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Prospect and Markowitz Stochastic Dominance

Abstract

Levy and Wiener (1998), Levy and Levy (2002, 2004) develop the Prospect and Markowitz stochastic dominance theory with S-shaped and reverse S-shaped utility functions for investors. In this paper, we extend their work on Prospect Stochastic Dominance theory (PSD) and Markowitz Stochastic Dominance theory (MSD) to the first three orders and link the corresponding S-shaped and reverse S-shaped utility functions to the first three orders. We also provide experiments to illustrate each case of the MSD and PSD to the first three orders and demonstrate that the higher order MSD and PSD cannot be replaced by the lower order MSD and PSD. Furthermore, we formulate the following PSD and MSD properties: hierarchy exists in both PSD and MSD relationships; arbitrage opportunities exist in the first orders of both PSD and MSD; and for any two prospects under certain conditions, their third order MSD preference will be 'the opposite of' or 'the same as' their counterpart third order PSD preference. By extending the work of Levy and Wiener and Levy and Levy, we provide investors with more tools to identify the first and third order PSD and MSD prospects and thus they could make wiser choices on their investment decision.

Keywords: Prospect stochastic dominance, Markowitz stochastic dominance, risk seeking, risk averse, S-shaped utility function, reverse S-shaped utility function

JEL Classification: D81, C91

1 Introduction

According to the von Neuman and Morgenstern (1944) expected utility theory, the functions for risk averters and risk seekers are concave and convex respectively, and both are increasing functions. Comparing the utility functions and the stochastic dominance (SD) theory has generated great interest among academics. Linking the SD theory to the selection rules for risk averters under different restrictions on the utility functions include Quirk and Saposnik (1962), Fishburn (1964), Hanoch and Levy (1969), Whitmore (1970), Hammond (1974) and Tesfatsion (1976). Linking the SD theory to the selection rules for risk seekers include Hammond (1974), Meyer (1977), Stoyan (1983), Levy and Wiener (1998), Wong and Li (1999) and Anderson (2004).

Examining the relative attractiveness of various forms of investments, Friedman and Savage (1948) claim that the strictly concave functions may not be able to explain the behavior why investors buy insurance or lottery tickets. Markowitz (1952), the first to address Friedman and Savage's concern, proposes a utility function which has convex and concave regions in both the positive and the negative domains[1] while Gneezy, et al. (2006) suggest that there are choice situations in which decision makers discount lotteries for uncertainty in a manner that cannot be accommodated by standard models of risky choice. To support Markowitz's proposed utility function, Williams (1966) reports data where a translation of outcomes produces a dramatic shift from risk aversion to risk seeking while Fishburn and Kochenberger (1979) document the prevalence of risk seeking in choices between negative prospects. Kahneman and Tversky (1979) and Tversky and Kahneman (1992) claim that the (value) utility function[2] is concave for gains and convex for losses, yielding an S-shaped function. They also develop a formal theory of loss aversion

[1]Ng (1965) and Machina (1982) also provide other explanations to Friedman and Savage's paradox.

[2]Kahneman and Tversky (1979) and Tversky and Kahneman (1992) call it value function. For simplicity, we call it utility function. There will be more discussion in Section 2.

called prospect theory in which investors can maximize the expectation of the S-shaped utility function. It is one of the most popular decision-making theories about risk-taking and has gained much attention from economists and professionals in the financial sector.

Thereafter, a stream of papers building economic or financial models on the prospect theory has been written, for example, Shefrin and Statman (1993), Benartzi and Thaler (1995), Levy and Wiener (1998), Levy and Levy (2002, 2004) and Wang and Fischbeck (2004). There have also been many empirical and experimental attempts to test the prospect theory, for example, the equity premium puzzle by Benartzi and Thaler (1995) and the buying strategies of hog farmers by Pennings and Smidts (2003). Most of these studies support the prospect theory. The prospect theory has also been widely applied in Economics and Finance, see for example, Myagkov and Plott (1997), and Levy and Levy (2004).

Noticing the presence of risk seeking in preferences among positive as well as negative prospects, Markowitz (1952) also proposes another type of utility functions different from the pure S-shaped utility functions used in the prospect theory. He suggests a utility which is first concave, then convex, then concave, and finally convex to explain Friedman and Savage's question about why investors buy insurance and buy lotteries tickets. Using sequential gambles technique, Thaler and Johnson (1990) obtain experimental evidence to show that prior outcomes affect subsequent behavior in a way that is contrary to the static version of the prospect theory. In particular, subjects are more risk seeking following gains and more risk averse following losses. This implies that in a dynamic context, a reverse S-shaped utility function may be more descriptive of actual behavior. Levy and Wiener (1998) further develop the theory for the reverse S-shaped utility functions for investors. Levy and Levy (2002) are the first to extend the work of Markowitz (1952), Kahneman and Tversky (1979), Tversky and Kahneman (1992), Thaler and Johnson (1990) and Levy and Wiener (1998). They develop a new criterion called Markowitz

Stochastic Dominance (MSD) to determine the dominance of one investment alternative over another for all reverse S-shaped functions, and another criterion called Prospect Stochastic Dominance (PSD) to determine the dominance of one investment alternative over another for all prospect theory S-shaped utility functions.

Working along similar lines as Whitmore (1970) who extends the second order SD developed by Quirk and Saposnik (1962) and others to the third order SD for risk averters, in this paper, we first extend the work of Levy and Levy to take the PSD and MSD to the first three orders SD and link the corresponding S-shaped and reverse S-shaped utility functions to the first three orders.

Another contribution of Levy and Levy to the literature is to prove the second order PSD and MSD satisfy the expected utility paradigm which is an important issue in the literature. Arrow (1971) first points out that an individual with unbounded utility must violate either the completeness or the continuity axiom of the expected utility theory while Machina (1982) suggests that the expected utility analysis is too theoretical and may not be empirically valid. Swalm (1966), Kahneman and Tversky (1979), Kahneman et al. (1990), and Barberis, Huang, and Santos (2001) also mount a critique of expected utility theory. Rabin (2000) also points out that the expected utility cannot explain loss aversion which accounts for the modest-scale risk aversion for both large and small stakes typically observed in empirical studies. To circumvent this problem, Kahneman and Tversky (1979) suggest employing the certainty equivalent approach to study the negative and the positive domains separately. Nonetheless, the PSD and MSD developed in Levy and Wiener (1998) and Levy and Levy (2002, 2004) bypass the above problems. Moreover, they show that both MSD and PSD satisfy the expected utility paradigm. Following Levy and Levy, another contribution of this article is to examine the compatibility of both the extended MSD and PSD with the expected utility theory and proves that both MSD and PSD of any order are consistent with the expected utility theory.

In addition, we provide experiments to illustrate each case of the MSD and PSD to the first three orders and demonstrate that the higher order MSD and PSD cannot be replaced by the lower order MSD and PSD. We also develop some other properties for the extended MSD and PSD as follows: hierarchy exists in both PSD and MSD; arbitrage opportunities exist for the first orders of both PSD and MSD; and for any two prospects under certain conditions, their third order MSD preference will be 'the opposite of' or 'the same as' their third order counterpart PSD preference. In terms of empirical analysis, our approach is superior to Levy and Levy's as the definitions of the extended PSD and MSD developed in our paper enable investors to identify the MSD and PSD prospects to the first three orders. With more information, investors can make wiser decisions with their investments. For example, when an investor has identified the first order PSD and MSD prospects, the arbitrage opportunities are revealed. In addition, by identifying the third order PSD and MSD prospects, an investor can make wiser choices about these prospects. However, Levy and Levy's approach only allows investors to identity the MSD and PSD to the second order. Without the extended PSD and MSD definitions, Levy and Levy's investors would not have as much information as ours to make their investment decisions.

The paper is organized as follows. We begin by introducing definitions and notations in the next section. Section 3 develops several theorems and properties for the extended MSD and PSD. Section 4 provides illustrations for MSD and PSD to the first three orders and demonstrates that the higher order MSD and PSD cannot be replaced by the lower order MSD and PSD. Section 5 concludes our findings.

2 Definitions and Notations

Let $\mathbb{R}$ be the set of real numbers and $\overline{\mathbb{R}}$ be the set of extended real numbers. $\Omega = [a, b]$ is a subset of $\overline{\mathbb{R}}$ in which $a < 0$ and $b > 0$ and they can be finite or infinite. Let $\mathbb{B}$ be the

Borel σ-field of Ω and μ be a *measure* on $(\Omega, \mathbb{B})$. We first define the functions F and F^D of the measure μ on the support Ω as

$$F(x) = \mu[a, x] \quad \text{and} \quad F^D(x) = \mu[x, b] \qquad \text{for all} \quad x \in \Omega\,. \tag{1}$$

Function F is a *probability distribution function* and μ is a *probability measure* if $\mu(\Omega) = 1$. In this paper, the definition of F is slightly different from the 'traditional' definition of a distribution function. We follow the basic probability theory that for any random variable X and for any probability measure P, there exists a unique induced probability measure μ on $(\Omega, \mathbb{B})$ and a probability distribution function F such that F satisfies (1) and

$$\mu(B) = P(X^{-1}(B)) = P(X \in B) \qquad \text{for any} \quad B \in \mathbb{B}\,.$$

An integral written in the form of $\int_A f(t)\, d\,\mu(t)$ or $\int_A f(t)\, d\,F(t)$ is a Lebesgue integral for any integrable function $f(t)$. If the integral has the same value for any set A which is equal to $(c, d]$, $[c, d)$ or $[c, d]$, then we use the notation $\int_c^d f(t)\, d\,\mu(t)$ instead. In addition, if μ is a Borel measure with $\mu(c, d] = d - c$ for any $c < d$, then we write the integral as $\int_c^d f(t)\, dt$.

