

## The differential analytic index in Simons-Sullivan differential K-theory

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*Published in:*  
Annals of Global Analysis and Geometry

*DOI:*  
[10.1007/s10455-012-9325-1](https://doi.org/10.1007/s10455-012-9325-1)

Published: 01/12/2012

[Link to publication](#)

*Citation for published version (APA):*  
HO, M. H. (2012). The differential analytic index in Simons-Sullivan differential K-theory. *Annals of Global Analysis and Geometry*, 42(4), 523-535. <https://doi.org/10.1007/s10455-012-9325-1>

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# THE DIFFERENTIAL ANALYTIC INDEX IN SIMONS-SULLIVAN DIFFERENTIAL $K$ -THEORY

MAN-HO HO

*Dedicated to my father Kar-Ming Ho*

ABSTRACT. We define the Simons-Sullivan differential analytic index by translating the Freed-Lott differential analytic index via explicit ring isomorphisms between Freed-Lott differential  $K$ -theory and Simons-Sullivan differential  $K$ -theory. We prove the differential Grothendieck-Riemann-Roch theorem in Simons-Sullivan differential  $K$ -theory using a theorem of Bismut.

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## 1. INTRODUCTION

As explained in [3], [4], [7], [10], the physics motivation for differential  $K$ -theory is a quantum field theory whose Lagrangian has differential form field strength. This leads to a generalized cohomology theory with a map to ordinary cohomology that implements charge quantization. In [7] Freed argued that there should be a similar extension of topological  $K$ -theory. We refer to [8, §1.4] for a historical discussion. The mathematical motivation for differential  $K$ -theory can be traced to Cheeger-Simons differential characters [6], the unique differential extension of ordinary cohomology [14], and to the work of Karoubi [11]. It is thus natural to look for differential extensions of generalized cohomology theories, for example topological  $K$ -theory. The

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2010 *Mathematics Subject Classification.* Primary 19L50; Secondary 19K56.

differential extension of topological  $K$ -theory is now known as differential  $K$ -theory. Roughly speaking, differential  $K$ -theory is topological  $K$ -theory combined with differential form data in a complicated way, just as differential characters combine ordinary cohomology with differential form data. Various definitions of differential  $K$ -theory have been proposed, notably by Bunke-Schick [3], Freed-Lott [8], Hopkins-Singer [10] and Simons-Sullivan [15]. Axioms for differential extensions of generalized cohomology theories are given in [4], where it is shown that two differential extensions of a fixed generalized cohomology theory satisfying certain conditions are uniquely isomorphic. In particular the four models of differential  $K$ -theory mentioned above are isomorphic by this abstract result. For more details and an introduction to differential  $K$ -theory, see [5].

The Atiyah-Singer family index theorem can be formulated as the equality of the analytic and topological pushforward maps  $\text{ind}^{\text{an}} = \text{ind}^{\text{top}} : K(X) \rightarrow K(B)$ . Applying the Chern character, we get the Grothendieck-Riemann-Roch theorem, the commutativity of the following diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{\text{ch}} & H^{\text{even}}(X; \mathbb{Q}) \\ \text{ind}^{\text{an}} \downarrow & & \downarrow \int_{X/B} \text{Todd}(X/B) \cup (\cdot) \\ K(B) & \xrightarrow[\text{ch}]{} & H^{\text{even}}(B; \mathbb{Q}) \end{array}$$

Analogous theorems hold in differential  $K$ -theory. Bunke-Schick prove the differential Grothendieck-Riemann-Roch theorem (dGRR) [3, Theorem 6.19], i.e., for a proper submersion  $\pi : X \rightarrow B$  of even relative dimension, the following diagram is commutative:

$$\begin{array}{ccc} \widehat{K}_{\text{BS}}(X) & \xrightarrow{\widehat{\text{ch}}_{\text{BS}}} & \widehat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q}) \\ \text{ind}_{\text{BS}}^{\text{an}} \downarrow & & \downarrow \widehat{\int}_{X/B} \widehat{\text{Todd}}(\widehat{\nabla}^{T^V X})_* (\cdot) \\ \widehat{K}_{\text{BS}}(B) & \xrightarrow[\widehat{\text{ch}}_{\text{BS}}]{} & \widehat{H}^{\text{even}}(B; \mathbb{R}/\mathbb{Q}) \end{array}$$

where  $\widehat{H}(X; \mathbb{R}/\mathbb{Q})$  is the ring of differential characters [6],  $\widehat{\text{ch}}_{\text{BS}}$  is the differential Chern character [3, §6.2],  $\text{ind}_{\text{BS}}^{\text{an}}$  is the Bunke-Schick differential analytic index [3, §3] and  $\widehat{\int}_{X/B} \widehat{\text{Todd}}(\widehat{\nabla}^{T^V X})_*$  is a modified pushforward of differential characters [3, §6.4]. The notation is explained more fully in later sections. Freed-Lott prove the differential family index theorem [8, Theorem 7.32]  $\text{ind}_{\text{FL}}^{\text{an}} = \text{ind}_{\text{FL}}^{\text{top}} : \widehat{K}_{\text{FL}}(X) \rightarrow \widehat{K}_{\text{FL}}(B)$ , where  $\text{ind}_{\text{FL}}^{\text{an}}$  and  $\text{ind}_{\text{FL}}^{\text{top}}$  are the Freed-Lott differential analytic index [8, Definition 3.11] and the differential topological index [8, Definition 5.33]. Applying the differential Chern character  $\widehat{\text{ch}}_{\text{FL}}$  yields the dGRR [8, Corollary 8.23]. Since  $\text{ind}_{\text{BS}}^{\text{an}} = \text{ind}_{\text{FL}}^{\text{an}}$  [3, Corollary 5.5], the two dGRR theorems are essentially the same. See [9] for

a short proof of the dGRR.

To this point, the differential index theorem formulated in Simons-Sullivan differential  $K$ -theory has not appeared. The purpose of this paper is to fill this gap by both defining the differential analytic index and proving the dGRR in Simons-Sullivan differential  $K$ -theory.

