Grain rotations and distortions in the asymptotic variance of vacancy of the Boolean model

Rau, Christian; CHIU, Sung Nok

Published in:
Journal of Mathematical Analysis and Applications

DOI:
10.1016/j.jmaa.2011.06.007

Published: 15/12/2011

Document Version:
Peer reviewed version

Grain Rotations and Distortions in the Asymptotic Variance of Vacancy of the Boolean Model

Christian Rau[a,1,*], Sung Nok Chiu[b,2]

[a] *School of Mathematical Sciences, Monash University, Clayton VIC 3800, Australia*

[b] *Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong*

Abstract

We consider the asymptotic variance of vacancy (AVV) in the high-intensity small-grain Boolean model. Subjecting the grains to rotations or, more generally, linear distortions gives rise to a function which maps distortion distributions to the AVV of the corresponding Boolean model. We mainly study continuity properties of this function, where we use the L^1 Wasserstein metric on distortion distributions. An important role in the formulation and derivation of our results is played by notions of symmetry commonly used in multivariate analysis and stochastic simulation, such as conjugation-invariance and group models.

Keywords: Boolean model, coverage, rotations, set covariance function, vacancy, Wasserstein distance.

2000 MSC: 60D05, 60E99, 60K30

1. Introduction

Consider a Boolean model $\Xi = \cup_i(\boldsymbol{\xi}_i + \mathcal{S}_i) \subset \mathbb{R}^k$ $(k \geq 2)$, where $\{\boldsymbol{\xi}_i\}$ is a homogeneous Poisson process with intensity $\lambda > 0$. Here and below we write $y + B = \{y + z : z \in B\}$

*Corresponding author.

[1] Tel: (+61 3) 9905 4404, Fax: (+61 3) 9905 4403, Email: Christian.Rau@monash.edu

[2] Email: snchiu@hkbu.edu.hk

for $y \in \mathbb{R}^k$ and $B \subset \mathbb{R}^k$. Random grains $\mathcal{S}_i$ are independent and identically distributed (i.i.d.) as the random closed set $\mathcal{S}$, and $\{\mathcal{S}_i\}$ is independent of $\{\boldsymbol{\xi}_i\}$. Throughout this paper we assume that $\mathrm{E}(|\mathcal{S}|^2) < \infty$, where $|\cdot|$ denotes k-dimensional volume. The vacancy of Ξ in a fixed Riemann measurable set $\mathcal{R} \subset \mathbb{R}^k$ of finite positive content is defined as $V = |\mathcal{R} \setminus \Xi|$. (As in [1], we define a Riemann measurable set as a set whose indicator function is directly Riemann integrable, hence not necessarily bounded.) As the distribution of V is complicated, it is of interest to study the limiting behaviour under either a 'high intensity' or 'moderate intensity' regime. Specifically, consider the limit of $\lambda \mathrm{Var}(V)$, that is the asymptotic variance of vacancy (abbreviated to AVV hereafter), as $\lambda \to \infty$ and the grains are distributed (and scaled) as $\delta\mathcal{S}$, where $\delta = \delta(\lambda) \searrow 0$ such that $\delta^k \lambda \to \rho \in [0, \infty]$. Section 2.3 displays the limiting AVV for the cases $\rho < \infty$ and $\rho = \infty$, as derived in [1]. It is also known [1, pp. 144–145 and pp. 189–190] that in both cases the AVV does not increase when the grains are subjected to i.i.d. 'random, uniformly distributed rotations'. More precisely, in the setup of [1] the grains are i.i.d. and distributed as $\mathcal{S}^* = \boldsymbol{T}\mathcal{S} = \{\boldsymbol{T}\boldsymbol{s} : \boldsymbol{s} \in \mathcal{S}\}$, where $\boldsymbol{T}$ is distributed as the unique probability measure on $SO(k)$ such that $Q\boldsymbol{T} \stackrel{\mathrm{d}}{=} \boldsymbol{T}$, for any fixed $Q \in SO(k)$. Here and below $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution. Throughout this paper, random vectors and matrices are distinguished from deterministic ones by boldfacing. We write $M(k)$ for the set of all real $k \times k$ matrices, $O(k) = \{A \in M(k) : A^T A = I_k\}$, where I_k denotes the $k \times k$ identity matrix, and $SO(k) = \{A \in O(k) : \det A = 1\}$. Note that $\mathcal{S}^*$ is *isotropic* (e.g. [1, p. 124]; [2, pp. 39 and 41]), in the sense that

$$Q\mathcal{S}^* \stackrel{\mathrm{d}}{=} \mathcal{S}^* \qquad \text{for any fixed } Q \in SO(k). \tag{1}$$

We will usually assume that $\boldsymbol{T}$ is independent of $\mathcal{S}$ (and does not depend on λ).

In the present paper we investigate continuity properties of the AVV when $\boldsymbol{T}$ is a rotation with an *arbitrary* distribution (on occasion with additional restrictions, such as symmetry properties specified later), and further in the case where $\boldsymbol{T}$ is a random matrix in the space of (as we shall say) *η-near volume-preserving* linear maps, or $NV_\eta(k)$ maps for short. Roughly speaking, the parameter $\eta > 0$ measures the deviation of $\boldsymbol{T}$ from an orthogonal map; see Section 2.2 for the precise definitions. There we also explain the reason for considering $NV_\eta(k)$, rather than $SL(k)$, as domains. Recall that the set of (exactly) area preserving linear maps from $\mathbb{R}^k$ into itself is identified with the set $SL(k) = \{A \in M(k) : \det A = 1\}$. The set $SL(2)$ contains shearing matrices of the form

$$T_w = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \qquad w \in \mathbb{R}. \tag{2}$$

For any $\eta > 0$, the set $NV_\eta(2)$ contains 'not too large' (as determined by η) shearing maps; see Section 2.2. The maps in the set $NV_\eta(k)$ will be called *distortions*.

Let $\mathscr{P}(NV_\eta(k))$ denote the set of all Borel probability measures on $(NV_\eta(k), \mathscr{B})$, where $\mathscr{B}$ is the trace σ-field on $NV_\eta(k)$ of the Borel σ-field on $M(k)$ with the natural Euclidean topology (identifying $M(k)$ with $\mathbb{R}^{k^2}$). The space $\mathscr{P}(NV_\eta(k))$ is endowed with the L^1 Wasserstein metric defined at (5) below. Let further $\tau^2_\infty(F_{\boldsymbol{T}})$ and $\tau^2_{\text{fin}}(F_{\boldsymbol{T}})$ denote the factor in the AVV which depends on the distribution $F_{\boldsymbol{T}}$ of $\boldsymbol{T}$, in the respective cases $\rho = \infty$ and $\rho < \infty$, associated with grains $\{\mathcal{S}_i^* = \boldsymbol{T}_i\mathcal{S}_i\,,\, i \geq 1\}$, where the $\{\boldsymbol{T}_i\}$ are i.i.d. distortions independent of $\{\mathcal{S}_i\}$, with common distribution equal to $F_{\boldsymbol{T}}$. Sections 3 and 4 discuss, respectively, optimality and continuity properties of the function

$$\tau^2 : \mathscr{P}(NV_\eta(k)) \to (0, \infty)\,, \qquad F_{\boldsymbol{T}} \mapsto \tau^2(F_{\boldsymbol{T}})\,, \tag{3}$$

where τ^2 stands for either τ^2_∞ or τ^2_{fin}. In each of the cases $\rho = \infty$ and $\rho < \infty$, we also present results for the case where τ^2 is restricted to a set of probability measures on $O(k)$ possessing a certain symmetry property, and then we will use the supremum norm distance on induced densities on the unit sphere instead.

We emphasise that our focus in this paper is not on asymptotic properties of vacancy, but rather an exposition of a relatively under-researched aspect of stochastic geometry, namely that of stability of a stochastic functional, for an important example of such a functional. Stability could, of course, be investigated in situations where asymptotics play no role. In particular, we could consider continuity of the variance of vacancy in the non-asymptotic case. However, such an analysis is severely complicated by the presence of edge effects and hence not pursued here. Some further discussion of open problems implied by this paper is given in Section 5.