Random variables, denoted by X and Y defined on Ω are considered together with their corresponding probability distribution functions F and G and their corresponding probability density functions f and g respectively. The following notations will be used throughout this paper:

$$\begin{aligned} \mu_F = \mu_X = E(X) &= \int_a^b x\, d\,F(x), & \mu_G = \mu_Y = E(Y) &= \int_a^b x\, d\,G(x)\,; \\ f(x) = F_0^A(x) &= F_0^D(x), & g(x) = G_0^A(x) &= G_0^D(x) \\ H_n^A(x) &= \int_a^x H_{n-1}^A(y)\, dy\,, & H_n^D(x) &= \int_x^b H_{n-1}^D(y)\, dy \quad n = 1, 2, 3; \end{aligned} \tag{2}$$

where $H = F$ or G. The above definitions have been commonly used in the literature, see for example, Wong and Li (1999), Li and Wong (1999) and Anderson (2004). All functions are assumed to be measurable and all random variables are assumed to satisfy:

$$F_1^A(a) = 0 \quad \text{and} \quad F_1^D(b) = 0. \tag{3}$$

Condition (3) will hold for any random variable except a random variable with positive probability at the points of negative infinity or positive infinity. For $H = F$ or G, we define the following functions for MSD and PSD:

$$\begin{aligned}
H_1^a(x) &= H(x) = H_1^A(x), \quad H_1^d(x) = 1 - H(x) = H_1^D(x); \\
& \quad \text{Dear Raymond, should we define } H_1^a \text{ and } H_1^d \text{ as below?} \qquad (4) \\
H_1^a(x) &= H_1^A(x) - H_1^A(0) \quad \text{for} \quad x \geq 0 \\
H_1^d(x) &= H_1^D(x) - H_1^D(0) \quad \text{for} \quad x \leq 0 \\
H_i^d(y) &= \int_y^0 H_{i-1}^d(t)dt, \quad y \leq 0; \quad \text{and} \\
H_i^a(x) &= \int_0^x H_{i-1}^a(t)dt, \quad x \geq 0 \quad \text{for } \; i = 2, 3 \; . \qquad (5)
\end{aligned}$$

In order to make the computation easier, we further define

$$\begin{aligned}
H_i^M(x) &= \begin{cases} H_i^A(x) & x \leq 0 \\ H_i^D(x) & x > 0 \;\; ; \end{cases} \\
H_i^P(x) &= \begin{cases} H_i^d(x) & x \leq 0 \\ H_i^a(x) & x > 0 \;\; ; \end{cases}
\end{aligned} \tag{6}$$

where $H = F$ and G and $i = 1, 2$ and 3.

We note that the definition of H_i^A can be used to develop the stochastic dominance theory for risk averters (see, for example, Quirk and Saposnik 1962, Fishburn 1964, Hanoch and Levy 1969) and thus we could call this type of SD Ascending Stochastic Dominance

(ASD) and call H_i^A to be the i^{th} order ASD integral or the i^{th} order cumulative probability as H_i^A is integrated in ascending order from the leftmost point of downside risk. On the other hand, H_i^D can be used to develop the stochastic dominance theory for risk seekers (see, for example, Meyer 1977, Stoyan 1983, Levy and Wiener 1998, Wong and Li 1999, and Anderson 2004) and thus we could call this type of SD Descending Stochastic Dominance (DSD) and call H_i^D to be the i^{th} order DSD integral or the i^{th} order reversed cumulative probability as H_i^D is integrated in descending order from the rightmost point of upside profit. Typically, risk averters prefer assets that have a smaller probability of losing, especially in downside risk while risk seekers prefer assets that have a higher probability of gaining, especially in upside profit. To make a choice between two assets F or G, risker averters will compare their corresponding i^{th} order ASD integrals F_i^A and G_i^A and choose F if F_i^A is smaller since it has a smaller probability of losing. On the other hand, risk seekers will compare their corresponding i^{th} order DSD integrals F_i^D and G_i^D and choose F if F_i^D is bigger since it has a higher probability of gaining.

As pointed out by Markowitz (1952) and many others, investors' behaviors can be different in the positive and negative domains of the return. Without loss of generality, in this paper, 'upside profit' refers to the positive domain of the return and 'downside risk' the negative domain of return. We first consider the function H_i^M which is equal to H_i^A in downside risk and equal to H_i^D in upside profit. By comparing the F_i^M and G_i^M of the two assets F and G, we study whether we could choose an asset which shows a smaller probability in downside risk and a bigger probability in upside profit. Once we find F such that it has a smaller ASD integral in downside risk and a higher DSD integral in upside profit, one may believe that F has the best of both worlds – a smaller probability of losing in downside risk and a larger probability to gain in upside profit. On the other hand, in this paper we also study the properties of the function H_i^P which is equal to ASD integral (H_i^a) in upside profit and equal to the DSD integral (H_i^d) in downside risk. As shown in next section, our paper shows that H_i^M can be used to develop the MSD theory while H_i^P can be used to develop the PSD theory.[3] We first state in the following definitions for this purpose:

Definition 1. *Given two random variables X and Y with F and G as their respective probability distribution functions, X is at least as large as Y and F is at least as large as G in the sense of:*

a. *FMSD, denoted by $X \succeq_1^M Y$ or $F \succeq_1^M G$, if and only if $F_1^M(-x) \leq G_1^M(-x)$ and $F_1^M(x) \geq G_1^M(x)$ for each $x \geq 0$;*

b. *SMSD, denoted by $X \succeq_2^M Y$ or $F \succeq_2^M G$, if and only if $F_2^M(-x) \leq G_2^M(-x)$ and $F_2^M(x) \geq G_2^M(x)$ for each $x \geq 0$;*

c. *TMSD, denoted by $X \succeq_3^M Y$ or $F \succeq_3^M G$, if and only if $F_3^M(-x) \leq G_3^M(-x)$ and $F_3^M(x) \geq G_3^M(x)$ for each $x \geq 0$;*

where FMSD, SMSD, and TMSD stand for the first, second and third order Markowitz Stochastic Dominance (MSD) respectively.

If, in addition, there exists an x in $[a, b]$ such that $F_i^M(x) < G_i^M(x)$ with $x < 0$ or $F_i^M(x) > G_i^M(x)$ with $x > 0$ for $i = 1, 2$ and 3, we say that X is larger than Y and F is larger than G in the sense of SFMSD, SSMSD, and STMSD, denoted by $X \succ_1^M Y$ or $F \succ_1^M G, X \succ_2^M Y$ or $F \succ_2^M G$, and $X \succ_3^M Y$ or $F \succ_3^M G$ respectively, where SFMSD, SSMSD, and STMSD stand for strictly first, second and third order Markowitz Stochastic Dominance respectively.

Definition 2. *Given two random variables X and Y with F and G as their respective probability distribution functions, X is at least as large as Y and F is at least as large as G in the sense of:*

[3]Thus, we call the function H_i^M the i^{th} order MSD integral and call the function H_i^P the i^{th} order PSD integral.

a. *FPSD, denoted by* $X \succeq_1^P Y$ *or* $F \succeq_1^P G$*, if and only if* $F_1^P(-x) \geq G_1^P(-x)$ *and* $F_1^P(x) \leq G_1^P(x)$ *for each* $x \geq 0$*;*

b. *SPSD, denoted by* $X \succeq_2^P Y$ *or* $F \succeq_2^P G$*, if, and only if,* $F_2^P(-x) \geq G_2^P(-x)$ *and* $F_2^P(x) \leq G_2^P(x)$ *for each* $x \geq 0$*;*

c. *TPSD, denoted by* $X \succeq_3^P Y$ *or* $F \succeq_3^P G$*, if and only if* $F_3^P(-x) \geq G_3^P(-x)$ *and* $F_3^P(x) \leq G_3^P(x)$ *for each* $x \geq 0$*;*

where FPSD, SPSD, and TPSD stand for the first, second and third order Prospect Stochastic Dominance (PSD) respectively.

If, in addition, there exists an x *in* $[a, b]$ *such that* $F_i^P(x) > G_i^P(x)$ *with* $x < 0$ *or* $F_i^P(x) < G_i^P(x)$ *with* $x > 0$ *for* $i = 1, 2$ *and 3, we say that* X *is larger than* Y *and* F *is larger than* G *in the sense of SFPSD, SSPSD, and STPSD, denoted by* $X \succ_1^P Y$ *or* $F \succ_1^P G, X \succ_2^P Y$ *or* $F \succ_2^P G$*, and* $X \succ_3^P Y$ *or* $F \succ_3^P G$ *respectively, where SFPSD, SSPSD, and STPSD stand for strictly first, second and third order Prospect Stochastic Dominance respectively.*

Levy and Levy (2002) define the MSD and PSD functions as:

$$
\begin{aligned}
H^M(x) &= \begin{cases} \int_a^x H(t)\,dt & x < 0 \\ \int_x^b H(t)\,dt & x > 0 \end{cases} \\
H^P(x) &= \begin{cases} \int_x^0 H(t)\,dt & x < 0 \\ \int_0^x H(t)\,dt & x > 0 \end{cases}
\end{aligned}
\tag{7}
$$

where $H = F$ and G. MSD and PSD are expressed in the following definition:

Definition 3.

a. $F \succeq_{MSD} G$ *if* $F^M(x) \leq G^M(x)$ *for all* x*; and*

b. $F \succeq_{PSD} G$ *if* $F^P(x) \leq G^P(x)$ *for all* x*.*

One can easily show that $F \succeq_{MSD} G$ if and only if $F \succeq_2^M G$ and $F \succeq_{PSD} G$ if and only if $F \succeq_2^P G$. Hence, the MSD and PSD defined in Levy and Levy are the same as the second order MSD and PSD defined in our paper. We note that Levy and Wiener (1998) and Levy and Levy (2004) define PSD as $F \succeq_{PSD} G$ if and only if

$$0 \leq \int_{x_1}^{x_2} [G(z) - F(z)]\, dz \quad \text{for all} \quad x_1 \leq 0 \leq x_2$$

with at least one strict inequality.

