Supplementary Material: "On high-dimensional misspecified mixed model analysis in genome-wide association studies"

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S1. Expressions for $\kappa_{k,l}(\gamma,\tau)$ and $\eta_k(\gamma,\tau)$. Let $d_\gamma := 1+\tau\gamma h_{1,0}(\gamma)$ (dependence on τ is suppressed for notational convenience). Then define

$$
\begin{aligned}
\bar{a}_1^{(1)}(\gamma,\tau) &= \frac{1}{d_\gamma^2},\\
\bar{a}_1^{(2)}(\gamma,\tau) &= \frac{-2\tau\gamma h_{2,0}(\gamma)}{d_\gamma^3}, \quad \bar{a}_2^{(2)}(\gamma,\tau) = \frac{1}{d_\gamma^2}\\
\bar{a}_1^{(3)}(\gamma,\tau) &= \frac{3(\tau\gamma)^2(h_{2,0}(\gamma))^2}{d_\gamma^4} - \frac{2\tau\gamma h_{3,0}(\gamma)}{d_\gamma^3},\\
\bar{a}_2^{(3)}(\gamma,\tau) &= \frac{-2\tau\gamma h_{2,0}(\gamma)}{d_\gamma^3}, \quad \bar{a}_3^{(3)}(\gamma,\tau) = \frac{1}{d_\gamma^2}.
\end{aligned}
\tag{S.1}
$$

and

$$
\begin{aligned}
\bar{b}_1^{(1)}(\gamma,\tau) &= \frac{1}{d_\gamma}\\
\bar{b}_1^{(2)}(\gamma,\tau) &= \frac{-\tau\gamma h_{2,0}(\gamma)}{d_\gamma^2}, \quad \bar{b}_2^{(2)}(\gamma,\tau) = \frac{1}{d_\gamma}\\
\bar{b}_1^{(3)}(\gamma,\tau) &= \frac{(\tau\gamma)^2(h_{2,0}(\gamma))^2}{d_\gamma^3} - \frac{\tau\gamma h_{3,0}(\gamma)}{d_\gamma^2},\\
\bar{b}_2^{(3)}(\gamma,\tau) &= \frac{-\tau\gamma h_{2,0}(\gamma)}{d_\gamma^2}, \quad \bar{b}_3^{(3)}(\gamma,\tau) = \frac{1}{d_\gamma}.
\end{aligned}
\tag{S.2}
$$

Let

$$
\kappa_{k,l}(\gamma,\tau) = \sum_{q_1=1}^{k}\sum_{q_2=1}^{l} \bar{a}_{q_1}^{(k)}(\gamma,\tau)\bar{a}_{q_2}^{(l)}(\gamma,\tau)h_{q_1+q_2,0}(\gamma), \qquad 1 \le k,l \le 3. \tag{S.3}
$$

Also, let

$$
\begin{aligned}
\text{(S.4)}\ \eta_k(\gamma,\tau) &= \sum_{q_1=1}^{k} \bar{b}_{q_1}^{(k)}(\gamma,\tau) h_{q_1,0}(\gamma), \quad 1 \le k \le 3 \\
\eta_k(\gamma,\tau) &= \sum_{q_1=1}^{3}\sum_{q_2=1}^{k-3} \bar{b}_{q_1}^{(3)}(\gamma,\tau)\bar{b}_{q_2}^{(k-3)}(\gamma,\tau) h_{q_1+q_2,0}(\gamma), \quad 4 \le k \le 6.
\end{aligned}
$$

An alternative expression for $\eta_k(\gamma,\tau)$ is as follows: for all $k \ge 1$,

$$
\begin{aligned}
\eta_k(\gamma,\tau) &= \sum_{q_1=0}^{k-1} \chi^{(q_1)}(\gamma,\tau) h_{k-q_1,0}(\gamma) \\
\text{where } \chi^{(l)}(\gamma,\tau) &= -\tau\gamma \sum_{q_2=0}^{l-1} \chi^{(0)}(\gamma,\tau)\chi^{(q_2)}(\gamma,\tau) h_{l-q_2+1,0}(\gamma), \quad \text{for } l \ge 1, \\
\text{and } \chi^{(0)}(\gamma,\tau) &= \frac{1}{1+\tau\gamma h_{1,0}(\gamma)}.
\end{aligned}
$$

S2. Technical lemmas.

LEMMA S.1 (Bai and Silverstein (2010), Lemma 8.10). *Let Q be an $n \times n$ non-random matrix and $W = (W_1, \dots, W_n)'$ be a random vector with independent entries. Assume that $\mathbb{E}(W_j) = 0$, $\mathbb{E}(W_j^2) = 1$ and $\mathbb{E}|W_j|^\ell \le \nu_\ell$ for $\ell \ge 2$, for all j. Then, for any integer $r \ge 1$,*

$$
\text{(S.5)} \quad \mathbb{E}\left[|W'QW - tr(Q)|^r\right] \le C_r \left[\nu_4^{r/2}(tr(QQ'))^{r/2} + \nu_{2r} tr((QQ')^{r/2})\right],
$$

where C_r is a constant depending on r only, and for any real-valued function f on $\mathbb{R}$, $tr(f(QQ')) = \sum_{i=1}^n f(\lambda_i(QQ'))$ where $\lambda_i(QQ')$ denotes the i-th largest eigenvalue of QQ'.