The first main result of this paper (Theorem 1) is the construction of explicit ring isomorphisms between Simons-Sullivan differential  $K$ -theory and Freed-Lott differential  $K$ -theory. While these theories must be isomorphic by [4, Theorem 3.10], the explicit isomorphisms have not been appeared in literature as far as we know. Moreover, it follows from Corollary 1 that the flat parts of these differential  $K$ -theories are also isomorphic via the restriction of the explicit ring isomorphisms in Theorem 1. This result is a more explicit version of [4, Theorem 5.5] in this case. The advantage of these explicit ring isomorphisms is that we see which elements in these differential  $K$ -groups correspond to each other.

The second main result of this paper is the dGRR in Simons-Sullivan differential  $K$ -theory. We first define the Simons-Sullivan differential analytic index by translating the Freed-Lott analytic index via the explicit isomorphisms in Theorem 1. To be precise, we study the special case where the family of kernels  $\ker(D^E)$  forms a superbundle. The general case follows from a standard perturbation argument as in [8, §7]. The Simons-Sullivan differential analytic index of an element  $\mathcal{E} = [E, h^E, [\nabla^E]] \in \widehat{K}_{\text{SS}}(X)$ , in the special case, is given by

$$\text{ind}_{\text{SS}}^{\text{an}}(\mathcal{E}) = [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]],$$

where  $[V, h^V, [\nabla^V]] := \widehat{\text{CS}}^{-1}(\widehat{\eta}(\mathcal{E}))$ , and all the terms will be explained below. The general case of  $\text{ind}_{\text{SS}}^{\text{an}}(\mathcal{E})$  is given by a similar formula. This formula is considerably more complicated than the Freed-Lott differential analytic index. This indicates that Simons-Sullivan differential  $K$ -theory is not the easiest setting for differential index theory, although the Simons-Sullivan construction of the differential  $K$ -group is perhaps the simplest among all the existing ones. We then prove the dGRR (Theorem 2) in the special case, i.e., the commutativity of the following diagram

$$\begin{array}{ccc} \widehat{K}_{\text{SS}}(X) & \xrightarrow{\widehat{\text{ch}}_{\text{SS}}} & \widehat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q}) \\ \text{ind}_{\text{SS}}^{\text{an}} \downarrow & & \downarrow \widehat{\int}_{X/B} \widehat{\text{Todd}}(\widehat{\nabla}^{T^V X})_*(\cdot) \\ \widehat{K}_{\text{SS}}(B) & \xrightarrow{\widehat{\text{ch}}_{\text{SS}}} & \widehat{K}_{\text{SS}}(B) \end{array}$$

in Simons-Sullivan differential  $K$ -theory, using a theorem of Bismut [1, Theorem 1.15]. The general case of the dGRR follows by a similar argument,

since [1, Theorem 1.15] can be extended to the general case.

In principle all the theorems and proofs can be transported from Freed-Lott differential  $K$ -theory to Simons-Sullivan differential  $K$ -theory by the explicit isomorphisms given by Theorem 1. However, with [1, Theorem 1.15] the proof of the dGRR is easier.

The paper is organized as follows: the next two sections contain all the necessary background material. Section 2 reviews Simons-Sullivan differential  $K$ -theory. Section 3 reviews Freed-Lott differential  $K$ -theory and the construction of the Freed-Lott differential analytic index. The main results of the paper are proved in Section 5, including the explicit isomorphisms between Simons-Sullivan differential  $K$ -theory and Freed-Lott differential  $K$ -theory, the formula for the differential analytic index in Simons-Sullivan differential  $K$ -theory and the dGRR.

#### ACKNOWLEDGEMENT

The author would like to thank Steven Rosenberg for many stimulating discussions of this problem and the referee for many helpful suggestions.

## 2. SIMONS-SULLIVAN DIFFERENTIAL $K$ -THEORY

In this section we review Simons-Sullivan differential  $K$ -theory [15]. For our purpose, we use the Hermitian version of structured bundles instead of the complex version. Consider a triple  $(V, h, \nabla)$ , where  $V \rightarrow X$  is a Hermitian vector bundle over a compact manifold  $X$  with a Hermitian metric  $h$  and a unitary connection  $\nabla$ . Recall that the Chern character form  $\text{ch}(\nabla) \in \Omega^{\text{even}}(X; \mathbb{R})$  and the Chern-Simons transgression form  $\text{cs}(\nabla^t) \in \Omega^{\text{odd}}(X; \mathbb{R})$  of two connections  $\nabla^0, \nabla^1$  on  $V \rightarrow X$  joined by a smooth curve  $\nabla^t$  of connections are related by the equality

$$d \text{cs}(\nabla^t) = \text{ch}(\nabla^1) - \text{ch}(\nabla^0). \quad (1)$$

Define

$$\text{CS}(\nabla^0, \nabla^1) := \text{cs}(\nabla^t) \pmod{\text{Im}(d : \Omega^{\text{even}}(X) \rightarrow \Omega^{\text{odd}}(X))},$$

where  $\nabla^t$  is a smooth curve joining the connection  $\nabla^1$  and  $\nabla^0$ . Since  $\text{cs}(\nabla^t)$  only depends on the curve joining the connections up to an exact form [15, Proposition 1.6],  $\text{CS}(\nabla^0, \nabla^1)$  is well defined <sup>1</sup>.

For two connections  $\nabla^0, \nabla^1$  on  $V \rightarrow X$ , we set  $\nabla^0 \sim \nabla^1$  if and only if  $\text{CS}(\nabla^0, \nabla^1) = 0$ .  $\sim$  is an equivalence relation.

The triple  $\mathcal{V} = (V, h, [\nabla])$  is called a (Hermitian) structured bundle. Two

<sup>1</sup>It follows from (1) that  $d \text{CS}(\nabla^0, \nabla^1) = \text{ch}(\nabla^1) - \text{ch}(\nabla^0)$ . There are other sign convention, for example see [8]. We will use the convention  $d \text{CS}(\nabla^0, \nabla^1) = \text{ch}(\nabla^1) - \text{ch}(\nabla^0)$  in this paper.

structured bundles  $\mathcal{V} = (V, h^V, [\nabla^V])$  and  $\mathcal{W} = (W, h^W, [\nabla^W])$  are isomorphic if there exists a vector bundle isomorphism  $\sigma : V \rightarrow W$  such that  $\sigma^*h^W = h^V$  and  $\sigma^*([\nabla^W]) = [\nabla^V]$ . Denote by  $\text{Struct}(X)$  the set of all isomorphism classes of structured bundles. Direct sum and tensor product of structured bundles are well-defined [15], so  $\text{Struct}(X)$  is an abelian semi-ring.

