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# $L(j,k)$-labeling numbers of square of paths[☆]

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**Abstract**

For $j \leq k$, the $L(j,k)$-labeling arose from code assignment problem. That is, let $j$, $k$ and $m$ be positive numbers, an $m$-$L(j,k)$-labeling of a graph $G$ is a mapping $f : V(G) \to [0,m]$ such that $|f(u)-f(v)| \geq j$ if $d(u,v) = 1$, and $|f(u)-f(v)| \geq k$ if $d(u,v) = 2$. The span of $f$ is the difference between the maximum and the minimum numbers assigned by $f$. The $L(j,k)$-labeling number of $G$, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labelings of $G$. The $k$th power $G^k$ of an undirected graph $G$ is the graph with the vertex set of $G$ in which two vertices are adjacent when their distance in $G$ is at most $k$. In this paper, the $L(j,k)$-labeling numbers of $P_n^2$ are determined for $j \leq k$.


*Keywords:* $L(j,k)$-labeling; Path; Square of path; Code assignment

## 1. Introduction

The rapid growth of computer wireless networks highlighted the scarcity of available codes (such as radio frequencies) for communication with minimum interference. For example, the *Packet Radio Network* (PRN) is a computer network that uses radio frequencies to transmit packet among computers. The two major types of interference in PRN are *Direct collision (or interference)*, which is caused by the transmission of adjacent stations (computers), and *Hidden terminal collision (or interference)*, which is caused by distance-two stations that transmit to the same receiving station or receive the data from the same transmitting station.

Let $G$ be a graph and let $V(G)$ and $E(G)$ be its vertex set and edge set, respectively. For any two vertices $u$ and $v$, let $d_G(u,v)$ (or simply $d(u,v)$) denote the distance (length of a shortest path) between $u$ and $v$ in $G$. Noted that all graphs considered in this article are simple connected and undirected. All notation not defined in this article can be found in the book [1].

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For positive numbers $j$ and $k$, an $L(j,k)$-*labeling* $f$ of $G$ is an assignment of numbers to vertices of $G$ such that $|f(u)-f(v)| \geq j$ if $uv \in E(G)$, and $|f(u)-f(v)| \geq k$ if $d(u,v)=2$. The *span* of $f$ is the difference between the maximum and the minimum numbers assigned by $f$. In other words, if we list the image of $f$ as a non-decreasing sequence $\{f(u_i)\}_{i=1}^n$, then the span of $f$ is $f(u_n)-f(u_1)$ which is called the *span of the sequence*, where $n$ is the order of $G$. The $L(j,k)$-*labeling number* of $G$, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labeling of $G$. Without loss of generality, we may assume the minimum value of each labeling $f$ is 0.

In a computer network, suppose direct collision is so weak that it can be ignored, and two distance-two stations can generate a hidden terminal collision. In order to avoid the hidden terminal collision, Bertossi and Bonuccelli [2] introduced an optimal code assignment, that is, two distance-two stations have different codes. By corresponding codes to labels, this code assignment problem is equivalent to the $L(0,1)$-labeling problem, that is, two distance-two vertices must be assigned different labels.

In general, the direct collision cannot be ignored. Based on this premise, Jin and Yeh [3] generalized the code assignment problem to $L(j,k)$-labeling problem with $j \leq k$. That is, to avoid direct collision, any two adjacent stations are required to be assigned at least $j$ apart codes. Additionally, to avoid hidden terminal collision, any two distance-two stations need to be assigned at least $k$ apart codes. Therefore, we face on the $L(j,k)$-labeling problem with $j \leq k$.

On the other hand, $L(j,k)$-labeling numbers of graphs for $j \geq k$ have been studied in many articles. Interested readers were referred to the surveys [4,5].

By now, about the $L(j,k)$-labeling numbers of graphs for $j \leq k$, there already exist some results. For example, Jin and Yeh determined $L(0,1)$, $L(1,1)$, $L(1,2)$-labeling numbers of paths, cycles and grids in [3]. Furthermore, Niu [6] introduced $L(j,k)$-labeling numbers of paths and cycles, and Griggs and Jin studied $L(j,k)$-labeling numbers of lattices (grids) in [7]. Moreover, Jayasree and Nicholas [8] mentioned $L(1,2)$-labeling numbers of certain generalizes Petersen graphs and $n$-star. In [9], the authors introduced the $L(j,k)$-labeling numbers of trees and stars with maximum degree. Lam, Lin and Wu [10] worked on $L(j,k)$-labeling numbers of product of completed graphs. Recently, Shiu and Wu determined $L(j,k)$-labeling numbers of direct and Cartesian product of path and cycle in [11] and [12], respectively. Moreover, the authors studied circular $L(j,k)$-labeling numbers of tree and Cartesian products of graphs, direct product of path and cycle, and square of paths in [13–15], respectively.

The $k$th *power* $G^k$ of a graph $G$ is the graph with the vertex set of $G$ in which two vertices are adjacent when their distance in $G$ is at most $k$. $G^2$ is called the *square* of $G$.

**Lemma 1.1.** *Let $j$ and $k$ be two positive numbers with $j \leq k$. Suppose $G$ is a graph and $H$ is an induced subgraph of $G$. Then $\lambda_{j,k}(G) \geq \lambda_{j,k}(H)$.*

Note that Lemma 1.1 is not true if $H$ is not an induced subgraph. Throughout this paper, $P_n = v_0v_1\cdots v_{n-1}$ denotes the path of order $n$.

## 2. $L(j,k)$-labeling numbers of $P_4^2$ and $P_5^2$

**Theorem 2.1.** *For $j \leq k$, $\lambda_{j,k}(P_4^2) = \max\{k, 3j\}$.*

**Proof.** Let $\lambda = \max\{k, 3j\}$. Let $f$ be the labeling of $P_4^2$ defined by $f(v_0)=0$, $f(v_1)=j$, $f(v_2)=2j$ and $f(v_3)=\lambda$. It is easy to verify that $f$ is a $\lambda$-$L(j,k)$-labeling of $P_4^2$. Hence $\lambda_{j,k}(P_4^2) \leq \lambda$.

On the other hand, since any two vertices of $P_4^2$ are adjacent or of distance two, $\lambda_{j,k}(P_4^2) \geq 3j$. Moreover, since $d(v_0,v_3)=2$, $\lambda_{j,k}(P_4^2) \geq k$. It implies that $\lambda_{j,k}(P_4^2) \geq \max\{k,3j\} = \lambda$. Hence, $\lambda_{j,k}(P_4^2) = \max\{k,3j\}$. □

**Theorem 2.2.** *For $j \leq k$, $\lambda_{j,k}(P_5^2) = \max\{j+k, 4j\}$.*

**Proof.** Let $\eta = \max\{k, 3j\}$ and $\lambda = \eta + j$. Let $f$ be the labeling of $P_5^2$ defined by $f(v_0)=0$, $f(v_1)=j$, $f(v_2)=2j$, $f(v_3)=\eta$ and $f(v_3)=\eta+j$. It is easy to verify that $f$ is a $\lambda$-$L(j,k)$-labeling of $P_5^2$. Hence $\lambda_{j,k}(P_5^2) \leq \lambda$.