LEMMA S.2. *Let Q_1 and Q_2 be $n \times n$ non-random matrices and $Y = (Y_1, \dots, Y_n)^T$ and $W = (W_1, \dots, W_n)'$ be independent random vectors with $\mathbb{E}(Y_j) = \mathbb{E}(W_j) = 0$, $\mathbb{E}(Y_j^2) = \mathbb{E}(W_j^2) = 1$ and $\mathbb{E}|W_j|^\ell \le \nu_{\ell,w}$ and $\mathbb{E}|W_j|^\ell \le \nu_{\ell,y}$, for $\ell \ge 2$, for all j. Then, for any integer $r \ge 1$,*

$$
\begin{aligned}
\text{(S.6)} \quad & \mathbb{E}\left[|\frac{1}{n}W'Q_1YY'Q_2W - \frac{1}{n}Y'Q_2Q_1Y|^r\right] \\
& \le 2^{r-1}C_r(\nu_{4,w}^{r/2} + \nu_{2r,w})\|Q_1\|^r\|Q_2\|^r \left(1 + C_r\frac{\nu_{4,y}^{r/2}}{n^{r/2}} + C_r\frac{\nu_{2r,y}}{n^{r-1}}\right),
\end{aligned}
$$

where C_r is as in Lemma S.1.

Proof of Lemma S.2. Notice that $Q_1YY'Q_2$ is a rank 1 matrix, and hence

$$\mathrm{tr}(Q_1YY'Q_2Q_2'YY'Q_1) = (Y'Q_2'Q_2Y)(Y'Q_1'Q_1Y) = \lambda_1(Q_1YY'Q_2Q_2'YY'Q_1),$$

and $\lambda_j(Q_1YY'Q_2Q_2'YY'Q_1) = 0$ for $2 \le j \le n$. Using this, together with Lemma S.1, we obtain

$$\begin{aligned}
&\mathbb{E}\left[|\frac{1}{n}W'Q_1YY'Q_2W - \frac{1}{n}Y'Q_2Q_1Y|^r|Y\right]\\
\le\ & C_r(\nu_{4,w}^{r/2} + \nu_{2r,w})\left(\frac{1}{n}Y'Q_2'Q_2Y\right)^{r/2}\left(\frac{1}{n}Y'Q_1'Q_1Y\right)^{r/2}.
\end{aligned}$$

Now, another application of Lemma S.1 yields

$$\begin{aligned}
&\mathbb{E}|Y'Q_2'Q_2Y - \mathrm{tr}(Q_2'Q_2)|^r\\
\le\ & C_r\left(\nu_{4,y}^{r/2}(\mathrm{tr}(Q_2'Q_2Q_2'Q_2))^{r/2} + \nu_{2r,y}\mathrm{tr}((Q_2'Q_2Q_2'Q_2)^{r/2})\right)\\
=\ & C_r\left(\nu_{4,y}^{r/2}(\mathrm{tr}((Q_2'Q_2)^2))^{r/2} + \nu_{2r,y}\mathrm{tr}((Q_2'Q_2)^r)\right)\\
\le\ & C_r\|Q_2\|^{2r}\left(\nu_{4,y}^{r/2}n^{r/2} + \nu_{2r,y}n\right).
\end{aligned}$$

Then, (S.6) follows from last two displays, Cauchy-Schawarz inequality, and the fact that $|a+b|^r \le 2^{r-1}(|a|^r + |b|^r)$.

S3. Proofs of some auxiliary results from Section 7.

S3.1. *Proof of Proposition 7.1.* Let $\bar{n} = n - q$. We prove that for $j = 1, 2$,

$$\frac{1}{\bar{n}}b_j(\gamma) - b_{j,\infty}(\gamma) = o_{\mathrm{P}}(n^{-1/2}), \quad \frac{1}{\bar{n}}c_j(\gamma) - c_{j,\infty}(\gamma) = o_{\mathrm{P}}(n^{-1/2}),$$

In view of (54) and (55) it suffices to show that

$$\frac{1}{\bar{n}}\mathrm{tr}(\Sigma^{-1}\bar{U}\Sigma^{-1}\bar{U}_{(1)}) - \omega\frac{\tau h_{1,0}^2(\gamma) + h_{2,1}(\gamma)}{(1+\tau\gamma h_{1,0}(\gamma))^2} = o_{\mathrm{P}}(n^{-1/2}), \tag{S.7}$$

$$\frac{1}{\bar{n}}\mathrm{tr}(\Sigma^{-2}\bar{U}_{(1)}) - \omega\frac{h_{2,0}(\gamma)}{1+\tau\gamma h_{1,0}(\gamma)} = o_{\mathrm{P}}(n^{-1/2}), \tag{S.8}$$

$$\frac{1}{\bar{n}}\mathrm{tr}(\Sigma^{-1}\bar{U}^s) - h_{1,s}(\gamma) = o_{\mathrm{P}}(n^{-1/2}),\ s = 0, 1. \tag{S.9}$$

