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Published in:
Stochastic Processes and their Applications

DOI:
10.1016/S0304-4149(96)00113-5

Published: 01/02/1997

A central limit theorem for linear Kolmogorov's birth–growth models

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Abstract

A Poisson process in space–time is used to generate a linear Kolmogorov's birth–growth model. Points start to form on $[0, L]$ at time zero. Each newly formed point initiates two bidirectional moving frontiers of constant speed. New points continue to form on not-yet passed over parts of $[0, L]$. The whole interval will eventually be passed over by moving frontiers. Let N_L be the total number of points formed. Quine and Robinson (1990) showed that if the Poisson process is homogeneous in space–time , the distribution of $(N_L - \mathbf{E}[N_L])/\sqrt{\mathbf{var}[N_L]}$ converges weakly to the standard normal distribution. In this paper a simpler argument is presented to prove this asymptotic normality of N_L for a more general class of linear Kolmogorov's birth–growth models.

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60G55
SECONDARY 60F05, 60D05

CENTRAL LIMIT THEOREM * COVERAGE * INHOMOGENEOUS POISSON PROCESS * JOHNSON–MEHL TESSELLATION * KOLMOGOROV'S BIRTH–GROWTH MODEL

1. Introduction

Consider the following linear random birth–growth model. Points are *formed* on $[0, L]$ according to a spatial–temporal Poisson process $\Psi_L \equiv \{(x_i, t_i) \in [0, L] \times [0, \infty)\}$. (Points are *born* at the locations x_i at times t_i, $i = 0, 1, 2, \ldots$) Its intensity measure is $\ell \times \Lambda$, where ℓ is the Lebesgue measure in $\mathbb{R}$, while Λ is an arbitrary locally finite measure on $[0, \infty)$ such that

$$\mu \equiv \int_0^\infty \exp\left\{-\int_0^t 2v(t-s)\Lambda(ds)\right\}\Lambda(dt) \quad < \quad \infty, \text{ and} \tag{1.1}$$

$$\Lambda([0,t]) \quad > \quad 0 \qquad \text{for all } t > 0. \tag{1.2}$$

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The role of these two regularity conditions will be explained the next section.

Points start to form on $[0, L]$ at time zero. Once the point (x_i, t_i) is formed (born), two bidirectional *moving frontiers* commence at x_i. Each frontier moves at a constant speed v until it meets an opposing one. The intervals passed over by moving frontiers are regarded as *covered*. New points continue to form on uncovered parts of $[0, L]$ until the whole interval is covered.

Such a birth–growth process in two-dimensions was first developed by Kolmogorov (1937) to study the growth of crystal aggregates, and then was proved to be very useful (see e.g. Chiu (1995), Cowan *et al* (1995), Evans (1945), Gilbert (1962), Johnson and Mehl (1939), Meijering (1953), Møller (1992), Okabe *et al* (1992), Quine and Robinson (1990, 1992), Stoyan *et al* (1995) and Vanderbei and Shepp (1988)).

Denote by Φ_L the spatial–temporal process of the points formed. For ease of presentation, we consider Φ_L both as a random set in $[0, L] \times [0, \infty)$ and a random measure. Let N_L be the total number of points in the set Φ_L. Quine and Robinson (1990) proved that the distribution of $(N_L - \mathbf{E}[N_L])/\sqrt{\mathbf{var}[N_L]}$ converges weakly to the standard normal distribution for $\Lambda(dt) = \lambda dt$. In the following a simpler proof of the asymptotic normality of N_L as $L \to \infty$ will be given for the following two cases: (i) $\Lambda(\{0\}) > 0$ and (ii) $\Lambda \ll \ell$, $\Lambda([0, \varepsilon]) = O(\varepsilon)$ as $\varepsilon \to 0$ and $\Lambda([0, L]) = O(L)$ as $L \to \infty$. The latter case is a generalisation of the model of Quine and Robinson (1990). Nevertheless, the proof presented here is inspired by their approach.

2. Mean and variance

Although the following results hold only almost surely, for simplicity this will not be said explicitly hereinafter. Note that condition (1.2) guarantees that $N_L \geq 1$ as $L \to \infty$ and that the birth–growth process starts at time zero.