Definition 4. *$n = 1, 2, 3, U_n^A, U_n^{SA}, U_n^D$ and U_n^{SD} are the sets of the utility functions u such that:*

$$\begin{aligned}
U_n^A(U_n^{SA}) &= \{u : (-1)^{i+1} u^{(i)} \geq (>)\, 0\,,\ i = 1, \cdots, n\}\,; \\
U_n^D(U_n^{SD}) &= \{u : u^{(i)} \geq (>)\, 0\,,\ i = 1, \cdots, n\}\,; \\
U_n^S(U_n^{SS}) &= \{u : u^+ \in U_n^A(U_n^{SA}) \quad \textit{and} \quad u^- \in U_n^D(U_n^{SD})\,,\ i = 1, \cdots, n\}\,; \\
U_n^R(U_n^{SR}) &= \{u : u^+ \in U_n^D(U_n^{SD}) \quad \textit{and} \quad u^- \in U_n^A(U_n^{SA})\,,\ i = 1, \cdots, n\}\,.
\end{aligned}$$

where $u^{(i)}$ is the i^{th} derivative of the utility function u, $u^+ = u$ restricted for $x \geq 0$ and $u^- = u$ restricted for $x \leq 0$.

Note that the theory can be easily extended to satisfy utilities defined in Definition 4 to be non-differentiable. In this paper, we will skip the discussion of non-differentiable utilities. It is noted that investors in U_n^A is risk averse while investors in U_n^D is risk seeking. Investors in U_n^R with reversed S-shaped utility functions are risk seeking for gains but risk aversion for losses while investors in U_n^S with S-shaped utility functions are risk averse for gains but risk seeking for losses. Refer to Figure 1 for the shape of utility functions in U_2^A, U_2^D, U_2^R and U_2^S and refer to Figure 2 for the shape of the first derivatives of the utility functions in U_3^A, U_3^D, U_3^R and U_3^S respectively.

An individual choosing between F and G in accordance with a consistent set of preferences will satisfy the Von Neumann-Morgenstern (1944) consistency properties. Accordingly, F is (strictly) preferred to G, or equivalently, X is (strictly) preferred to Y if

$$\Delta Eu \equiv u(F) - u(G) \equiv u(X) - u(Y) \geq 0 (> 0), \tag{8}$$

where $u(F) \equiv u(X) \equiv \int_a^b u(x)dF(x)$ and $u(G) \equiv u(Y) \equiv \int_a^b u(x)dG(x)$.

There is an ongoing debate in the literature regarding the shape of the utility functions. The utility functions U_2^A and U_2^{SA} advocated in the literature depict the concavity of the utility function, which is equivalent to risk aversion, according to the notion of decreasing marginal utility. The prevalence of risk aversion is the best known generalization regarding risky choices and was popular among the early decision theorists of the twentieth century (Pratt 1964, Arrow 1971).

Noticing the presence of risk seeking in preferences among positive as well as negative prospects, Markowitz (1952) proposes a utility function which has convex and concave regions in both the positive and the negative domains. The regions are first concave, then convex, then concave, and finally convex. This utility function could be used to explain the purchasing of both insurance and lotteries observed by Friedman and Savage (1948). The portion of this utility function that has convex and concave regions in the negative and the positive domains respectively is equivalent to U_2^S defined in our paper and forms a S-shaped utility function. Later, Kahneman and Tversky (1979) and Tversky and Kahneman (1992) formally develop the prospect theory to link up the S-shaped utility functions. Similarly, the portion that has concave and convex regions in the negative and the positive domains respectively is equivalent to U_2^R defined in our paper and forms a reverse S-shaped utility function (Thaler and Johnson 1990; Levy and Wiener 1998; Levy and Levy 2002, 2004).

Whitmore (1970) extends the second order SD developed by Quirk and Saposnik (1962) and others to the third order SD and improves the linkage of SD to the utility functions for risk averse investors up to U_3^A. In this paper, we extend PSD and MSD developed by Levy and Levy to the first three orders and improve the linkage of PSD and MSD to the utility functions up to U_3^S and U_3^R. Details of these linkages are discussed in the next section. One can easily show that U_1^S and U_1^R are equivalent to U_1^A and U_1^D; all of these are simply sets of increasing utility functions. The set U_2^S containing S-shaped utility functions, with concave and convex regions in the positive and the negative domains respectively, and the set U_n^R containing reverse S-shaped utility functions, with convex and concave regions in the positive and the negative domains respectively, have been discussed in detail in the literature, for example, see Markowitz (1952), Thaler and Johnson (1990), Levy and Wiener (1998) and Levy and Levy (2002, 2004). A utility in U_3^S is increasing with its marginal utility decreasing in the positive domain but increasing in the negative domain, and is graphically convex in both the positive and negative domains. On the other hand, a utility in U_3^R is increasing with its marginal utility increasing in the positive domain but decreasing in the negative domain, and is graphically convex in both the positive and negative domains. In order to draw a clearer picture for both the second and third orders SD, we define the following Pratt-Arrow risk aversion at ω for an individual with the utility function u:

$$r(\omega) = -\frac{u^{(2)}(\omega)}{u^{(1)}(\omega)} = -\frac{d \log u^{(1)}(\omega)}{d\,\omega}. \tag{9}$$

where $u^{(i)}$ is the i^{th} derivative of the utility function u.

With the definition of risk aversion, one can easily show the relationship between risk aversion and the sets of utility functions defined in Definition 4. For example, if $u \in U_2^S$, then its risk aversion will be positive in the positive domain and negative in the negative domain. Similarly, if $u \in U_2^R$, then its risk aversion will be negative in the position domain and positive in the negative domain. In addition, if the risk aversion is positively decreasing in the positive domain and negatively decreasing in the negative domain, then the utility function belongs to $u \in U_3^S$. On the other hand, if the risk aversion is negatively decreasing in the positive domain but positively decreasing in the negative domain, then the utility function belongs to $u \in U_3^R$. Investors with utility u is well-known to have Decreasing Absolute Risk Aversion (DARA) behavior if $u^{(1)} > 0$, $u^{(2)} < 0$ and $u^{(3)} > 0$, see for example, Falk and Levy (1989). We can say that investors with utility functions $u \in U_3^S$ have DARA behavior in the positive domain and investors with utility functions $u \in U_3^R$ have DARA behavior in the negative domain.

Let us turn to the empirical evidence on the S-shaped or reverse S-shaped utility functions. It is well-known that under the expected utility theory, convexity of utility is equivalent to risk seeking while concavity is equivalent to risk aversion. Empirical measurements generally corroborate with the concavity in the utility for gains, for example, see Fishburn and Kochenberger (1979) and Fennema and van Assen (1999). However, the behavior of gamblers reveals convexity for gains (Friedman and Savage 1948; Markowitz 1952). For the utility for losses, some studies find convexity while some find concavity. For example, Fishburn and Kochenberger (1979) and Pennings and Smidts (2003) find convex utility for losses for the majority of cases and concave utility for losses for a sizable minority of subjects. Despite the studies of Currim and Sarin (1989) and Myagkov and Plott (1997), no conclusive evidence in favor of convex utility for losses is provided, which would have supported the reverse S-shaped utility functions. On the other hand, Wu and Gonzalez (1996) propose to use preference 'ladders' to test for concavity and convexity of the weighting function. They validate the findings of an S-shaped weighting function, concave up to $p < .40$, and convex beyond that probability. Nevertheless, using sequential gambles technique, Thaler and Johnson (1990) obtain experimental evidence to show that prior outcomes affect subsequent behavior in a way that subjects are more risk seeking following gains and more risk averse following losses. This supports the reverse S-shaped utility function behavior.

Finally, we note that in the prospect theory developed by Kahneman and Tversky (1979) and Tversky and Kahneman (1992), the S-shaped utility function is called the value function as it is attuned to the evaluation of changes or differences of wealth rather than the evaluation of absolute magnitudes. In this paper, we simply call it utility function as we do not restrict its applications to total wealth or the changes or differences of wealth. We also note that Levy and Wiener (1998) define U_p and Levy and Levy (2002) define V_{KT} as the class of all prospect theory value S-shaped functions with an inflection point at $x = 0$ where the subscripts KT denote Kahneman and Tversky. This is the same as our U_2^S. They also define V_M as the class of all Markowitz utility functions which are reverse S-shaped, with an inflection point at $x = 0$, where the subscript M denotes Markowitz. This is the same as our U_2^R. In addition, prospect theory assumes loss aversion which reflects the observed behavior that agents are more sensitive to losses than to gains, resulting in the value functions for losses are usually restricted to be steeper than their shapes for gains.[4] In another words, the investors are downside risk averse and could be measured by loss aversion.[5] In Definition 4, we do not include this restriction in the definition of U_2^S. However, U_2^S is a more general class of S-shaped utility functions containing all the value functions with this restriction and hence the theory of loss aversion and value function satisfy the theory developed in this paper.

3 Theory

In this section we develop the basic theorems and some basic properties for MSD and PSD. We first introduce the basic theorem linking the MSD of the first three orders to

[4] see for example, Kahneman and Tversky (1979), Tversky and Kahneman (1992), Barberis, Huang and Santos (2001) and Wakker (2003).