The Simons-Sullivan differential  $K$ -group is defined to be

$$\widehat{K}_{\text{SS}}(X) = K(\text{Struct}(X)).$$

Thus, Simons-Sullivan differential  $K$ -theory is a  $K$ -theory of vector bundles with connections.

To be precise,  $[\mathcal{V}_1] - [\mathcal{W}_1] = [\mathcal{V}_2] - [\mathcal{W}_2]$  in  $\widehat{K}_{\text{SS}}(X)$  if and only if there exists a structured bundle  $(G, h^G, [\nabla^G]) \in \text{Struct}(X)$  such that  $V_1 \oplus W_2 \oplus G \cong W_1 \oplus V_2 \oplus G$  and  $\text{CS}(\nabla^{V_1} \oplus \nabla^{W_2} \oplus \nabla^G, \nabla^{V_2} \oplus \nabla^{W_1} \oplus \nabla^G) = 0$ .

Define

$$\text{Struct}_{\text{ST}}(X) = \{\mathcal{V} \in \text{Struct}(X) | V \text{ is stably trivial}\}$$

$$\text{Struct}_{\text{SF}}(X) = \{\mathcal{V} \in \text{Struct}(X) | \mathcal{V} \oplus \mathcal{F} \cong \mathcal{H}\}$$

where  $\mathcal{F} \rightarrow X$  and  $\mathcal{H} \rightarrow X$  are flat structured bundles. Elements in  $\text{Struct}_{\text{SF}}(X)$  are said to be stably flat. Let  $U := \varinjlim U(n)$ . Denote by  $\theta \in \Omega^1(U, \mathfrak{u})$  the canonical left invariant  $\mathfrak{u}$ -valued form on  $U$ . Define

$$b_j = \frac{1}{(j-1)!} \left( \frac{1}{2\pi i} \right)^j \int_0^1 (t^2 - t)^{j-1} dt, j \in \mathbb{N}$$

$$\Theta = \sum_{j=1} b_j \text{tr} \left( \overbrace{\theta \wedge \cdots \wedge \theta}^{2j-1} \right) \in \Omega^{\text{odd}}(U)$$

Then define

$$\Omega_U(X) = \{g^*(\Theta) + d\beta | g : X \rightarrow U, \beta \in \Omega^{\text{even}}(X)\}$$

$$\Omega_{\text{BU}}^\bullet(X) = \{\omega \in \Omega_{d=0}^\bullet(X) | [\omega] \in \text{Im}(\text{ch} : K^{-(\bullet \bmod 2)}(X) \rightarrow H^\bullet(X; \mathbb{Q}))\}.$$

where  $\bullet \in \{\text{even}, \text{odd}\}$ . The so-called Venice lemma in [15] shows that the map  $\widehat{\text{CS}} : \frac{\text{Struct}_{\text{ST}}(X)}{\text{Struct}_{\text{SF}}(X)} \rightarrow \frac{\Omega^{\text{odd}}(X)}{\Omega_U(X)}$  defined by<sup>2</sup>

$$\widehat{\text{CS}}(\mathcal{V}) := \text{CS}(\nabla^V \oplus \nabla^F, \nabla^H) \pmod{\frac{\Omega_U(X)}{\Omega_{\text{exact}}^{\text{odd}}(X)}}$$

is an isomorphism, where  $F \rightarrow X$  and  $H \rightarrow X$  are trivial bundles over  $X$  such that  $H \cong V \oplus F$  and  $\nabla^F, \nabla^H$  are flat connections on  $F, H$ , respectively.

<sup>2</sup>This definition differs from the one in [15, Proposition 2.4] by a sign.

Also, the homomorphism

$$\Gamma : \frac{\text{Struct}_{\text{ST}}(X)}{\text{Struct}_{\text{SF}}(X)} \rightarrow \widehat{K}_{\text{SS}}(X)$$

defined by  $\Gamma(\mathcal{V}) = [\mathcal{V}] - [\dim(\mathcal{V})]$  is injective, for  $\dim(\mathcal{V})$  the trivial structured bundle of rank  $V$  with the trivial metric and connection. Thus the homomorphism

$$i : \frac{\Omega^{\text{odd}}(X)}{\Omega_{\text{U}}(X)} \rightarrow \widehat{K}_{\text{SS}}(X)$$

defined by  $i(\phi) = \Gamma \circ \widehat{\text{CS}}^{-1}(\phi)$  is injective. If we pick  $\mathcal{V} \in \widehat{\text{CS}}^{-1}(\phi)$ , then  $\mathcal{V}$  is a stably trivial structured bundle and

$$d\phi = d\text{CS}(\nabla^V \oplus \nabla^F, \nabla^H) = \text{ch}(\nabla^V) - \text{rank}(V) \pmod{\frac{\Omega_{\text{U}}(X)}{\Omega_{\text{exact}}^{\text{odd}}(X)}}$$

is independent of the choice of  $\mathcal{V}$ .

In the following hexagon the diagonal and the off-diagonal sequences are exact, and every square and triangle commutes:

$$\begin{array}{ccccc}
 0 & & & & 0 \\
 & \searrow & & & \nearrow \\
 & & K_{\text{SS}}^{-1}(X; \mathbb{R}/\mathbb{Z}) & \xrightarrow{B} & K(X) & & 0 \\
 & & \nearrow j & & \nearrow \delta & & \\
 H^{\text{odd}}(X; \mathbb{R}) & \xrightarrow{i \text{ deR}} & & & & \xrightarrow{\text{ch}} & H^{\text{even}}(X; \mathbb{R}) \\
 & \searrow \text{deR} & & & \searrow \text{ch}_{\widehat{K}_{\text{SS}}} & & \\
 & & \widehat{K}_{\text{SS}}(X) & & \widehat{K}_{\text{SS}}(X) & & \\
 & & \nearrow i & & \nearrow \text{ch}_{\widehat{K}_{\text{SS}}} & & \\
 \frac{\Omega^{\text{odd}}(X)}{\Omega_{\text{U}}(X)} & \xrightarrow{d} & \Omega_{\text{BU}}^{\text{even}}(X) & & \Omega_{\text{BU}}^{\text{even}}(X) & & \\
 & \nearrow & & & \nearrow \text{deR} & & \\
 0 & & & & & & 0
 \end{array} \tag{2}$$

In [15] the homomorphism  $\text{ch}_{\widehat{K}_{\text{SS}}} : \widehat{K}_{\text{SS}}(X) \rightarrow \Omega_{\text{BU}}^{\text{even}}(X)$  is just denoted by  $\text{ch}$ , which is a well defined lift of the Chern character form of a connection on a vector bundle to elements in  $\widehat{K}_{\text{SS}}(X)$ . We use the notation  $\text{ch}_{\widehat{K}_{\text{SS}}}$  in order to keep track of the Chern character in different usage.