Before proving these assertions, we introduce some notations. We write Σ_{-C} to denote $\Sigma - \gamma p^{-1}\sum_{i\in C}\zeta_i\zeta_i'$ where $C \subset \{1, \ldots, p\}$. For convenience, we use $i_1 i_2 \cdots i_r$ to denote the set $\{i_1, i_2, \ldots, i_r\}$. Also, for $1 \le i, j \le p$, define

$\beta^{(l)}_{ij;C} = \zeta_i'\Sigma^{-l}_{-C}\zeta_i$ and $\mu^{(l)}_C = \bar{n}^{-1}\mathrm{tr}(\Sigma^{-l}_{-C})$, for $l \geq 1$. Further, use $\mathbb{E}_C$ to mean the conditional expectation given $\{Z_j : j \in \{1,\ldots,p\}\setminus C\}$. Finally, we define $\tau_p = \bar{n}/p$. Also, for further calculations, define $\bar{\zeta}_j \equiv \bar{\zeta}_j(\gamma) = \gamma^{1/2}p^{-1/2}\zeta_j$, and $\bar{\beta}^{(l)}_{ij;C} = \bar{\zeta}_i'\Sigma^{-l}_{-C}\bar{\zeta}_j$ for $l \geq 1$. Notice that $\Sigma \equiv \Sigma_\gamma = I_{\bar{n}} + \sum_{j=1}^p \bar{\zeta}_j\bar{\zeta}_j'$. Finally, use the notation iC to mean $\{i\}\cup C$. Using this notation, we have the following elementary result.

$$\text{(S.10)}\qquad \Sigma^{-1}_{-C} = \Sigma^{-1}_{-iC} - \frac{1}{1+\bar{\beta}^{(1)}_{ii;iC}}\Sigma^{-1}_{-iC}\bar{\zeta}_i\bar{\zeta}_i'\Sigma^{-1}_{-iC}, \quad \text{for } i \in \{1,\ldots,p\}\setminus C,$$

where $C \subset \{1,\ldots,p\}\setminus\{i\}$.

Notice that (S.9) easily follows from (54) and (55). So we focus our attention to proving (S.7) and (S.8). From (37), we can write

$$\text{(S.11)}\qquad \frac{1}{\bar{n}}\mathrm{tr}(\Sigma^{-1}\bar{U}\Sigma^{-1}\bar{U}_{(1)}) \;=\; \frac{1}{p}\sum_{i=1}^m \frac{(\bar{n}/p)\bar{u}_i^2 + \gamma^{-1}(\bar{u}_i - \bar{w}_i)}{(1+\gamma(\bar{n}/p)\bar{u}_i)^2}\,,$$

where $\bar{u}_i = \bar{n}^{-1}\zeta_i'\Sigma^{-1}_{-i}\zeta_i = \bar{n}^{-1}\beta^{(1)}_{ii;i}$ and $\bar{w}_i = \bar{n}^{-1}\zeta_i'\Sigma^{-2}_{-i}\zeta_i = \bar{n}^{-1}\beta^{(2)}_{ii;i}$.

For future calculations, we state the following proposition.

PROPOSITION S.1. *Suppose that the elements $\{Z_{i'j'} : i' = 1,\ldots,n; j' = 1,\ldots,p\}$ are independent, sub-Gaussian with $\mathbb{E}(Z_{i'j'}) = 0$, $\mathbb{E}(Z^2_{i'j'}) = 1$ for all i', j' and $\max_{1\leq i'\leq n; 1\leq j'\leq p}\|Z_{i'j'}\|_{\psi_2} \leq K_0 < \infty$, where $\|\cdot\|_{\psi_2}$ is the sub-Gaussian norm defined in (14). Then the following statements are true.*

(i) *There exists a constant $c_0 > 0$ such that, for any integer $l \geq 1$, for any subset $C \subset \{1,\ldots,p\}$ such that $i \in C$, and any $t > 0$,*

$$\text{(S.12)}\ \ \mathbb{E}(|\frac{1}{n}\beta^{(l)}_{ii;C} - \frac{1}{n}tr(\Sigma^{-l}_{-C})| > t) \leq 2\exp\left\{-c_0\min\left(\frac{t^2}{nK_0^4}, \frac{t}{K_0^2}\right)\right\}.$$

(ii) *For any integer $r \geq 1$, for any integer $l \geq 1$, for any subset $C \subset \{1,\ldots,p\}$ such that $i \in C$,*

$$\text{(S.13)}\qquad \mathbb{P}\left(|\frac{1}{n}\beta^{(l)}_{ii;C} - \frac{1}{n}tr(\Sigma^{-l}_{-C})|^r\right) \leq C_r K_0^{2r}\left(\frac{4^r}{n^{r/2}} + \frac{(2r)^r}{n^{r-1}}\right).$$