[5] see for example, Chateauneuf and Cohen (1994), Rabin (2000), Rizzo and Zeckhauser (2004) and Kobberling and Wakker (2005).

investors with reverse S-shaped utility functions to the first three orders:

Theorem 1. *Let X and Y be random variables with probability distribution functions F and G respectively. Suppose u is a utility function. For $i = 1, 2$ and 3, we have*

$F \succeq_i^M (\succ_i^M) G$ *if and only if* $u(F) \geq (>) u(G)$ *for any* u *in* U_i^R (U_i^{SR}).

Refer to Appendix A for the proof of Theorem 1. We next introduce the theorem linking PSD to investors with S-shaped utility functions to the first three orders:

Theorem 2. *Let X and Y be random variables with probability distribution functions F and G respectively. Suppose u is a utility function. For $i = 1, 2$ and 3, we have*

$F \succeq_i^P (\succ_i^P) G$ *if and only if* $u(F) \geq (>) u(G)$ *for any* u *in* U_i^S (U_i^{SS}).

See Appendix B for the proof of Theorem 2. The SD results for risk averters and risk seekers similar to the above two theorems have been well explored. Linking the SD theory to risk averters, there are Hadar and Russell (1971) and Bawa (1975) who prove that the stochastic dominance results for continuous density functions are linked with continuously differentiable utility functions; Hanoch and Levy (1969) and Tesfatsion (1976) who prove the validity of the first and second order stochastic dominance for general distribution functions; and Whitmore (1970) who extends their results and shows that the third order stochastic dominance for risk averters holds true. Broadening the scope, Meyer (1977) discusses the validity of the second order stochastic dominance for risk seekers and risk averters while Stoyan (1983) proves that the first and second order stochastic dominance results are applicable to risk seekers as well as risk averters. Furthermore, Levy and Wiener (1998) and Levy and Levy (2002, 2004) develop the second order PSD and MSD theories and link them to the second order S-shaped and reverse S-shaped utility functions. We extend their work and link PSD and MSD of any order to the S-shaped and reverse S-shaped utility functions as shown in the above two theorems.

Many studies claim that the prospect theory is a critique of the expected utility theory as the convexity of the value function is different in the positive domain from that in the negative domain. Levy and Levy (2002) prove that both the second order MSD and PSD satisfy the expected utility paradigm. In this paper, we extend Levy and Levy's results to examine the compatibility of the MSD and PSD of any order with the expected utility theory and prove that the MSD and PSD of any order are consistent with the expected utility paradigm as shown in the above two theorems.

Whitmore (1970) extends the second order SD to the third order SD for risk averters and thereafter many academics demonstrate the usefulness of the third order SD, see for example, Whitmore and Findley (1978), Shorrocks and Foster (1987), Gotoh and Hiroshi (2000) and Ng (2000). In addition, Hammond (1974) generalizes the SD theory to the n-order for any integer n. Both the MSD and PSD theories can be extended to any order in similar ways. However, we focus our discussion up to the first three orders in this paper as the first three orders SD are of most importance in theory as well as empirical applications.

It is well-known that hierarchy exists in SD relationships for risk averters and risk seekers: the first order SD implies the second order SD which in turn implies the third order SD in the SD rules for risk averters as well as risk seekers (Falk and Levy 1989). Thus, the following hierarchical relationships for MSP and PSD are obtained:

Corollary 1. *For any random variables X and Y, for $i = 1$ and 2, we have the following:*

a. *if $X \succeq_i^M (\succ_i^M) Y$, then $X \succeq_{i+1}^M (\succ_{i+1}^M) Y$; and*

b. *if $X \succeq_i^P (\succ_i^P) Y$, then $X \succeq_{i+1}^P (\succ_{i+1}^P) Y$.*

The proof of Corollary 1 is straightforward. The results of this corollary suggest that practitioners report the MSD and PSD results to the lowest order in empirical analyses.

Levy and Levy (2002) show that it is possible for MSD to be 'the opposite' of PSD in their second orders and that F dominates G in SPSD, but G dominates F in SMSD. In the following corollary, we extend their result to include MSD and PSD to the second and third orders:

Corollary 2. *For any random variables X and Y, if F and G have the same mean which is finite, then we have*

a.
$$F \succeq_2^M (\succ_2^M) G \quad \textit{if and only if} \quad G \succeq_2^P (\succ_2^P) F \, ; \textit{and} \tag{10}$$

b. *if, in addition, either $F \succeq_2^M (\succ_2^M) G$ or $G \succeq_2^P (\succ_2^P) F$ holds, we have*

$$F \succeq_3^M (\succ_3^M) G \quad \textit{and} \quad G \succeq_3^P (\succ_3^P) F \, . \tag{11}$$

The proof of (10) follows the paper by Levy and Levy, while (11) follows Corollary 1. However, there are cases when distributions F and G have the same mean and do not satisfy (10) yet satisfying (11) as shown in the following example:

Example 1: Consider the distribution functions

$$F(t) = \begin{cases} 0 & -1 \le t \le -7/8, \\ 1/6 & -7/8 \le t \le -3/4, \\ 2(t+1)/3 & -3/4 \le t \le -1/2, \\ 1/3 & -1/2 \le t \le -1/4, \\ 1/2 & -1/4 \le t \le 0, \\ 1 - G(-t) & 0 \le t \le 1, \end{cases} \quad \text{and} \quad G(t) = \begin{cases} 2(t+1)/3 & -1 \le t \le -3/4, \\ 1/6 & -3/4 \le t \le -5/8, \\ 1/3 & -5/8 \le t \le -1/2, \\ 1/2 + t/3 & -1/2 \le t \le 0, \\ 1 - F(-t) & 0 \le t \le 1. \end{cases}$$

In this example, one can easily show that there is no SMSD and no SPSD dominance but

$F \succeq_3^M G$ and $G \succeq_3^P F$.[6] The above corollary provides the conditions in which F is 'the opposite' of G and the above example shows that there exist pairs of distributions which are 'opposites' in the third order but not in the second order. On the other hand, we find that under some regularities, F becomes 'the same' as G in the sense of TMSD and TPSD as shown in the corollary below:

Corollary 3. *If F and G satisfy*

$$F_2^A(0) = G_2^A(0),\ F_3^A(0) = G_3^A(0),\ F_2^a(b) = G_2^a(b),\ \text{and } F_3^a(b) = G_3^a(b), \tag{12}$$

then

$$F \succeq_3^M (\succ_3^M) G \quad \textit{if and only if} \quad F \succeq_3^P (\succ_3^P) G\,.$$

The proof of Corollary 3 is straightforward.[7] One should note that the assumptions in (12) are very restrictive. In fact, if some of the assumptions are not satisfied, there exists F and G such that $G \succeq_3^P F$ but neither $F \succeq_3^M G$ nor $G \succeq_3^M F$ holds, as shown in the following example:

Example 2: Consider

$$F(t) = \begin{cases} 4(t+1)/5 & -1 \leq t \leq -3/4, \\ 2t/5 + 1/2 & -3/4 \leq t \leq -1/4, \\ (4t+3)/5 & -1/4 \leq t \leq 0, \\ 1 - G(-t) & 0 \leq t \leq 1, \end{cases} \quad \text{and} \quad G(t) = \begin{cases} 0 & -1 \leq t \leq -3/4, \\ 2/5 & -3/4 \leq t \leq 0, \\ 1 - F(-t) & 0 \leq t \leq 1. \end{cases}$$

In this example, one can easily show that we do not have $F \succeq_3^M G$ or $G \succeq_3^M F$ but we

[6] The working is available on request.

[7] The proof is available on request.

have $G \succeq_3^P F$.[8] The above corollary and example show that under some regularities, F is 'the same' as G in the sense of TMSD and TPSD. One may wonder whether this 'same direction property' could appear in FMSD vs FPSD and SMSD vs SPSD. In the following corollary, we show that this is possible.

Corollary 4.

If the random variable $X = p + qY$ and if $p + qx \geq (>) x$ for all $x \in [a, b]$, then we have $X \succeq_i^M (\succ_i^M) Y$ and $X \succeq_i^P (\succ_i^P) Y$ for $i = 1$, 2 and 3.

The proof of the above corollary is trivial. One may first modify the proofs from Theorem 4 of Hadar and Russel (1971), Theorem 1' of Tesfatsion (1976) or Theorem 8(a) in Li and Wong (1999) to obtain the proof of Corollary 4 for $i = 1$. Thereafter, apply the hierarchical property in Corollary 1 to obtain the proof of Corollary 4 for $i = 2$ and 3.

As shown by Levy and Levy (2002), MSD is generally not 'the opposite' of PSD. In other words, if F dominates G in PSD, it does not necessarily mean that G dominates F in MSD. This is easy to see because having a higher mean is a necessary condition for dominance by both rules. Therefore, if F dominates G in the sense of PSD, and F has a higher mean than G, G cannot possibly dominate F in the sense of MSD. The above corollary goes one step further and shows that they could be 'the same' in the sense of MSD and PSD. In addition, we derive the following corollary to show the relationship between the first order MSD and PSD.

Corollary 5. *For any random variables X and Y, we have:*

$$X \succeq_1^M (\succ_1^M) Y \quad \textit{if and only if} \quad X \succeq_1^P (\succ_1^P) Y \ .$$

The proof of Corollary 5 is straightforward.[9] In addition, one can easily show that X

[8] The working is available on request.
[9] The proof is available on request.

stochastically dominates Y in the sense of FMSD or FPSD if and only if X stochastically dominates Y in the sense of the first order SD (FSD). Incorporating this into the Arbitrage versus SD theorem in Jarrow (1986) will yield the following corollary:

Corollary 6. *If the market is complete, then for any random variables X and Y, $X \succ_1^M Y$ or if $X \succ_1^P Y$ if and only if there is an arbitrage opportunity between X and Y such that one will increase one's wealth as well as one's utility if one shifts the investments from Y to X.*

Jarrow (1986) defines a 'complete' market as 'an economy where all contingent claims on the primary assets trade.' The Arbitrage versus SD theorem in Jarrow (1986) says that if the market is complete, then X stochastically dominates Y in the sense of FSD if and only if there is an arbitrage opportunity between X and Y. As $X \succeq_1^M Y$ is equivalent to $X \succeq_1^P Y$ (see Corollary 5), both are equivalent to X stochastically dominates Y in the sense of FSD. Corollary 6 holds when the Arbitrage versus SD theorem in Jarrow is applied.

