0.01$	D	3	100	3	$r_{cl} = 0.03$	D	90	15	14
$r_{SSI} = 0.03$	F	0	100	0	$r_{SSI} = 0.05$	F	2	18	0
	J	0	100	0		J	51	16	7
$n_1 = 25$	A	100	72	72	$n_1 = 25$	A	9	97	9
$n_c = 10$	K	7	71	3	$n_c = 15$	K	0	97	0
$r_{cl} = 0.02$	D	89	14	11	$r_{cl} = 0.01$	D	0	50	0
$r_{SSI} = 0.05$	F	3	3	0	$r_{SSI} = 0.03$	F	0	19	0
	J	75	16	10		J	0	53	0
$n_1 = 50$	A	36	100	36	$n_1 = 50$	A	61	100	61
$n_c = 5$	K	0	100	0	$n_c = 5$	K	2	100	2
$r_{cl} = 0.01$	D	13	100	13	$r_{cl} = 0.02$	D	31	100	31
$r_{SSI} = 0.03$	F	0	100	0	$r_{SSI} = 0.05$	F	0	100	0
	J	0	100	0		J	0	100	0
$n_1 = 50$	A	20	100	20	$n_1 = 50$	A	33	53	14
$n_c = 25$	K	0	100	0	$n_c = 40$	K	2	52	0
$r_{cl} = 0.02$	D	5	96	5	$r_{cl} = 0.02$	D	11	28	2
$r_{SSI} = 0.05$	F	0	81	0	$r_{SSI} = 0.05$	F	1	19	0
	J	0	97	0		J	4	34	2
$n_1 = 75$	A	28	100	28	$n_1 = 75$	A	31	100	31
$n_c = 10$	K	0	100	0	$n_c = 10$	K	0	100	0
$r_{cl} = 0.01$	D	5	100	5	$r_{cl} = 0.03$	D	12	100	12
$r_{SSI} = 0.03$	F	0	100	0	$r_{SSI} = 0.05$	F	0	100	0
	J	0	100	0		J	0	100	0
$n_1 = 75$	A	14	100	14	$n_1 = 75$	A	11	100	11
$n_c = 25$	K	0	100	0	$n_c = 50$	K	0	100	0