(iii) *For any integer $r \geq 1$, for any integers $k, l \geq 1$, for any subset $C \subset \{1,\ldots,p\}$ such that $i, j \in C$ with $i \neq j$,*

$$\text{(S.14)}\qquad \mathbb{E}\left(|\frac{1}{n}\beta^{(k)}_{ij;C}\beta^{(l)}_{ij;C} - \frac{1}{n}\beta^{(k+l)}_{jj;C}|^r\right)$$
$$\leq\ C_r 2^{r-1}K_0^{2r}(4^r + (2r)^r)\left[1 + C_r K_0^{2r}\left(\frac{4^r}{n^{r/2}} + \frac{(2r)^r}{n^{r-1}}\right)\right].$$

The positive constant C_r in (ii) and (iii) depends only on r.

From (S.10), we obtain

$$\text{(S.15)} \qquad \mathrm{tr}(\Sigma^{-2}) = \mathrm{tr}(\Sigma_{-i}^{-2}) - 2\frac{\bar\beta_{ii;i}^{(3)}}{1+\bar\beta_{ii;i}^{(1)}} + \left(\frac{\bar\beta_{ii;i}^{(2)}}{1+\bar\beta_{ii;i}^{(1)}}\right)^2 .$$

Noticing that $\bar\beta_{ii;i}^{(1)}/(1+\bar\beta_{ii;i}^{(1)}) \le 1$ and $\bar\beta_{ii;i}^{(3)} \le \|\Sigma_{-i}^{-2}\|\bar\beta_{ii;i}^{(1)} \le \bar\beta_{ii;i}^{(1)}$, we have

$$\text{(S.16)} \qquad \max_{1\le i\le p} |\mathrm{tr}(\Sigma_{-i}^{-2}) - \mathrm{tr}(\Sigma^{-2})| = O(1).$$

Define $\mu_i^{(l)} = \bar n^{-1}\mathrm{tr}(\Sigma_{-i}^{-l})$, $l = 1, 2$. By (S.13), we have

$$\text{(S.17)} \qquad \max_{1\le i\le m} |\bar u_i - \mu_i^{(1)}| = O_{\mathrm{P}}(\sqrt{\log p/n}), \qquad \max_{1\le i\le m} |\bar w_i - \mu_i^{(2)}| = O_{\mathrm{P}}(\sqrt{\log p/n}).$$

Since $m/p-\omega = o_{\mathrm{P}}(n^{-1/2})$ and $n/p-\tau = o_{\mathrm{P}}(n^{-1/2})$, we deduce from (S.11), (S.17), (42), (S.16), (54) and (55), by a second order Taylor series expansion, that in order to prove (S.7), it is enough to show that

$$\text{(S.18)} \qquad \frac{1}{m}\sum_{i=1}^m (\bar u_i - \mu_i^{(1)}) = o_{\mathrm{P}}(n^{-1/2}),$$

$$\text{(S.19)} \qquad \frac{1}{m}\sum_{i=1}^m (\bar w_i - \mu_i^{(2)}) = o_{\mathrm{P}}(n^{-1/2}).$$

Observe that, for any $1 \le i \ne j \le m$,

$$\begin{aligned}
&\bar u_i - \mu_i^{(1)} \\
= &\ \frac{1}{\bar n}\zeta_i'\Sigma_{-ij}^{-1}\zeta_i - \frac{1}{\bar n}\mathrm{tr}(\Sigma_{-ij}^{-1}) - \frac{\bar n}{p^2}\gamma\left(\frac{\bar n^{-1}(\zeta_i'\Sigma_{-ij}^{-1}\zeta_j)^2 - \bar n^{-1}\zeta_j'\Sigma_{-ij}^{-2}\zeta_j}{1+\bar\beta_{jj;ij}^{(1)}}\right) \\
= &\ \left(\frac{1}{\bar n}\beta_{ii;ij}^{(1)} - \mu_{ij}^{(1)}\right) - \tau_p\gamma\frac{1}{p}\left(\frac{\bar n^{-1}(\beta_{ij;ij}^{(1)})^2 - \bar n^{-1}\beta_{jj;ij}^{(2)}}{1+\bar\beta_{jj;ij}^{(1)}}\right).
\end{aligned}$$