**Remark 1.** We show that  $\Omega_{\text{U}}(X) = \Omega_{\text{BU}}^{\text{odd}}(X)$ , and we will use this identification throughout this paper. This is implicitly stated in [13, Diagram 1]. We include the easy proof here for completeness. Let  $d$  be the trivial connection on the trivial bundle  $X \times \mathbb{C}^N \rightarrow X$  for some  $N \in \mathbb{N}$ . By the proof of [15, Lemma 2.3], the connection  $d + g^*(\theta)$  on  $X \times \mathbb{C}^N \rightarrow X$ , where  $g : X \rightarrow \text{U}$  is an arbitrary but fixed smooth map, has trivial holonomy. Following the proof of [15, Lemma 2.3], we have  $g^*(\Theta) = \text{CS}(d, d + g^*(\theta)) = \text{CS}(d, d + g^{-1}dg) =: \text{ch}^{\text{odd}}([g])$ , so  $\Omega_{\text{U}}(X) = \Omega_{\text{BU}}^{\text{odd}}(X)$ .

3. FREED-LOTT DIFFERENTIAL  $K$ -THEORY

In this section we review Freed-Lott differential  $K$ -theory [8]. If

$$0 \longrightarrow E_1 \xrightarrow{i} E_2 \begin{array}{c} \xrightarrow{j} \\ \xleftarrow{s} \end{array} E_3 \longrightarrow 0 \quad (3)$$

is a split short exact sequence of complex vector bundles with connections  $\nabla_i$  on  $E_i \rightarrow X$ , for  $i = 1, 2, 3$ , we define the relative Chern-Simons transgression form  $\text{CS}(\nabla_1, \nabla_2, \nabla_3) \in \frac{\Omega^{\text{odd}}(X)}{\text{Im}(d)}$  by

$$\text{CS}(\nabla_1, \nabla_2, \nabla_3) := \text{CS}((i \oplus s)^* \nabla_2, \nabla_1 \oplus \nabla_3),$$

noting that  $i \oplus s : E_1 \oplus E_3 \rightarrow E_2$  is a vector bundle isomorphism.

The Freed-Lott differential  $K$ -group  $\widehat{K}_{\text{FL}}(X)$  is defined to be the abelian group with the following generators and relation: a generator of  $\widehat{K}_{\text{FL}}(X)$  is a quadruple  $\mathcal{E} = (E, h, \nabla, \phi)$ , where  $(E, h, \nabla)$  is as before and  $\phi \in \frac{\Omega^{\text{odd}}(X)}{\text{Im}(d)}$ . The only relation is  $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$  if and only if there exists a short exact sequence of Hermitian vector bundles (3) and

$$\phi_2 = \phi_1 + \phi_3 - \text{CS}(\nabla_1, \nabla_2, \nabla_3).$$

For  $\mathcal{E}_1, \mathcal{E}_2 \in \widehat{K}_{\text{FL}}(X)$ , the addition

$$\mathcal{E}_1 + \mathcal{E}_2 := (E_1 \oplus E_2, h^{E_1} \oplus h^{E_2}, \nabla^{E_1} \oplus \nabla^{E_2}, \phi_1 + \phi_2)$$

is well defined. Note that  $\mathcal{E}_1 = \mathcal{E}_2$  if and only if there exists  $(F, h^F, \nabla^F, \phi^F) \in \widehat{K}_{\text{FL}}(X)$  such that

- (1)  $E_1 \oplus F \cong E_2 \oplus F$ , and
- (2)  $\phi_1 - \phi_2 = \text{CS}(\nabla^{E_2} \oplus \nabla^F, \nabla^{E_1} \oplus \nabla^F)$ ,

The Freed-Lott differential Chern character  $\widehat{\text{ch}}_{\text{FL}} : \widehat{K}_{\text{FL}}(X) \rightarrow \widehat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$  is defined by

$$\widehat{\text{ch}}_{\text{FL}}(\mathcal{E}) = \widehat{\text{ch}}(E, \nabla) + i_2(\phi),$$

where  $\widehat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$  is the  $\mathbb{R}/\mathbb{Q}$  Cheeger-Simons differential characters [6],  $\mathcal{E} = (E, h, \nabla, \phi) \in \widehat{K}_{\text{FL}}(X)$ ,  $\widehat{\text{ch}}(E, \nabla)$  is the differential Chern character defined in [6, §4], and  $i_2 : \frac{\Omega^{\text{odd}}(X)}{\Omega_{\mathbb{Q}}^{\text{odd}}(X)} \rightarrow \widehat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$  is an injective homo-

morphism defined by  $i_2(\omega)(z) := \int_z \omega \pmod{\mathbb{Q}}$  for  $z \in Z_{\text{even}}(X)$  [6, Theorem 1.1].