Slivnyak (1962) showed that the reduced Palm distribution of a Poisson process is the same as the distribution of the Poisson process (see e.g. Stoyan *et al* (1995)). Applying this result on the Mecke–Campbell (or called refined Campbell) theorem (Mecke, 1967) yields

$$\mathbf{E} \sum_{(x_0,t_0),\ldots,(x_n,t_n)\in\Psi_L}^{\neq} f((x_0, t_0), \ldots, (x_n, t_n), \Psi_L)$$
$$= \int\int \cdots \int\int \mathbf{E} f((x_0, t_0), \ldots, (x_n, t_n), \Psi_L \cup \{(x_0, t_0), \ldots, (x_n, t_n)\})$$
$$\times \ell(dx_0)\Lambda(dt_0) \cdots \ell(dx_n)\Lambda(dt_n), \tag{2.1}$$

where $\sum^{\neq}$ denotes the summation over $(n + 1)$-tuples of $n + 1$ distinct spatial–temporal points (see also Møller (1992) Eq. (3.1)). This equation will be used to obtain the mean and variance of N_L.

Let $\Xi(\Psi_L, t)$ denote the random region in $(-\infty, \infty) \times [0, \infty)$ which is *covered* just before time t by the Ψ_L-generated birth–growth process. It can be written as

$$\Xi(\Psi_L, t) \equiv \bigcup_{(x,s)\in\Psi_L, s<t} \{(y, r) : y \in [x - (r - s)v, x + (r - s)v], r \geq s\}.$$

For each point (x, t) in Ψ_L, these three events $\{(x, t) \notin \Xi(\Psi_L, t)\}$, $\{(x, t) \notin \Xi(\Psi_L \setminus \{(x, t)\}, t)\}$ and $\{(x, t) \in \Phi_L\}$ are equivalent, since the first two events imply that at time t the position x has not yet been covered by the Ψ_L-generated birth–growth process, and consequently a point is formed at (x, t). Therefore, we have

$$\mathbf{E}[N_L] = \mathbf{E}\left[\sum_{(x,t)\in\Psi_L} \mathbf{1}\left((x,t) \notin \Xi(\Psi_L, t)\right)\right],$$

where $\mathbf{1}(\cdot)$ denotes the indicator function, and so by (2.1),

$$\mathbf{E}[N_L] = \int_0^\infty \int_0^L \mathbf{E}\left[\mathbf{1}\left((x,t) \notin \Xi(\Psi_L, t)\right)\right] \ell(dx)\Lambda(dt). \tag{2.2}$$

To evaluate this integral, one has to note that no point of Ψ_L lies outside the region $[0, L] \times [0, \infty)$, and so the equality

$$\mathbf{E}\left[\mathbf{1}\left((x,t) \notin \Xi(\Psi_L, t)\right)\right] = \exp\left\{-\int_0^t 2v(t-s)\Lambda(ds)\right\}$$

holds only when $vt \le x \le L - vt$ and $0 \le t \le L/2v$. But nevertheless, when $L \to \infty$,

$$\mathbf{E}[N_L] \sim \mu L, \tag{2.3}$$

where μ has been defined and assumed to be finite in condition (1.1).

Remark. Suppose that the birth–growth process took place on the whole real line $(-\infty, \infty)$ instead of the interval $[0, L]$. Denote the collection of all points that would be born at uncovered positions by Φ. Then μ is the intensity of the point process which is the projection of Φ onto $(-\infty, \infty)$.