On the other hand, since any two vertices of $P_5^2$ are adjacent or of distance two, $\lambda_{j,k}(P_5^2) \geq 4j$. Moreover, since two adjacent vertices $v_3$ and $v_4$ are at distance two from vertex $v_0$, $\lambda_{j,k}(P_5^2) \geq j+k$. It implies that $\lambda_{j,k}(P_5^2) \geq \max\{j+k, 4j\}$. Hence, $\lambda_{j,k}(P_5^2) = \max\{j+k, 4j\}$. □

## 3. $L(j,k)$-labeling numbers of $P_n^2$ for $n \geq 6$

In this section, we shall study the $L(j,k)$-number of $P_n^2$ by separating the condition $j \leq k$ into three cases which are $j \leq k < 2j$, $3j \leq k$ and $2j \leq k < 3j$, where $n \geq 6$.

We consider $j \leq k < 2j$ first. Define a labeling $f$ for $P_n^2$ by $f(v_i) = [i]_6 j$ for $0 \leq i \leq n$. Clearly, $f$ is a $(5j)$-$L(j,k)$-labeling for $P_n^2$ when $j \leq k < 2j$. So

$$\lambda_{j,k}(P_n^2) \leq 5j \text{ for } n \geq 6. \tag{3.1}$$

**Lemma 3.1.** *Let $j$ and $k$ be two positive numbers with $j \leq k < 3j$. Then $\lambda_{j,k}(P_6^2) \geq \min\{5j, 3j+k\}$.*

**Proof.** Suppose $f$ is a $\lambda$-$L(j,k)$-labeling of $P_6^2$ and $\lambda < 5j$. Let $I_0 = [0, j)$, $I_1 = [j, (\lambda+j)/3]$, $I_2 = ((\lambda+j)/3, (2\lambda-j)/3)$, $I_3 = [(2\lambda-j)/3, \lambda-j]$ and $I_4 = (\lambda-j, \lambda]$. Here, each $I_i$ is of length less than $j$, $0 \leq i \leq 4$. Since any two of vertices $v_0, v_1, v_2, v_3, v_4$ are adjacent or of distance two, each interval $I_i$ contains exactly one labels of $f(v_0), f(v_1), f(v_2), f(v_3), f(v_4)$. Similarly, each interval $I_i$ contains exactly one labels of $f(v_1), f(v_2), f(v_3), f(v_4), f(v_5)$. By pigeonhole principle, two labels among $f(v_0), f(v_1), f(v_2), f(v_3), f(v_4), f(v_5)$ fall into the same interval. By considering the distance between those vertices, we can see that only $f(v_0)$ and $f(v_5)$ lie in the same interval. By symmetry of the graph, it suffices to consider $f(v_0), f(v_5) \in I_i$ for $0 \leq i \leq 2$. Let $A = \{v_0, v_5\}$.

**Case 1**. Suppose $f(v_0), f(v_5) \in I_0$. Here $f(v_1), f(v_2), f(v_3), f(v_4) \in [k, \lambda]$. Since $k \geq j$, by Theorem 2.1 we have $\lambda - k \geq 3j$. Hence $\lambda \geq 3j + k$.

**Case 2**. Suppose $f(v_0), f(v_5) \in I_1$. We have $f(v_1), f(v_2), f(v_3), f(v_4) \in [0, (\lambda+j)/3 - k) \cup [j+k, \lambda]$. Since $k \geq j$ and $\lambda < 5j$, the length of $[0, (\lambda+j)/3 - k)$ is less than $j$. Thus, $[j+k, \lambda]$ contains three of $f(v_1)$, $f(v_2)$, $f(v_3)$ and $f(v_4)$. Now $\lambda - k - j \geq 2j$, i.e., $\lambda \geq 3j + k$.

**Case 3**. Suppose $f(v_0), f(v_5) \in I_2$. Let $f(w_i) \in I_i$ for some $w_i$, where $i = 0, 1, 3, 4$. Hence $\{w_0, w_1, w_3, w_4\} = \{v_1, v_2, v_3, v_4\}$. There exists $v \in A$ such that $d(v, w_1) = 2$. So $f(v) - f(w_1) \geq k$. Then the span of the increasing sequence $f(w_0) < f(w_1) < f(v) < f(w_3) < f(w_4)$ is at least $3j + k$. Hence $\lambda \geq 3j + k$.

Thus $\lambda_{i,j}(P_6^2) \geq \min\{5j, 3j+k\}$. $\square$

**Theorem 3.2.** *Suppose $6 \leq n \leq 10$. Let $j$ and $k$ be two positive numbers. If $j \leq k < 2j$, then $\lambda_{j,k}(P_n^2) = 3j + k$.*

**Proof.** Define a labeling $f$ for $P_{10}^2$ as follows:
$f(v_0) = f(v_5) = 0$, $f(v_1) = k$, $f(v_2) = j+k$, $f(v_3) = 2j+k$, $f(v_4) = 3j+k$, $f(v_6) = j$, $f(v_7) = 2j$, $f(v_8) = 3j$, $f(v_9) = 4j$. It is easy to verify that $f$ is a $(3j+k)$-$L(j,k)$-labeling of $P_{10}^2$ when $j \leq k < 2j$.

By Lemma 1.1, we have $\lambda_{j,k}(P_n^2) \leq 3j + k$ for $n \leq 10$.

Since $P_6^2$ is an induced subgraph of $P_n^2$, it suffices to show that $\lambda = \lambda_{j,k}(P_6^2) \geq 3j + k$. By Lemma 3.1 and $j \leq k < 2j$, we have $\lambda_{i,j}(P_6^2) \geq 3j+k$. Thus $\lambda_{i,j}(P_6^2) = 3j+k$. By Lemma 1.1 we get that $\lambda_{j,k}(P_n^2) \geq \lambda_{i,j}(P_6^2) = 3j+k$ for $n \geq 6$.

Combining the discussion above, we have $\lambda(P_n^2) = 3j + k$ for $6 \leq n \leq 10$. $\square$

For any integer $a$, $[a]_m \in \{0, 1, \ldots, m-1\}$ denotes the residue of $a$ modulo $m$, where $m$ is a positive integer greater than 1. For convenience, we let $V_i = \{v_l \in V(P_n^2) \mid l \equiv i \pmod 5\}$, $0 \leq i \leq 4$, where $n \geq 11$. And also let $E(A, B)$ be the set of edges from $A$ to $B$ and $f(A) = \{f(v) \mid v \in A\}$ for a labeling $f$ of $P_n^2$, where $A$ and $B$ are subsets of $V(P_n^2)$.

**Theorem 3.3.** *Suppose $11 \leq n \leq 15$. Let $j$ and $k$ be two positive numbers with $j \leq k < 2j$. Then $\lambda_{j,k}(P_n^2) = \min\{2j+2k, 5j\}$.*

**Proof.** Let $(0, k, 2k, j+2k, 2j+2k, 0, k, j+k, 2j+k, 2j+2k, 0, j, 2j, 2j+k, 2j+2k)$ be the list of the values of $(g(v_i))_{0 \leq i \leq 14}$. Hence this defines a $(2j+2k)$-$L(j,k)$-labeling $g$ for $P_{15}^2$. By Lemma 1.1 and (3.1), we have $\lambda_{j,k}(P_n^2) \leq \min\{2j+2k, 5j\}$ for $11 \leq n \leq 15$.