Section 2 contains notation and basic concepts, as well as the results on the AVV obtained in [1]. As already mentioned, Sections 3 and 4 discuss, separately for the cases $\rho = \infty$ and $\rho < \infty$, optimality and continuity properties for distortions, and continuity of rotations under symmetry assumptions. Section 5 concludes the paper.

2. Preliminaries

2.1. Notation and basic concepts

As usual, the open ball in $\mathbb{R}^k$ with radius $r > 0$ centred at x is denoted by $B_r(x)$. The notation $\sigma_{k-1}(\cdot)$ denotes non-normalised Lebesgue surface measure on $\mathbb{S}^{k-1}$, the unit sphere in $\mathbb{R}^k$ with total surface area $\omega_k = \sigma_{k-1}(\mathbb{S}^{k-1}) = 2\pi^{k/2}/\Gamma(k/2)$. The volume of the unit ball $B_1(0)$ is denoted by $\kappa_k = \omega_k/k$.

If $(E_1, \mathcal{E}_1, \nu)$ is a measure space, $(E_2, \mathcal{E}_2)$ a measurable space and $f : E_1 \to E_2$ a measurable map, then the image measure νf^{-1} is defined by $\big(\nu f^{-1}\big)(A) = \nu(f^{-1}(A))$ for $A \in \mathcal{E}_2$. In particular, if $f(x) = \Pi x$ for some $\Pi \in M(k)$, then the resulting image measure is denoted by $\nu\Pi^{-1}$. The set of all probability measures on (the Borel σ-field of) a metric space $(\mathcal{X}, d)$ is denoted by $\mathscr{P}(\mathcal{X})$. The vectors $e^{(1)}, \ldots, e^{(k)}$ are the columns of I_k, and $x \cdot y$ is the dot product of $x, y \in \mathbb{R}^k$. For vectors $u, v \in \mathbb{S}^{k-1}$ we use the geodesic (great circle) distance $d(u, v) = \cos^{-1}(u \cdot v)$, and let $u^\perp = \{x \in \mathbb{R}^k : u \cdot x = 0\}$. The p-norms $(1 \leq p < \infty)$ and the supremum norm of $h : \mathbb{S}^{k-1} \to \mathbb{R}$ are defined as

$$\|h\|_p = \left(\int_{\mathbb{S}^{k-1}} h^p(w)\, \sigma_{k-1}(dw) \right)^{1/p} \qquad \text{and} \qquad \|h\|_\infty = \sup_{w \in \mathbb{S}^{k-1}} |h(w)|\,,$$

respectively. The spectral norm of $A \in M(k)$ is $\|A\|_2 := \sup_{\|x\|=1} \|Ax\|$. The group of all invertible $k \times k$ matrices, and the subset of $k \times k$ upper triangular matrices with strictly positive diagonal entries are denoted by $GL(k)$ and $R(k)$ respectively. These sets are endowed with a Lebesgue measure $\lambda_{GL(k)}$ and $\lambda_{R(k)}$, which is the restriction of the Lebesgue measure $\lambda_{M(k)}$ on $k \times k$ matrices; the latter is the k^2-fold product measure of ordinary Lebesgue measure on $\mathbb{R}$. (No confusion of these Lebesgue measures with the driving intensity λ of the Boolean model is possible.) The space $GL(k)$ is locally compact and hence (e.g. [3, Chapter 5]) admits a Haar measure (a left as well as right Haar measure) $\mu_{GL(k)}$, which is unique up to scaling. The transformation formula (see e.g. [4, p. 117] or [3, p. 78])

$$\mu_{GL(k)} = g_k \cdot \lambda_{GL(k)} \qquad \text{with} \qquad g_k(M) = |\det M|^{-k} \tag{4}$$

shows in particular that $\mu_{GL(k)}$ and $\lambda_{GL(k)}$ are equivalent.

We shall generally assume that all probabilistic objects are defined on a common probability space $(\Omega, \mathscr{F}, \mathrm{P})$, with $\mathrm{E}(\cdot)$ denoting the expectation with respect to P. The Minkowski sum of the sets $A, B \subset \mathbb{R}^k$ is $A \oplus B = \{a + b : \ a \in A\,, b \in B\}$. For $\varepsilon > 0$, the ε-inner parallel body $C_{-\varepsilon}$ of the set $C \subset \mathbb{R}^k$ is the largest set A such that $A \oplus B_\varepsilon(0) \subseteq C$.

We shall need the so-called (compact) *Stiefel manifolds*, which are defined as

$$St(k, m) = \{A \in \mathbb{R}^{k \times m} : A^T A = I_m\}\,, \quad 1 \le m \le k\,,$$

so that in particular $St(k, k) = O(k)$. The $O(k)$-action on $St(k, m)$ defined by left matrix multiplication is transitive, and hence its invariant probability measure $\mu_{k,m} = \mu_{St(k,m)}$ is well defined (see e.g. [4, Theorem 4.6 (iii), p. 66]).

To measure closeness of elements of $\mathscr{P}(\mathcal{X})$ in the context of (3) and its variants considered in this paper, where $(\mathcal{X}, d)$ is a given metric space, we shall use the L^1 Wasserstein distance, also known as the Kantorovich-Rubinstein distance, which may be expressed by the *duality formula* (see [5, p. 152]):

$$\mathcal{W}_1(\nu_1, \nu_2) = \sup_{\|\phi\|_{Lip} \leq 1} \left\{ \int \phi \, d\nu_1 - \int \phi \, d\nu_2 \right\}, \qquad \nu_1, \nu_2 \in \mathscr{P}(\mathcal{X}), \tag{5}$$

where $\|\phi\|_{Lip}$ is the Lipschitz seminorm of the function $\phi : \mathcal{X} \to \mathbb{R}$. If ϕ is Lipschitz continuous with constant L, then $\phi/\|\phi\|_\infty$ is Lipschitz with constant 1, and therefore

$$\sup_{\|\phi\|_{Lip} \leq L} \left\{ \int \phi \, d\nu_1 - \int \phi \, d\nu_2 \right\} \leq L \cdot \mathcal{W}_1(\nu_1, \nu_2). \tag{6}$$

For functions $f, g : \mathbb{R}_+ \to \mathbb{R}$, the notation $f \lesssim g$ means that $\limsup_{x\to\infty} f(x)/g(x) \leq 1$.

2.2. Near volume preserving maps and decomposition

In this section we define the class of $NV_\eta(k)$ matrices which, as noted, includes rotations and shearings as defined at (2).

Definition 1. *For $W \subseteq M(k)$, let*

$$K_W(r) = \{A \in W : 0 \leq \|A\|_2 \leq r\}, \quad r \geq 0.$$

For $\eta > 0$, the space $NV_\eta(k) = K_{GL(k)}(\mathrm{e}^\eta) \setminus K_{GL(k)}(\mathrm{e}^{-\eta})$ is the space of η-near-volume preserving matrices.

Note that for the matrices T_w as defined at (2),

$$T_w \in NV_\eta(2) \qquad \text{for} \quad \eta \geq \sqrt{\frac{2 + w^2 + |w|\sqrt{4 + w^2}}{2}}. \tag{7}$$

Clearly $NV_\eta(k) \supset O(k)$ and $NV_\eta(k) \not\supset SL(k)$ for all $\eta \geq 0$. We think of η as a small perturbation parameter. For example, it may relate to the maximum shearing in k dimensions, analogous to (7).

The class $NV_\eta(k)$ is natural to consider both for the formulation of our results, and also for simulation purposes. While it does not constitute one the main purposes of the paper, we pause to comment on the latter aspect. If the random distortion $\boldsymbol{T}$ is not only $NV_\eta(k)$-valued but also $O(k)$-invariant in the sense that $Q\boldsymbol{T} \stackrel{\mathrm{d}}{=} \boldsymbol{T}$ for all $Q \in O(k)$, or even 'bi-invariant' in the sense that $Q_1\boldsymbol{T}Q_2^{-1} \stackrel{\mathrm{d}}{=} \boldsymbol{T}$ for all $Q_1, Q_2 \in O(k)$, then the distribution of $\boldsymbol{T}$ may be simulated efficiently, regardless of the size of η; see [4, pp. 134ff]. This simulation algorithm utilises the QR (or Gram-Schmidt) decomposition

$$\iota : O(k) \times R(k) \to GL(k)\,, \qquad (Q, R) \mapsto QR\,, \tag{8}$$

which is a diffeomorphism (see [4, pp. 118]), with the corresponding decomposition of measures

$$\lambda_{GL(k)} = \left(\mu_{O(k)} \otimes h_{R(k)} \cdot \lambda_{R(k)}\right) \iota^{-1}\,, \tag{9}$$

where the density $h_{R(k)}$ satisfies $h_{R(k)}\left((r_{ij})_{1\le i\le j\le k}\right) \propto \prod_{\ell=1}^{k} r_{\ell\ell}^{k-\ell}$. Formula (4) may be used to switch between Haar and Lebesgue densities.