**3.1. The Freed-Lott differential analytic index.** In this subsection we review the construction of the Freed-Lott differential analytic index. Consider the following diagram:

$$\begin{array}{ccccccc}
 (E, h, \nabla) & & (S^V X, \widehat{\nabla}^{T^V X}) & & (L^V X, \nabla^{L^V X}) & & \pi_* E \\
 \swarrow & & \downarrow & & \swarrow & & \downarrow \\
 (TX, g^{TX}, \nabla^{TX}) & & (T^V X, g^{T^V X}, \nabla^{T^V X}) & & (T^H X, \pi^* g^{TB}) & & (\ker(D^V), h^{\ker(D^V)}, \nabla^{\ker(D^V)}) \\
 \searrow & & \downarrow & & \searrow & & \downarrow \\
 & & X & \xrightarrow{\pi} & & & (B, g^{TB})
 \end{array}$$

In this diagram,  $\pi : X \rightarrow B$  is a proper submersion with closed fibers of even relative dimension and  $T^V X \rightarrow X$  is the vertical tangent bundle, which is assumed to have a metric  $g^{T^V X}$ .  $T^H X \rightarrow X$  is a horizontal distribution,  $g^{TB}$  is a Riemannian metric on  $B$ , the metric on  $TX \rightarrow X$  is defined by  $g^{TX} := g^{T^V X} \oplus \pi^* g^{TB}$ ,  $\nabla^{TX}$  is the corresponding Levi-Civita connection, and  $\nabla^{T^V X} := P \circ \nabla^{TX} \circ P$  is a connection on  $T^V X \rightarrow X$ , where  $P : TX \rightarrow T^V X$  is the orthogonal projection.  $T^V X \rightarrow X$  is assumed to have a  $\text{Spin}^c$  structure. Denote by  $S^V X \rightarrow X$  the  $\text{Spin}^c$ -bundle associated to the characteristic Hermitian line bundle  $L^V \rightarrow X$  with a unitary connection. The connections on  $T^V X \rightarrow X$  and  $L^V X \rightarrow X$  induce a connection  $\widehat{\nabla}^{T^V X}$  on  $S^V X \rightarrow X$ . Define an even form  $\text{Todd}(\widehat{\nabla}^{T^V X}) \in \Omega^{\text{even}}(X)$  by

$$\text{Todd}(\widehat{\nabla}^{T^V X}) = \widehat{A}(\nabla^{T^V X}) \wedge e^{\frac{1}{2}c_1(\nabla^{L^V X})}.$$

The modified pushforward of forms  $\pi_* : \Omega^{\text{odd}}(X) \rightarrow \Omega^{\text{odd}}(B)$  is defined by

$$\pi_*(\phi) = \int_{X/B} \text{Todd}(\widehat{\nabla}^{T^V X}) \wedge \phi.$$

The Freed-Lott differential analytic index  $\text{ind}^{\text{an}} : \widehat{K}_{\text{FL}}(X) \rightarrow \widehat{K}_{\text{FL}}(B)$  [8, Definition 3.11] is defined by

$$\text{ind}^{\text{an}}(\mathcal{E}) = (\ker(D^E), h^{\ker(D^E)}, \nabla^{\ker(D^E)}, \pi_*(\phi) + \widetilde{\eta}(\mathcal{E})),$$

where  $\mathcal{E} = (E, h, \nabla, \phi) \in \widehat{K}_{\text{FL}}(X)$ ,  $\widetilde{\eta}(\mathcal{E})$  is the Bismut-Cheeger eta form [2] characterized, up to exact form, by

$$d\widetilde{\eta}(\mathcal{E}) = \int_{X/B} \text{Todd}(\widehat{\nabla}^{T^V X}) \wedge \text{ch}(\nabla) - \text{ch}(\nabla^{\ker(D^E)}),$$

$D^E$  is the family of Dirac operators on  $S^V X \otimes E$ , and  $\ker(D^E)$  is assumed to form a superbundle over  $B$ .

## 4. MAIN RESULTS

**4.1. Explicit isomorphisms between  $\widehat{K}_{\text{FL}}$  and  $\widehat{K}_{\text{SS}}$ .** In this subsection we construct explicit isomorphisms between the Simons-Sullivan differential  $K$ -group and the Freed-Lott differential  $K$ -group.

**Theorem 1.** *Let  $X$  be a compact manifold. The maps*

$$f : \widehat{K}_{\text{SS}}(X) \rightarrow \widehat{K}_{\text{FL}}(X), \quad g : \widehat{K}_{\text{FL}}(X) \rightarrow \widehat{K}_{\text{SS}}(X)$$

*defined by*

$$\begin{aligned} f([E, h^E, [\nabla^E]] - [F, h^F, [\nabla^F]]) &= (E, h^E, \nabla^E, 0) - (F, h^F, \nabla^F, 0), \\ g(E, h^E, \nabla^E, \phi) &= [E, h^E, [\nabla^E]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]], \end{aligned}$$

*where  $\mathcal{V} = (V, h^V, [\nabla^V]) \in \widehat{\text{CS}}^{-1}(\phi)$ , are well defined ring isomorphisms, with  $f^{-1} = g$ . Moreover,  $f$  is natural and unique [4, Theorem 3.10], and is compatible with the structure maps  $i, j, \delta$  and  $\text{ch}_{\widehat{K}_{\text{SS}}}$  in (2).*

*Proof.* First we show that the maps  $f$  and  $g$  are well defined. For the map  $f$ , if  $[E_1, h^{E_1}, [\nabla^{E_1}]] - [F_1, h^{F_1}, [\nabla^{F_1}]] = [E_2, h^{E_2}, [\nabla^{E_2}]] - [F_2, h^{F_2}, [\nabla^{F_2}]]$  in  $\widehat{K}_{\text{SS}}(X)$ , then

$$(E_1, h^{E_1}, \nabla^{E_1}, 0) - (F_1, h^{F_1}, \nabla^{F_1}, 0) = (E_2, h^{E_2}, \nabla^{E_2}, 0) - (F_2, h^{F_2}, \nabla^{F_2}, 0),$$

since there exists  $(G, h^G, [\nabla^G]) \in \text{Struct}(X)$  such that  $E_1 \oplus F_2 \oplus G \cong F_1 \oplus E_2 \oplus G$  and

$$0 = \text{CS}(\nabla^{E_1} \oplus \nabla^{F_2} \oplus \nabla^G, \nabla^{F_1} \oplus \nabla^{E_2} \oplus \nabla^G) = \text{CS}(\nabla^{E_1} \oplus \nabla^{F_2}, \nabla^{F_1} \oplus \nabla^{E_2}).$$

It follows that the map  $f$  is well defined.