Let $1 \leq i \neq j \leq m$. Then,

$$
\begin{aligned}
\text{(S.20)} \quad & \mathbb{E}[(\bar{u}_i - \mu_i^{(1)})(\bar{u}_j - \mu_j^{(1)})] \\
= \; & \mathbb{E}[(\bar{n}^{-1}\beta_{ii;ij} - \mu_{ij}^{(1)})(\bar{n}^{-1}\beta_{jj;ij}^{(1)} - \mu_{ij}^{(1)})] \\
& -\tau_p\gamma\frac{1}{p}\mathbb{E}\left(\frac{(\bar{n}^{-1}\beta_{jj;ij}^{(1)} - \mu_{ij}^{(1)})(\bar{n}^{-1}(\beta_{ij;ij}^{(1)})^2 - \bar{n}^{-1}\beta_{jj;ij}^{(2)})}{1+\bar{\beta}_{jj;ij}^{(1)}}\right) \\
& -\tau_p\gamma\frac{1}{p}\mathbb{E}\left(\frac{(\bar{n}^{-1}\beta_{ii;ij}^{(1)} - \mu_{ij}^{(1)})(\bar{n}^{-1}(\beta_{ij;ij}^{(1)})^2 - \bar{n}^{-1}\beta_{ii;ij}^{(2)})}{1+\bar{\beta}_{ii;ij}^{(1)}}\right) \\
& +\tau_p^2\gamma^2\frac{1}{p^2}\mathbb{E}\left(\frac{(\bar{n}^{-1}(\beta_{ij;ij}^{(1)})^2 - \bar{n}^{-1}\beta_{jj;ij}^{(2)})(\bar{n}^{-1}(\beta_{ij;ij}^{(1)})^2 - \bar{n}^{-1}\beta_{ii;ij}^{(2)})}{(1+\bar{\beta}_{jj;ij}^{(1)})(1+\bar{\beta}_{ii;ij}^{(1)})}\right).
\end{aligned}
$$

Notice that, $\mathbb{E}_i(\bar{u}_i) = \bar{n}\mu_i^{(1)}$, $\mathbb{E}_{ij}(\beta_{ij;ij}^{(1)}) = 0$, $\mathbb{E}_i[(\beta_{ij;ij}^{(1)})^2] = \beta_{jj;ij}^{(2)}$ and $\mathbb{E}_{ij}(\beta_{ii;ij}^{(1)}) = \bar{n}\mu_{ij}^{(1)}$. Thus, the first term on the RHS of (S.20) vanishes since, by independence of Z_i and Z_j,

$$
\begin{aligned}
& \mathbb{E}_{ij}[(\bar{n}^{-1}\beta_{ii;ij} - \mu_{ij}^{(1)})(\bar{n}^{-1}\beta_{jj;ij}^{(1)} - \mu_{ij}^{(1)})] \\
= \; & \mathbb{E}_{ij}(\bar{n}^{-1}\beta_{ii;ij} - \mu_{ij}^{(1)})\mathbb{E}_{ij}(\bar{n}^{-1}\beta_{jj;ij}^{(1)} - \mu_{ij}^{(1)}) \; = \; 0.
\end{aligned}
$$

Second term on the RHS of (S.20) vanishes because $1 + \tau_p\gamma\bar{n}^{-1}\beta_{jj;ij}^{(1)}$ is a function of $\{Z_{i'} : i' \neq i\}$ and

$$
\begin{aligned}
& \mathbb{E}_j\left[(\bar{n}^{-1}\beta_{jj;ij}^{(1)} - \mu_{ij}^{(1)})(\bar{n}^{-1}(\beta_{ij;ij}^{(1)})^2 - \bar{n}^{-1}\beta_{jj;ij}^{(2)})\right] \\
= \; & (\bar{n}^{-1}\beta_{jj;ij}^{(1)} - \mu_{ij}^{(1)})\mathbb{E}_j(\bar{n}^{-1}(\beta_{ij;ij}^{(1)})^2 - \bar{n}^{-1}\beta_{jj;ij}^{(2)}) \; = \; 0.
\end{aligned}
$$

Similarly, the third term on the RHS of (S.20) is zero. Finally, since $(1 + \bar{\beta}_{jj;ij}^{(1)}) \geq 1$ and $\|\Sigma_{-ij}^{-1}\| \leq 1$ for all $i \neq j$, by an application of Cauchy-Schwarz inequality, followed by (S.14), we obtain that the last term on the RHS of (S.20) is uniformly (in i, j) bounded by Kp^{-2} for some constant K, and hence

$$
\begin{aligned}
\operatorname{var}(\sum_{i=1}^{m}(\bar{u}_i - \mu_i^{(1)})) \; &= \; \sum_{i=1}^{m}\mathbb{E}[(\bar{u}_i - \mu_i^{(1)})^2] + \sum_{1\leq i\neq j\leq m}\mathbb{E}[(\bar{u}_i - \mu_i^{(1)})(\bar{u}_j - \mu_j^{(1)})] \\
&\leq \; K'\frac{m}{\bar{n}} + K\frac{m(m-1)}{p^2} \; = \; O(1).
\end{aligned}
$$

for some constants $K, K' > 0$. Here, we bounded $\mathrm{E}[(\bar{u}_i - \mu_i^{(1)})^2]$ by making use of (S.13). Consequently, (S.18) is proved.