$r_{cl} = 0.02$	D	1	100	1	$r_{cl} = 0.01$	D	3	100	3
$r_{SSI} = 0.05$	F	0	100	0	$r_{SSI} = 0.03$	F	0	100	0
	J	0	100	0		J	0	100	0
$n_1 = 100$	A	17	100	17	$n_1 = 100$	A	6	100	6
$n_c = 20$	K	0	100	0	$n_c = 20$	K	0	100	0
$r_{cl} = 0.01$	D	0	100	0	$r_{cl} = 0.04$	D	0	100	0
$r_{SSI} =$ —	F	0	100	0	$r_{SSI} =$ —	F	0	100	0
	J	0	100	0		J	0	100	0
$n_1 = 100$	A	2	100	2	$n_1 = 100$	A	0	100	0
$n_c = 50$	K	0	100	0	$n_c = 80$	K	0	99	0
$r_{cl} = 0.04$	D	0	100	0	$r_{cl} = 0.04$	D	0	91	0
$r_{SSI} =$ —	F	0	100	0	$r_{SSI} =$ —	F	0	90	0
	J	0	100	0		J	0	94	0

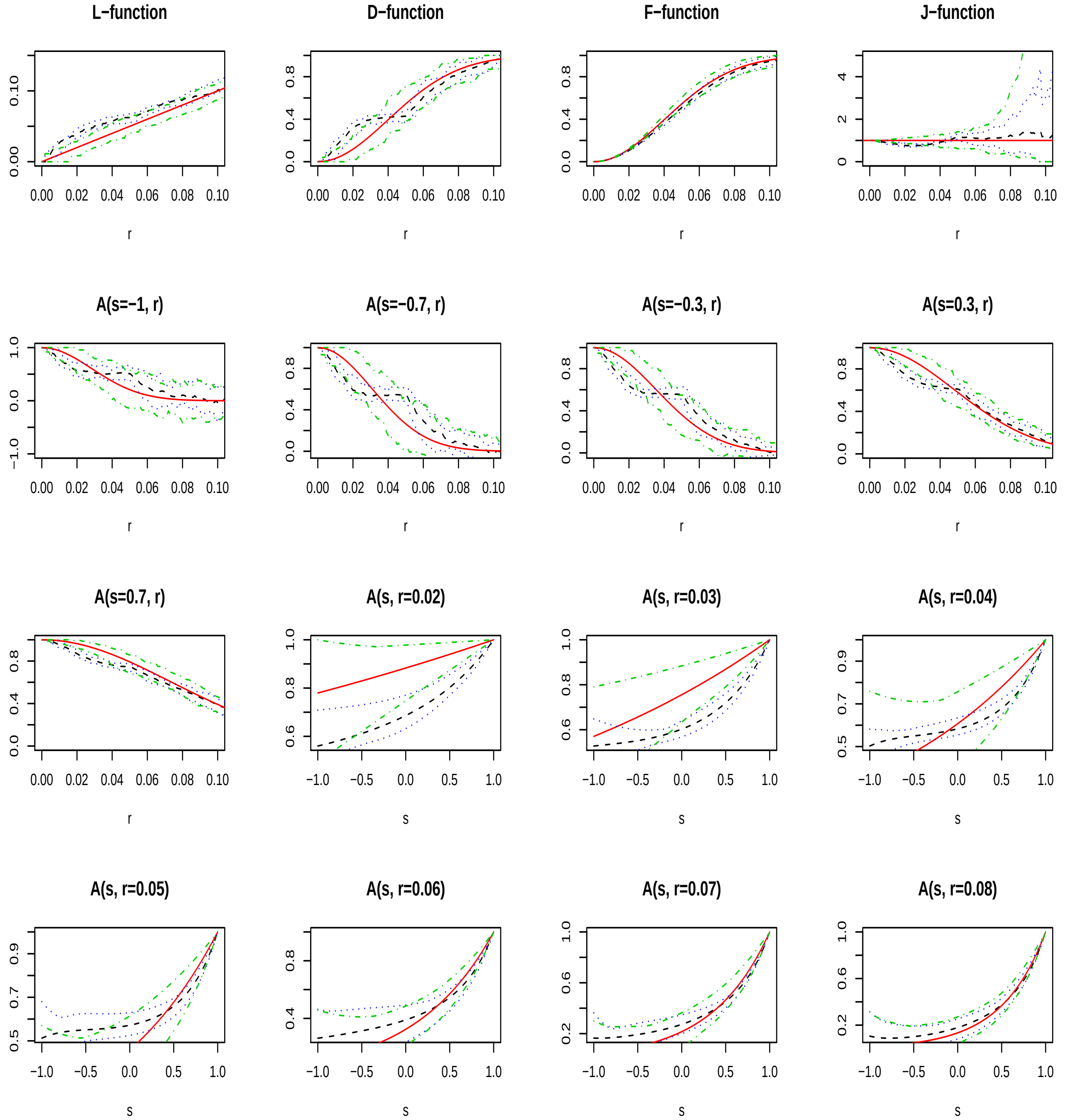

Figure 1: The means (dashed lines) and the envelopes (dotted lines) of the estimates of L, D, F, J and cross-sections of A from 10 independent realisations of the mixed process with $n = 100$, $n_1 = 45$, $n_c = 20$, $r_{cl} = 0.025$ and $r_{SSI} = 0.05$. The solid lines and the dotted-dashed lines are, respectively, the corresponding theoretical means and the envelopes of 99 independent realisations of the binomial process with the same number of points in the unit square.