Similar to the proof of Theorem 3.2, in order to obtain the theorem it suffices to show that $\lambda = \lambda_{j,k}(P_{11}^2) \geq \min\{2j+2k, 5j\}$. Thus we have to show that "if $\lambda < 5j$, then $\lambda \geq 2j + 2k$".

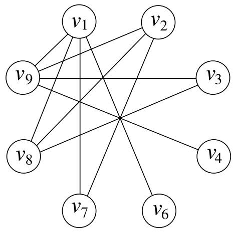


**Fig. 1.** The graph $H - V_0$.

Now suppose $\lambda < 5j$. Let $f$ be a $\lambda$-$L(j,k)$-labeling of $P_{11}^2$. Let $I_i$ be intervals defined in the proof of Theorem 3.2. Note that the length of each interval is less than $j$. By pigeonhole principle, at least one interval contains three vertex labels. Note that such labels may be the same. Let $H$ be a graph with the vertex set $V(P_{11}^2)$ in which two vertices are adjacent if they are of distance at least 3 in $P_{11}^2$. Note that, $H$ is a compatibility graph, in which two vertices are adjacent if and only if their assigned labels can lie in the same interval $I_i$ for some $i$. We can see that $H$ contains only one 3-cycle which is $v_0v_5v_{10}v_0$. Thus, only $f(v_0)$, $f(v_5)$ and $f(v_{10})$ lie in the same interval $I_{h_0}$ for some $h_0$. Thus, each of other interval contains exactly two labels. By symmetry of the graph, we may assume that $0 \le h_0 \le 2$.

By considering the subgraph induced by $\{v_l \mid 0 \le l \le 5\}$ and the same argument in the proof of Theorem 3.2, each of $f(v_1)$, $f(v_2)$, $f(v_3)$, $f(v_4)$ lies in exactly one different interval. Let $f(v_i) \in I_{h_i}$ for $1 \le i \le 4$. Now $\{h_0, h_1, h_2, h_3, h_4\} = \{0, 1, 2, 3, 4\}$. Similarly, by considering the subgraph induced by $\{v_l \mid 5 \le l \le 10\}$, each of $f(v_6)$, $f(v_7)$, $f(v_8)$, $f(v_9)$ lies in exactly one different interval $I_{h_i}$, $1 \le i \le 4$.

By considering another compatibility graph $H - V_0$ (Fig. 1), we can see that $f(V_1) \subset I_{h_1}$ and $f(V_4) \subset I_{h_4}$. This forces that $f(V_2) \subset I_{h_2}$ and $f(V_3) \subset I_{h_3}$.

**Case 1**. Suppose $h_0 = 0$. We want to determine the span of the set $S = \{f(v_i) \mid 1 \le i \le 9, i \ne 5\}$, i,e., the maximum difference between each pair of labels in $S$. For each $w \in V_i$, $1 \le i \le 4$, there is a $v \in V_0$ such that $d(w, v) = 2$. Thus, $S \subset [k, \lambda]$. We shall face on all permutations of $h_1h_2h_3h_4$. For example, suppose $h_1h_2h_3h_4 = 1234$. That means $f(V_1) \subset I_1$, $f(V_2) \subset I_2$, $f(V_3) \subset I_3$ and $f(V_4) \subset I_4$. Considering the path $v_6v_2v_3v_4$ at the graph $H_2$ shown in Fig. 2, we have an increasing subsequence $f(v_6) < f(v_2) < f(v_3) < f(v_4)$. Thus the span of $S$ is at least $k + 2j$. The reflection case of this case is $h_1h_2h_3h_4 = 4321$. By means of reflection there are $4!/2 = 12$ permutations we have to deal with.

Combining all cases, we have $\lambda - k \ge 2j + k$. Hence $\lambda \ge 2j + 2k$.

**Case 2**. Suppose $h_0 = 1$. Similar to Case 2 of the proof of Theorem 3.2, $[j + k, \lambda]$ contains three of $f(V_1)$, $f(V_2)$, $f(V_3)$ and $f(V_4)$. No matter which case, the span of the union of these three subsets is at least $k + j$. So $\lambda - j - k \ge k + j$. Hence $\lambda \ge 2j + 2k$.

**Case 3**. Suppose $h_0 = 2$. Now $\{h_1, h_2, h_3, h_4\} = \{0, 1, 3, 4\}$. Consider the graph $H_2$. There is always a hard edge in $E(V_s, V_t)$, where $1 \le s < t \le 4$. Moreover, each vertex in $H_2$ is of distance either 1 or 2 to $v_5$ in $P_{11}^2$. Thus, for each permutation of $h_1h_2h_3h_4$, there always exists an increasing subsequence of $f(V(P_{11}^2))$ involving $f(v_5)$ with the span at least $2j + 2k$. For example, when $h_1h_2h_3h_4 = 0314$, the required subsequence is $f(v_6) < f(v_3) < f(v_5) < f(v_7) < f(v_4)$. So the span of $f$ is at least $2j + 2k$.

Combining the discussion above, we have $\lambda = 2j + 2k$ for $11 \le n \le 15$. $\square$

**Theorem 3.4.** *Suppose $16 \le n \le 20$. Let $j, k$ be two positive numbers with $j \le k < 2j$. Then $\lambda_{j,k}(P_n^2) = \min\{j + 3k, 5j\}$.*

**Proof.** Let $(0, k, 2k, 3k, j + 3k, 0, k, 2k, j + 2k, 2j + 2k, 0, k, j + k, 2j + k, 2j + 2k, 0, j, 2j, 2j + k, 2j + 2k)$ be the list of the values of $(g(v_i))_{0 \le i \le 19}$. Hence this defines a $(j + 3k)$-$L(j,k)$-labeling $g$ for $P_n^2$ if $16 \le n \le 20$. By Lemma 1.1 and (3.1), we have $\lambda_{j,k}(P_n^2) \le \min\{j + 3k, 5j\}$ for $16 \le n \le 20$.

Conversely, we consider $\lambda = \lambda_{j,k}(P_{16}^2)$. Similar to the proof of Theorem 3.3 we assume $\lambda < 5j$ and show that $\lambda \ge j + 3k$ in the following.

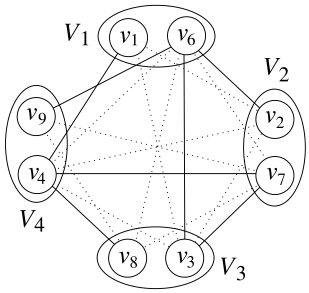


**Fig. 2.** Graph $H_2$: Dot and hard edges indicate distance 1 and 2 between the involved vertices in $P_{11}^2$, respectively.