Similarly, for all (x_0, t_0) and (x_1, t_1) in Ψ_L, consider

$$\begin{aligned} f((x_0,t_0),(x_1,t_1),\Psi_L) &= \mathbf{1}((x_0,t_0) \notin \Xi(\Psi_L,t_0))\mathbf{1}((x_1,t_1) \notin \Xi(\Psi_L,t_1)) \\ &= \mathbf{1}((x_0,t_0) \in \Phi_L)\mathbf{1}((x_1,t_1) \in \Phi_L) \end{aligned}$$

Then obviously,

$$\mathbf{E}[N_L(N_L-1)] = \mathbf{E}\sum_{(x_0,t_0),(x_1,t_1)\in\Psi_L}^{\neq} f((x_0,t_0),(x_1,t_1),\Psi_L)$$

By applying (2.1) we have

$$\begin{aligned} \mathbf{E}[N_L(N_L-1)] = \int_0^\infty \int_0^\infty \int_0^L \int_0^L & \mathbf{E} f((x_0,t_0),(x_1,t_1),\Psi_L \cup \{(x_0,t_0),(x_1,t_1)\}) \\ & \times \ell(dx_0)\ell(dx_1)\Lambda(dt_0)\Lambda(dt_1). \end{aligned}$$

By definition of Ξ,

$$\begin{aligned} \{(x_1,t_1) \notin \Xi(\Psi_L \cup \{(x_0,t_0),(x_1,t_1)\},t)\} &= \{(x_1,t_1) \notin \Xi(\Psi_L \cup \{(x_0,t_0)\},t)\} \\ &\subseteq \{(x_1,t_1) \notin \Xi(\Psi_L,t)\}. \end{aligned}$$

So

$$\begin{aligned} &\mathbf{E} f((x_0,t_0),(x_1,t_1),\Psi_L \cup \{(x_0,t_0),(x_1,t_1)\}) \\ &\qquad \le \mathbf{E}\left[\mathbf{1}\left((x_0,t_0) \notin \Xi(\Psi_L,t)\right)\mathbf{1}\left((x_1,t_1) \notin \Xi(\Psi_L,t)\right)\right]. \end{aligned}$$

Consequently from (2.2) and (2.3), $\mathbf{E}[N_L(N_L-1)] \le \mathbf{E}[N_L]^2$ and

$$\sigma^2 \equiv \lim_{L\to\infty} \frac{\mathbf{var}[N_L]}{L} \le \mu.$$

More directly, σ^2 can be obtained from the following equation:

$$\begin{aligned}&(\sigma^2+\mu^2-\mu)\\ &= \lim_{L\to\infty}\int_0^\infty\int_0^L\int_0^\infty\int_{|x_1-x_0|<v|t_1-t_0|} \exp\{-J(x_0,t_0,x_1,t_1)\}\,\ell(dx_0)\Lambda(dt_0)\ell(dx_1)\Lambda(dt_1)/L,\end{aligned}$$

where

$$\begin{aligned}&J(x_0,t_0,x_1,t_1)\\ &\quad= \int_0^{\max(t_0,t_1)} 2v\left([t_1-s]_+ + [t_0-s]_+ - [\frac{|x_0-x_1|+v(t_0+t_1)}{2v}-s]_+\right)\Lambda(ds),\end{aligned}$$

and $[x]_+ = \max(x,0)$.

3. Central Limit Theorem

Throughout this section it is assumed that either (i) $\Lambda(\{0\})>0$ or (ii) $\Lambda \ll \ell$, $\Lambda([0,\varepsilon]) = O(\varepsilon)$ as $\varepsilon\to 0$ and $\Lambda([0,L]) = O(L)$ as $L\to\infty$.

Let $T_1 \equiv \inf\{t_i : (x_i,t_i)\in\Psi_L\}$ denote the birth-time of the first born (formed) point. Since $N_L \ge 1$ as $L\to\infty$, to show the asymptotic normality of N_L, we can assume without loss of generality that $\Psi_L \ne \emptyset$, and so $T_1<\infty$ is well-defined. Let U follow the uniform distribution on $[0,L]$. Thus, when there is a unique first point in Ψ_L, the spatial–temporal coordinates of this point follows the same distribution as (U,T_1). When there are more than one points born at T_1 (e.g. when Λ is concentrated at $\{0\}$, and so $T_1=0$), (U,T_1) can be regarded as the spatial–temporal coordinates of a randomly chosen one of them.