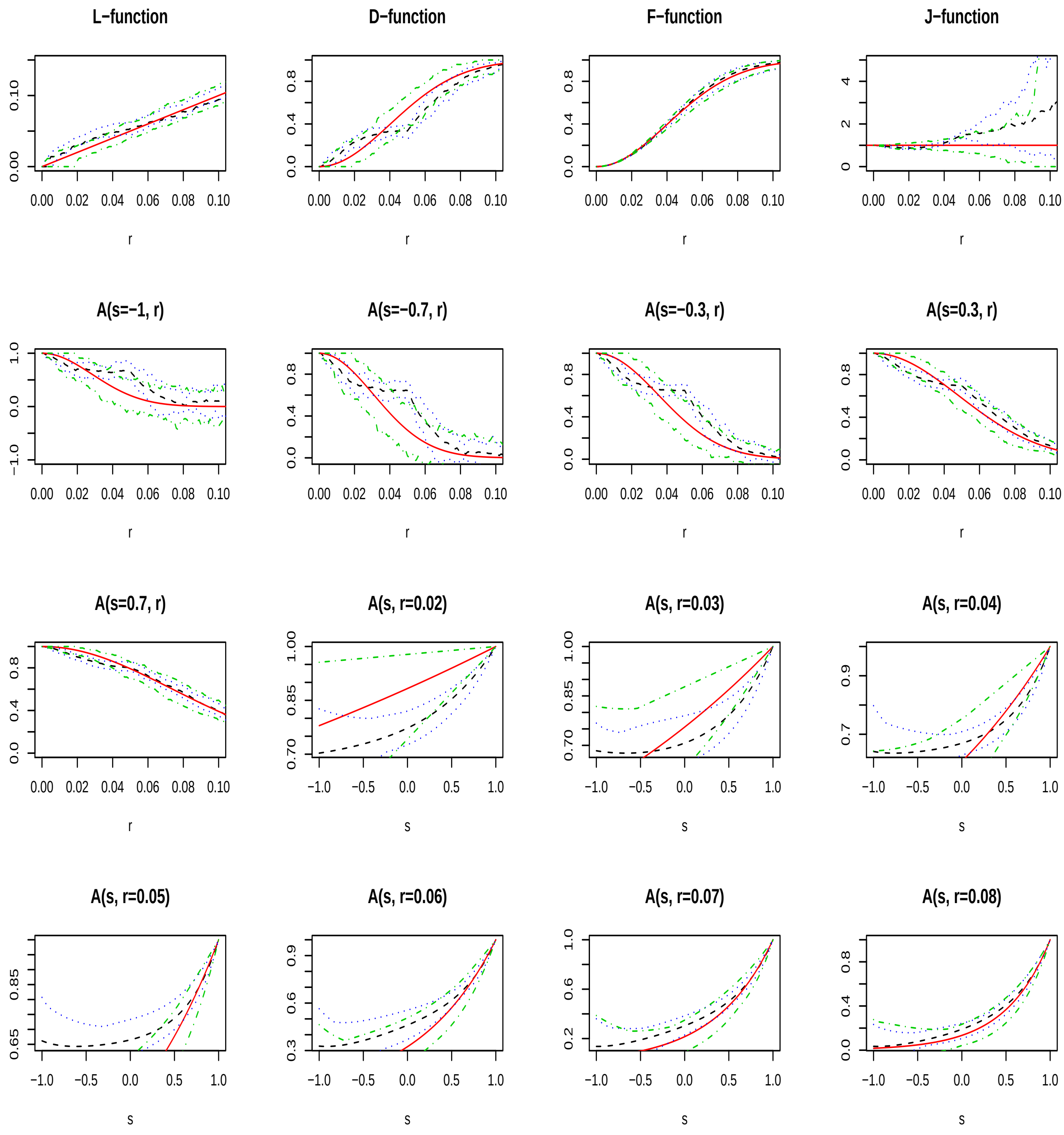

Figure 2: The means (dashed lines) and the envelopes (dotted lines) of the estimates of L, D, F, J and cross-sections of A from 10 independent realisations of the mixed process with $n = 100$, $n_1 = 36$, $n_c = 16$, $r_{cl} = 0.025$ and $r_{SSI} = 0.05$. The solid lines and the dotted-dashed lines are, respectively, the corresponding theoretical means and the envelopes of another sequence of 99 independent realisations of the binomial process with the same number of points in the unit square.