For the map  $g$ , if  $(E, h^E, \nabla^E, \phi) = (F, h^F, \nabla^F, \omega)$  in  $\widehat{K}_{\text{FL}}(X)$ , then there exists  $(G, h^G, \nabla^G, \phi^G) \in \widehat{K}_{\text{FL}}(X)$  such that  $E \oplus G \cong F \oplus G$  and  $\phi - \omega = \text{CS}(\nabla^F \oplus \nabla^G, \nabla^E \oplus \nabla^G)$ . We want

$$\begin{aligned} &[E, h^E, [\nabla^E]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]] \\ &= [F, h^F, [\nabla^F]] + [W, h^W, [\nabla^W]] - [\dim(W), h, [d]], \end{aligned}$$

where  $\widehat{\text{CS}}(\mathcal{V}) = \phi$  and  $\widehat{\text{CS}}(\mathcal{W}) = \omega$ . We need to show that there exists  $(G', h^{G'}, [\nabla^{G'}]) \in \text{Struct}(X)$  such that

$$\begin{aligned} &(E, h^E, [\nabla^E]) + (V, h^V, [\nabla^V]) + (\dim(W), h, [d]) + (G', h^{G'}, [\nabla^{G'}]) \\ &= (F, h^F, [\nabla^F]) + (W, h^W, [\nabla^W]) + (\dim(V), h, [d]) + (G', h^{G'}, [\nabla^{G'}]), \end{aligned} \quad (4)$$

and  $\text{CS}(\nabla^E \oplus \nabla^V \oplus \nabla^{G'}, \nabla^F \oplus \nabla^W \oplus \nabla^{G'}) = 0$ . (4) is equivalent to

$$\begin{aligned} &(E \oplus V \oplus \dim(W) \oplus G', h^E \oplus h^V \oplus h \oplus h^{G'}, [\nabla^E \oplus \nabla^V \oplus d \oplus \nabla^{G'}]) \\ &= (F \oplus W \oplus \dim(V) \oplus G', h^F \oplus h^W \oplus h \oplus h^{G'}, [\nabla^F \oplus \nabla^W \oplus d \oplus \nabla^{G'}]). \end{aligned}$$

Since  $V$  and  $W$  are stably trivial, there exist trivial bundles  $V'$  and  $W'$  with connections  $\nabla^{V'}$  and  $\nabla^{W'}$  such that

$$H^V := \dim(V) \oplus V' = V \oplus V', \quad H^W := \dim(W) \oplus W' = W \oplus W',$$

and

$$\phi = \text{CS}(\nabla^V \oplus \nabla^{V'}, \nabla^{H^V}), \quad \omega = \text{CS}(\nabla^W \oplus \nabla^{W'}, \nabla^{H^W}).$$

By taking  $G' = G \oplus V' \oplus W'$ , we have

$$\begin{aligned}
& (E \oplus V \oplus \dim(W)) \oplus (G \oplus V' \oplus W') \\
& \cong (E \oplus G) \oplus (V \oplus V') \oplus (\dim(W) \oplus W') \\
& \cong (F \oplus G) \oplus (\dim(V) \oplus V') \oplus (W \oplus W') \\
& \cong (F \oplus W \oplus \dim(V)) \oplus (G \oplus V' \oplus W')
\end{aligned} \tag{5}$$

and for  $d^V, d^W$  the trivial connections on  $\dim(V), \dim(V)$ , respectively,

$$\begin{aligned}
& \text{CS}(\nabla^E \oplus \nabla^V \oplus d^W \oplus \nabla^G \oplus \nabla^{V'} \oplus \nabla^{W'}, \nabla^F \oplus \nabla^W \oplus d^V \oplus \nabla^G \oplus \nabla^{V'} \oplus \nabla^{W'}) \\
& = \text{CS}(\nabla^E \oplus \nabla^G, \nabla^F \oplus \nabla^G) + \text{CS}(\nabla^V \oplus \nabla^{V'}, d^V \oplus \nabla^{V'}) + \text{CS}(d^W \oplus \nabla^{W'}, \nabla^W \oplus \nabla^{W'}) \\
& = -\phi + \omega + \text{CS}(\nabla^V \oplus \nabla^{V'}, \nabla^{H^V}) + \text{CS}(\nabla^{H^W}, \nabla^W \oplus \nabla^{W'}) \\
& = -\phi + \omega + \phi - \omega \\
& = 0
\end{aligned} \tag{6}$$

(5) and (6) imply (4), so the map  $g : \widehat{K}_{\text{FL}}(X) \rightarrow \widehat{K}_{\text{SS}}(X)$  is well defined.

We now show that  $f$  and  $g$  are inverses. Note that

$$\begin{aligned}
(g \circ f)([E, h^E, [\nabla^E]] - [F, h^F, [\nabla^F]]) &= g((E, h^E, \nabla^E, 0) - (F, h^F, \nabla^F, 0)) \\
&= [E, h^E, [\nabla^E]] - [F, h^F, [\nabla^F]]
\end{aligned}$$

as  $\widehat{\text{CS}}^{-1}(0) = 0 \in \frac{\text{Struct}_{\text{ST}}(X)}{\text{Struct}_{\text{SF}}(X)}$ . For the other direction, we consider

$$(f \circ g)(E, h^E, \nabla^E, \phi) = (E, h^E, \nabla^E, 0) + (V, h^V, \nabla^V, 0) - (\dim(V), h, d, 0),$$

where  $\widehat{\text{CS}}(\mathcal{V}) = \phi$  for  $\mathcal{V} := (V, h^V, [\nabla^V]) \in \text{Struct}_{\text{ST}}(X)$ . It suffices to show

$$(E, h^E, \nabla^E, \phi) + (\dim(V), h, d^V, 0) = (E, h^E, \nabla^E, 0) + (V, h^V, \nabla^V, 0),$$

which is equivalent to

$$(E \oplus \dim(V), h^E \oplus h, \nabla^E \oplus d^V, \phi) = (E \oplus V, h^E \oplus h^V, \nabla^E \oplus \nabla^V, 0). \tag{7}$$

To see this, since  $\mathcal{V} = (V, h^V, [\nabla^V])$  is stably trivial, there exist trivial structured bundles  $\mathcal{F} = (F, h, [d^F])$  and  $\mathcal{H} = (H, h, [d^H])$  such that  $V \oplus F \cong H$  and  $\phi = \text{CS}(d^H, \nabla^V \oplus d^F)$ . Thus  $E \oplus \dim(V) \oplus \dim(F) \cong E \oplus \dim(H) \cong E \oplus V \oplus F$ , and

$$\begin{aligned}
\text{CS}(\nabla^E \oplus \nabla^V, \nabla^E \oplus d^V) &= \text{CS}(\nabla^E \oplus \nabla^V \oplus d^F, \nabla^E \oplus d^V \oplus d^F) \\
&= \text{CS}(\nabla^E \oplus \nabla^V \oplus d^F, \nabla^E \oplus d^H) = \phi.
\end{aligned}$$

This proves (7).