Denote 'has the same distribution as' by '$\stackrel{d}{=}$'. As it can be seen from Figure 1, when the first point is born at (U,T_1), $[0,L]$ is divided into two intervals of length U and $L-U$. A moving frontier commences at an endpoint of each interval. Therefore, N_L is equal to one plus the total number of points of Φ_L in the two right-angled triangles with bases length U and $L-U$ on $[0,L]$ shown in Figure 1, i.e.

$$N_L \stackrel{d}{=} 1 + R_U^{(T_1)} + R_{L-U}^{(T_1)}, \tag{3.1}$$

where $R_U^{(T_1)}$ denotes the total number of points of Φ_L which are born in the right-angled triangle based on $[0,U)\times\{T_1\}$ and height U/v. Denote this right-angled triangle by $A_U^{(T_1)}$. Thus, $R_U^{(T_1)} = \Phi_L(A_U^{(T_1)})$. Moreover, moving frontiers in one of these two triangles cannot pass into the other one, and Ψ_L is a Poisson process. Thus, conditional on $\{U=u\}$, where $0<u<L$, the random variables $R_U^{(T_1)}$ and $R_{L-U}^{(T_1)}$ in (3.1) are independent. Note that only the position on the time-axis $\{T_1\}$ and the length U of the base of the triangle but not the exact position of this interval on $[0,L]$ are essential to the distribution of $R_U^{(T_1)}$, with the understanding that the open endpoint of the base $[0,U)$ (i.e. $\{U\}$) contains a point of Φ_L.

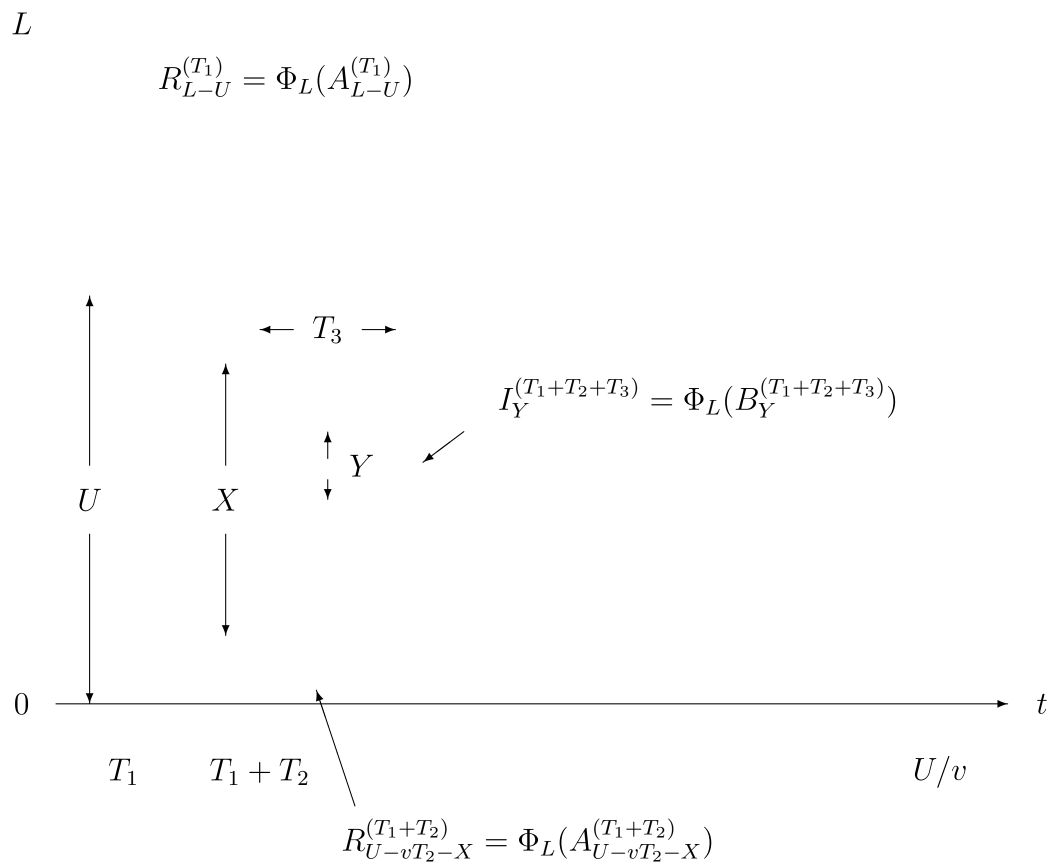

Figure 1 (after Quine and Robinson (1990)).