The safety-first rule is first introduced by Roy (1952) for decision making under uncertainty. It stipulates choosing an alternative that provides a target mean return while minimizing the probability of the return falling below some threshold of disaster. Bawa (1978) takes the idea and examines the relationships between the SD and generalized safety-first rules for arbitrage distributions. Jarrow (1986) first studies the relationship between SD and arbitrage pricing and discovers the existence of the arbitrage opportunities in the SD rules. In this paper, we extend Jarrow's work on arbitrage pricing to both MSD and PSD.

Using the results in Theorems 1 and 2, we can call a person a first-order-MSD (FMSD) investor if his/her utility function u belongs to U_1^R, and a first-order-PSD (FPSD) investor if his/her utility function U belongs to U_1^S. A second-order-MSD (SMSD) risk investor, a second-order-PSD (SPSD) risk investor, a third-order-MSD (TMSD) risk investor and a third-order-PSD (TPSD) risk investor can be defined in the same way. From Definition 4 and the definition of risk aversion defined in (9), one can tell that the risk aversion of a SPSD investor is positive in the positive domain and negative in the negative domain and a SMSD investor's risk aversion is negative in the positive domain and positive in the negative domain. If one's risk aversion is positive and decreasing in the positive domain and negative and decreasing in the negative domain, then one is a TPSD investor; but the reverse is not true. Similarly, if one's risk aversion is negative and decreasing in the positive domain and positive and decreasing in the negative domain, then one is a TMSD investor. We summarize these results in the following corollary:

Corollary 7. *For an investor with an increasing utility function u and risk aversion r,*

a. *s/he is a SPSD investor if and only if her/his risk aversion r is positive in the positive domain and negative in the negative domain;*

b. *s/he is a SMSD investor if and only if her/his risk aversion r is negative in the positive domain and positive in the negative domain;*

c. *if her/his risk aversion r is always decreasing and is positive in the positive domain and negative in the negative domain, then s/he is a TPSD investor; and*

d. *if her/his risk aversion r is always decreasing and is negative in the positive domain and positive in the negative domain, then s/he is a TMSD investor.*

The proof of Corollary 7 is straightforward.[10] Corollary 7 states the relationships between different types of investors and their risk aversions. We note that the converse of (c) and (d) are not true.

[10] The proof is available on request.

4 Illustration

In this section we illustrate each case of MSD and PSD to the first three orders by using examples from Levy and Levy (2002) and modifying them. We first use Task III of Experiment 3 in Levy and Levy (2002) which is a replication of the tasks in Kahneman and Tversky (1979). In the experiment, \$10,000 is invested in either stock F or Stock G with the following dollar gain one month later and with probabilities f and g respectively, as shown in Table 1.

We use the MSD and PSD integrals H_i^M and H_i^P for $H = F$ and G and $i = 1, 2$ and 3 as defined in (6). To make the comparison easier, we define their differentials

$$GF_i^M = G_i^M - F_i^M \quad \text{and} \quad GF_i^P = G_i^P - F_i^P \tag{13}$$

for $i = 1, 2$ and 3 and present the results of the MSD and PSD integrals with their differentials for the first three orders in Tables 2 and 3.

In this example, Levy and Levy conclude that $F \succeq_{MSD} G$ but $G \succeq_{PSD} F$ while our results show that $F \succeq_i^M G$ and $G \succeq_i^P F$ for $i = 2$ and 3. From Corollary 1, we know that hierarchy exists in both MSD and PSD such that $F \succeq_2^M G$ implies $F \succeq_3^M G$ while $G \succeq_2^P F$ implies $G \succeq_3^P F$. Hence, one only has to report the lowest SD order. Our findings shows that $F \succeq_2^M G$ and $G \succeq_2^P F$, same as the findings in Levy and Levy. Our approach has no advantage over Levy and Levy's in this example. However, Levy and Levy's approach can only detect the second order MSD and PSD while our approach, by incorporating the extended PSD and MSD, enables investors to compare MSD and PSD to any order. In order to show the superiority of our approach, we modify the above experiment by adjusting the probabilities f and g for investments F and G respectively. Reported in Tables 4–6 are all other orders of both MSD and PSD. For simplicity, we only report the differentials GF_i^M and GF_i^P and skip reporting their integrals. For easy comparison, we also report the MSD and PSD computation based on Levy and Levy's formula:

$$GF^M = G^M - F^M \quad \text{and} \quad GF^P = G^P - F^P \quad . \tag{14}$$

Note that Levy and Levy define $F \succeq_{MSD} G$ if $GF^M(x) \geq 0$ for all x and $F \succeq_{PSD} G$ if $GF^P(x) \geq 0$ for all x with some strict inequality.

In Table 4, if one adopts Levy and Levy's approach, one will conclude that $F \succeq_{MSD} G$ and $F \succeq_{PSD} G$. However, if one applies our approach, one will conclude that $F \succeq_1^M G$ and $F \succeq_1^P G$, which is different from the conclusion drawn from Levy and Levy's approach. From Corollary 1, we know that hierarchy exists in both MSD and PSD such that $F \succeq_1^M G$ implies $F \succeq_2^M G$ while $G \succeq_1^P F$ implies $G \succeq_2^P F$. Hence, one only has to report the lowest SD order. However, reporting the first order MSD and PSD obtained by using our approach should be more appropriate.

In Table 5, if one uses Levy and Levy's approach, one will conclude that neither $F \succeq_{MSD} G$ nor $F \succeq_{MSD} G$, instead $G \succeq_{PSD} F$. However, if one applies our approach, one will conclude that $G \succeq_2^P F$ but $F \succeq_3^M G$, which is different from the conclusion drawn from Levy and Levy's approach. Similarly, in Table 6, if one uses Levy and Levy's approach, one will conclude that neither $F \succeq_{PSD} G$ nor $F \succeq_{PSD} G$, instead $G \succeq_{MSD} F$. However, if one applies our approach, one will conclude that $F \succeq_2^M G$ but $G \succeq_3^P F$, which is different from the conclusion drawn from Levy and Levy's approach. Thus, our approach reveals more information on both MSD and PSD.

The results from our illustrations are more informative for investors than Levy and Levy's because we identify the MSD and PSD prospects for the first three orders while Levy and Levy only identify MSD and PSD for the second order, which may not truly present the MSD and PSD nature of these prospects. As our approach can provide investors with more information about investments opportunities, our approach could enable investors to make wiser decisions on investments. For example, in Table 4, using Levy and Levy's approach, SMSD and SPSD (also TMSD and TPSD) investors will choose to invest on F rather than G and will increase their expected utilities but not their wealth when shifting their investments from G to F. For FMSD and FPSD investors, they will not be able to obtain any useful information at all. However, if investors adopt our approach, it will be a completely different story. FMSD, SMSD, TMSD, FPSD, SPSD and TPSD investors will choose to invest on F rather than G and all of them will increase their expected utilities as well as their wealth when shifting their investments from G to F. What's more, our approach enables investors to identify that there is an arbitrage opportunity between F and G and one could long F and short G and making good profit.

Furthermore, Levy and Levy's approach will not be able to reveal any TMSD or TPSD prospect, while ours will enable investors to identify them, which in turn provides useful information for the TMSD and TPSD investors. If the approach by Levy and Levy is applied, one will conclude neither MSD nor PSD. For the TMSD and TPSD investors, they will not know about the relationships between these prospects and will miss these investment opportunities. For example, referring to Table 5, TMSD investors will not be able to decide which prospect to invest if they apply Levy and Levy's approach. However, if they apply our approach, they will invest in F rather than G and if they have invested in G, our approach will tell them that they will increase their expected utilities if they shift their investments from G to F. Similar conclusion can be made by TPSD investors about the investment choices presented in Table 6. We note that SD for both risk averters and risk seekers can be extended to any order. Our approach can also be easily extended to any order. Hence if investors need to identify any prospect of MSP or PSD of an order higher than three, they could easily extend our theory to meet their needs.

5 Concluding Remarks

In this paper, we extend the MSD and PSD theory by first defining the MSD and PSD of the first three orders and link them to the corresponding S-shaped and reverse S-shaped utility functions to the first three orders. We then provide experiments to illustrate each case of the MSD and PSD to the first three orders and demonstrate that the higher order of MSD and PSD cannot be replaced by the lower order MSD and PSD. In addition, we develop some properties for the extended MSD and PSD including the hierarchy that exists in both PSD and MSD relationships; arbitrage opportunity that exists for the first orders of both PSD and MSD; and for any two prospects satisfying certain conditions, their third order of MSD preference will be 'the opposite' of or 'the same' as their third order counterpart PSD preference.

Prospect theory is a paradigm challenging the expected utility theory. The main controversy is the prospect theory's S-shaped value function which describes preferences. This has been discussed in our paper in detail and our conclusion is that it is consistent with the expected utility theory. The next allegation is that the prospect theory invalidates the expected utility theory as being "subjectively distorted probabilities" (Levy and Wiener 1998)[11]. This was later corrected by what is now known as Cumulative Prospect Theory, see Starmer (2000) for the review of the subject. We suggest incorporating the Bayesian approach (Matsumura, et al. 1990) and distribution-free statistics (see for example, Wong and Miller 1990) into the subjective probability (see for example, Anscombe and Aumann 1963 and Machina and Schmeidler 1992) to estimate the subjectively distorted probabilities. Prospect theory will satisfy the Bayesian expected utility maximization. Thus, the problem that the prospect theory violates the expected utility theory could be circumvented.