An alternative approach would be to consider for the distortions the space $SL(k)$. The version of (9) with the Lebesgue replaced by the Haar measures does not apply in this case because $SL(k)$ is a (k^2-1)-dimensional manifold. The results in [4, Section 4.6] hold for decompositions of Lie groups with respect to maximal compact subgroups, and in the case of $SL(k)$ this maximal compact subgroup is equal to $SO(k)$; see [6, p. 270]. There is a decomposition $SL(k) = SO(k)\mathscr{SP}_k$, where $\mathscr{SP}_k$ is the set of positive definite $k \times k$ matrices with determinant 1; see [7, p. 269]. However, this decomposition seems to be of limited help for simulation purposes. Moreover, while $\mu_{SL(k)}$ has an expression in terms of (k^2-1)-dimensional Hausdorff measure (see [8]), it is non-trivial to simulate by other means than those discussed in [4]. These problems could be removed by considering a neighbourhood of $SL(k)$ of full (i.e. k^2-dimensional) Lebesgue measure, similar to the definition of $NV_\eta(k)$. However, the fact that a map in $SL(k)$ can have arbitrarily large or small eigenvalues makes the formulation of the corresponding results in this paper much more cumbersome.

2.3. Non-distorted grains and expected geometric covariogram

The following formulae for the AVV in the absence of rotation were derived in [1, p. 142, (3.14) and p. 190, (3.A)]. (The latter case is only an exercise, but more details are given as part of the proof of Proposition 2 below.) In the case $\rho < \infty$,

$$\begin{aligned}
\lambda \mathrm{Var}(V) \to \tau_{\mathrm{fin}}^2 &= c_{\mathcal{S},\rho} \int_0^\infty r^{k-1} J_{\mathcal{S}}(r)\, dr\,, \\
c_{\mathcal{S},\rho} &= \rho |\mathcal{R}| \exp\{-2\rho \mathrm{E}(|\mathcal{S}|)\}\,, \\
J_{\mathcal{S}}(r) &= \int_{\mathbb{S}^{k-1}} \{\exp\left(\rho H_{\mathcal{S}}(u,r)\right) - 1\}\ \sigma_{k-1}(du)\,, \quad r \geq 0\,, \\
H_{\mathcal{S}}(u,r) &= \mathrm{E}\{|(ru + \mathcal{S}) \cap \mathcal{S}|\}\,, \qquad (u,r) \in \left(\mathbb{R}^k \setminus \{0\}\right) \times [0,\infty)\,,
\end{aligned}$$

while in the case $\rho = \infty$,

$$\begin{aligned}
\mathrm{Var}(V) &\sim \lambda^{-1} (\delta^k \lambda)^{1-k} |\mathcal{R}|\, \tau_\infty^2\,, \\
\tau_\infty^2 &= \Gamma(k) \exp\{-\delta^k \lambda \mathrm{E}(|\mathcal{S}|)\} \int_{\mathbb{S}^{k-1}} \{q_{\mathcal{S}}(u)\}^{-k}\, \sigma_{k-1}(du)\,, \qquad (10)
\end{aligned}$$

where the functions $H_{\mathcal{S}}$ and $q_{\mathcal{S}} : \mathbb{S}^{k-1} \to [0,\infty)$ are related through the following regularity condition given in [1, p. 190]:

$$H_{\mathcal{S}}(u,r) = \mathrm{E}(|\mathcal{S}|) - q_{\mathcal{S}}(u) r + o(r) \qquad (11)$$

uniformly in $u \in \mathbb{S}^{k-1}$ as $r \searrow 0$, where $q_{\mathcal{S}}$ is bounded away from zero. On occasion we shall use the notation $H_{\mathcal{S}}(v,r)$ for any $v \in \mathbb{R}^k \setminus \{0\}$, by letting $H_{\mathcal{S}}(v,r) = H_{\mathcal{S}}(v/\|v\|, \|v\| r)$. Note that unlike in [1], in formula (10) we include the factor $\exp\{-\delta^k \lambda \mathrm{E}(|\mathcal{S}|)\}$ in the definition of τ_∞^2 (called τ^2 in [1]). This is because in Proposition 2 of Section 3.1, which deals with the case of distorted grains, the distribution of the distortion will appear in the exponential.

Throughout this paper we will assume that (11) holds and that $\mathcal{S}$ is a *regular convex set*, that is, convex and equal to the closure of its interior, with probability 1. We also assume the standard conditions (see e.g. [9]) that $\mathrm{E}|\mathcal{S} \oplus K| < \infty$ for all compact $K \subset \mathbb{R}^k$, and that there exists an $\varepsilon > 0$ such that $\mathrm{E}|\mathcal{S}_{-\varepsilon}| > 0$. The function $H_{\mathcal{S}}$ is the expected value of the so-called *geometric covariogram* $\gamma_{\mathcal{S}}(u, r) = |(ru + S) \cap \mathcal{S}|$ of $\mathcal{S}$. For $u \in \mathbb{S}^{k-1}$, let $p_{u^\perp}$ denote the orthogonal projection on $u^\perp$ and $\nu_{u^\perp}$ the $(k-1)$-dimensional Lebesgue measure on $u^\perp$. The following facts about $H_{\mathcal{S}}$ and $q_{\mathcal{S}}$ are given in ([10, Section 4-3]; [9]).

(I) $H_{\mathcal{S}}(u, \cdot)$ is convex on $[0, W_u(\mathcal{S}))$, where $W_u(\mathcal{S}) = \sup_{x,y\in\mathcal{S}}\langle y - x, u\rangle$ is the width of $\mathcal{S}$ in direction u, and is identically zero outside this interval;

(II) $H_{\mathcal{S}}(u, \cdot)$ is continuously differentiable on $[0, W_u(\mathcal{S}))$, and is differentiable from the right at the origin with

$$q_{\mathcal{S}}(u) = -\left.\frac{\partial H_{\mathcal{S}}(u, r)}{\partial r}\right|_{r=0+} = \mathrm{E}\left\{\nu_{u^\perp}(p_{u^\perp}(\mathcal{S}))\right\} . \tag{12}$$

The statement of (I) also holds for $\gamma_{\mathcal{S}}$ in place of $H_{\mathcal{S}}$ and gives

$$\gamma_{\mathcal{S}}(u, r) \le \left(1 - \frac{r}{W_u(\mathcal{S})}\right) \cdot |\mathcal{S}| ,$$

hence

$$H_{\mathcal{S}}(u, r) \le \mathrm{E}(|\mathcal{S}|) - r\mathrm{E}\left\{\frac{|\mathcal{S}|}{W_u(\mathcal{S})}\right\} \le \mathrm{E}(|\mathcal{S}|) - r\zeta_{\mathcal{S}} , \quad \text{where} \quad \zeta_{\mathcal{S}} = \min_{u\in\mathbb{S}^{k-1}} \mathrm{E}\left\{\frac{|\mathcal{S}|}{W_u(\mathcal{S})}\right\} . \tag{13}$$

Clearly $\zeta_{\boldsymbol{Q}\mathcal{S}} = \zeta_{\mathcal{S}}$ for any random rotation $\boldsymbol{Q}$. Hence the effect of the distribution of $\mathcal{S}$ 'modulo' its rotated part is separated from the effect of the rotation in our results to be presented later. Note that $\zeta_{\mathcal{S}} < q_{\mathcal{S}}$. Let $\psi_\eta(r)$ denote the Lipschitz constant of $H_{\mathcal{S}}$ within the annulus $r \pm \mathrm{e}^\eta$.