**Table 1**
Lower bound of the span $S$.

| $h_1h_2h_3h_4$ | Subsequence of labels | Lower bound of the span |
|---|---|---|
| 1234 | $f(v_6) < f(v_2) < f(v_3) < f(v_4)$ | $2j + k$ |
| 1243 | $f(v_6) < f(v_7) < f(v_4) < f(v_8)$ | $j + 2k$ |
| 1324 | $f(v_6) < f(v_3) < f(v_7) < f(v_4)$ | $3k$ |
| 1342 | $f(v_1) < f(v_4) < f(v_7) < f(v_3)$ | $3k$ |
| 1423 | $f(v_6) < f(v_3) < f(v_4) < f(v_7)$ | $3k$ |
| 1432 | $f(v_1) < f(v_4) < f(v_3) < f(v_7)$ | $j + 2k$ |
| 2134 | $f(v_2) < f(v_6) < f(v_3) < f(v_4)$ | $j + 2k$ |
| 2143 | $f(v_2) < f(v_6) < f(v_9) < f(v_8)$ | $j + 2k$ |
| 2314 | $f(v_3) < f(v_6) < f(v_2) < f(v_4)$ | $j + 2k$ |
| 2413 | $f(v_3) < f(v_6) < f(v_9) < f(v_7)$ | $j + 2k$ |
| 3124 | $f(v_7) < f(v_3) < f(v_6) < f(v_9)$ | $3k$ |
| 3214 | $f(v_3) < f(v_7) < f(v_6) < f(v_9)$ | $j + 2k$ |

Let $I_i$ be defined in Theorem 3.2. By considering the subgraphs induced by $\{v_i \mid 0 \le i \le 10\}$ and $\{v_i \mid 5 \le i \le 15\}$, we obtain that $f(V_0) \subset I_{h_0}$ for some $h_0 \in \{0, 1, 2\}$ (without loss of generality), $f(V_i) \subset I_{h_i}$, $1 \le i \le 4$, where $\{h_0, h_1, h_2, h_3, h_4\} = \{0, 1, 2, 3, 4\}$.

**Case 1**. Suppose $h_0 = 0$. By Table 1, we only need to consider the case when $h_1h_2h_3h_4 = 1234$. In this case, there is a subsequence $f(v_{11}) < f(v_7) < f(v_3) < f(v_4)$. Now the span of $S$ is at least $j + 2k$.

**Case 2**. Suppose $h_0 = 1$. Similar to Case 2 of the proof of Theorem 3.2, $[j + k, \lambda]$ contains three of $f(V_1)$, $f(V_2)$, $f(V_3)$ and $f(V_4)$. No matter which case (see Fig. 3), the span of the union of these three subsets is at least $2k$. So $\lambda - j - k \ge 2k$. Hence $\lambda \ge j + 3k$.

**Case 3**. Suppose $h_0 = 2$. Now $\{h_1, h_2, h_3, h_4\} = \{0, 1, 3, 4\}$. We have to deal with the 12 permutations of $h_1h_2h_3h_4$. We only provide the discussion of the case when $h_1h_2h_3h_4 = 0314$ here. Other cases are similarly to show. For this case, we have an increasing subsequence $f(v_{11}) < f(v_8) < f(v_5) < f(v_7) < f(v_4)$. So the span of $f$ is at least $j + 3k$.

Combining the discussion above, we have $\lambda = j + 3k$ for $16 \le n \le 20$. $\square$

**Lemma 3.5.** *Let $W_i$ be a set consisting of 4 vertices of a graph $G$, $1 \le i \le 4$. Assume that $E(W_i, W_{i+1})$ contains at least 3 disjoint edges, for $1 \le i \le 3$. Then there is a path $w_1w_2w_3w_4$ with $w_i \in W_i$.*

**Proof.** Let $w_{1j}w_{2j} \in E(W_1, W_2)$ for $1 \le j \le 3$, $w_{ij} \in W_i$. Since at most one vertex in $W_2$ is not incident with edge of $E(W_1, W_2)$, at least two edges of $E(W_2, W_3)$ are adjacent with $w_{1j}w_{2j}$, $1 \le j \le 3$. After renaming if necessary, we may assume that such two edges of $E(W_2, W_3)$ are $w_{21}w_{31}$ and $w_{22}w_{32}$. Since only one vertex in $W_4$ is not incident edges in $E(W_3, W_4)$, there is an edge, say $w_{31}w_{41} \in E(W_3, W_4)$. Now we have a path $w_{11}w_{21}w_{31}w_{41}$. $\square$

**Corollary 3.6.** *Let $W_i$ be a set consisting of 4 vertices of a graph $G$, $1 \le i \le 3$. Assume that $E(W_i, W_{i+1})$ contains at least 3 disjoint edges, for $1 \le i \le 2$. Then there are two disjoint paths $w_1w_2w_3$ with $w_i \in W_i$.*

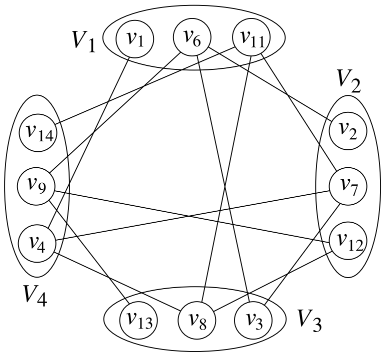


**Fig. 3.** Graph $H_3$: Two vertices are adjacent if they are of distance 2 in $P_{16}^2$.

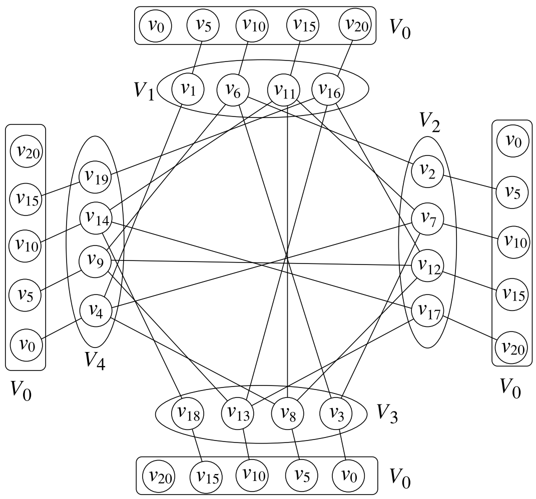


**Fig. 4.** Graph $H_4$.

**Theorem 3.7.** *Suppose $n \ge 21$. Let $j, k$ be two positive numbers with $j \le k < 2j$. Then $\lambda_{j,k}(P_n^2) = \min\{4k, 5j\}$.*

**Proof.** Define a labeling $g$ for $P_n^2$ by $g(v_i) = [i]_5 k$. Clearly $g$ is a $(4k)$-$L(j,k)$-labeling of $P_n^2$. By (3.1), we have $\lambda_{j,k}(P_n^2) \le \min\{4k, 5j\}$.

Consider the graph $P_{21}^2$ and assume $\lambda < 5j$. Let $I_i$ be defined in Theorem 3.2. By a similar argument of the proof of Theorem 3.4, we have $f(V_0) \subset I_{h_0}$ for some $h_0 \in \{0, 1, 2\}$, $f(V_i) = \{f(v_1), f(v_6), f(v_{11}), f(v_{16})\} \subset I_{h_i}$, $1 \le i \le 4$, where $\{h_0, h_1, h_2, h_3, h_4\} = \{0, 1, 2, 3, 4\}$. Let $H_4$ be the graph in Fig. 4. Two vertices are adjacent if they are of distance 2 in $P_{21}^2$.

