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THE TIME OF COMPLETION OF A LINEAR BIRTH-GROWTH MODEL

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Abstract

Consider the following birth-growth model in $\mathbb{R}$. Seeds are born randomly according to an inhomogeneous space-time Poisson process. A newly formed point immediately initiates a bi-directional coverage by sending out a growing branch. Each frontier of a branch moves at a constant speed until it meets an opposing one. New seeds continue to form on the uncovered parts on the line. We are interested in the time until a bounded interval is completely covered. The exact and limiting distributions as the length of interval tends to infinity are obtained for this completion time by considering a related Markov process. Moreover, some strong limit results are also established.

Keywords: Completion time; coverage; inhomogeneous Poisson process; Johnson-Mehl model; linear birth-growth model; Markov process; strong limit theorem

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1. Introduction

Consider the following linear random birth-growth model. Points arrive indepen-

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX dently on a line at random positions and times according to a space-time Poisson process $\Phi \equiv \{(x_i, t_i) \in \mathbb{R} \times [0, \infty)\}$ with intensity measure $\mathrm{d}x\lambda(t)\mathrm{d}t$. The first arrived point (x_1, t_1) immediately initiates a bi-directional coverage by sending out a growing branch centered at x_1. Each frontier of the branch moves with a constant speed v until it meets an opposing one. Other points continue to arrive according to Φ. If a point arrives at a position that has already been covered by a branch, it will be deleted (or thinned); otherwise, another bi-directional coverage will be initiated by the same mechanism as that initiated by the first point. Applications of such processes can be found in cell biology (Wolk [13]), molecular biology (Vanderbei and Shepp [12]; Cowan *et al.* [4]) and neurobiology (Quine and Robinson [10, 11]) as well as other more obvious areas such as crystal growth (Kolmogorov [8]; Johnson and Mehl [7]; Meijering [9]). The distributions of random variables such as the number of unthinned points and the time required to cover an interval of a given length have been studied under various assumptions on the arrival regime. Quine and Robinson [10], Holst *et al.* [6], Chiu [2] and Chiu and Quine [3] showed the asymptotic normality of the number of unthinned points. Chiu [1] and Erhardsson [5] proved that the number of uncovered components has an asymptotic Poisson distribution. Vanderbei and Shepp [12] and Cowan *et al.* [4] studied the limiting distributions, by different means, of the completion time of the birth-growth model with $\lambda(x) = \lambda$ and $\lambda(x) = \frac{\gamma}{\mu}\mathrm{e}^{-\frac{x}{\mu}}$, respectively, where λ, γ and μ are positive finite constants. Weak limit theorems have also been proved. The general model considered by Holst *et al.* [6] incorporates both these models as special cases. For limit theorems of the completion time in higher dimensional cases see Chiu [1]. The current paper deals with the linear birth-growth model. We use the Markov process approach suggested by Vanderbei and Shepp [12] (see also Erhardsson [5] and Holst *et al.* [6]) to establish, under more general conditions than Holst *et al.* [6], the exact and limiting distributions and strong limit theorems for the time of complete coverage of a sufficiently long interval.

2. Laplace transform of the completion time

Assume that the space-time Poisson process Φ with intensity measure $\mathrm{d}x\lambda(t)\mathrm{d}t$ is defined in the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $\lambda(\cdot)$ is integrable and such that for all $t > 0$,

$$0 < \Lambda(t) := \int_0^t \lambda(y)\mathrm{d}y < \infty.$$

Because the process is homogeneous in space, the growth velocity v of the seed can be taken as $\frac{1}{2}$ by a change of scale. After a shear transformation $(x,t) \longrightarrow (x + \frac{t}{2}, t)$, a stationary Markov process $\{\xi_x, -\infty < x < \infty\}$ with the filtration $\{\mathcal{F}_x, -\infty < x < \infty\}$ is obtained, where $\mathcal{F}_x$ is the σ-algebra generated by the points $\{(x_i, t_i) \in \Phi : -\infty < x_i \leq x\}$ (for details see Holst *et al.* [6]). Denote by T_L the lowest time level at which the interval $(0, L)$ is completely covered and by τ_z the value of x at which the process $\{\xi_x\}$ first hits the level z (note that the level of $\{\xi_x\}$ is the time and the parameter space is the positions). Then for $t < z$,

$$\mathbf{P}_t(T_L < z) = \mathbf{P}_t(\tau_z > L),$$

where $\mathbf{P}_t$ denotes the conditional probability given that the initial level is $\xi_0 = t$.