If $R_U^{(T_1)} \geq 1$, i.e. there is another point $(X, T_1 + T_2)$ of Φ_L in the triangle $A_U^{(T_1)}$, then by a similar argument as above, $R_U^{(T_1)}$ equals one plus the total number of points in a right-angled and an isosceles triangle. More precisely,

$$R_U^{(T_1)} \stackrel{d}{=} \mathbf{1}(T_2 < \frac{U}{v}) \left(1 + R^{(T_1+T_2)}_{U-vT_2-X} + I_X^{(T_1+T_2)}\right), \tag{3.2}$$

where $I_X^{(T_1+T_2)}$ denotes the total number of points of Φ_L which are born in the isosceles triangle with base $(0, X) \times \{T_1 + T_2\}$ and height $X/(2v)$ shown in Figure 1. Denote this isosceles triangle by $B_X^{(T_1+T_2)}$, i.e. $I_X^{(T_1+T_2)} = \Phi_L(B_X^{(T_1+T_2)})$. Define $\inf\{t_i : (x_i, t_i) \in \emptyset\} = \infty$. Then X in (3.2) is uniform on $(0, U - vT_2)$, where $T_2 \equiv \min\{U/v, \inf\{t_i : (x_i, t_i) \in \Psi_L \cap A_U^{(T_1)}\}\}$. Furthermore, conditional on $\{U = u, X = x, T_2 = t_2\}$, where $0 < u < L$, $0 < x < u - vt_2$ and $0 \leq t_2 < u/v$, the random variables $R^{(T_1+T_2)}_{U-vT_2-X}$ and $I_X^{(T_1+T_2)}$ are independent. Similar to $R_U^{(T_1)}$, only $T_1 + T_2$ and X but not the exact position of the interval on $[0, L]$ on which the isosceles triangle based are essential to the distribution of $I_X^{(T_1+T_2)}$, with the understanding that each of the two endpoints of the base $(0, X)$ (i.e. $\{0\}$ and $\{X\}$) contains a point of Φ_L.

Applying this kind of argument again to the isosceles triangle $B_X^{(T_1+T_2)}$ yields

$$I_X^{(T_1+T_2)} \stackrel{d}{=} \mathbf{1}(T_3 < \frac{X}{2v}) \left(1 + I_Y^{(T_1+T_2+T_3)} + I^{(T_1+T_2+T_3)}_{X-2vT_3-Y}\right), \tag{3.3}$$

where Y is uniform on $(0, X - 2vT_3)$, $T_3 \equiv \min\{X/(2v), \inf\{t_i : (x_i, t_i) \in \Psi_L \cap B_X^{(T_1+T_2)}\}\}$, and conditional on $\{U = u, X = x, Y = y, T_2 = t_2, T_3 = t_3\}$, where $0 < u < L$, $0 < x < u - vt_2$, $0 < y < x - 2vt_3$, $0 \le t_2 < u/v$ and $0 \le t_3 < x/(2v)$, the random variables $I_Y^{(T_1+T_2+T_3)}$ and $I_{X-2vT_3-Y}^{(T_1+T_2+T_3)}$ are independent.

Lemma 3.1 *Suppose Z_L is a positive and finite random variable with finite mean and variance for each L. If the Laplace transform has the form*

$$\mathbf{E}[\exp\{-\xi Z_L\}] = \exp\{L\alpha(\xi) + \beta(\xi)\}$$

for some real-valued functions α and β such that $\alpha(\xi)$ and $\beta(\xi)$ are bounded for each fixed $\xi \in [0, \infty)$, then the distribution of $(Z_L - \mathbf{E}[Z_L])/\sqrt{\mathbf{var}[Z_L]}$ converges weakly to the standard normal distribution as $L \to \infty$.