**Remark 1.** For $1 \le s < t \le 4$, there are at least 3 disjoint edges in $E(V_s, V_t) \subset E(H_4)$. For each $w \in V_i$ with $1 \le i \le 4$, there is a unique $v \in V_0$ such that $wv \in E(V_0, V_i)$.

**Case 1.** Suppose $h_0 = 0$. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ h_1 & h_2 & h_3 & h_4 \end{pmatrix}$ be a permutation of $\{1, 2, 3, 4\}$. Let $k_i = \sigma^{-1}(i)$. By Remark 1 and Lemma 3.5, there is a path $w_1 w_2 w_3 w_4$ in which $w_i \in V_{k_i}$. By Remark 1 again, there is $w_0 \in V_0$ such that $d(w_0, w_1) = 2$. Hence the path $w_0 \cdots w_5$ induces a subsequence $f(w_0) < f(w_1) < f(w_2) < f(w_3) < f(w_4)$ and the span of this subsequence is $4k$. Hence we have $\lambda \ge 4k$.

**Case 2.** Suppose $h_0 = 1$. Let $\sigma = \begin{pmatrix} 0 & 2 & 3 & 4 \\ h_1 & h_2 & h_3 & h_4 \end{pmatrix}$ be a permutation of $\{0, 2, 3, 4\}$. Let $k_i = \sigma^{-1}(i)$, $2 \le i \le 4$ and $\sigma^{-1}(0) = \ell$. By Corollary 3.6, there are two disjoint paths $w_2 w_3 w_4$ and $u_2 u_3 u_4$ such that $w_i, u_i \in V_{k_i}$,

$2 \le i \le 4$. By Remark 1 there are $u_0, w_0 \in V_0$ such that $w_0w_2$ and $u_0u_2$ are edges of $E(V_0, V_{k_2})$. Since only one vertex in $V_0$ is not adjacent to vertex of $V_\ell$, there is a $w_1 \in V_\ell$ such that either $w_1w_0 \in E(V_\ell, V_0)$ or $w_1u_0 \in E(V_\ell, V_0)$. Let us say $w_1w_0 \in E(V_\ell, V_0)$. There is a path $w_1w_0w_2w_3w_4$ that induces a subsequence $f(w_1) < f(w_0) < f(w_2) < f(w_3) < f(w_4)$. Hence we have $\lambda \ge 4k$.

**Case 3**. Suppose $h_0 = 2$. Let $\sigma = \begin{pmatrix} 0 & 1 & 3 & 4 \\ h_1 & h_2 & h_3 & h_4 \end{pmatrix}$ be a permutation of $\{0, 1, 3, 4\}$. Let $k_i = \sigma^{-1}(i)$, $i \in \{0, 1, 3, 4\}$. Note that $E(V_{k_0}, V_{k_1})$ and $E(V_{k_3}, V_{k_4})$ contain 3 disjoint edges, respectively; $E(V_{k_1}, V_0)$ and $E(V_0, V_{k_3})$ contain 4 disjoint edges, respectively. By a similar argument as above, there are three paths $u_1u_2u_0$ with $u_1 \in V_{k_1}$, $u_2 \in V_{k_1}$ and $u_0 \in V_0$ and three paths $w_0u_3u_4$ with $u_3 \in V_{k_3}$, $u_4 \in V_{k_4}$ and $w_0 \in V_0$. By pigeonhole principle, there are two paths $u_1u_2u_0$ and $w_0u_3u_4$ such that $u_0 = w_0$. Here we have a path $u_1u_2u_0u_3u_4$. Hence we have $\lambda \ge 4k$. □

Now, we consider the case when $k \ge 3j$.

**Theorem 3.8.** *Let $j$ and $k$ be two positive numbers. If $k \ge 3j$, then $\lambda_{j,k}(P_6^2) = 2j + k$.*

**Proof.** Let $\lambda = \lambda_{j,k}(P_6^2)$. Define $g(v_0) = 0$, $g(v_1) = j$, $g(v_2) = 2j$, $g(v_3) = k$, $g(v_4) = j + k$, $g(v_5) = 2j + k$. It is easy to verify that $g$ is a $(2j + k)$-$L(j, k)$-labeling for $P_6^2$. Hence $\lambda \le 2j + k$.

On the other hand, let $f$ be a $\lambda$-$L(j, k)$-labeling of $P_6^2$. Since $v_3, v_4$ are distance two apart from $v_0$, $f(v_3), f(v_4) \in [0, f(v_0) - k] \cup [f(v_0) + k, \lambda]$. If $f(v_3) < f(v_0) < f(v_4)$ or $f(v_4) < f(v_0) < f(v_3)$, then $\lambda \ge 2k > 2j + k$. Since we have just known that $\lambda \le 2j + k$, this is not a case. So both $f(v_3)$ and $f(v_4)$ are either greater than or less than $f(v_0)$. Without loss of generality, we may assume $f(v_3)$ and $f(v_4)$ are greater than $f(v_0)$, otherwise consider the labeling $\overline{f} = \lambda - f$. That is, $f(v_3), f(v_4) \in [f(v_0) + k, \lambda]$. Hence $\lambda - k \ge f(v_0)$. Moreover, since $d(v_1, v_4) = 2$, $f(v_1) \in [0, \lambda - k]$. Similarly, since $d(v_1, v_5) = 2$, $f(v_5) \in [k, \lambda]$. Since $d(v_2, v_5) = 2$, $f(v_2) \in [0, \lambda - k]$. Now, we can conclude that $f(v_0), f(v_1), f(v_2) \in [0, \lambda - k]$. Since $v_0, v_1, v_2$ are adjacent to each other, $\lambda - k \ge 2j$. Hence $\lambda \ge 2j + k$. Hence $\lambda_{j,k}(P_6^2) = 2j + k$. □

**Theorem 3.9.** *Suppose $n \ge 7$. Let $j, k$ be two positive numbers. If $k \ge 3j$, then $\lambda_{j,k}(P_n^2) = 2k$.*

**Proof.** Let $\lambda = \lambda_{j,k}(P_n^2)$. Define $g(v_0) = 0$, $g(v_1) = j$, $g(v_2) = 2j$, $g(v_3) = k$, $g(v_4) = j + k$, $g(v_5) = 2j + k$, $g(v_6) = 2k$ and $g(v_i) = g(v_{[i]_7})$ for $i \ge 7$. It is easy to verify that $g$ is a $2k$-$L(j, k)$-labeling of $P_n^2$. Hence $\lambda \le 2k$.

Since $P_7^2$ is an induced subgraph of $P_n^2$, it suffices to show that $\lambda = \lambda_{j,k}(P_7^2) \ge 2k$. Let $f$ be a $\lambda$-$L(j, k)$-labeling of $P_7^2$. Consider the labels of $v_0$, $v_3$ and $v_4$. By the same argument of the proof of Theorem 3.8, we only need to consider the case when $f(v_3), f(v_4) \in [f(v_0) + k, \lambda]$. By the proof of Theorem 3.8 again, we have $f(v_0), f(v_1), f(v_2) \in [0, \lambda - k]$.

Now, if $f(v_3) < f(v_6)$, then $f(v_0) < f(v_3) < f(v_6)$ implies $\lambda \ge 2k$. The remaining case is $f(v_6) < f(v_3)$. But this implies that $f(v_6) \in [0, \lambda - k]$. Now, $f(v_2), f(v_6) \in [0, \lambda - k]$. Hence $\lambda \ge 2k$.