Now, to prove (S.19), we make use of use of (S.15) and (S.10) to write

$$\begin{aligned}\bar{w}_i - \mu_i^{(2)} &= \left(\frac{1}{\bar{n}}\beta_{ii;ij}^{(2)} - \mu_{ij}^{(2)}\right) - 2\tau_p\gamma\frac{1}{p}\left(\frac{\bar{n}^{-1}\beta_{ij;ij}^{(1)}\beta_{ij;ij}^{(2)} - \bar{n}^{-1}\beta_{jj;ij}^{(3)}}{1+\bar{\beta}_{jj;ij}^{(1)}}\right)\\ &\quad +\gamma\frac{1}{p}\frac{\bar{\beta}_{jj;ij}^{(2)}}{(1+\bar{\beta}_{jj;ij}^{(1)})^2}(\bar{n}^{-1}(\beta_{ij;ij}^{(1)})^2 - \bar{n}^{-1}\beta_{jj;ij}^{(2)})\\ &= \delta_{1,i,ij} + \delta_{2,i,ij} + \delta_{3,i,ij}, \quad \text{say.}\end{aligned}$$

It is easy to see that $\mathbb{E}(\delta_{1,i,ij}\delta_{1,j,ij}) = 0$, $\mathbb{E}(\delta_{1,i,ij}\delta_{2,j,ij}) = 0$ and $\mathbb{E}(\delta_{1,i,ij}\delta_{3,j,ij}) = 0$ for any $i \neq j$. Moreover, $\max_{i\neq j}\max\{\mathbb{E}(\delta_{2,i,ij}^2), \mathbb{E}(\delta_{3,i,ij}^2)\} \leq Kp^{-2}$ for some constant $K > 0$. Thus, the proof of (S.19) follows by arguments similar to those used in proving (S.18).

Next, to prove (S.8), we note that by Lemma S.1,

$$\begin{aligned}\frac{1}{\bar{n}}\mathrm{tr}(\Sigma^{-2}\bar{U}_{(1)}) &= \frac{1}{p\bar{n}}\sum_{i=1}^{m}\zeta_i'\Sigma^{-2}\zeta_i = \frac{1}{p}\sum_{i=1}^{m}\frac{\bar{n}^{-1}\beta_{ii;i}^{(2)}}{(1+\tau_p\gamma\bar{n}^{-1}\beta_{ii;i}^{(1)})^2}\\ &= \frac{1}{p}\sum_{i=1}^{m}\frac{\bar{w}_i}{(1+\tau_p\gamma\bar{u}_i)^2}.\end{aligned}$$

Then (S.8) follows from (S.17), (S.18) and (S.19) using arguments similar to those used above. This completes the proof of Proposition 7.1.

S3.2. *Proof of Proposition S.1.* For proving each of the three statements, we use the fact that for any $C \subset \{1,\ldots,p\}$, $\Sigma_{-C} = I_{n-q} + \gamma\bar{U}_{-C}$, and hence, for any $l \geq 1$, $\|\Sigma_{-C}^{-l}\| \leq \|\Sigma_{-C}^{-1}\|^l \leq 1$. The latter also implies that $\|\Sigma_{-C}^{-l}\|_2 \leq \sqrt{n-q}\|\Sigma_{-C}^{-l}\| \leq \sqrt{n}$. Now, using Proposition 2.1 with $Q = A\Sigma_{-C}^{-l}A'$ and $\boldsymbol{\xi} = Z_i$, and noticing that $A'A = I_{n-q}$, so that, $\|A\| \leq 1$ and $\mathrm{tr}(A\Sigma_{-C}^{-l}A') = \mathrm{tr}(\Sigma_{-C}^{-l})$, we have (S.12).

Next, the definition of the sub-Gaussian norm $\|\cdot\|_{\psi_2}$, and the condition $\|Z_{i'j'}\|_{\psi_2} \leq K_0$ imply that,

$$\max_{1\leq i'\leq n; 1\leq j'\leq p}\mathbb{E}|Z_{i'j'}|^s \leq K_0^s s^{s/2} \qquad \text{for any integer } s \geq 1.$$

So, in particular, $(\max_{i',j'}\mathbb{E}|Z_{i'j'}|^4)^{r/2} \leq K_0^{2r}4^r$ and $\max_{i',j'}\mathbb{E}|Z_{i'j'}|^{2r} \leq K_0^{2r}(2r)^r$. Thus, (S.13) follows from Lemma S.1 by taking $Q = A\Sigma_{-C}^{-l}A'$ and $W = Z_i$, while (S.14) follows from Lemma S.2 by taking $W = Z_i$, $Y = Z_j$, $Q_1 = A\Sigma_{-C}^{-k}A'$ and $Q_2 = A\Sigma_{-C}^{-l}A'$.

Proof of Proposition 3.1 relies heavily on the following lemma, which is obtained by repeated applications of (S.10).