3. Optimality properties for distorted grains

In contrast to Section 2.3, in the remainder of the paper we regard τ^2_{fin} and τ^2_{∞} as deterministic functionals, taking a distribution $F_{\boldsymbol{T}} \in \mathscr{P}(NV_\eta(k))$, which defines the distorted typical grain $\mathcal{S}^* = \boldsymbol{T}\mathcal{S}$, as its argument. Note that the marginals of $F_{\boldsymbol{T}}\iota = F_{\boldsymbol{T}}\big(\iota^{-1}\big)^{-1}$, where ι is defined at (8), are probability measures on $O(k)$ and $R(k) \cap NV_\eta(k)$ respectively. Hereinafter we use the letters $\boldsymbol{Q}$ and $\boldsymbol{R}$ (possibly subscripted) if we are dealing with random matrices taking values in the two respective spaces.

3.1. The case $\rho = \infty$

We prove the following generalisation of (10) to the 'distorted' case.

Proposition 2.

$$\tau^2_{\infty}(F_{\boldsymbol{T}}) \sim \exp\{-\delta^k \lambda \mathrm{E}(|\det \boldsymbol{T}|)\mathrm{E}(|\mathcal{S}|)\} \times \Gamma(k) \int_{\mathbb{S}^{k-1}} \frac{\sigma_{k-1}(du)}{\left[\mathrm{E}\left\{\|\boldsymbol{T}^{-1}u\| \cdot |\det \boldsymbol{T}| \cdot q_{\mathcal{S}}\left(\frac{\boldsymbol{T}^{-1}u}{\|\boldsymbol{T}^{-1}u\|}\right)\right\}\right]^k} . \tag{14}$$

Proof. We follow the outline of the arguments given in [1, p. 190]. From [1, p. 129], we have

$$\begin{aligned}\mathrm{Var}(V) &\leq \delta^k \exp\{-2\delta^k \lambda \mathrm{E}(|\mathcal{S}^*|)\} \\ &\quad \times \int_{\mathbb{R}^k} \int_{\mathcal{R}-w} \big(\exp[\delta^k \lambda \mathrm{E}\{|(v - w + \mathcal{S}^*) \cap \mathcal{S}^*|\}] - 1\big)\, dv\, dw \\ &= \delta^k \exp\{-2\delta^k \lambda \mathrm{E}(|\mathcal{S}^*|)\}|\mathcal{R}| \int_{\mathbb{R}^k} \big(\exp[\delta^k \lambda \mathrm{E}\{|(x + \mathcal{S}^*) \cap \mathcal{S}^*|\}] - 1\big)\, dx\,,\end{aligned}$$

and if $\mathcal{R} = \cup_{1\le i\le n}\mathcal{D}_i$ is a disjoint union of cubes in $\mathbb{R}^k$ with respective side lengths $2t_i$, then

$$\begin{aligned}\mathrm{Var}(V) &\ge \delta^k \exp\{-2\delta^k\lambda \mathrm{E}(|\mathcal{S}^*|)\} \\ &\qquad\times \int_{\mathcal{R}_{-\varepsilon}} \int_{\|w\|_\infty \le \delta^{-1}\varepsilon t} \left(\exp[\delta^k \lambda \mathrm{E}\{|(v-w+\mathcal{S}^*)\cap \mathcal{S}^*|\}]-1\right) dv\, dw \\ &= \delta^k \exp\{-2\delta^k\lambda\mathrm{E}(|\mathcal{S}^*|)\}(1-\varepsilon)^k|\mathcal{R}| \int_{\mathbb{R}^k} \left(\exp[\delta^k\lambda\mathrm{E}\{|(x+\mathcal{S}^*)\cap\mathcal{S}^*|\}]-1\right)\, dx\end{aligned}$$

for any $0<\varepsilon<1$, where $t=\min t_i$. For $r\ge 0$ and $u\in\mathbb{S}^{k-1}$, we have

$$\begin{aligned}H_{\mathcal{S}^*}(u,r) &= \mathrm{E}\{|(ru+\mathcal{S}^*)\cap\mathcal{S}^*|\} = \mathrm{E}[\mathrm{E}\{|(ru+\mathcal{S}^*)\cap\mathcal{S}^*| \mid \boldsymbol{T}\}] \\ &= \mathrm{E}\left[\mathrm{E}\left\{|\det \boldsymbol{T}|\cdot|(r\boldsymbol{T}^{-1}u+\mathcal{S})\cap\mathcal{S}| \;\middle|\; \boldsymbol{T}\right\}\right] \\ &= \mathrm{E}\left[|\det \boldsymbol{T}|\left\{\mathrm{E}(|\mathcal{S}|) - q_{\mathcal{S}}\left(\frac{\boldsymbol{T}^{-1}u}{\|\boldsymbol{T}^{-1}u\|}\right)\|\boldsymbol{T}^{-1}u\|r + o(\|\boldsymbol{T}^{-1}u\|r)\right\}\right], \qquad (15)\end{aligned}$$

where the $o(\cdot)$ term is uniformly small as $r\searrow 0$. For $s\in(0,\infty]$, we thus obtain

$$\begin{aligned}&\int_{\|x\|<s} \left(\exp[\delta^k\lambda\mathrm{E}\{|(x+\mathcal{S}^*)\cap\mathcal{S}^*|\}]-1\right)\, dx \\ &\quad= \int_0^s r^{k-1}\int_{\mathbb{S}^{k-1}} \left(\exp\left\{\delta^k\lambda H_{S^*}(u,r)\right\}-1\right)\sigma_{k-1}(du)\, dr \\ &\quad\sim (\delta^k\lambda)^{-k}\exp\{\delta^k\lambda\mathrm{E}(|\det\boldsymbol{T}|)\mathrm{E}(|\mathcal{S}|)\} \\ &\qquad\quad\times\int_{\mathbb{S}^{k-1}}\int_0^s r^{k-1}\exp\left[\mathrm{E}\left\{-r\delta^k\lambda\cdot\|\boldsymbol{T}^{-1}u\|\cdot|\det\boldsymbol{T}|\cdot q_{\mathcal{S}}\left(\frac{\boldsymbol{T}^{-1}u}{\|\boldsymbol{T}^{-1}u\|}\right)\right\}\right]\sigma_{k-1}(du)\, dr\,,\end{aligned}$$

which completes the proof. □

In [1, pp. 189–190, Problem 3.11 (ii)] it is stated that $\mu_{O(k)}$ minimises $\tau^2_\infty(F_{\boldsymbol{Q}})$ over all $F_{\boldsymbol{Q}}\in\mathscr{P}(O(k))$. We give the proof for completeness. From (15) it follows that if $\boldsymbol{Q}\sim\mu_{O(k)}$ then $q_{\boldsymbol{Q}S}(u)=\omega_k^{-1}\|q_{\mathcal{S}}\|_1$ is constant for $u\in\mathbb{S}^{k-1}$. Noting that the function $\{x\mapsto x^{-k}\}$ is convex on $(0,\infty)$, we can apply Jensen's inequality to the integral in (14), to obtain

$$\int_{\mathbb{S}^{k-1}}\frac{\sigma_{k-1}(du)}{\{q_{\boldsymbol{Q}S}(u)\}^k}\left(=\frac{\omega_k^{k+1}}{\|q_{\mathcal{S}}\|_1^k}\right)\le\omega_k^{-1}\int_{\mathbb{S}^{k-1}}\sigma_{k-1}(du)\int_{\mathbb{S}^{k-1}}\frac{\sigma_{k-1}(dw)}{q^k_{\mathcal{S}}(w)}=\int_{\mathbb{S}^{k-1}}\frac{\sigma_{k-1}(dw)}{q^k_{\mathcal{S}}(w)}\,,$$

which verifies the assertion. It seems impossible to simplify (14).