This completes the proof. □

Finally, we consider the case when $2j \le k < 3j$.

**Theorem 3.10.** *Let $j$ and $k$ be two positive numbers. If $2j \le k < 3j$, then $\lambda_{j,k}(P_6^2) = 5j$.*

**Proof.** Define a labeling $g$ for $P_6^2$ by $g(v_i) = [i]_6 j$. Clearly $g$ is a $(5j)$-$L(j, k)$-labeling of $P_6^2$. Thus, we have $\lambda_{j,k}(P_6^2) \le 5j$.

Moreover, by Lemma 3.1, we have $\lambda_{j,k}(P_6^2) \ge \min\{5j, 3j + k\} = 5j$. Hence $\lambda_{j,k}(P_6^2) = 5j$. □

**Theorem 3.11.** *Let $j$ and $k$ be two positive numbers. If $2j \le k < 3j$ and $7 \le n \le 12$, then $\lambda_{j,k}(P_n^2) = 3j + k$.*

**Proof.** Let $g(v_0) = 0$, $g(v_1) = j$, $g(v_2) = k$, $g(v_3) = j + k$, $g(v_4) = 2j + k$, $g(v_5) = 3j + k$, $g(v_6) = 0$, $g(v_7) = j$, $g(v_8) = 2j$, $g(v_9) = 3j$, $g(v_{10}) = 4j$, $g(v_{11}) = 5j$. It is easy to check that $g$ is a $(3j + k)$-$L(j, k)$-labeling of $P_{12}^2$. Thus, By Lemma 1.1, $\lambda_{j,k}(P_n^2) \le 3j + k$ for $7 \le n \le 12$.

Let $\lambda = \lambda_{j,k}(P_7^2)$ and let $f$ be a $\lambda$-$L(j, k)$-labeling of $P_7^2$. We have $\lambda \le 3j + k$. As the proofs of those previous theorems, we only need to show $\lambda \ge 3j + k$.

Let $J_0 = [0, j)$, $J_1 = [j, 2j)$, $J_2 = [2j, \lambda/2]$, $J_3 = (\lambda/2, \lambda - 2j]$, $J_4 = (\lambda - 2j, \lambda - j]$ and $J_5 = (\lambda - j, \lambda]$. Since $\lambda \le 3j + k$ and $k < 3j$, the length of each interval is less than $j$. Thus, if $f(u), f(w) \in J_i$ for some $i$, then $d(u, w) > 2$. Hence $\{u, w\}$ is either $A_0 = \{v_0, v_5\}$, $A_1 = \{v_1, v_6\}$ or $A_2 = \{v_0, v_6\}$. Also, each $J_i$ cannot contain more than 2 vertices. Since the length of each $J_i \cup J_{i+1}$ is less then $2j < k$, for $0 \le i \le 4$, if $\{u, w\}$ is none of $A_0$, $A_1$ and $A_2$ but $f(\{u, w\}) \subset J_i \cup J_{i+1}$ for some $i$, then $d(u, w) = 1$. For this case, $f(\{u, w\}) \subset J_i \cup J_{i+1}$ associates a path $uw$ of length 1. Furthermore, $J_i \cup J_{i+1}$ cannot contain three labels of $\{f(v_0), f(v_1), f(v_5), f(v_6)\}$.

Following we want to find an increasing sequence of labels with span at least $3j + k$. It is easy to get the following claim.

**Claim 1.** *Suppose $v \in \{v_2, v_3, v_4\}$. For each $i$, $i = 0, 1, 2$, there exists $w_i \in A_i$ such that $d(v, w_i) = 2$.*

Now, by pigeonhole principle, there is at least one $J_q$ containing two labels. By symmetry we may assume that $q = 0, 1, 2$, otherwise consider the labeling $\overline{f} = \lambda - f$. Thus, $J_q$ contains either $f(A_0)$, $f(A_1)$ or $f(A_2)$.

**Case A.** Suppose there are two intervals, say $J_q$ and $J_r$, containing 2 labels, where $0 \le q \le 2$ and $q < r$. In this case, $f(A_2)$ does not contain in $J_q \cup J_r$. By renumbering the vertex if necessary, we may assume that $f(A_1) \subset J_q$ and $f(A_0) \subset J_r$. Let $\{f(u_1), f(u_2), f(u_3)\} = \{f(v_2), f(v_3), f(v_4)\}$, where $f(u_1) < f(u_2) < f(u_3)$.

- **A-1.** Suppose $f(v_0) < f(u_1)$. By Claim 1 there is $w_0 \in A_0$ such that $d(w_0, u_1) = 2$. Now we have $f(v_1) < f(w_0) < f(u_1) < f(u_2) < f(u_3)$ with span at least $3j + k$.
- **A-2.** Suppose $f(v_1) < f(u_1) < f(v_0)$ or $f(u_1) < f(v_1) < f(u_2)$. By Claim 1 there is $w_1 \in A_1$ such that $d(w_1, u_1) = 2$. Thus, $f(v_0)$, $f(u_2)$, $f(u_3)$ lie in $[k + j, \lambda]$. So $\lambda - (k + j) \ge 2j$. Hence $\lambda \ge 3j + k$.
- **A-3.** Suppose $f(u_2) < f(v_1)$ and $f(u_3) > f(v_1)$. In this case, we have $f(v_1) < f(u_3) < f(v_0)$ or $f(u_2) < f(v_0) < f(u_3)$. This the reflexive case of Case A-2.
- **A-4.** Suppose $f(u_3) < f(v_1)$. This is the reflexive case of Case A-1.