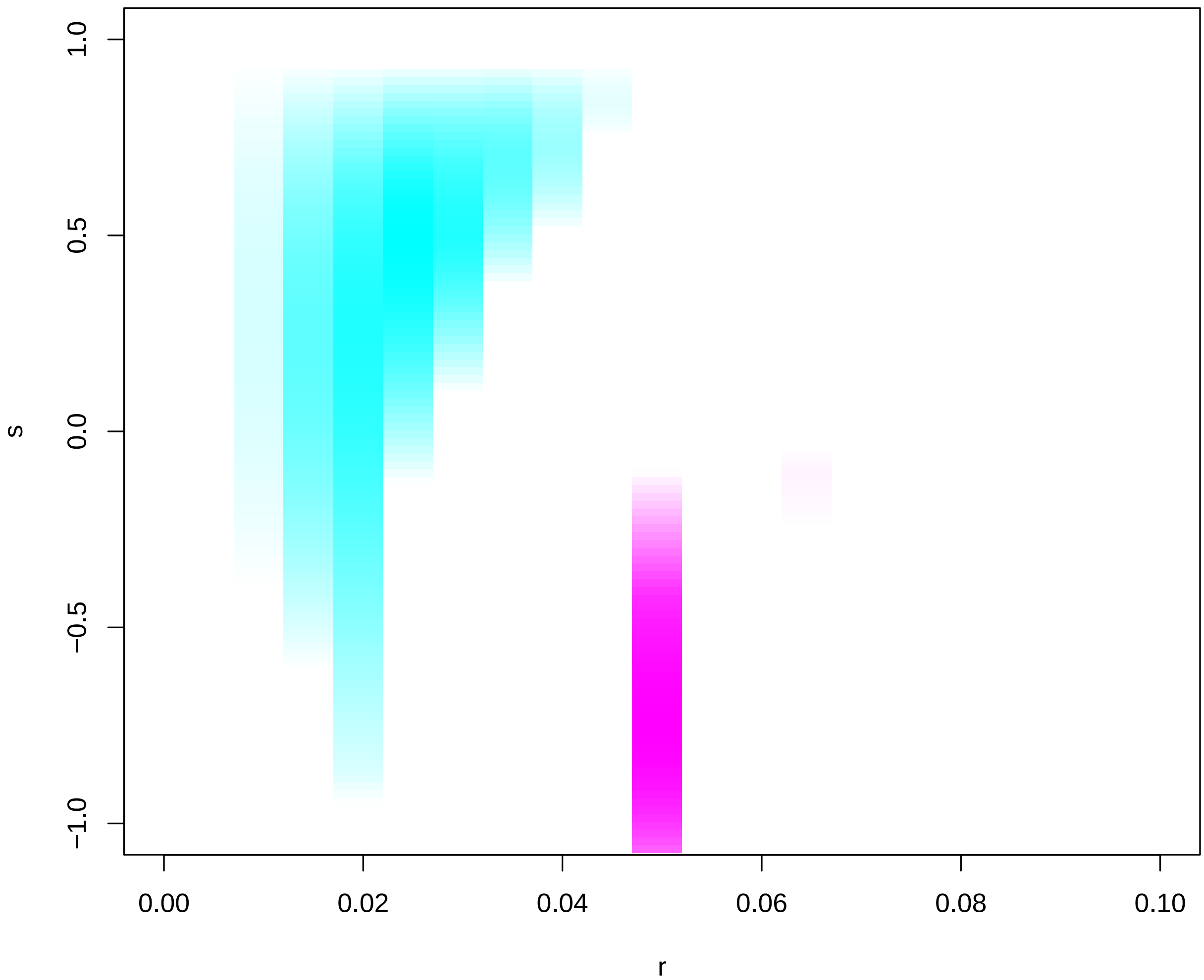

Figure 3: The contour plot for the differences between the mean of the empirical A from the ten realisations of the same mixed process as in Figure 1 and the lower and upper envelopes of 99 independent realisations of the binomial process. The white color is used to indicate where the mean of the empirical A lies entirely within the binomial envelopes. The contours close to $r = 0.02$ show how much the mean of the empirical A goes below the lower envelope (the darker the colour, the lower the mean of the empirical A). The contours close to $r = 0.05$ show how much the mean of the empirical A exceeds the upper envelope (the darker the colour, the higher the value of the mean of the empirical A).