3.2. The case $\rho < \infty$

From the computations in [1], it readily follows that the Haar measure $\mu_{O(k)}$ enjoys the same optimality property over $\mathscr{P}(O(k))$ as in the setup of Section 3.1. Here we present a more general result. To see what sort of condition we need, consider the case $k = 2$, choose any distribution for $\boldsymbol{Q}$, and define the distribution of $\boldsymbol{R}$ as follows,

$$\mathrm{P}\left(\boldsymbol{R} = \begin{pmatrix} \mathrm{e}^{-\eta} & 0 \\ 0 & \mathrm{e}^{\eta} \end{pmatrix}\right) = \mathrm{P}\left(\boldsymbol{R} = \begin{pmatrix} \mathrm{e}^{\eta} & 0 \\ 0 & \mathrm{e}^{-\eta} \end{pmatrix}\right) = \frac{1}{2}\,.$$

Then of course $\boldsymbol{T} = \boldsymbol{Q}\boldsymbol{R} \in NV_\eta(2)$. Let $\mathcal{S}$ be an ellipse with its two half-axes equal to $\mathrm{e}^{\eta/2}e^{(1)}$ and $\mathrm{e}^{-\eta/2}e^{(2)}$, and equal to $\mathrm{e}^{-\eta/2}e^{(1)}$ and $\mathrm{e}^{\eta/2}e^{(2)}$, with probability 1/2 each. If we allow dependence of $\boldsymbol{R}$ and $\mathcal{S}$ (and only then), we can achieve that

$$\boldsymbol{R}\mathcal{S} \stackrel{\mathrm{d}}{=} \mathcal{S}\,. \tag{16}$$

Returning to the general setup, condition (16) may be regarded as a condition on $\mathcal{S}$ for given $\boldsymbol{R}$, or *vice versa*. The arguments in [1, p. 144–145] then lead to the following result. We temporarily write $\tau^2_{\mathrm{fin}}(F_{\boldsymbol{T}}) = \tau^2_{\mathrm{fin}}(F_{\boldsymbol{Q}}, F_{\boldsymbol{R}}, F_{\mathcal{S}})$.

Proposition 3. *The function $\tau^2_{\mathrm{fin}}(\cdot)$ is constant over triplets $(F_{\boldsymbol{Q}}, F_{\boldsymbol{R}}, F_{\mathcal{S}})$ satisfying* (16) *and, on the domain of such triplets, attains its minima for $F_{\boldsymbol{Q}} = \mu_{O(k)}$.*

4. Continuity for distorted and rotated grains

As in Section 3, we consider the cases $\rho = \infty$ and $\rho < \infty$ in turn.

4.1. The case $\rho = \infty$

Proposition 2 yields the following continuity result. Note that here and in Theorems 5 and 9 below, the minimum over $O(k)$ arises due to the appearance of differences of integrals with respect to the surface measure σ_{k-1}, which is of course $O(k)$-invariant.

Theorem 4. *Let $\boldsymbol{T}_1, \boldsymbol{T}_2 \in NV_\eta(k)$. Then*

$$|\tau_\infty^2(F_{\boldsymbol{T}_1}) - \tau_\infty^2(F_{\boldsymbol{T}_2})| \lesssim k\Gamma(k)\omega_k \exp\{(k^2+4)\eta - \delta^k \lambda \mathrm{e}^{-k\eta}\mathrm{E}(|\mathcal{S}|)\} \\ \times \|q_{\mathcal{S}}\|_\infty^k \|q_{\mathcal{S}}^{-1}\|_\infty^{2k} \min_{\Pi \in O(k)} \mathcal{W}_1(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2}\Pi^{-1}).$$

Proof. From Proposition 2 and the inequalities

$$\mathrm{e}^{-\eta} \le \|\boldsymbol{T}_i^{-1}u\| \le \mathrm{e}^{\eta}, \quad \mathrm{e}^{-k\eta} \le |\det \boldsymbol{T}_i|^{-1} \le \mathrm{e}^{k\eta}, \quad \text{and} \quad q_{\mathcal{S}}(u) \ge \|q_{\mathcal{S}}^{-1}\|_\infty^{-1}$$

for $i = 1, 2$ and $u \in \mathbb{S}^{k-1}$, we obtain

$$|\tau_\infty^2(F_{\boldsymbol{T}_1}) - \tau_\infty^2(F_{\boldsymbol{T}_2})| \lesssim \Gamma(k)\|q_{\mathcal{S}}\|_\infty^k \|q_{\mathcal{S}}^{-1}\|_\infty^{2k} \exp\{(k+2)\eta - \delta^k\lambda \mathrm{e}^{-k\eta}\mathrm{E}(|\mathcal{S}|)\} \\ \times \left| \int_{\mathbb{S}^{k-1}} \left\{ \left(\mathrm{E}\|\boldsymbol{T}_1^{-1}u\|\right)^k - \left(\mathrm{E}\|\boldsymbol{T}_2^{-1}u\|\right)^k \right\} \sigma_{k-1}(du) \right|. \tag{17}$$

For any $S, T \in NV_\eta(k)$ we have $T(T^{-1} - S^{-1})S = S - T$ and hence, for any $u \in \mathbb{S}^{k-1}$,

$$|\|T^{-1}u\| - \|S^{-1}u\|| \le \|T^{-1} - S^{-1}\| \le \mathrm{e}^{2\eta}\|S - T\|, \tag{18}$$

which proves that the map $\{T \mapsto \|T^{-1}u\|\}$ is Lipschitz continuous on $NV_\eta(k)$ with constant $\mathrm{e}^{2\eta}$. Therefore, for any $\Pi \in O(k)$, the integral on the right-hand side of (17) is

$$\left| \int_{\mathbb{S}^{k-1}} \left\{ \left(\mathrm{E}\|\boldsymbol{T}_1^{-1}u\|\right)^k - \left(\mathrm{E}\|\boldsymbol{T}_2^{-1}\Pi u\|\right)^k \right\} \sigma_{k-1}(du) \right| \\ \le k \exp\{k(k-1)\eta\} \int_{\mathbb{S}^{k-1}} \left|\mathrm{E}\|\boldsymbol{T}_1^{-1}u\| - \mathrm{E}\|\boldsymbol{T}_2^{-1}\Pi u\|\right| \sigma_{k-1}(du) \\ \le \omega_k k \exp[\{k(k-1)+2\}\eta]\, \mathcal{W}_1(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2}\Pi^{-1}),$$

where the last step follows from (6). This concludes the proof. $\square$

For the last results in this section, we assume that $\boldsymbol{Q}$ follows a group model, as defined in (ii) below formula (20). We shall further require that the distribution of $\boldsymbol{Q}$ be (strongly) *conjugation-invariant*, which means that

$$\boldsymbol{Q} \overset{\mathrm{d}}{=} P^T \boldsymbol{Q} P \qquad \text{for any fixed } P \in O(k). \tag{19}$$

'Strongly' here refers to the fact that $P \in O(k)$, not just $SO(k)$. Lastly, we assume that $\boldsymbol{Q}$ has a continuous density $f_{\boldsymbol{Q}}$ with respect to Haar measure. Note that (19) holds for all $P \in O(k)$ if and only if $f_{\boldsymbol{Q}}$ (which need not be continuous) depends on the eigenvalues of $\boldsymbol{Q}$ only through elementary symmetric functions; see [11, p. 585]. There is a simple relationship between the densities $f_{\boldsymbol{Q}v}$ and $f_{\boldsymbol{Q}}^{(k)} = f_{\boldsymbol{Q}e^{(k)}}$, for $v \in \mathbb{S}^{k-1}$, taken with respect to normalised surface measure $\mu_{k,1} = \omega_k^{-1} \sigma_{k-1}$. Indeed, let $\Pi \in O(k)$ be such that $\Pi e^{(k)} = v$; then

$$\omega_k^{-1} f_{\boldsymbol{Q}v}(y)\, \sigma_{k-1}(dy) = \mathrm{P}(\boldsymbol{Q}v \in dy) = \mathrm{P}(\Pi \boldsymbol{Q} e^{(k)} \in dy) = \omega_k^{-1} f_{\boldsymbol{Q}}^{(k)}(\Pi^{-1} y)\, \sigma_{k-1}(dy)\,,$$

that is,

$$f_{\boldsymbol{Q}v} = f_{\boldsymbol{Q}}^{(k)} \circ \Pi^{-1} \qquad \text{for any } \Pi \in O(k) \text{ with } \Pi e^{(k)} = v\,. \tag{20}$$

The computation leading to (20), with Π replaced by $M_0^T \Pi$, shows that the subsequent results hold for any family of densities $\big\{ f_{\boldsymbol{Q}}(\cdot \mid M) \big\}$ on $SO(k)$, parametrised by 'modal' matrices in $M \in SO(k)$, which is such that

(i) There exists M_0 such that $f_{\boldsymbol{Q}}(\cdot \mid M_0)$ is conjugation-invariant (typically $M_0 = I_k$),

(ii) $f_{\boldsymbol{Q}}(M^T \cdot \mid M) = f_{\boldsymbol{Q}}(\cdot \mid I_k)$ for all M.