**Case B.** Suppose there is only one interval $J_q$ containing 2 labels, where $0 \le q \le 2$.

- **B-1.** Suppose $J_0$ contains two labels.
  - **a.** $f(A_0) \subset J_0$. The span of the set $\{f(v_1), f(v_2), f(v_3), f(v_4)\}$ is at least $3j$. No matter which label is the minimum, there always exists a vertex $w \in A_0$ such that $f(w)$ is less than this minimum by at least $k$. Hence, the span of the set $\{f(w), f(v_1), f(v_2), f(v_3), f(v_4)\}$ is at least $3j + k$.
  - **b.** $f(A_1) \subset J_0$. Consider the set $\{f(v_2), f(v_3), f(v_4), f(v_5)\}$. Similar to Case a, we will get the same result.
  - **c.** $f(A_2) \subset J_0$. Since the length of $J_0 \cup J_1$ is less than $k$, only $f(v_1)$ or $f(v_5)$ lies in $J_1$. Renaming the vertex if necessary, we may assume $f(v_1) \in J_1$. By the same reason, only $f(v_2)$ or $f(v_3)$ lies in $J_2$. Suppose $f(v_4) \in J_3$. Let $f(w_4) \in J_4$ and $f(w_5) \in J_5$. We have $f(v_0) < f(v_1) < f(v_4) < f(w_4) < f(w_5)$ with span at least $3j + k$. We will get the same result, if we replace $v_4$ by $v_5$. Now the remaining cases are $f(v_2) < f(v_3) < f(v_4) < f(v_5)$, $f(v_2) < f(v_3) < f(v_5) < f(v_4)$, $f(v_3) < f(v_2) < f(v_4) < f(v_5)$ and $f(v_3) < f(v_2) < f(v_5) < f(v_4)$. The span of the last three sequences are at least $j + k$. So combining with the sequence $f(v_0) < f(v_1)$ we have a sequence with span $3j + k$. Finally, we consider the case $f(v_0) < f(v_1) < f(v_2) < f(v_3) < f(v_4) < f(v_5)$. Since $f(v_6)$ also in $J_0$, we have the sequence $f(v_6) < f(v_2) < f(v_3) < f(v_4) < f(v_5)$ with span $3j + k$.
- **B-2.** Suppose $J_1$ contains two labels. Let $f(A_i) \subset J_1$. Then $f(A_l) \cap (J_0 \cup J_2) = \varnothing$ for all $l$. Let $f(w_r) \in J_r$, for $0 \le r \le 5$ and $r \ne 1$. There is a $v \in A_i$ such that $d(v, w_2) = 2$. Hence the sequence $f(w_0) < f(v) < f(w_2) < f(w_3) < f(w_4) < f(w_5)$ is of span at least $3j + k$ (as the span of $\{f(w_2), f(w_3), f(w_4), f(w_5)\}$ is at least $2j$, $f(v) - f(w_0) \ge j$ and $f(w_2) - f(v) \ge k$).
- **B-3.** Suppose $J_2$ contains two labels. Let $f(A_i) \subset J_2$. Then $f(A_l) \cap (J_1 \cup J_3) = \varnothing$ for all $l$. Let $f(w_r) \in J_r$, for $0 \le r \le 5$ and $r \ne 2$. There is a $v \in A_i$ such that $d(v, w_1) = 2$. Since $w_1 \in \{v_2, v_3, v_4\}$, $1 \le d(w_0, w_1) \le 2$. This implies that $f(w_1) - f(w_0) \ge j$. Since $w_1 \in \{v_2, v_3, v_4\}$, $f(w_3) - f(v) \ge j$. Note that the span of $\{f(w_3), f(w_4), f(w_5)\}$ is at least $j$. Hence the sequence $f(w_0) < f(w_1) < f(v) < f(w_3) < f(w_4) < f(w_5)$ is of span at least $3j + k$.

Combining the above cases, we have $\lambda \ge 3j + k$. Hence the proof is completed. □

**Theorem 3.12.** *Let $j$ and $k$ be two positive numbers with $2j \le k < 3j$. If $n \ge 13$, then $\lambda_{j,k}(P_n^2) = \min\{j + 2k, 6j\}$.*

**Proof.** Suppose $6j \le j + 2k$. Define $g(v_i) = [i]_7 j$ for all $i$. It is easy to verify that $g$ is a $(6j)$-$L(j,k)$-labeling for $P_n^2$.

Suppose $6j > j + 2k$. Define $g(v_0) = 0$, $g(v_1) = j$, $g(v_2) = k$, $g(v_3) = j + k$, $g(v_4) = 2k$, $g(v_5) = j + 2k$ and $g(v_i) = g(v_{[i]_6})$ for $7 \le i \le n$. It is easy to verify that $g$ is a $(j + 2k)$-$L(j,k)$-labeling for $P_n^2$.

Let $f$ be a $\lambda$-$L(j,k)$-labeling for $P_{13}^2$. It suffices to show that $\lambda \ge \min\{j + 2k, 6j\}$. Now, we assume $\lambda < j + 2k$. We want to show that $\lambda \ge 6j$. We may assume $f(v_0) < f(v_3)$, otherwise consider the labeling $\overline{f} = \lambda - f$.

**Case A.** Suppose $f(v_3) < f(v_6)$. Since $f(v_0) < f(v_3) < f(v_6)$, $f(v_0) \in [0, \lambda - 2k]$, $f(v_3) \in [k, \lambda - k]$ and $f(v_6) \in [2k, \lambda]$. Since $d(v_3, v_7) = 2$, $f(v_7) \in [0, \lambda - 2k] \cup [2k, \lambda]$. Since the length $[2k, \lambda]$ is less than $j$, $f(v_7) \in [0, \lambda - 2k]$. This implies that $f(v_7) < f(v_3)$. As the length of $[2k, \lambda]$ is less than $j$, we have $f(v_2)$, $f(v_4)$ and $f(v_5)$ are not in $[2k, \lambda]$. By considering the distance apart from $v_6$, we have $f(v_4), f(v_5) \in [0, \lambda - j]$ and $f(v_2) \in [0, \lambda - k]$. By $f(v_0), f(v_7) \in [0, \lambda - 2k]$, we have $f(v_1), f(v_2), f(v_5) \in [j, \lambda]$ and $f(v_4) \in [k, \lambda]$. As the length of $[k, \lambda - k]$ is less than $j$, we have $f(v_1)$, $f(v_2)$, $f(v_4)$ and $f(v_5)$ are not in this interval. Now we summarize the range of some labels: $f(v_1) \in [j, k) \cup (\lambda - k, \lambda]$; $f(v_2) \in [j, k)$; $f(v_3) \in [k, \lambda - k]$; $f(v_4) \in (\lambda - k, \lambda - j]$ and $f(v_5) \in [j, k) \cup (\lambda - k, \lambda - j]$. Since the length of $[j, k)$ is less than $k$, $f(v_5) \notin [j, k)$ and hence $f(v_5) \in (\lambda - k, \lambda - j] \subset (\lambda - k, \lambda]$. Also since the length of $(\lambda - k, \lambda]$ is less than $k$, $f(v_1) \notin (\lambda - k, \lambda]$. Hence $f(v_1) \in [j, k)$. Now we have $f(v_0) < f(v_1), f(v_2), f(v_3) < f(v_5)$. So $f(v_5) \ge 2j + k$. That is, $f(v_5) \in [2j + k, \lambda - j]$. Up to now we have $f(v_0)$, $f(v_1)$, $f(v_2)$, $f(v_3)$, $f(v_4)$ and $f(v_5)$ are at most $\lambda - j$. By Theorem 3.10, $\lambda - j \ge 5j$. Hence $\lambda \ge 6j$.

**Case B.** Suppose $f(v_3) > f(v_6)$. We have $f(v_6) \in [0, \lambda - k]$. Suppose $f(v_7) > f(v_3)$. Since $f(v_0) < f(v_3) < f(v_7)$, $f(v_0) \in [0, \lambda - 2k]$, $f(v_3) \in [k, \lambda - k]$ and $f(v_7) \in [2k, \lambda]$. Since the lengths of $[0, \lambda - 2k]$ and $[2k, \lambda]$ are less than $j$, $f(v_4)$ does not lie in these two intervals. This implies that $f(v_0) < f(v_4) < f(v_7)$. Now, $f(v_3), f(v_4) \in [k, \lambda - k]$ which is impossible. That means $f(v_7) < f(v_3)$ and $f(v_7) \in [0, \lambda - k]$. Since $v_3$, $v_{10}$ are of distance two from $v_6$ and $v_7$ and the length of $[0, \lambda - k]$ is less than $j + k$, $f(v_3), f(v_{10}) \ge \max\{f(v_6), f(v_7)\} + k \ge j + k$ or $f(v_3), f(v_{10}) \le \min\{f(v_6), f(v_7)\} - k \le (\lambda - k - j) - k < 0$. So the last case is impossible. Hence $f(v_3), f(v_{10}) \in [j + k, \lambda]$. Hence $f(v_6), f(v_7) < f(v_3), f(v_{10})$. Similarly, we have $f(v_{11}) \in [0, \lambda - 2k] \cup [k, \lambda]$.