$f$  is obviously a natural ring homomorphism. Since  $g = f^{-1}$ ,  $g$  is also a ring homomorphism.  $\square$

The following corollary follows from the compatibility of  $f$  and  $\text{ch}_{\widehat{K}_{\text{SS}}}$ .

**Corollary 1.** *Let  $X$  be a compact manifold. The following diagram is commutative.*

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{\text{SS}}^{-1}(X; \mathbb{R}/\mathbb{Z}) & \xrightarrow{j} & \widehat{K}_{\text{SS}}(X) & \xrightarrow{\text{ch}} & \Omega_{\text{BU}}(X) \longrightarrow 0 \\ & & \bar{f} \downarrow & & f \downarrow & & = \downarrow \\ 0 & \longrightarrow & K_{\text{L}}^{-1}(X; \mathbb{R}/\mathbb{Z}) & \xrightarrow{j'} & \widehat{K}_{\text{FL}}(X) & \xrightarrow{\omega} & \Omega_{\text{BU}}(X) \longrightarrow 0 \end{array}$$

where  $K_{\text{L}}^{-1}(X; \mathbb{R}/\mathbb{Z})$  is the flat part of  $\widehat{K}_{\text{FL}}(X)$  (see also [12]),  $\bar{f}$  is the restriction of  $f$  to  $K_{\text{SS}}^{-1}(X; \mathbb{R}/\mathbb{Z})$ . Here  $\omega : \widehat{K}_{\text{FL}}(X) \rightarrow \Omega_{\text{BU}}(X)$  is defined by  $\omega(E, h^E, \nabla^E, \phi) = \text{ch}(\nabla^E) + d\phi$ .

Note that the horizontal sequences are exact by [8], [15].

**4.2. The differential analytic index in  $\widehat{K}_{\text{SS}}$ .** In this subsection we give the formula for the differential analytic index in Simons-Sullivan differential  $K$ -theory.

Let  $\pi : X \rightarrow B$  be a proper submersion of even relative dimension and its fibers are assumed to be  $\text{Spin}^c$ . The differential analytic index in Simons-Sullivan differential  $K$ -theory is given by forcing the following diagram to be commutative:

$$\begin{array}{ccc} \widehat{K}_{\text{SS}}(X) & \xrightarrow{f} & \widehat{K}_{\text{FL}}(X) \\ \text{ind}_{\text{SS}}^{\text{an}} \downarrow & & \downarrow \text{ind}_{\text{FL}}^{\text{an}} \\ \widehat{K}_{\text{SS}}(B) & \xleftarrow{g} & \widehat{K}_{\text{FL}}(B) \end{array}$$

Let  $\mathcal{E} := [E, h^E, [\nabla]] \in \widehat{K}_{\text{SS}}(X)$ . Since

$$\begin{aligned} (g \circ \text{ind}_{\text{FL}}^{\text{an}} \circ f)(\mathcal{E}) &= [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] \\ &\quad + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]], \end{aligned}$$

where  $\mathcal{V} := (V, h^V, [\nabla^V]) \in \frac{\text{Struct}_{\text{ST}}(B)}{\text{Struct}_{\text{SF}}(B)}$  is uniquely determined by the condition  $\widehat{\text{CS}}(\mathcal{V}) = \tilde{\eta}(\mathcal{E}) \pmod{\frac{\Omega_{\text{U}}(B)}{\Omega_{\text{exact}}^{\text{odd}}(B)}}$ , it follows that the differential analytic index in the Simons-Sullivan differential  $K$ -theory  $\text{ind}_{\text{SS}}^{\text{an}} : \widehat{K}_{\text{SS}}(X) \rightarrow \widehat{K}_{\text{SS}}(B)$  is given by

$$\text{ind}_{\text{SS}}^{\text{an}}(\mathcal{E}) = [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]], \quad (8)$$

where  $\ker(D^E)$  is assumed to form a superbundle over  $B$ . Although  $\mathcal{V} := \widehat{\text{CS}}^{-1}(\tilde{\eta}(\mathcal{E}))$  is uniquely determined up to a stably flat structured bundle, its class  $[\mathcal{V}] \in \widehat{K}_{\text{SS}}(B)$  is unique since the differential  $K$ -theory class of a stably flat structured bundle is zero. Moreover, since  $\text{ind}_{\text{FL}}^{\text{an}}$  is well defined (see [9, Proposition 1] for a proof which does not use the differential family index

theorem), it follows that  $\text{ind}_{\text{SS}}^{\text{an}}$  is well defined too.

If one defines the Simons-Sullivan differential analytic index  $\text{ind}_{\text{SS}}^{\text{an}}$  without considering the other differential analytic indices, a natural candidate would be, say, in the special case when  $\ker(D^E) \rightarrow B$  is a superbundle,

$$\text{ind}_{\text{SS}}^{\text{an}}(\mathcal{E}) = [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]].$$

This definition coincides with (8) if and only if  $\mathcal{V} \in \text{Struct}_{\text{SF}}(B)$ , which is equivalent to saying that  $\tilde{\eta}(\mathcal{E}) \in \Omega_{\text{U}}(B) = \Omega_{\text{BU}}^{\text{odd}}(B)$ . However, this is not true since

$$d\tilde{\eta}(\mathcal{E}) = \int_{X/B} \text{Todd}(\widehat{\nabla}^{T^V X}) \wedge \text{ch}(\nabla) - \text{ch}(\nabla^{\ker(D^E)}),$$

which shows that  $\tilde{\eta}(\mathcal{E})$  is not closed in general.