It is known that the Laplace transform of τ_z can be obtained by considering the transition semigroup of operators $\{T_x\}$ defined by

$$
\begin{aligned}
T_x f(t) := {} & \mathbf{E}_t f(\xi_x) \\
= {} & \left(1 - \int_0^x \Lambda(t+u)\mathrm{d}u\right) f(t+x) + \left(\int_0^x \Lambda(t+u)\mathrm{d}u\right) \int_0^t f(u)\frac{\lambda(u)}{\Lambda(t)}\mathrm{d}u + o(x) \\
= {} & (1 - x\Lambda(t+\delta_x))\, f(t+x) + x\Lambda(t+\delta_x) \int_0^t f(u)\frac{\lambda(u)}{\Lambda(t)}\mathrm{d}u + o(x),
\end{aligned}
$$

for some δ_x in $(0,x)$, where $\mathbf{E}_t$ denotes the conditional expectation given $\xi_0 = t$ and f is a bounded measurable real-valued function on $[0,\infty)$. Thus, the infinitesimal generator $\mathcal{A}$ is given by

$$
\mathcal{A} f(t) = f'(t) - \Lambda(t) f(t) + \int_0^t f(u)\lambda(u)\mathrm{d}u.
$$

The Laplace transform $f(t) = \mathbf{E}_t \mathrm{e}^{-\alpha\tau_z}$ is the solution of

$$
\begin{cases}
\mathcal{A} f(t) = \alpha f(t), \quad 0 < t < z, \\
f(z) = 1.
\end{cases}
\tag{2.1}
$$

Holst *et al.* [6, p. 908] derived the same system of equations by a regenerative argument and obtained explicitly the Laplace transform

$$
\mathbf{E}_t \mathrm{e}^{-\alpha\tau_z} = \frac{1 + \alpha\int_0^t \mathrm{e}^{\alpha u + \Delta(u)}\mathrm{d}u}{1 + \alpha\int_0^z \mathrm{e}^{\alpha u + \Delta(u)}\mathrm{d}u},
$$

where $\Delta(u) = \int_0^u \Lambda(t)\mathrm{d}t$.

In principle the inverse Laplace transform can always be found, but it is in the form of a Bromwich integral. Even for the simplest case in which $\lambda(t) = \lambda$, the Bromwich integral is difficult to calculate (see Vanderbei and Shepp [12, p. 308]). Only limit theorems have been derived in Holst *et al.* [6]. In the next section we obtain the exact distribution for τ_z.

3. Exact distribution

Since the Laplace transform

$$\mathbf{E}_t \exp(-\alpha\tau_z) = 1 - \alpha \int_0^\infty \mathrm{e}^{-\alpha L}\mathbf{P}_t(\tau_z > L)\mathrm{d}L$$

satisfies system (2.1), $\mathbf{P}_t(\tau_z > L) = q(t,L)$ is the unique solution of the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial q(t,L)}{\partial L} = \mathcal{A}q(t), & L > 0, 0 < t < z, \\ \lim\limits_{t\to z} q(t,L) = 0, & L > 0, \\ \lim\limits_{L\to 0} q(t,L) = 1, & 0 < t < z, \end{cases} \tag{3.1}$$

where $q(t) = q(t,L)$.

Theorem 3.1. *Let T_L denote the earliest time that the interval $(0,L)$ is completely covered. For $0 \le t < z$ and $L > 0$,*

$$\mathbf{P}_t(T_L < z) = \sum_k C_k \left(1 + a_k(z)\int_0^t \exp(a_k(z)u + \Delta(u))du\right) \mathrm{e}^{a_k(z)L},$$

where $a_1(z) > a_2(z) > a_3(z) > \cdots$ are all negative zeros of $g(a) := 1 + a\int_0^z \exp(au + \Delta(u))du$, and

$$C_k = -\frac{1}{a_k^2(z)\int_0^z u\mathrm{e}^{a_k(z)u+\Delta(u)}du - 1}.$$

Proof. Setting $q(t,L) = U(t)V(L)$ yields two equations:

$$V'(L) = aV(L), \quad L > 0, \tag{3.2}$$

$$\mathcal{A}U(t) = aU(t), \quad 0 < t < z, \tag{3.3}$$

where a is a separation constant. The boundary condition of $q(t, L)$ leads to

$$\lim_{t\to z} U(t) = 0. \tag{3.4}$$

For $a \geq 0$ the only solution of (3.3) with boundary condition (3.4) is zero. Next, assume $a < 0$. The solution of equation (3.3) is of the form