[11] We note that Wu et al. (2005) develop a critical test of the two prospect theories based on their respective probability tradeoff consistency conditions.

The advantage of the stochastic dominance approach is that we have a decision rule which holds for all utility functions of certain class. Specifically, PSD (MSD) of any order is a criterion which is valid for all S-shaped (reverse S-shaped) utility functions of the corresponding order. Moreover, the SD rules for S-shaped and reverse S-shaped utility functions can be employed with mixed prospects. We note that in our paper we do not restrict the S-shaped utility functions to be steeper than their shapes for gains as the restricted set in value functions defined by Kahneman and Tversky (1979) and Tversky and Kahneman (1992). However, the class of S-shaped utility functions defined in our paper is more general and contains all the value functions with this restriction. Wakka (2003) claims that some examples in Levy and Levy (2002) violate this curvature restriction on value function posited by prospect theory. In this paper, as we follow Levy and Levy (2002)'s definition of S-shaped utility function without this curvature restriction. Our examples could be set without this curvature restriction. However, one could easier show that all examples with this curvature restriction will fit our theory well.

The MSD and PSD developed by Levy and Wiener (1998), Levy and Levy (2002, 2004) and the extensions in our paper only link to the S-shaped and reverse S-shaped utility functions. These utility functions are simplified version of the utility functions proposed by Markowitz (1952) which have convex and concave regions in both the positive and the negative domains. Empirical studies reveal a more complex behavior. For example, people are mostly risk averse to prospects yielding a best outcome with a low probability but they are mostly risk seeking to prospects yielding a worst outcome with a low probability (Starmer 2000 and Luce 2000). Further research includes extension of the MSD and PSD to link to this more complicated patterns of behavior.

These days, it is popular to apply SD to explain financial theories and anomalies, for example, McNamara (1998), Post and Levy (2005), Post (2003), Kuosmanen (2004), Fong et al. (2005) and Broll et al. (2006). Some apply the Levy and Levy approach to study risk averse and risk seeking behaviors. For example, Post and Levy (2005) study risk seeking behaviors in order to explain the cross-sectional patterns of stock returns and suggest that the reverse S-shaped utility functions can explain stock returns, with risk aversion to losses and risk seeking for gains reflecting investors' twin desire for downside protection in bear markets and upside potential in bull markets. Using the second order PSD and MSD introduced by Levy and Levy is too restrictive. We recommend that financial analysts and investors apply the approach introduced in this paper and examine the MSD and PSD relationships of different orders so that they can make wiser decisions about their investments.

Appendices

We only prove the necessary condition for both Theorems 1 and 2. The sufficient condition can be proved by contradiction. Huang and Litzenberger (1988) and others have proved the sufficeint condition of SD for risk averters. One could easily modify their proofs to obtain the proof of sufficeint conditions in Theorems 1 and 2 of our paper.

Appendix A – Proof of Theorem 1:

Levy and Levy (2002) have proved the second order of Theorem 1. Suppose $[a, b]$ is the support with negative a and positive b, we modify and extend their proof to include the first three orders of the MSD as follows:

$$
\begin{aligned}
\Delta Eu &\equiv u(F) - u(G) \equiv \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) \\
&= [F(x) - G(x)]u(x)\big|_a^b - \int_a^b [F(x) - G(x)]u^{(1)}(x)\,dx \\
&= \int_a^b [G(x) - F(x)]u^{(1)}(x)\,dx \\
&= \int_a^0 [G(x) - F(x)]u^{(1)}(x)\,dx + \int_0^b [G(x) - F(x)]u^{(1)}(x)\,dx \\
&= \int_a^0 [G_1^A(x) - F_1^A(x)]u^{(1)}(x)\,dx + \int_0^b [F_1^D(x) - G_1^D(x)]u^{(1)}(x)\,dx \qquad (15) \\
&= \int_a^0 u^{(1)}(x)\,d\,[G_2^A(x) - F_2^A(x)] - \int_0^b u^{(1)}(x)\,d\,[F_2^D(x) - G_2^D(x)] \\
&= [G_2^A(x) - F_2^A(x)]u^{(1)}(x)\big|_a^0 - \int_a^0 [G_2^A(x) - F_2^A(x)]u^{(2)}(x)\,dx - \\
&\quad [F_2^D(x) - G_2^D(x)]u^{(1)}(x)\big|_0^b + \int_0^b [F_2^D(x) - G_2^D(x)]u^{(2)}(x)\,dx \\
&= B_1 + \int_a^0 [F_2^A(x) - G_2^A(x)]u^{(2)}(x)\,dx + \int_0^b [F_2^D(x) - G_2^D(x)]u^{(2)}(x)\,dx \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&= B_1 + \int_a^0 u^{(2)}(x)\, d\,[F_3^A(x) - G_3^A(x)] - \int_0^b u^{(2)}(x)\, d\,[F_3^D(x) - G_3^D(x)] \\
&= B_1 + [F_3^A(x) - G_3^A(x)]u^{(2)}(x)\Big|_a^0 - \int_a^0 [F_3^A(x) - G_3^A(x)]u^{(3)}(x)\, dx - \\
&\quad [F_3^D(x) - G_3^D(x)]u^{(2)}(x)\Big|_0^b + \int_0^b [F_3^D(x) - G_3^D(x)]u^{(3)}(x)\, dx \\
&= B_1 + B_2 + \int_a^0 [G_3^A(x) - F_3^A(x)]u^{(3)}(x)\, dx + \int_0^b [F_3^D(x) - G_3^D(x)]u^{(3)}(x)\, dx \quad (17)
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= [G_2^A(0) - F_2^A(0) + F_2^D(0) - G_2^D(0)]u^{(1)}(0) \quad \text{and} \\
B_2 &= [F_3^A(0) - G_3^A(0) + F_3^D(0) - G_3^D(0)]u^{(2)}(0)\,. \qquad (18)
\end{aligned}
$$

From (15), we have if $F \succeq_1^M G$ then $F_1^D(x) \geq G_1^D(x)$ for $x \geq 0$ and $F_1^A(x) \leq G_1^A(x)$ for $x \leq 0$. If $u \in U_1^R$ then $u^{(1)} \geq 0$. Hence $\Delta Eu = u(F) - u(G) \geq 0$.

If $F \succeq_2^M G$, then $F_2^D(x) \geq G_2^D(x)$ for $x \geq 0$ and $F_2^A(x) \leq G_2^A(x)$ for $x \leq 0$. If in addition, $u \in U_2^S$ then $u^{(1)} \geq 0$, $u^{(2)}(x) \geq 0$ for $x \geq 0$ and $u^{(2)}(x) \leq 0$ for $x \leq 0$. From (18), $B_1 \geq 0$, and hence from (16), $\Delta Eu = u(F) - u(G) \geq 0$.

If $F \succeq_3^M G$, then $F_3^D(x) \geq G_3^D(x)$ for $x \geq 0$ and $F_3^A(x) \leq G_3^A(x)$ for $x \leq 0$. If in addition, $u \in U_3^R$ then $u^{(1)} \geq 0$, $u^{(2)}(x) \geq 0$ for $x \geq 0$, $u^{(2)}(x) \leq 0$ for $x \leq 0$, $u^{(2)}(0) = 0$ and $u^{(3)} \geq 0$. From (18), we have $B_2 = 0$ and hence from (17), $\Delta Eu = u(F) - u(G) \geq 0$.

Appendix B – Proof of Theorem 2:

Levy and Levy (2002) have proved the second order of Theorem 2. Suppose $[a, b]$ is the support with negative a and positive b, we modify and extend their proof to include the first three orders of the PSD as follows:

$$
\begin{aligned}
\Delta Eu &\equiv u(F) - u(G) \equiv \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) \\
&= \int_a^0 [G(x) - F(x)]u^{(1)}(x)\, dx + \int_0^b [G(x) - F(x)]u^{(1)}(x)\, dx \\
&= \int_a^0 [F_1^d(y) - G_1^d(y)]u^{(1)}(y)\, dy + \int_0^b [G_1^a(x) - F_1^a(x)]u^{(1)}(x)\, dx \qquad (19) \\
&= \int_a^0 u^{(1)}(y)\, d\,[G_2^d(y) - F_2^d(y)] + \int_0^b u^{(1)}(x)\, d\,[G_2^a(x) - F_2^a(x)] \\
&= [G_2^d(y) - F_2^d(y)]u^{(1)}(y)\Big|_a^0 + \int_a^0 [F_2^d(y) - G_2^d(y)]u^{(2)}(y)\, dy \\
&\quad + [G_2^a(x) - F_2^a(x)]u^{(1)}(x)\Big|_0^b + \int_0^b [F_2^a(x) - G_2^a(x)]u^{(2)}(x)\, dx \\
&= B_2 + \int_a^0 [F_2^d(y) - G_2^d(y)]u^{(2)}(y)\, dy + \int_0^b [F_2^a(x) - G_2^a(x)]u^{(2)}(x)\, dx \qquad (20) \\
&= B_2 + \int_a^0 u^{(2)}(y)\, d\,[G_3^d(y) - F_3^d(y)] + \int_0^b u^{(2)}(x)\, d\,[F_3^A(x) - G_3^A(x)] \\
&= B_2 + [G_3^d(y) - F_3^d(y)]u^{(2)}(y)\Big|_a^0 + \int_a^0 [F_3^d(y) - G_3^d(y)]u^{(3)}(y)\, dx \\
&\quad + [F_3^a(x) - G_3^a(x)]u^{(2)}(x)\Big|_0^b + \int_0^b [G_3^a(x) - F_3^a(x)]u^{(3)}(x)\, dx \\
&= B_2 + B_3 + \int_a^0 [F_3^d(y) - G_3^d(y)]u^{(3)}(y)\, dy + \int_0^b [G_3^a(x) - F_3^a(x)]u^{(3)}(x)\, dx \qquad (21)
\end{aligned}
$$

where

$$
\begin{aligned}
B_2 &= u^{(1)}(a)[F_2^d(a) - G_2^d(a)] + u^{(1)}(b)[G_2^a(b) - F_2^a(b)] \\
&\quad + u^{(1)}(0)[G_2^d(0) - F_2^d(0)] + u^{(1)}(0)[F_2^a(0) - G_2^a(0)] \quad \text{and} \\
B_3 &= u^{(2)}(a)[F_3^d(a) - G_3^d(a)] + u^{(2)}(b)[F_3^a(b) - G_3^a(b)] \\
&\quad + u^{(2)}(0)[G_3^d(0) - F_3^d(0)] + u^{(2)}(0)[G_3^a(0) - F_3^a(0)]
\end{aligned}
\tag{22}
$$