Proof. As $\mathbf{E}[Z_L]$ and $\mathbf{var}[Z_L]$ exist and are finite,

$$\begin{aligned} \alpha(\xi) &= \alpha'(0)\xi + \frac{\alpha''(0)}{2}\xi^2 + o(\xi^2), \\ \beta(\xi) &= \beta'(0)\xi + \frac{\beta''(0)}{2}\xi^2 + o(\xi^2). \end{aligned}$$

Then for each fixed ξ,

$$\mathbf{E}\left[\exp\left\{-\xi\left(\frac{Z_L - \mathbf{E}[Z_L]}{\sqrt{\mathbf{var}[Z_L]}}\right)\right\}\right] = \exp\left\{\frac{\xi^2}{2} + Lo\left(\frac{\xi^2}{L}\right)\right\} \to \exp\left\{\frac{\xi^2}{2}\right\}$$

as $L \to \infty$, and the result follows. □

Lemma 3.2 *(a) As $L \to \infty$, T_1, T_2 and T_3 converge in probability to zero.*

(b) As $L \to \infty$, for each $0 < u, x, y \le 1$, $R_{uL}^{(T_1)} - R_{uL}^{(0)}$, $R_{x(uL-vT_2)}^{(T_1+T_2)} - R_{xuL}^{(0)}$, $I_{x(uL-vT_2)}^{(T_1+T_2)} - I_{xuL}^{(0)}$ and $I_{y[x(uL-vT_2)-2vT_3]}^{(T_1+T_2+T_3)} - I_{yxuL}^{(0)}$ converge in probability to zero.

Proof. (a) For each $\varepsilon > 0$, $\{T_1 > \varepsilon\}$ is equivalent to $\{\Psi_L([0, L] \times [0, \varepsilon]) = 0\}$. Thus

$$\mathbf{P}\{T_1 > \varepsilon\} = \exp\{-L\Lambda([0, \varepsilon])\} \to 0$$

as $L \to \infty$. The convergence in probability to zero for T_2 and T_3 can be proved similarly.

(b) For $\Lambda(\{0\}) > 0$, the statement is obvious, since T_1, T_2, T_3 converge almost surely to zero. Hence it suffices to consider only the case that $\Lambda \ll \ell$, $\Lambda([0, \varepsilon]) = O(\varepsilon)$ as $\varepsilon \to 0$ and $\Lambda([0, L]) = O(L)$ as $L \to \infty$.

Clearly

$$\mathbf{E}\left[\left|R_{uL}^{(T_1)} - R_{uL}^{(0)}\right|\right] \le \mathbf{E}[uL\Lambda([0, T_1])] + \mathbf{E}[vT_1\Lambda([0, uL/v])].$$

Thus, if $\mathbf{E}[LT_1]$ converges to zero, the result follows. Since LT_1 converges to zero in probability and is uniformly integrable, it converges in mean to zero.

The convergence in probability to zero of the other random variables can be proved similarly. □

Remark. Lemma 3.2(b) is quite similar to Slutzky's theorem for convergence in probability. However, $R_{uL}^{(\cdot)}$ is not a Borel function on $\mathbb{R}$ but a random variable, and so a proof for the statement is necessary. Moreover, Lemma 3.2(b) implies that the conditional distribution of N_L, conditional on the event $\{T_1 = 0, T_2 = 0, T_3 = 0\}$, converges weakly to the unconditional distribution of N_L as $L \to \infty$.

Lemma 3.3 *For each real ξ and positive x and L denote*

$$\mathcal{L}_{xL}^{(I)}(\xi) \equiv \mathbf{E}[\exp\{-\xi(I_{xL}^{(0)} + 1)\}].$$

Conditional on the event $\{T_1 = 0, T_2 = 0, T_3 = 0, U = uL, X = xL\}$, where $0 < u < 1$ and $0 < x < u$,

$$\mathcal{L}_{xL}^{(I)}(\xi) = \exp\{xL\alpha(\xi)\}$$

for some real-valued function α such that $\alpha(\xi)$ is finite for each $\xi \in [0, \infty)$.