**Lemma 1.** *Let  $\mathcal{E} = [E, h, [\nabla]] \in \widehat{K}_{\text{SS}}(X)$ . Then*

$$\text{ch}_{\widehat{K}_{\text{SS}}}(\text{ind}_{\text{SS}}^{\text{an}}(\mathcal{E})) = \text{ch}(\nabla^{\ker(D^E)}) + d\tilde{\eta}(\mathcal{E}).$$

It follows from Lemma 1 and the local family index theorem that

$$\begin{aligned} \text{ch}_{\widehat{K}_{\text{SS}}}(\text{ind}_{\text{SS}}^{\text{an}}(\mathcal{E})) &= \text{ch}(\nabla^{\ker(D^E)}) + d\tilde{\eta}(\mathcal{E}) \\ &= \int_{X/B} \text{Todd}(\widehat{\nabla}^{T^V X}) \wedge \text{ch}(\nabla^E) \\ &= \pi_*(\text{ch}_{\widehat{K}_{\text{SS}}}(\mathcal{E})). \end{aligned}$$

We define the Simons-Sullivan differential Chern character  $\widehat{\text{ch}}_{\text{SS}} : \widehat{K}_{\text{SS}}(X) \rightarrow \widehat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$  by

$$\widehat{\text{ch}}_{\text{SS}}(\mathcal{E}) := \widehat{\text{ch}}(E, \nabla),$$

where  $\mathcal{E} = [E, h, [\nabla]]$ .

It is instructive to note that the following diagram commute,

$$\begin{array}{ccc} & & \widehat{K}_{\text{SS}}(X) \\ & \nearrow i & \downarrow f \\ \frac{\Omega^{\text{odd}}(X)}{\Omega_{\text{BU}}^{\text{odd}}(X)} & & \widehat{K}_{\text{FL}}(X) \\ & \searrow j & \end{array}$$

where  $f : \widehat{K}_{\text{SS}}(X) \rightarrow \widehat{K}_{\text{FL}}(X)$  is the isomorphism given by Theorem 1.

**4.3. The differential Grothendieck-Riemann-Roch theorem.** In this subsection we prove the dGRR in Simons-Sullivan differential  $K$ -theory. To be precise, we first prove the special case that the family of kernels  $\ker(D^E)$  forms a superbundle by a theorem of Bismut reviewed below. The general case follows from the standard perturbation argument as in [8, §7].

We now recall Bismut's theorem. For the geometric construction of the analytic index given in §4.2, with the fibers assumed to be Spin, and  $\ker(D^E) \rightarrow B$  assumed to form a superbundle, we have

$$\widehat{\text{ch}}(\ker(D^E), \nabla^{\ker(D^E)}) + i_2(\tilde{\eta}) = \int_{X/B} \widehat{A}(T^V X, \nabla^{T^V X}) * \widehat{\text{ch}}(E, \nabla^E) \quad (9)$$

[1, Theorem 1.15], where  $\int_{X/B}$  is the pushforward of differential characters for proper submersion [8, §8.3],  $*$  is the multiplication of differential characters [6, §1], and  $\widehat{A}(T^V X, \nabla^{T^V X}) \in \widehat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$  is the differential character associated to the  $\widehat{A}$ -class (see [6, §2]). If the fibers are assumed to be  $\text{Spin}^c$ , (9) has the obvious modification, and in our notation becomes

$$\widehat{\text{ch}}(\ker(D^E), \nabla^{\ker(D^E)}) + i_2(\tilde{\eta}) = \int_{X/B} \widehat{\text{Todd}}(T^V X, \widehat{\nabla}^{T^V X}) * \widehat{\text{ch}}(E, \nabla^E), \quad (10)$$

for  $\widehat{\text{Todd}}(T^V X, \widehat{\nabla}^{T^V X}) \in \widehat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$  the differential character associated to the Todd class (see [6, §2]). We will write  $\widehat{\text{Todd}}(T^V X, \widehat{\nabla}^{T^V X})$  as  $\widehat{\text{Todd}}(\widehat{\nabla}^{T^V X})$  in the sequel. Note that (9) and (10) extend to the general case where  $\ker(D^E) \rightarrow B$  does not form a bundle [1, p. 23].

**Theorem 2 (Differential Grothendieck-Riemann-Roch theorem).**

Let  $\pi : X \rightarrow B$  be a proper submersion with closed  $\text{Spin}^c$ -fibers of even dimension. The following diagram is commutative:

$$\begin{array}{ccc} \widehat{K}_{\text{SS}}(X) & \xrightarrow{\widehat{\text{ch}}_{\text{SS}}} & \widehat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q}) \\ \text{ind}_{\text{SS}}^{\text{an}} \downarrow & & \downarrow \int_{X/B} \widehat{\text{Todd}}(\widehat{\nabla}^{T^V X}) * (\cdot) \\ \widehat{K}_{\text{SS}}(B) & \xrightarrow{\widehat{\text{ch}}_{\text{SS}}} & \widehat{H}^{\text{even}}(B; \mathbb{R}/\mathbb{Q}) \end{array}$$

i.e., if  $\mathcal{E} = [E, h, [\nabla^E]] \in \widehat{K}_{\text{SS}}(X)$ , then

$$\widehat{\text{ch}}_{\text{SS}}(\text{ind}_{\text{SS}}^{\text{an}}(\mathcal{E})) = \int_{X/B} \widehat{\text{Todd}}(\widehat{\nabla}^{T^V X}) * \widehat{\text{ch}}_{\text{SS}}(\mathcal{E}).$$

*Proof.*

$$\begin{aligned}
& \widehat{\text{ch}}_{\text{SS}}(\text{ind}_{\text{SS}}^{\text{an}}(\mathcal{E})) \\
&= \widehat{\text{ch}}_{\text{SS}}([\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]]) \\
&= \widehat{\text{ch}}(\ker(D^E), \nabla^{\ker(D^E)}) + i_2(\widetilde{\eta}(\mathcal{E})) \\
&= \int_{X/B} \widehat{\text{Todd}}(\widehat{\nabla}^{TVX}) * \widehat{\text{ch}}(E, \nabla^E) \\
&= \int_{X/B} \widehat{\text{Todd}}(\widehat{\nabla}^{TVX}) * \widehat{\text{ch}}_{\text{SS}}(\mathcal{E})
\end{aligned}$$

where the second equality follows from Proposition ?? and the third equality follows from (10).  $\square$

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