$$U(t) = B_1 \left(1 + a \int_0^t \exp(au + \Delta(u))\mathrm{d}u\right),$$

where B_1 is a constant and a can be determined by (3.4), that is,

$$1 + a \int_0^z \exp(au + \Delta(u))\mathrm{d}u = 0.$$

Let $a_1(z), a_2(z), a_3(z), \cdots$ denote its all negative roots, which are all simple, and so without loss of generality assume that $0 > a_1(z) > a_2(z) > a_3(z) > \cdots$. Then the solutions to equations (3.2) and (3.3) are, respectively,

$$V_k(L) = B_{2k}\mathrm{e}^{a_k(z)L},$$

$$U_k(t) = B_{1k} \left(1 + a_k(z) \int_0^t \exp(a_k(z)u + \Delta(u))\mathrm{d}u\right),$$

for $k = 1, 2, \cdots$, where B_{1k} and B_{2k} are constants. Hence the general solution to problems (3.1) is of the form

$$q(t, L) = \sum_k C_k \left(1 + a_k(z) \int_0^t \exp\left(a_k(z)u + \Delta(u)\right)\mathrm{d}u\right) \mathrm{e}^{a_k(z)L}, \tag{3.5}$$

where C_k's are constants. The initial condition given in (3.1) yields

$$C_k = -\frac{1}{a_k^2(z) \int_0^z u\mathrm{e}^{a_k(z)u + \Delta(u)}\mathrm{d}u - 1},$$

and the result follows.

4. Limiting distributions

Theorem 4.1. *Let T_L denote the earliest time that the interval $(0, L)$ is completely covered. For $z > 0$ and $0 \leq t < z$,*

$$\lim_{L\to\infty} \frac{1}{L} \log \mathbf{P}_t(T_L < z) = a_1(z), \tag{4.1}$$

where $a_1(z)$ is the principal zero of

$$g(a) = 1 + a \int_0^z e^{au+\Delta(u)} du.$$

Proof. From Theorem 3.1

$$\begin{aligned}\mathbf{P}_t(T_L < z) &= \mathrm{e}^{a_1(z)L} C_1 \left(1 + a_1(z) \int_0^t \exp(a_1(z)u + \Delta(u))\mathrm{d}u\right) \\ &\quad + \mathrm{e}^{a_1(z)L} \sum_{k\geq 2} C_k \left(1 + a_k(z) \int_0^t \exp(a_k(z)u + \Delta(u))\mathrm{d}u\right) \mathrm{e}^{(a_k(z)-a_1(z))L}.\end{aligned} \tag{4.2}$$

The Abel test suggests that the series above is uniformly convergent with respect to L on $[L_0, \infty)$, where $L_0 > 0$. Since $C_1 \left(1 + a_1(z) \int_0^t \exp(a_1(z)u + \Delta(u))\mathrm{d}u\right) > 0$, and $a_k(z)$'s are decreasing, the result follows.

Theorem 4.2. *For each real u, let $L = L(z)$ be a function of z. If $L \to \infty$ as $z \to \infty$ in such a manner that $a_1(z)L \to -\mathrm{e}^{-u}$ as $z \to \infty$, then*

$$\lim_{z\to\infty} \mathbf{P}_t(T_L < z) = \exp(-e^{-u}). \tag{4.3}$$

Proof. For each $a < 0$, $g(a) = 1 + a \int_0^z \mathrm{e}^{au+\Delta(u)}\mathrm{d}u < 0$ as z is large enough. This implies that $\lim_{z\to\infty} a_k(z) = -\infty, k \geq 2$ and $\lim_{z\to\infty} a_1(z) = 0$. Moreover, the initial condition given in (3.1) leads to $\lim_{z\to\infty} C_1 = 1$. The result follows.

Remark 4.1. This proof fills in the gap mentioned in Vanderbei and Shepp [12, p. 311].