Proof. Since

$$\mathbf{P}\left\{I_{xL}^{(0)} = k\right\} \le \frac{\lambda_L^k e^{-\lambda_L}}{k!}$$

for $k = 0, 1, 2, \ldots$, where $\lambda_L = \int_0^{L/(2v)}(L - 2vt)\Lambda(dt)$, $\mathcal{L}_{xL}^{(I)}(\cdot)$ exists and is bounded above by 1 on $[0, \infty)$.

From (3.3), conditional on $\{T_1 = 0, T_2 = 0, T_3 = 0, U = uL, X = xL\}$ where $0 < u < 1$ and $0 < x < u$,

$$I_{xL}^{(0)} \stackrel{d}{=} 1 + I_Y^{(0)} + I_{xL-Y}^{(0)}, \tag{3.4}$$

where Y is uniform on $(0, xL)$, and conditional further on $\{Y = yL\}$, where $0 < y < x$, the random variables $I_{yL}^{(0)}$ and $I_{xL-yL}^{(0)}$ are independent. Moreover, denote

$$I(x) \equiv I_{xL}^{(0)} + 1$$

for $0 < x < u$. For each fixed L,

$$\mathbf{E}\left[|I(x+\delta) - I(x)|\right] \le 2v|\delta|\Lambda([0, uL/(2v)]) + uL\Lambda([0, |\delta|]) \to 0$$

for any $x \in (0, u)$ as $\delta \to 0$, and so $\mathbf{E}[\exp\{-\xi I(\cdot)\}]$ is uniformly continuous on $(0, u)$.

From (3.4), we have

$$\mathcal{L}_{xL}^{(I)}(\xi) = \int_0^x \mathcal{L}_{yL}^{(I)}(\xi)\mathcal{L}_{(x-y)L}^{(I)}(\xi)\frac{dy}{x}. \tag{3.5}$$

For each x and ξ define $g : [-x/2, x/2] \mapsto \mathbb{R}$ by

$$g(z) \equiv \mathcal{L}_{xL}^{(I)}(\xi) - \mathcal{L}_{(2^{-1}x+z)L}^{(I)}(\xi)\mathcal{L}_{(2^{-1}x-z)L}^{(I)}(\xi),$$

and rewrite (3.5) as

$$\int_{-2^{-1}x}^{2^{-1}x} g(z)\frac{dz}{x} = 0. \tag{3.6}$$

Since g is continuous and symmetric about $g(0)$, (3.6) implies that $g(0) = 0$. Thus for all integers $n \in \{k \in \mathbb{Z} : 0 < 2^{-k}x < u\}$, where $\mathbb{Z}$ denotes the set of all integers,

$$\mathcal{L}_{xL}^{(I)}(\xi) = \mathcal{L}_{2^{-n}xL}^{(I)}(\xi)^{2^n}.$$

As $\mathcal{L}_1^{(I)}$ is bounded on $[0, \infty)$, there exists a real-valued function α such that $\mathcal{L}_1^{(I)}(\xi) \equiv \exp\{\alpha(\xi)\}$, which is finite for each nonnegative ξ. Define $f(w) \equiv \exp\{-w\alpha(\xi)\}\mathcal{L}_w^{(I)}(\xi)$, for all $0 < w < uL$, and 1 otherwise. Then $f(1) = 1$ and $f(2^{-n}) = 1$ for every positive or negative integer n such that $2^{-n} < uL$. Since f is continuous on $(0, uL)$, $f = 1$. Therefore, for all $0 < x < u$,

$$\mathcal{L}_{xL}^{(I)}(\xi) = \exp\{xL\alpha(\xi)\}.$$

□

Remark. As Quine and Robinson (1990) remarked, if instead on an interval, the growth process took place on a circle of perimeter length L, the number of points formed would be distributed as $1 + I_L^{(T_1)}$. Let N_L^* be a random variable with the same distribution as $1 + I_L^{(T_1)}$. Lemmas 3.2(b) and 3.3 mean that in this case the Laplace transform $\mathbf{E}[\exp\{-\xi N_L^*\}] \sim \exp\{L\alpha(\xi)\}$. By Lemma 3.1, asymptotic normality holds, and by Theorem 3.1 below, N_L^* and N_L have the same asymptotic distribution as $L \to \infty$.