As $a \leq 0$, $F_2^d(a) \geq G_2^d(a)$. Similarly $G_2^a(b) \geq F_2^a(b)$ as $b \geq 0$. Since $u^{(1)}(a)$, $u^{(1)}(b)$ are nonnegative; $H_2^d(0) = H_2^a(0) = 0$ for $H = F$ and G, we see that $B_2 \geq 0$. Also, as $a \leq 0$, $F_3^d(a) \geq G_3^d(a)$ and also we have $u^{(2)}(a) \geq 0$. Similarly $G_2^a(b) \geq F_2^a(b)$ as $b \geq 0$, but we have $u^{(2)}(b) \leq 0$. In addition, $H_3^d(0) = H_3^a(0) = 0$ for $H = F$ and G, We see that $B_3 \geq 0$.

Hence, from (19), if $X \succeq_1^P Y$ or $F \succeq_1^P G$, then we have $\Delta Eu = u(F) - u(G) \geq 0$; from (20), we have if $X \succeq_2^P Y$ or $F \succeq_2^P G$, then $\Delta Eu = u(F) - u(G) \geq 0$; and from (21), we have if $X \succeq_3^P Y$ or $F \succeq_3^P G$, then $\Delta Eu = u(F) - u(G) \geq 0$.

References

Anderson, G.J. (2004). "Toward an Empirical Analysis of Polarization," *Journal of Econometrics* 122, 1-26.

Anscombe, F.J., and R.J. Aumann. (1963). "A Definition of Subjective Probability," *Annals of Mathematical Statistics* 34(1), 199-205.

Arrow, K.J. (1971). *Essays in the Theory of Risk-Bearing,* Chicago: Markham.

Barberis, N., M. Huang, and T. Santos. (2001). "Prospect Theory and Asset Prices," *Quarterly Journal of Economics* 116, 1-53.

Bawa, V.S. (1975). "Optimma Rules for Ordering Uncertain Prospects," *Journal of Financial Economics* 2, 95-121.

Bawa, V.S. (1978). "Safety-first, Stochastic Dominance, and Optimal Portfolio Choice," *Journal of Financial and Quantitative Analysis* 13, 255-271.

Benartzi, S., and R. Thaler. (1995). "Myopic Loss Aversion and the Equity Premium Puzzle," *Quarterly Journal of Economics* 110(1), 73-92.

Broll, U., J.E. Wahl, and W.K. Wong. (2006). "Elasticity of Risk Aversion and International Trade," *Economics Letters* 91(1), 126-130.

Chateauneuf, A., and M. Cohen. (1994). "Risk Seeking with Diminishing Marginal Utility in a Non-Expected Utility Model," *Journal of Risk Uncertainty* 9, 77-91.

Currim, I.S., and R.K. Sarin. (1989). "Prospect Versus Utility," *Management Science* 35, 22-41.

Falk, H., and H. Levy. (1989). "Market Reaction to Quarterly Earnings' Announcements: A Stochastic Dominance Based Test of Market Efficiency," *Management Science* 35(4), 425-446.

Fennema, H., and H. van Assen. (1999). "Measuring the Utility of Losses by Means of the Trade-Off Method," *Journal of Risk and Uncertainty* 17, 277-295.

Fishburn, P.C. (1964). *Decision and Value Theory,* New York.

Fishburn, P.C., and G.A. Kochenberger. (1979). "Two-piece Von Neumann-Morgenstern Utility Functions," *Decision Sciences* 10, 503-518.

Fong, W.M., W.K. Wong, and H.H. Lean. (2005). "International Momentum Strategies: A Stochastic Dominance Approach," *Journal of Financial Markets* 8, 89-109.

Friedman, M., and L.J. Savage. (1948). "The Utility Analysis of Choices Involving Risk," *Journal of Political Economy* 56, 279-304.

Gotoh, J.Y., and K. Hiroshi. (2000). "Third Degree Stochastic Dominance and Mean-Risk Analysis," *Management Science* 46(2), 289-301.

Gneezy, U., J.a. List, and G. Wu. (2006). "The Uncertainty Effect: When a Risky Prospect is Valued Less Than Its Worst Possible Outcome," *Quarterly Journal of Economics,* (forthcoming).

Hadar, J., and W.R. Russel. (1969). "Rules for Ordering Uncertain Prospects," *American Economic Review* 59, 25-34.

Hadar, J., and W.R. Russel. (1971). "Stochastic Dominance and Diversification," *Journal of Economic Theory* 3, 288-305.

Hammond, J.S. (1974). "Simplifying the Choice between Uncertain Prospects where Preference is Nonlinear," *Management Science* 20(7), 1047-1072.

Hanoch G., and H. Levy. (1969). "The Efficiency Analysis of Choices Involving Risk," *Review of Economic studies* 36, 335-346.

Huang, Chi-fu, and Robert H. Litzenberger. (1988). *Foundations for Financial Economics,* New York : North-Holland.

Jarrow, R. (1986). "The Relationship between Arbitrage and First Order Stochastic Dominance," *Journal of Finance* 41, 915-921.

Kahneman, D., and A. Tversky. (1979). "Prospect Theory of Decisions under Risk," *Econometrica* 47(2), 263-291.

Kahneman, D., J. L. Knetsch, and R. H. Thaler. (1990). "Experimental Test of the Endowment Effect and the Coase Theorem," *Journal of Political Economy* 98, 1325-1348.

Kobberling, Veronika, and Peter P. Wakker. (2005). "An Index of Loss Aversion," *Journal of Economic Theory* 122, 119-131.

Kuosmanen, T. (2004). "Efficient Diversification According to Stochastic Dominance Criteria," *Management Science* 50(10), 1390-1406.

Levy, H. (1998). *Stochastic Dominance: Investment Decision Making under Uncertainty,* Kluwer Academic Publishers Boston/Dordrecht/London.

Levy, H., and M. Levy. (2004). "Prospect Theory and Mean-Variance Analysis," *Review of Financial Studies* 17(4), 1015-1041.

Levy, M., and H. Levy. (2002). "Prospect Theory: Much Ado About Nothing?" *Management Science* 48(10), 1334-1349.

Levy, H., and Z. Wiener. (1998). "Stochastic Dominance and Prospect Dominance with Subjective Weighting Functions," *Journal of Risk and Uncertainty* 16(2), 147-163.

Li, C.K., and W.K. Wong. (1999). "A Note on Stochastic Dominance for Risk Averters and Risk Takers," *RAIRO Recherche Operationnelle* 33, 509-524.

Luce, R. Duncan. (2000). *Utility of Gains and Losses: Measurement Theoretical and Experimental Approaches,* Lawrence Erlbaum Publishers, London. UK.

Machina, Mark J. (1982). "'Expected Utility' Analysis without the Independence Axiom," *Econometrica* 50(2), 277-324.

Machina, Mark J., and David Schmeidler. (1992). "A More Robust Definition of Subjective Probability," *Econometrica* 60(4), 745-780.

Markowitz, H.M. (1952). "The Utility of Wealth," *Journal of Political Economy* 60, 151-156.

Matsumura, E.M., K.W. Tsui, and W.K. Wong. (1990). "An Extended Multinomial-Dirichlet Model for Error Bounds for Dollar-Unit Sampling," *Contemporary Accounting Research* 6(2-I), 485-500.

McNamara, J.R. (1998). "Portfolio Selection Using Stochastic Dominance Criteria," *Decision Sciences* 29(4), 785-801.

Meyer, J. (1977). "Second Degree Stochastic Dominance with Respect to a Function," *International Economic Review* 18, 476-487.

Myagkov, M., and C.R. Plott. (1997). "Exchange Economies and Loss Exposure: Experiments Exploring Prospect Theory and Competitive Equilibria in Market Environments," *American Economic Review* 87, 801-828.

Ng, M.C. (2000). "A Remark on Third Degree Stochastic Dominance," *Management Science* 46(6), 870-873.

Ng, Y.K. (1965). "Why do People Buy Lottery Tickets? Choices Involving Risk and the Indivisibility of Expenditure," *Journal of Political Economy* 73(5), 530-535.

Pennings, J.M.E., and A. Smidts. (2003). "The Shape of Utility Functions and Organizational Behavior," *Management Science* 49(9), 1251 -1263.

Quirk J.P., and R. Saposnik. (1962). "Admissibility and Measurable Utility Functions," *Review of Economic Studies* 29, 140-146.

Post, T. (2003). "Stochastic Dominance in Case of Portfolio Diversification: Linear Programming Tests," *Journal of Finance* 58(5), 1905-1931.