In the Holst *et al.* [6] they assumed that $\lambda(\cdot)$ satisfies (Λ_1) $\lim_{t\to\infty}\Lambda(t)<\infty$, (Λ_2) $\Lambda(t)\to\infty$ and $\frac{t\lambda(t)}{\Lambda(t)}\to\rho$ for some $0\le\rho<\infty$ as $t\to\infty$, or (Λ_3) $\Lambda(t)\to\infty$ and $\frac{\lambda(t)}{\Lambda(t)}\to c$ with $0<c<\infty$ as $t\to\infty$. For these three classes of $\lambda(\cdot)$, the condition $a_1(z)L\to-\mathrm{e}^{-u}$ is equivalent to

$$\Delta(z)-\log\Lambda(z)=\log L+u+o(1). \tag{4.4}$$

(see Holst *et al.* [6, p. 902 equation (2.2)]), which is very useful in finding an explicit expression for $L(z)$ in Theorem 4.2. However, the equivalence between (4.4) and $a_1(z)L\to-\mathrm{e}^{-u}$ does not hold for general $\lambda(\cdot)$. The following theorem gives a sufficient condition, which includes $(\Lambda_1)-(\Lambda_3)$, for this equivalence being true.

Theorem 4.3. *Suppose that* $\lim_{z\to\infty}\frac{\lambda(z)}{\Lambda^2(z)}=0$. *Then the condition* $\lim_{z\to\infty}a_1(z)L=-\mathrm{e}^{-u}$ *is equivalent to*

$$\Delta(z)-\log\Lambda(z)=\log L+u+o(1), \tag{4.5}$$

where u is a real number.

Proof. From $\lim_{z\to\infty}\frac{\lambda(z)}{\Lambda^2(z)}=0$, one can obtain

$$g'(0)=\int_0^z \mathrm{e}^{\Delta(u)}\mathrm{d}u\sim\frac{1}{\Lambda(z)}\mathrm{e}^{\Delta(z)},\ \text{as } z\to\infty.$$

Using one step of Newton's method yields

$$a_1(z)\sim-\frac{1}{g'(0)}\sim-\Lambda(z)\mathrm{e}^{-\Delta(z)},\ \text{as } z\to\infty,$$

and the equivalence follows.

The following example shows that $(\Lambda_1) - (\Lambda_3)$ do not include all cases.

Example. Suppose $\lambda(z) = \frac{1}{2}(2z^2+1)\mathrm{e}^{z^2}$, so that $\Lambda(z) = \frac{1}{2}z\mathrm{e}^{z^2}$ and $\Delta(z) = \mathrm{e}^{z^2} - 1$. Moreover, $\lim_{z\to\infty} \frac{\lambda(z)}{\Lambda(z)} = \infty$ and $\lim_{z\to\infty} \frac{\lambda(z)}{\Lambda^2(z)} = 0$. Hence $\lambda(\cdot)$ does not satisfy $(\Lambda_1) - (\Lambda_3)$, but satisfies the condition in Theorem 4.3. It follows from (4.5) that

$$\mathrm{e}^{z^2} = 1 + \log\frac{1}{2}z + \log \mathrm{e}^{z^2} + \log L + u + o(1)$$

which is equivalent to

$$z = \sqrt{\log\log L} + \frac{1 - \log 2 + \frac{1}{2}\log\log\log L + \log\log L + u}{\sqrt{\log\log L}\log L} + o\left(\frac{1}{\sqrt{\log\log L}\log L}\right),$$

and hence (4.3) gives

$$\lim_{L\to\infty} \mathbf{P}_t\left(\log L\ \sqrt{\log\log L}T_L - G(L) < u\right) = \exp(-\mathrm{e}^{-u}),$$

where $G(L) = \log L\ \log\log L + \frac{1}{2}\log\log\log L + \log\log L + 1 - \log 2$.

5. Strong Limit Theorems

For $\lambda(x) = \lambda$, Vanderbei and Shepp [12] proved that

$$\mathbf{E}T_L^n \sim \left(\lambda^{-1}\log\lambda L^2\right)^{\frac{n}{2}}, \text{ as } L \to \infty,$$

and Cowan *et al.* [4] showed that

$$\frac{\sqrt{\lambda}T_L}{\sqrt{\log\lambda L^2}} \to 1 \text{ in probability as } L \to \infty.$$

Actually, a stronger version can be obtained.