Lemma 3.4 *For each real ξ and positive u and L denote*

$$\mathcal{L}_{uL}^{(R)}(\xi) \equiv \mathbf{E}[\exp\{-\xi(R_{uL}^{(0)} + 1)\}].$$

Conditional on $\{T_1 = 0, T_2 = 0, T_3 = 0, U = uL\}$, where $0 < u < 1$,

$$\mathcal{L}_{uL}^{(R)}(\xi) = \exp\{uL\alpha(\xi) + \beta(\xi)\}$$

for some real-valued function β such that $\beta(\xi)$ is bounded for each $\xi \in [0, \infty)$, where α is the same as in Lemma 3.3.

Proof. Similar to the proof of Lemma 3.3, $\mathcal{L}_{uL}^{(R)}(\cdot)$ exists and is bounded, and conditional on $\{T_1 = 0, T_2 = 0, T_3 = 0, U = uL\}$, where $0 < u < 1$ and X is uniform on $(0, uL)$, we have

$$R_{uL}^{(0)} \stackrel{d}{=} 1 + R_{uL-X}^{(0)} + I_X^{(0)}.$$

Thus,

$$\mathcal{L}_{uL}^{(R)}(\xi) = \int_0^u \mathcal{L}_{(u-x)L}^{(R)}(\xi)\mathcal{L}_{xL}^{(I)}(\xi)\frac{dx}{u}.$$

Conditional on $\{T_1 = 0, T_2 = 0, T_3 = 0, U = uL, X = xL\}$ where $0 < x < u$, $\mathcal{L}_{xL}^{(I)}(\xi) = \exp\{xL\alpha(\xi)\}$ (Lemma 3.3). Hence, dividing both sides by $\exp\{uL\alpha(\xi)\}$ yields

$$h(u) = \int_0^u h(u - x)\frac{dx}{u} = \int_0^u h(x)\frac{dx}{u}, \tag{3.7}$$

where $h(u) \equiv \mathcal{L}_{uL}^{(R)}(\xi)\exp\{-uL\alpha(\xi)\}$. The derivative (with respect to u) of the leftmost term of (3.7) exists and is equal to zero. Thus for each ξ, h is a constant on $(0, u]$, say $\exp\{\beta(\xi)\}$ where β is some real-valued function such that $\beta(\xi)$ is bounded for each $\xi \in [0, \infty)$. The result follows. □

Theorem 3.1 *The distribution of* $(N_L - \mathbf{E}[N_L])/\sqrt{\mathbf{var}[N_L]}$ *converges weakly to the standard normal distribution as* $L \to \infty$.

Proof. It can be seen from (3.1) that conditional on $\{T_1 = 0\}$,

$$N_L \stackrel{d}{=} 1 + R_U^{(0)} + R_{L-U}^{(0)},$$

where U is uniform on $(0, L)$. Moreover, conditional further on $\{U = uL\}$, where $0 < u < 1$, the random variables $R_{uL}^{(0)}$ and $R_{(1-u)L}^{(0)}$ above are independent. Thus, by letting

$$\mathcal{L}_L(\xi) \equiv \mathbf{E}[\exp\{-\xi(N_L + 1)\}],$$

we have

$$\mathcal{L}_L(\xi) = \int_0^1 \mathcal{L}_{uL}^{(R)}(\xi)\mathcal{L}_{(1-u)L}^{(R)}(\xi)du.$$

From Lemma 3.4, conditional further on $\{T_2 = 0, T_3 = 0\}$,

$$\mathcal{L}_L(\xi) = \exp\left\{L\alpha(\xi) + \beta(\xi)\right\}.$$

By Lemma 3.2(b), this conditional distribution of N_L converges weakly to the unconditional distribution of N_L as $L \to \infty$. The asymptotic normality follows from Lemma 3.1. □

Acknowledgements

I thank R. Cowan and D. Stoyan for their helpful discussions and valuable suggestions on earlier drafts. I also thank the referee for useful comments. This work started during my stay in Freiberg, which was supported by a scholarship from DAAD.

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