Post, T., and H. Levy. (2005). "Does Risk Seeking Drive Asset Prices? A

Stochastic Dominance Analysis of Aggregate Investor Preferences and Beliefs," *Review of Financial Studies* 18(3), 925-953.

Pratt, J.W. (1964). "Risk Aversion in the Small and in the Large," *Econometrica* 32, 122-136.

Rabin, M. (2000). "Risk Aversion and Expected-Utility Theory: A Calibration Theorem," *Econometrica* 68, 1281-1292.

Rizzo, J.A., and R.J. Zeckhauser. (2004). "Reference Incomes, Loss Aversion, and Physician Behavior," *Review of Economics and Statistics* 85, 909-922.

Roy, A.D. (1952). "Safety First and the Holding of Assets," *Econometrica* 20(3), 431-449.

Shefrin, H., and M. Statman. (1993). "Behavioral Aspect of the Design and Marketing of Financial Products,"*Financial Management* 22(2), 123-134.

Shorrocks, A., and J. Foster. (1987). "Transfer Sensitive Inequality Measures," *Review of Economic Studies* 54, 485-497.

Starmer, Chris. (2000). "Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk," *Journal of Economic Literature* 38, 332-382.

Stoyan, D. (1983). *Comparison Methods for Queues and Other Stochastic Models,* New York: Wiley.

Swalm, R.O. (1966). "Utility Theory-Insights into Risk Taking," *Harvard Business Review* 44, 123-136.

Tesfatsion, L. (1976). "Stochastic Dominance and Maximization of Expected Utility," *Review of Economic Studies* 43, 301-315.

Thaler, R.H., and E.J. Johnson. (1990). "Gambling with the House Money and Trying to Break Even: the Effects of Prior Outcomes on Risky Choice," *Management Science* 36, 643-660.

Tversky, A., and D. Kahneman. (1981). "The Framing of Decisions and the Psychology of Choice," *Science* 211, 453-480.

Tversky, A., and D. Kahneman. (1992). Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty* 5, 297-323.

von Neuman, J., and O. Morgenstern. (1944). *Theory of Games and Economic Behavior,* Princeton University Press, Princeton, NJ.

Wakker, Peter P. (2003). "The Data of Levy and Levy (2002) "Prospect Theory: Much Ado About Nothing?" Actually Support Prospect Theory," *Management Science* 49(7), 979-981.

Whitmore, G.A. (1970). "Third-degree Stochastic Dominance," *American Economic Reivew* 60, 457-459.

Whitmore, G.A., and M.C. Findley. (1978). *Stochastic Dominance: An approach to decision making under risk,* Heath, Lexington: Mars.

Williams, C.A.Jr. (1966). "Attitudes toward Speculative Risks as an Indicator of Attitudes toward Pure Risks," *Journal of Risk and Insurance* 33(4), 577-586.

Wang, M., and P.S. Fischbeck, (2004). "Incorporating Framing into Prospect Theory Modeling: A Mixture-Model Approach," *Journal of Risk and Uncertainty* 29(2), 181-197.

Wong, W.K., and C.K. Li. (1999). "A Note on Convex Stochastic Dominance Theory," *Economics Letters* 62, 293-300.

Wong, W.K., and R.B. Miller. (1990). "Analysis of ARIMA-Noise Models with Repeated Time Series," *Journal of Business and Economic Statistics* 8(2), 243-250.

Wu, G. and R. Gonzalez. (1996). "Curvature of the Probability Weighting Function," *Management Science,* 42, 1676-1690.

Wu, G., J.Z. Wu, and M. Abdellaoui. (2005). "Testing Prospect Theory using Probability Tradeoff Consistency," *Journal of Risk and Uncertainty,* 30, 107-131.

Table 1 : The Distributions for Investments F and G

Investment F		Investment G	
Gain	Probability (f)	Gain	Probability (g)
-1,500	$\frac{1}{2}$	-3,000	$\frac{1}{4}$
4,500	$\frac{1}{2}$	3,000	$\frac{3}{4}$

Table 2 : The MSD Integrals and their Differentials for F and G

Gain	First Order			Second Order			Third Order		
X	F_1^M	G_1^M	GF_1^M	F_2^M	G_2^M	GF_2^M	F_3^M	G_3^M	GF_3^M
-3	0	0.25	0.25	0	0	0	0	0	0
-1.5	0.5	0.25	-0.25	0	0.375	0.375	0	0.28125	0.28125
0^-	0.5	0.25	-0.25	0.75	0.75	0	0.5625	1.125	0.5625
0^+	0.5	0.75	0.25	2.25	2.25	0	5.0625	3.375	-1.6875
3	0.5	0.75	0.25	0.75	0	-0.75	0.5625	0	-0.5625
4.5	0.5	0	-0.5	0	0	0	0	0	0

Table 3 : The PSD Integrals and their Differentials for F and G

Gain	First Order			Second Order			Third Order		
X	F_1^P	G_1^P	GF_1^P	F_2^P	G_2^P	GF_2^P	F_3^P	G_3^P	GF_3^P
-3	1	1	0	2.25	2.25	0	2.8125	3.375	0.5625
-1.5	1	0.75	-0.25	0.75	1.125	0.375	0.5625	0.84375	0.28125
0^-	0.5	0.75	0.25	0	0	0	0	0	0
0^+	0.5	0.25	-0.25	0	0	0	0	0	0
3	0.5	1	0.5	1.5	0.75	-0.75	2.25	1.125	-1.125
4.5	1	1	0	2.25	2.25	0	5.0625	3.375	-1.6875

Table 4 : The MSP and PSD Differentials for F and G : Case 2

Gain	probability		MSD			PSD			Levy and Levy	
X	f	g	GF_1^M	GF_2^M	GF_3^M	GF_1^P	GF_2^P	GF_3^P	GF^M	GF^P
-3	0	0.25	0.25	0	0	0	-0.45	-0.45	0	0.45
-1.5	0.2	0	0.05	0.375	0.28125	-0.25	-0.075	-0.05625	0.375	0.075
0^-	0	0	0.05	0.45	0.9	-0.05	0	0	0.45	0
0^+	0	0	-0.05	-1.35	-4.785	0.05	0	0	1.35	0
3	0	0.75	-0.05	-1.2	-0.9	0.8	0.15	0.225	1.2	0.15
4.5	0.8	0	-0.8	0	0	0	1.35	1.35	0	1.35

Table 5 : The MSP and PSD Differentials for F and G : Case 3

Gain	probability		MSD			PSD			Levy and Levy	
X	f	g	GF_1^M	GF_2^M	GF_3^M	GF_1^P	GF_2^P	GF_3^P	GF^M	GF^P
-3	0	0.25	0.25	0	0	0	0.075	0.73125	0	-0.075
-1.5	0.55	0	-0.3	0.375	0.28125	-0.25	0.45	0.3375	0.375	-0.45
0^-	0	0	-0.3	-0.075	0.50625	0.3	0	0	-0.075	0
0^+	0	0	0.3	0.225	-1.18125	-0.3	0	0	-0.225	0
3	0	0.75	0.3	-0.675	-0.50625	0.45	-0.9	-1.35	0.675	-0.9
4.5	0.45	0	-0.45	0	0	0	-0.225	-2.19375	0	-0.225

Table 6 : The MSP and PSD Differentials for F and G : Case 4

Gain	probability		MSD			PSD			Levy and Levy	
X	f	g	GF_1^M	GF_2^M	GF_3^M	GF_1^P	GF_2^P	GF_3^P	GF^M	GF^P
-3	0	0.25	0.25	0	0	0	-0.15	0.225	0	0.15
-1.5	0.4	0	-0.15	0.375	0.28125	-0.25	0.225	0.16875	0.375	-0.225
0^-	0	0	-0.15	0.15	0.625	0.15	0	0	0.15	0
0^+	0	0	0.15	-0.45	-2.7	-0.15	0	0	0.45	0
3	0	0.75	0.15	-0.9	-0.625	0.6	-0.45	-0.675	0.9	-0.45
4.5	0.6	0	-0.6	0	0	0	0.45	-0.675	0	0.45

Figure 1: Functions in U_2^A, U_2^D, U_2^S and U_2^R

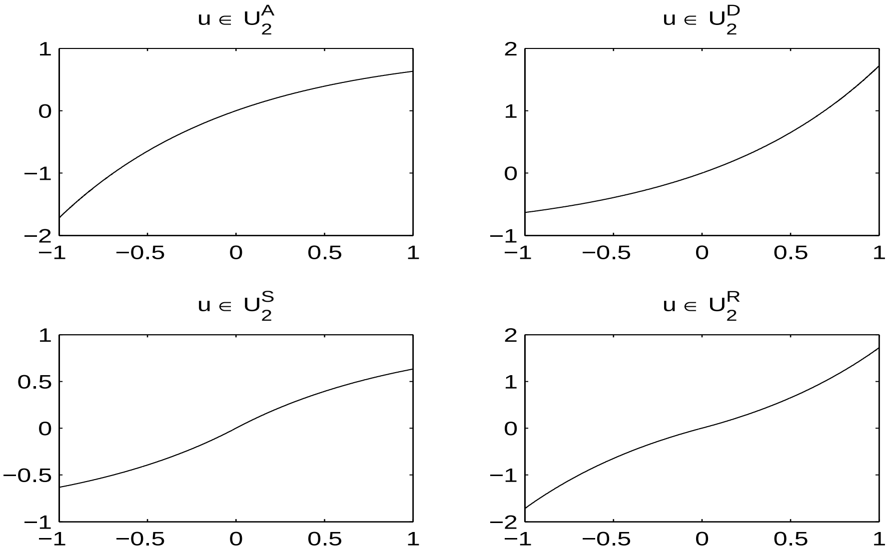

Figure 2: Derivatives of Functions in U_3^A, U_3^D, U_3^S and U_3^R