Theorem 5.1. *Let T_L denote the earliest time that the interval $(0, L)$ is completely covered.*

(1) If $\lambda(x) = \lambda$, where λ is a positive finite constant, then

$$\lim_{L\to\infty} \frac{\sqrt{\lambda}T_L}{\sqrt{\log \lambda L^2}} = 1, \quad \mathbf{P}_t\text{-almost surely.}$$

(2) If $\Lambda(x) = \gamma F(x)$, where F is a distribution function with support on $(0, \infty)$ and finite mean, and γ is a positive finite constant, then

$$\lim_{L\to\infty} \frac{\gamma T_L}{\log \gamma L} = 1, \quad \mathbf{P}_t\text{-almost surely.}$$

(3) If $\lambda(x) = e^x$, then

$$\lim_{L\to\infty} \frac{T_L}{\log\log L} = 1, \quad \mathbf{P}_t\text{-almost surely.}$$

(4) If $\lambda(x) = \frac{1}{2}(2x^2 + 1)e^{x^2}$, then

$$\lim_{L\to\infty} \frac{T_L}{\sqrt{\log\log L}} = 1, \quad \mathbf{P}_t\text{-almost surely.}$$

Proof. We prove (2) only, and the others can be proved in a similar way.

For any $0 < \rho < 1$ and for any $C > 1$,

$$\begin{aligned}
&\mathbf{P}_t\left(\liminf_{L\to\infty} \frac{\gamma T_L}{\log \gamma L} < \rho\right) \le \mathbf{P}_t\left(\inf_{C^n \le L \le C^{n+1}} \frac{\gamma T_L}{\log \gamma L} < \rho, \text{ i.o.}\right) \\
&\le \mathbf{P}_t(\gamma T_{C^n} \le \rho \log \gamma C^{n+1}, \text{i.o.}) \\
&= \mathbf{P}_t(\gamma T_{C^n} - \log \gamma C^n - \gamma\mu \le \rho \log \gamma C^{n+1} - \log \gamma C^n - \gamma\mu, \text{ i.o.}).
\end{aligned}$$

By Theorems 4.2 and 4.3 (see also Holst *et al.* [6, p. 909]),

$$\lim_{n\to\infty} \mathbf{P}_t(\gamma T_{C^n} - \log \gamma C^n - \gamma\mu \le u) = \exp(-e^{-u}). \tag{5.1}$$

Thus,

$$\mathbf{P}_t(\gamma T_{C^n} - \log \gamma C^n - \gamma\mu \le \rho \log \gamma C^{n+1} - \log \gamma C^n - \gamma\mu)$$
$$\sim \exp(-\exp(\mu\gamma - \log(\gamma^{\rho-1} C^{(\rho-1)n+\rho}))).$$

Note that

$$\sum_n \exp(-\exp(\mu\gamma - \log(\gamma^{\rho-1} C^{(\rho-1)n+\rho}))) < \infty.$$

Hence, by the Borel-Cantelli Lemma,

$$\mathbf{P}_t\left(\liminf_{L\to\infty} \frac{\gamma T_L}{\log \gamma L} < \rho\right) = 0,$$

which implies

$$\mathbf{P}_t\left(\liminf_{L\to\infty} \frac{\gamma T_L}{\log \gamma L} \ge 1\right) = 1. \tag{5.2}$$

On the other hand, for any $\epsilon > 0$ and $C > 1$,

$$\mathbf{P}_t\left(\limsup_{L\to\infty} \frac{\gamma T_L}{\log \gamma L} > 1+\epsilon\right) \le \mathbf{P}_t\left(\sup_{C^n \le L \le C^{n+1}} \frac{\gamma T_L}{\log \gamma L} > 1+\epsilon, \text{ i.o.}\right)$$
$$\le \mathbf{P}_t(\gamma T_{C^{n+1}} \ge (1+\epsilon)\log \gamma C^n, \text{i.o.})$$
$$= \mathbf{P}_t(\gamma T_{C^{n+1}} - \log \gamma C^{n+1} - \gamma\mu \ge (1+\epsilon)\log \gamma C^n - \log \gamma C^{n+1} - \gamma\mu, \text{ i.o.}).$$

Using (5.1) again yields

$$\mathbf{P}_t(\gamma T_{C^{n+1}} - \log \gamma C^{n+1} - \gamma\mu \ge (1+\epsilon)\log \gamma C^n - \log \gamma C^{n+1} - \gamma\mu) \sim \mathrm{e}^{\mu\gamma} C^{1-n\epsilon}.$$

The Borel-Cantelli Lemma leads to

$$\mathbf{P}_t\left(\limsup_{L\to\infty} \frac{\gamma T_L}{\log \gamma L} > 1+\epsilon\right) = 0.$$

By the arbitrariness of $\epsilon > 0$,

$$\mathbf{P}_t\left(\limsup_{L\to\infty}\frac{\gamma T_L}{\log\gamma L}\leq 1\right)=1,$$

and the result follows.

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