The class of distributions introduced in [12], for which

$$f(Q \mid M, \kappa) \propto \big| \det\big(I_k + M^T Q\big) \big|^{\kappa}\,, \tag{21}$$

where $\kappa \geq 0$ is a dispersion parameter, satisfies (i) and (ii). In this case, there exists even more symmetry in that if $f_{\boldsymbol{Q}} = f(\cdot \mid I_k, \kappa)$ then $\boldsymbol{Q}^T \stackrel{\mathrm{d}}{=} \boldsymbol{Q}$. The proof of the following proposition is quite similar to that of Theorem 4, and thus omitted.

Theorem 5. *Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2$ be random rotations which belong to a family of distributions satisfying* (i) *and* (ii) *above. Then*

$$|\tau^2_\infty(F_{\boldsymbol{Q}_1}) - \tau^2_\infty(F_{\boldsymbol{Q}_2})| \lesssim k\Gamma(k)\omega_k \exp\{-\delta^k \lambda \mathrm{e}^{-k\eta} \mathrm{E}(|\mathcal{S}|)\} \|q_{\mathcal{S}}\|^k_\infty \|q^{-1}_{\mathcal{S}}\|^{2k}_\infty \min_{\Pi \in O(k)} \left\| f^{(k)}_{\boldsymbol{Q}_1} - f^{(k)}_{\boldsymbol{Q}_2} \circ \Pi \right\|_\infty .$$

4.2. The case $\rho < \infty$

For the study of continuity properties, and similar to [1, p. 144], the behaviour of the following function is crucial:

$$J_{\boldsymbol{T}}(r) = \int_{\mathbb{S}^{k-1}} \left\{ \exp\left(\rho \int_{NV_\eta(k)} |\det T| \cdot H_{\mathcal{S}}(T^{-1}u, r)\, F_{\boldsymbol{T}}(dT) \right) - 1 \right\} \sigma_{k-1}(du)\,, \quad r \geq 0\,.$$

We use the notation

$$\Delta(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2}) = \left|\tau^2_{\text{fin}}(F_{\boldsymbol{T}_1}) - \tau^2_{\text{fin}}(F_{\boldsymbol{T}_2})\right|\,, \qquad \Delta_r(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2}) = |J_{\boldsymbol{T}_1}(r) - J_{\boldsymbol{T}_2}(r)|\,,$$

so that

$$\Delta(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2}) \leq c_{\mathcal{S},\rho} \int_0^\infty r^{k-1} \Delta_r(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2})\, dr\,. \tag{22}$$

Theorem 6. *For any distortions $\boldsymbol{T}_1, \boldsymbol{T}_2$,*

$$\Delta(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2}) \leq \omega_k \mathrm{e}^{(k+2)\eta} \rho c_{\mathcal{S},\rho} \exp\{\rho \mathrm{E}(|\mathcal{S}|)\} \left(\int_0^\infty r^{k-1} \psi_\eta(r)\, dr \right) \min_{\Pi \in O(k)} \mathcal{W}_1(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2}\Pi^{-1})\,.$$

Proof. For any $r \geq 0$, we have

$$\begin{aligned}
\Delta_r(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2}) \leq \int_{\mathbb{S}^{k-1}} \Bigg| &\exp\left\{\rho \int_{NV_\eta(k)} |\det T| \cdot H_{\mathcal{S}}(T^{-1}u, r) F_{\boldsymbol{T}_1}(dT)\right\} \\
&- \exp\left\{\rho \int_{NV_\eta(k)} |\det T| \cdot H_{\mathcal{S}}(T^{-1}u, r) F_{\boldsymbol{T}_2}(dT)\right\}\Bigg| \, \sigma_{k-1}(du) \\
\leq \rho \exp\{\rho \mathrm{E}(|\mathcal{S}|)\} \int_{\mathbb{S}^{k-1}} \Bigg| &\int_{NV_\eta(k)} |\det T| \cdot H_{\mathcal{S}}(T^{-1}u, r)\, F_{\boldsymbol{T}_1}(dT) \\
&- \int_{NV_\eta(k)} |\det T| \cdot H_{\mathcal{S}}(T^{-1}u, r)\, F_{\boldsymbol{T}_2}(dT)\Bigg| \, \sigma_{k-1}(du)\,. \qquad (23)
\end{aligned}$$

Here we have used the inequality $|\mathrm{e}^{ax} - \mathrm{e}^{ay}| \leq a|x-y| \exp\{a \max(x, y)\}$ valid for all $x, y, a \geq 0$, as well as the simple bound $H_{\mathcal{S}}(\cdot, \cdot) \leq \mathrm{E}(|\mathcal{S}|)$. From the discussion around (18), as well as the duality formula (6), the term inside the modulus $|\ldots|$ signs on the right-hand side of (23) is bounded from above by $\psi_\eta(r)\mathrm{e}^{(k+2)\eta}\, \mathcal{W}_1(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2})$. Using (22) we thus obtain the theorem. □

Remark 7. *Although the Wasserstein distance $\mathcal{W}_1(F_{\boldsymbol{T}_1}, F_{\boldsymbol{T}_2})$ appearing in Theorem 6 can almost never be computed exactly, the upper bound from [5, formula (7.1.5), p. 152] applies if the distributions $F_{\boldsymbol{T}_i}$ are generated from so-called transportation maps; specifically, the Wasserstein distance is bounded by the L^1 distance of these maps.*

Consider now a sequence of Boolean models

$$\{\mathscr{S}^{(n)} = \{\boldsymbol{\xi}_i + \mathcal{S}_i^{(n)},\, i \geq 1\}\,,\, n = 1, 2, \ldots\}\,.$$

Here $\mathcal{S}_i^{(n+1)} = \boldsymbol{Q}_n \mathcal{S}_i$ ($i \geq 1$), and the rotations $\{\boldsymbol{Q}_n,\, n \geq 1\}$ are defined recursively as follows. For indices $1 \leq i, j \leq n$ with $i \neq j$, let $G(i, j, \theta)$ be a Givens rotation in the plane spanned by $\left\{e^{(i)}, e^{(j)}\right\}$ (see [13, p. 215]) and let

$$\boldsymbol{Q}_0 = I_k\,, \qquad \boldsymbol{Q}_{n+1} = G(i_n, j_n, \theta_n)\boldsymbol{Q}_n\,, \quad n \geq 0\,,$$

where for each n, the pair of indices (i_n, j_n) with $1 \le i_n < j_n \le k$ is chosen uniformly with probability $1/\binom{k}{2}$ each, the angle θ_n is uniform on $[0, 2\pi)$, and the sequence $\big\{(i_n, j_n), \theta_n,\, n \ge 0\big\}$ is i.i.d. The sequence $\{\boldsymbol{Q}_n\}$ is known as Kac's random walk on $SO(k)$.

Corollary 8. *The sequence $\tau^2_{\text{fin}}(F_{\boldsymbol{Q}_n})$ converges to $\tau^2_{\text{fin}}(\mu_{SO(k)})$ with rate $\mathcal{O}(n^2 \log n)$.*

Proof. It was shown in [14] that Kac's random walk converges to $\mu_{SO(k)}$, with respect to the $\mathcal{W}_1$ metric, with rate $\mathcal{O}(n^2 \log n)$. Theorem 6 readily implies that the same convergence, and convergence rate, holds for the sequence $\tau^2_{\text{fin}}(F_{\boldsymbol{Q}_n})$. □

In the remainder of this section, we impose the same symmetry assumptions on $\boldsymbol{Q}_i$ as at the end of Section 4.1.