**B-1.** Suppose $f(v_{11}) \in [0, \lambda - 2k]$. Since the length of $[0, \lambda - 2k]$ is less than $j$, $f(v_7)$, $f(v_8)$, $f(v_9)$, $f(v_{10})$ and $f(v_{12})$ are greater than $f(v_{11})$. Combining with $f(v_7) < f(v_3), f(v_{10})$, we have $f(v_7) \in [k, \lambda - k]$, $f(v_3), f(v_{10}) \in [2k, \lambda]$. Since $f(v_8)$ cannot lie in $[0, \lambda - 2k] \cup [2k, \lambda]$, $f(v_{11}) < f(v_8) < f(v_{10})$. Now we have $f(v_8) \in [k, \lambda - j]$. Since $f(v_8)$ cannot lie in $[k, \lambda - k]$, $f(v_8) > f(v_7)$ and hence $f(v_8) \in [j + k, \lambda - j]$. Comparing $f(v_4)$ with $f(v_7)$, we have $f(v_4) \in [0, \lambda - 2k] \cup [2k, \lambda]$. From the range of $f(v_3)$, we have $f(v_4) \in [0, \lambda - 2k]$. From the range of $f(v_4)$, we have $f(v_5), f(v_6) \ge j$. Hence $f(v_i) \ge j$ for $5 \le i \le 10$. By Theorem 3.10, $\lambda - j \ge 5j$. Hence $\lambda \ge 6j$.

**B-2.** Suppose $f(v_{11}) \in [k, \lambda]$. Then $f(v_8) \in [0, \lambda - k] \cup [2k, \lambda]$. Suppose $f(v_8) \in [2k, \lambda]$. Since the length of $[2k, \lambda]$ is less than $j$, $f(v_i) \in [0, \lambda - j]$, where $4 \le i \le 12$ and $i \ne 8$. Moreover, since $d(v_8, v_{11}) = 2$, $f(v_{11}) \in [k, \lambda - k]$. Since the length of $[k, \lambda - k]$ is less than $j$, $f(v_6)$, $f(v_7)$, $f(v_9)$ and $f(v_{12})$ are less than $f(v_{11})$. Moreover, $f(v_6), f(v_7) \in [0, \lambda - 2k]$ and $f(v_9), f(v_{12}) \in [0, \lambda - k - j]$. But it is impossible, since the length of $[0, \lambda - k - j]$ is less than $k$. Thus, $f(v_8) \in [0, \lambda - k]$.

**a.** When $f(v_5) < f(v_8)$. This implies that $f(v_5) \in [0, \lambda - 2k]$ and hence $f(v_8) \in [k, \lambda - k]$ and $f(v_6), f(v_7) \in [j, \lambda - k]$. Since the length of $[k, \lambda - k]$ is less than $j$, $f(v_8)$ must be less than $f(v_{11})$ and $f(v_8)$ must be greater than $f(v_6)$ and $f(v_7)$. Thus, $f(v_{11}) \in [2k, \lambda]$. Now the length of $[2k, \lambda]$ is less than $j$, $f(v_9)$ and $f(v_{10})$ are less than $f(v_{11})$. That means, $f(v_5)$, $f(v_6)$, $f(v_7)$, $f(v_8)$, $f(v_9)$, $f(v_{10})$ lie in $[0, \lambda - j]$. By Theorem 3.10, we have $\lambda \ge 6j$.

**b.** When $f(v_5) > f(v_8)$. Hence $f(v_5) \in [k, \lambda]$.

**b-1.** Suppose $f(v_2) > f(v_5)$. Since $f(v_2) > f(v_5) > f(v_8)$, $f(v_8) \in [0, \lambda - 2k]$. Since the length of $[0, \lambda - 2k]$ is less than $j$, $f(v_4)$, $f(v_5)$, $f(v_6)$, $f(v_7)$ are greater than $j$. Combining with the ranges of $f(v_2)$ and $f(v_3)$ we have $f(v_i) \in [j, \lambda]$ for $2 \le i \le 7$. By Theorem 3.10, we have $\lambda \ge 6j$.

**b-2.** Suppose $f(v_2) < f(v_5)$. Then $f(v_2) \in [0, \lambda - k]$. Since $f(v_6) \in [0, \lambda - k]$ and $d(v_2, v_6) = 2$, $f(v_2) \in [0, \lambda - 2k]$ or $f(v_2) \in [k, \lambda - k]$. When $f(v_2) \in [0, \lambda - 2k]$. We have $f(v_6) \in [k, \lambda - k]$. Since $f(v_3)$ and $f(v_{10})$ are greater than $f(v_6)$, $f(v_3), f(v_{10}) \in [2k, \lambda]$. Since the length of $[2k, \lambda]$ is less than $j$, $f(v_i) \in [0, \lambda - j]$ for $4 \le i \le 9$. By Theorem 3.10, we have $\lambda \ge 6j$. When $f(v_2) \in [k, \lambda - k]$. Now $f(v_6) \in [0, \lambda - 2k]$ and $f(v_5) \in [2k, \lambda]$. By considering the distances from $v_9$ to $v_5$ and $v_6$, we have $f(v_9) \in [k, \lambda - k]$. Since the lengths of $[0, \lambda - 2k]$ and $[k, \lambda - k]$ are less than $j$, then $f(v_7), f(v_8) \in [j, \lambda - j - k]$. It implies that $f(v_{12}) \in [2k, \lambda]$ or $f(v_{12}) \in [0, \lambda - j - 2k]$ by considering the distances from $v_8$ and $v_9$. But the last case is impossible as $\lambda < j + 2k$. Now $j \le f(v_i)$ for $7 \le i \le 12$. By Theorem 3.10, we have $\lambda \ge 6j$. □

According to Theorems 3.2–3.12, we can obtain following conclusion.

**Corollary 3.13.** *Let $n \ge 6$ and $j, k$ be two positive numbers.*

1. *For $j \le k < 2j$,* $\lambda_{j,k}(P_n^2) = \begin{cases} 3j+k, & \text{if } 6 \le n \le 10, \\ \min\{2j+2k, 5j\}, & \text{if } 11 \le n \le 15, \\ \min\{j+3k, 5j\}, & \text{if } 16 \le n \le 20, \\ \min\{4k, 5j\}, & \text{if } n \ge 21. \end{cases}$
2. *For $2j \le k < 3j$,* $\lambda_{j,k}(P_n^2) = \begin{cases} 5j, & \text{if } n = 6, \\ 3j+k, & \text{if } 7 \le n \le 12, \\ \min\{j+2k, 6j\}, & \text{if } n \ge 13. \end{cases}$
3. *For $k \ge 3j$,* $\lambda_{j,k}(P_n^2) = \begin{cases} 2j+k, & \text{if } n = 6, \\ 2k, & \text{if } n \ge 7. \end{cases}$

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