LEMMA S.1. *For any $1 \le i \le p$, $C \subset \{1, \ldots, p\} \setminus \{i\}$, we have*

$$\begin{aligned}
\Sigma_{-C}^{-1}\zeta_i &= \frac{1}{1+\bar{\beta}_{ii;iC}^{(1)}}\Sigma_{-iC}^{-1}\zeta_i \\
\Sigma_{-C}^{-2}\zeta_i &= \frac{1}{1+\bar{\beta}_{ii;iC}^{(1)}}\Sigma_{-iC}^{-2}\zeta_i - \frac{\bar{\beta}_{ii;iC}^{(2)}}{(1+\bar{\beta}_{ii;iC}^{(1)})^2}\Sigma_{-iC}^{-1}\zeta_i \\
\Sigma_{-C}^{-3}\zeta_i &= \frac{1}{1+\bar{\beta}_{ii;iC}^{(1)}}\Sigma_{-iC}^{-3}\zeta_i - \frac{\bar{\beta}_{ii;iC}^{(2)}}{(1+\bar{\beta}_{ii;iC}^{(1)})^2}\Sigma_{-iC}^{-2}\zeta_i \\
&\quad + \left(\frac{(\bar{\beta}_{ii;iC}^{(2)})^2}{(1+\bar{\beta}_{ii;iC}^{(1)})^3} - \frac{\bar{\beta}_{ii;iC}^{(3)}}{(1+\bar{\beta}_{ii;iC}^{(1)})^2}\right)\Sigma_{-iC}^{-1}\zeta_i.
\end{aligned}$$

In general, for any $k \ge 1$,

$$\begin{aligned}
\Sigma_{-C}^{-k}\zeta_i &= \sum_{l=0}^{k-1}\chi_{i;iC}^{(l)}\Sigma_{-iC}^{-(k-l)}\zeta_i, \\
\textit{where,} \quad \chi_{i;iC}^{(l)} &= -\sum_{v=0}^{l-1}\chi_{i;iC}^{(0)}\chi_{i;iC}^{(v)}\bar{\beta}_{ii;iC}^{(l-v+1)}, \quad \textit{for } l \ge 1, \\
\textit{and} \quad \chi_{i;iC}^{(0)} &= \frac{1}{1+\bar{\beta}_{ii;iC}^{(1)}}.
\end{aligned}$$

S3.3. *Proof of Proposition 3.1.* The proof is long and rather tedious. So we just give a brief outline. First, since matrices $\bar{U}$ and $\Sigma = \Sigma_\gamma$ commute and since $\Sigma_{1,0} = I + \gamma_0\bar{U}_{(1)}$,

$$\begin{aligned}
\mathrm{tr}(\Sigma^{-k}\bar{U}^s\Sigma_{1,0}\Sigma^{-l}\bar{U}^t\Sigma_{1,0}) &= \mathrm{tr}(\Sigma^{-(k+l)}\bar{U}^{s+t}) + 2\gamma_0\mathrm{tr}(\Sigma^{-(k+l)}\bar{U}^{s+t}\bar{U}_{(1)}) \\
&\quad + \gamma_0^2\mathrm{tr}(\Sigma^{-k}\bar{U}^s\bar{U}_{(1)}\Sigma^{-l}\bar{U}^t\bar{U}_{(1)})
\end{aligned}$$

The term, $\bar{n}^{-1}\mathrm{tr}(\Sigma^{-(k+l)}\bar{U}^{s+t})$, converges in probability to $h_{k+l,s+t}(\gamma)$ by (54) and (55). Next, since $\bar{U} = -\gamma^{-1}(I - \Sigma)$, by binomial theorem, we have

$$\begin{aligned}
\mathrm{tr}(\Sigma^{-(k+l)}\bar{U}^{s+t}\bar{U}_{(1)}) &= \frac{(-1)^{s+t}}{\gamma^{s+t}}\sum_{j=0}^{s+t}(-1)^j\binom{s+t}{j}\mathrm{tr}(\Sigma^{-(k+l-j)}\bar{U}_{(1)}) \\
\mathrm{tr}(\Sigma^{-k}\bar{U}^s\bar{U}_{(1)}\Sigma^{-l}\bar{U}^t\bar{U}_{(1)}) &= \frac{(-1)^{s+t}}{\gamma^{s+t}}\sum_{j_1=0}^{s}\sum_{j_2=0}^{t}(-1)^{j_1+j_2} \\
&\quad \cdot \binom{s}{j_1}\binom{t}{j_2}\mathrm{tr}(\Sigma^{-(k-j_1)}\bar{U}_{(1)}\Sigma^{-(l-j_2)}\bar{U}_{(1)}).
\end{aligned}$$

So, the result follows once we show the following:

$$\frac{1}{\bar{n}}\mathrm{tr}(\Sigma^{-k}\bar{U}_{(1)}) \xrightarrow{\mathrm{P}} \omega\eta_k(\gamma), \quad k = 1, \ldots, 6. \tag{S.21}$$

$$\frac{1}{\bar{n}}\mathrm{tr}(\Sigma^{-k}\bar{U}_{(1)}\Sigma^{-l}\bar{U}_{(1)}) \xrightarrow{\mathrm{P}} \omega^2\kappa_{k,l}(\gamma) + \tau\omega\eta_k(\gamma)\eta_l(\gamma), \; 1 \le k, l \le 3. \tag{S.22}$$