Theorem 9. *Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in O(k)$ satisfy the same conditions as in Theorem 5, and let $p, q \in [1, \infty]$ be conjugate indices (that is, $p^{-1} + q^{-1} = 1$, where $\infty^{-1} = 0$). Then*

$$\Delta(F_{\boldsymbol{Q}_1}, F_{\boldsymbol{Q}_2}) \le c_{\mathcal{S},\rho} \left(\int_0^\infty r^{k-1} \exp\{-\rho \zeta_{\mathcal{S}} r\} \, \|H_{\mathcal{S}}(\cdot, r)\|_p \, dr \right) \exp\{\rho \mathrm{E}(|\mathcal{S}|)\}$$
$$\times \min_{\Pi \in O(k)} \left\| f^{(k)}_{\boldsymbol{Q}_1} - f^{(k)}_{\boldsymbol{Q}_2} \circ \Pi \right\|_q .$$

In particular,

$$\Delta(F_{\boldsymbol{Q}_1}, F_{\boldsymbol{Q}_2}) \le c_{\mathcal{S},\rho} \Gamma(k) \rho^k \zeta^k_{\mathcal{S}} \, \mathrm{E}(|\mathcal{S}|^2) \exp\{\rho \mathrm{E}(|\mathcal{S}|)\} \min_{\Pi \in O(k)} \left\| f^{(k)}_{\boldsymbol{Q}_1} - f^{(k)}_{\boldsymbol{Q}_2} \circ \Pi \right\|_\infty .$$

Proof. The second assertion follows from the first upon taking $p = 1, q = \infty$, bounding the exponential inside the integral by 1, and noting that (see [1, p. 143])

$$\int_0^\infty r^{k-1} \|H_{\mathcal{S}}(\cdot, r)\|_1 \, dr = \mathrm{E}(|\mathcal{S}|^2) .$$

To prove the first assertion, fix $m \in \{1, \ldots, k\}$ and let

$$\mathscr{P}_m(O(k)) = \left\{\nu \in \mathscr{P}(O(k)) : \nu \left(e^{(k-m+1)} \mid \ldots \mid e^{(k)}\right)^{-1} = \mu_{k,m}\right\},$$
$$\mathcal{C}_m = \left\{A \in O(k) : AU = U \text{ for } U = \operatorname{span}\left\{e^{(k-m+1)}, \ldots, e^{(k)}\right\}\right\},$$

where $\left(e^{(k-m+1)} \mid \ldots \mid e^{(k)}\right)(Q)$ removes the first $k-m$ columns of $Q \in O(k)$. By invariance of $\mu_{k,m}$, we obtain for any $\nu_m \in \mathscr{P}_m(O(k))$ and $r \geq 0$ that

$$\begin{aligned} J_{\boldsymbol{Q}}(r) &= \omega_k \int_{\mathcal{C}_m} \left\{\exp\left(\rho \int_{O(k)} H_{\mathcal{S}}(Q^{-1}Pe^{(k)}, r)\, F_{\boldsymbol{Q}}(dQ)\right) - 1\right\} \nu_m(dP) \\ &= \omega_k \int_{\mathcal{C}_m} \left\{\exp\left(\rho \int_{O(k)} H_{\mathcal{S}}(PQ^{-1}e^{(k)}, r)\, F_{\boldsymbol{Q}}(dQ)\right) - 1\right\} \nu_m(dP) \\ &= \omega_k \int_{\mathcal{C}_m} \left\{\exp\left(\rho\omega_k^{-1} \int_{\mathbb{S}^{k-1}} H_{\mathcal{S}}(Pw, r)\, f^{(k)}_{\boldsymbol{Q}^{-1}}(w)\, \sigma_{k-1}(dw)\right) - 1\right\} \nu_m(dP)\,. \end{aligned}$$

The reason why we display this formula with general m, rather than just with $\nu_k = \mu_{O(k)}$, is that we hope to get a tighter bound by choosing some $1 \leq m \leq k$. We further have

$$\begin{aligned} \Delta_r(F_{\boldsymbol{Q}_1}, F_{\boldsymbol{Q}_2}) \leq \omega_k \int_{\mathcal{C}_m} \Bigg| &\exp\left\{\rho\omega_k^{-1} \int_{\mathbb{S}^{k-1}} H_{\mathcal{S}}(Pw, r) f^{(k)}_{\boldsymbol{Q}_1^{-1}}(w)\, \sigma_{k-1}(dw)\right\} \\ &- \exp\left\{\rho\omega_k^{-1} \int_{\mathbb{S}^{k-1}} H_{\mathcal{S}}(Pw, r) f^{(k)}_{\boldsymbol{Q}_2^{-1}}(\Pi w)\, \sigma_{k-1}(dw)\right\}\Bigg| \nu_m(dP) \\ \leq \rho \exp\left\{\rho \mathrm{E}(|\mathcal{S}|) - r\rho\zeta_{\mathcal{S}}\right\} &\int_{\mathcal{C}_m} \Bigg| \int_{\mathbb{S}^{k-1}} H_{\mathcal{S}}(Pw, r) \\ &\times \left\{f^{(k)}_{\boldsymbol{Q}_1^{-1}}(w) - f^{(k)}_{\boldsymbol{Q}_2^{-1}}(\Pi w)\right\} \sigma_{k-1}(dw)\Bigg| \nu_m(dP) \end{aligned} \tag{24}$$

for arbitrary $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in O(k)$ and $\Pi \in \mathcal{C}_m$, where the last step follows from (13). Apply Hölder's inequality, take $m = k$ and use (22) to complete the proof. □

The computational burden in Theorem 9 that comes from the minimisation over $O(k)$ may be substantially reduced in case it is known that $f^{(k)}_{\boldsymbol{Q}_2}$ is $\mathcal{G}$-symmetric for some subgroup $\mathcal{G}$ of $O(k)$, as defined in [15]. Indeed, in that case, by definition there exists a fixed matrix

$R \in O(k)$ such that $f^{(k)}_{\boldsymbol{Q}_2}(PRu) = f^{(k)}_{\boldsymbol{Q}_2}(Ru)$ for all $P \in O(k)$, and thus the search for an optimal Π^* may then be restricted to any set $\mathcal{M}$ (not necessarily a subgroup of $O(k)$) such that $\mathcal{G}\mathcal{M} = \{HG : H \in \mathcal{G}, G \in \mathcal{M}\} = O(k)$.

Theorem 9 gives rise to an optimisation problem over $O(k)$ which is nonconvex and hence difficult. For $p = q = 2$, the optimisation problem may be stated as a constrained minimisation problem as follows,

$$G : M(k) \to \mathbb{R}\,, \qquad A \mapsto \tfrac{1}{2}\big\| f^{(k)}_{\boldsymbol{Q}_1} - f^{(k)}_{\boldsymbol{Q}_2} \circ A \big\|_2^2 \qquad \text{subject to } A \in O(k). \qquad (*)$$

If we assume that $f^{(k)}_{\boldsymbol{Q}_i}$ is continuously differentiable for $i = 1, 2$, then the function G in $(*)$ is twice continuously differentiable, and necessary and sufficient conditions for a local minimiser $A_0 \in O(k)$ are available; see [16, 17].

For $p = 1$ and $q = \infty$ we have a (nonconvex) minimax problem and techniques of nonlinear programming may be applied. We also obtain simplifications for other specific assumptions on the densities. We limit our discussion to the case $p = 1, q = \infty$. It was shown in [12, Proposition 3.3, p. 419] that if $\boldsymbol{Q}$ has the density $f(\cdot \mid I_k, \kappa)$ in the notation of (21), then $f^{(k)}_{\boldsymbol{Q}}(v \mid \kappa) \propto (1 + v_k)^{\kappa}$ for $v = (v_1, \ldots, v_k) \in \mathbb{S}^{k-1}$. Consider the case $k = 3$, where the latter density is expressed in spherical polar coordinates as $f^{(3)}_{\boldsymbol{Q}}(\varphi, \theta) = (\kappa + 1)\cos^{2\kappa}(\theta/2)$ for the 'latitude' $\varphi \in [0, 2\pi)$, and $\theta \in [0, \pi]$ denotes (geodesic) distance or the angle from the 'north pole' $e^{(3)}$. A simple argument shows that

$$\big\| f^{(3)}_{\boldsymbol{Q}}(\cdot \mid \kappa_1) - f^{(3)}_{\boldsymbol{Q}}(\cdot \mid \kappa_2) \big\|_\infty = |\kappa_1 - \kappa_2|\,.$$

Let $M^{(i)}$ denote the value of M in (i)–(ii) shown after (20) for the respective distributions of $\boldsymbol{Q}_i$, $i = 1, 2$. By choosing a suitable Π, we may assume that $M^{(1)} = M^{(2)} = I_k$ and hence the bound of Theorem 9 simplifies in this case. The cases $k \geq 4$ are more complex.

The following result shows that it is impossible to improve the bound by substituting values

$m < k$ in (24). Denote the right-hand side of (24) by $\Delta_r(F_{\boldsymbol{Q}_1}, F_{\boldsymbol{Q}_2} \mid \nu_m)$.

Proposition 10. *For any two conjugation-invariant* $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in O(k)$,

$$\min_{m=1,\ldots,k} \inf_{\nu_m \in \mathcal{P}_m(O(k))} \Delta_r(F_{\boldsymbol{Q}_1}, F_{\boldsymbol{Q}_2} \mid \nu_m) = \Delta_r(F_{\boldsymbol{Q}_1}, F_{\boldsymbol{Q}_2} \mid \mu_{O(k)}) \, .$$

Proof. We shall use the decomposition formula for $\mu_{O(k)}$ given in [11, p. 397]: for any measurable $f : O(k) \to [0, \infty]$ and $1 \leq m < k$,

$$\int_{O(k)} f(Q_1, Q_2) \, \mu_{O(k)}(dQ) = \int_{Q_1 \in St(k,m)} \int_{K \in O(k-m)} f(Q_1, GK) \, \mu_{O(k-m)}(dK) \, \mu_{k,m}(dQ_1) \, ,$$

where $Q_1 \in St(k, m)$ and $Q_2 \in St(k, k - m)$ respectively denote the first m columns and the last $k - m$ columns of $Q \in O(k)$, and $G = G(Q_1)$ is any matrix in $St(k, k - m)$ whose columns are orthogonal to those of Q_1 (so that $GG^T = I_k - Q_1 Q_1^T$). Equation (24) gives

$$\begin{aligned} \Delta_r(F_{\boldsymbol{Q}_1}, F_{\boldsymbol{Q}_2}) &\leq \| f^{(k)}_{\boldsymbol{Q}_1^{-1}} - f^{(k)}_{\boldsymbol{Q}_2^{-1}} \circ \Pi \|_\infty \\ &\times \int_{St(k,m)} \int_{\mathbb{S}^{k-1}} \int_{O(k-m)} H_{\mathcal{S}}\big((Q_1, GK)w, r\big) \vartheta_p(Q_1, GK) \mu_{O(k-m)}(dK) \, \sigma_{k-1}(dw) \mu_{k,m}(dQ_1) \, , \end{aligned}$$

for some Jacobian factor ϑ_p. The definition of $\mathcal{P}_m(O(k))$ implies that $\vartheta_p(Q_1, Q_2)$ is independent of Q_2. Thus the integrals simplify to yield the assertion. □

5. Concluding remarks

As mentioned in Section 1, the approach that we took to study the AVV may be transferred to other problems in stochastic geometry. We mention two scenarios to which the ideas of the present paper may be applied. The first of these is the size and structure of a 'typical' uncovered region $\mathcal{V} = \mathcal{V}_\lambda$ in a Boolean model with notation as in Section 1. 'Typical' is here taken to mean that we consider the conditional distribution of the vacant region containing the origin, conditional on the origin not being covered. It is known (see [1, Theorem 3.7,

p. 164]) that as $\lambda \to \infty$, the set $\lambda \mathcal{V}_\lambda$ converges essentially in distribution to a cell formed by a Poisson field of $(k-1)$-dimensional hyperplanes. In [18], by using integral geometric rather than analytic methods, this result was generalised to the case where $\mathcal{S}$ neither needs to be isotropic nor has a smooth boundary. The analogue of the problem in the present paper would be the study of continuity of the size and structure of the uncovered region as a function of the rotation or distortion. The approach proposed in the present paper seems to harmonise better with the geometric arguments in [1]; in particular, the Lie group structure and symmetry properties that we considered here seem difficult to exploit fully in the context of random set theory. In the latter context, it would be interesting to understand the nature of the interplay between random sets and associated probability metrics ([19]; [20, pp. 93ff]) on the one hand, and rotations or distortions on the other. A second problem to which the method of the present paper could be applied is the visibility function of the Boolean model for convex grains. Asymptotic results in this setup have recently been provided in [21].

The optimisation problems on $O(k)$ which arise from our results give new examples of such problems on the orthogonal group, and are quite different from the classical one of principal component analysis that was studied in [16, 17] as an application of his local and global optimality criteria on $St(k, m)$ and in particular $O(k)$. We leave a more extensive analysis of the optimisation aspect of our results to future endeavour.

Acknowledgements

We are grateful to the referee for numerous suggestions that considerably improved the paper. We acknowledge the support by grants from the Australian Research Council (Project No. DP0988483) and the Research Grants Council of the Hong Kong Special Administrative Region, China (Project Nos. HKBU200807 and HKBU200710).

References

[1] P. Hall, Introduction to the theory of coverage processes, Wiley, New York, 1988.

[2] R. Schneider, W. Weil, Stochastic and integral geometry, Springer, Berlin, 2008.

[3] J. Faraut, Analysis on Lie groups: an introduction, Cambridge University Press, Cambridge, 2008.

[4] W. Schindler, Measures with symmetry properties, Springer, Berlin, 2003.

[5] L. Ambrosio, N. Gigli, G. Savaré, Gradient flows in metric spaces and in the space of probability measures, Birkhäuser, Basel, second edition, 2008.

[6] H. Abbaspour, M. Moskowitz, Basic Lie theory, World Scientific, NJ, 2007.

[7] A. Terras, Harmonic analysis on symmetric spaces and applications. II, Springer, Berlin, 1988.

[8] L. Dedić, On Haar measure on $\mathrm{SL}(n, \mathbf{R})$, Publ. Inst. Math. (Beograd) (N.S.) 47 (1990) 56–60.

[9] C. Bordenave, Y. Gousseau, F. Roueff, The dead leaves model: a general tessellation modeling occlusion, Adv. in Appl. Probab. 38 (2006) 31–46.

[10] G. Matheron, Random sets and integral geometry, Wiley, New York, 1975.

[11] R. Muirhead, Aspects of multivariate statistical theory, Wiley, New York, 1982.

[12] C. León, J.-C. Massé, L.-P. Rivest, A statistical model for random rotations, J. Multivariate Anal. 97 (2006) 412–430.

[13] G. Golub, C. Van Loan, Matrix computations, Johns Hopkins University Press, Baltimore, third edition, 1996.

[14] R. Oliveira, On the convergence to equilibrium of Kac's random walk on matrices, Ann. Appl. Probab. 19 (2009) 1200–1231.

[15] L.-P. Rivest, On the information matrix for symmetric distributions on the hypersphere, Ann. Statist. 12 (1984) 1085–1089.

[16] T. Rapcsák, On minimization on Stiefel manifolds, European J. Oper. Res. 143 (2002) 365–376.

[17] T. Rapcsák, Some optimization problems in multivariate statistics, J. Global Optim. 28 (2004) 217–228.

[18] I. Molchanov, A limit theorem for scaled vacancies of the Boolean model, Stochastics Stochastics Rep. 58 (1996) 45–65.

[19] S. Rachev, Probability metrics and the stability of stochastic models, Wiley, Chichester, 1991.

[20] I. Molchanov, Theory of random sets, Springer, London, 2005.

[21] P. Calka, J. Michel, S. Porret-Blanc, Asymptotics of the visibility function in the Boolean model, 2009. ArXiv preprint 0905.4874v2. Available from `http://arxiv.org/abs/0905.4874v2`. Accessed on May 25, 2011.