Proof of (S.21). Since $\bar{U}_{(1)} = p^{-1}\sum_{i=1}^m \zeta_i\zeta_i'$,

$$\frac{1}{\bar{n}}\mathrm{tr}(\Sigma^{-k}\bar{U}_{(1)}) = \frac{1}{p}\sum_{i=1}^m \frac{1}{\bar{n}}\zeta_i'\Sigma^{-k}\zeta_i = \frac{1}{p}\sum_{i=1}^m \frac{1}{\bar{n}}\beta_{ii}^{(k)}.$$

So, the result follows by proving that $\max_{1\le i\le m}|\bar{n}^{-1}\beta_{ii}^{(k)} - \eta_k(\gamma,\tau)| \to 0$ in probability. By Lemma S.1, we can express

$$\begin{aligned}
\beta_{ii}^{(k)} &= \sum_{q_1=1}^k \bar{b}_{q_1;i}^{(k)}\beta_{ii;i}^{(q_1)} \quad k = 1, 2, 3 \\
\beta_{ii}^{(k)} &= \sum_{q_1=1}^3\sum_{q_2=1}^{k-3} \bar{b}_{q_1;i}^{(3)}\bar{b}_{q_2;i}^{(k-3)}\beta_{ii;i}^{(q_1+q_2)}, \quad \text{for } k = 4, 5, 6,
\end{aligned}$$

where

$$\begin{aligned}
\bar{b}_{1;i}^{(1)} &= \frac{1}{1+\bar{\beta}_{ii;i}^{(1)}} \\
\bar{b}_{1;i}^{(2)} &= \frac{-\bar{\beta}_{ii;i}^{(2)}}{(1+\bar{\beta}_{ii;i}^{(1)})^2}, \quad \bar{b}_{2;i}^{(2)} = \frac{1}{1+\bar{\beta}_{ii;i}^{(1)}} \\
\bar{b}_{1;i}^{(3)} &= \frac{(\bar{\beta}_{ii;i}^{(2)})^2}{(1+\bar{\beta}_{ii;i}^{(1)})^3} - \frac{\bar{\beta}_{ii;i}^{(3)}}{(1+\bar{\beta}_{ii;i}^{(1)})^2}, \quad \bar{b}_{2;i}^{(3)} = \frac{-\bar{\beta}_{ii;i}^{(2)}}{(1+\bar{\beta}_{ii;i})^2}, \quad \bar{b}_{3;i}^{(3)} = \frac{1}{1+\bar{\beta}_{ii;i}^{(1)}}.
\end{aligned}$$

Also, by (S.13), (54) and (55), for any $l \ge 1$, $\max_{1\le i\le p}|\bar{n}^{-1}\beta_{ii;i}^{(l)} - h_{l,0}(\gamma)| \to 0$ in probability. Recalling that $\bar{\beta}_{ii;i}^{(l)} = \tau_p\gamma\bar{n}^{-1}\beta_{ii;i}^{(l)}$, the result follows.
Proof of (S.22): We write

$$\begin{aligned}
&\frac{1}{\bar{n}}\mathrm{tr}(\Sigma^{-k}\bar{U}_{(1)}\Sigma^{-l}\bar{U}_{(1)}) \\
=\; &\frac{1}{p^2}\sum_{1\le i\ne j\le m}\frac{1}{\bar{n}}\zeta_i'\Sigma^{-k}\zeta_j\zeta_j'\Sigma^{-l}\zeta_i + \frac{\bar{n}}{p^2}\sum_{i=1}^m\left(\frac{1}{\bar{n}}\zeta_i'\Sigma^{-k}\zeta_i\right)\left(\frac{1}{\bar{n}}\zeta_i'\Sigma^{-l}\zeta_i\right) \\
=\; &\frac{m(m-1)}{p^2}\frac{1}{m(m-1)}\sum_{1\le i\ne j\le m}\frac{1}{\bar{n}}\beta_{ij}^{(k)}\beta_{ij}^{(l)} + \frac{\bar{n}}{p}\frac{m}{p}\frac{1}{m}\sum_{i=1}^m\left(\frac{1}{\bar{n}}\beta_{ii}^{(k)}\right)\left(\frac{1}{\bar{n}}\beta_{ii}^{(l)}\right).
\end{aligned}$$

It is obvious from the proof of (S.21) that the second term in the last line above converges to $\tau\omega\eta_k(\gamma,\tau)\eta_l(\gamma,\tau)$ in probability. Thus, it remains to show that for any $1 \le k, l \le 3$,

$$\frac{1}{m(m-1)} \sum_{1\le i\ne j\le m} \frac{1}{\bar{n}} \beta_{ij}^{(k)} \beta_{ij}^{(l)} \xrightarrow{\mathrm{P}} \kappa_{k,l}(\gamma,\tau). \tag{S.23}$$

As a first step, applying Lemma S.1 twice, we obtain that for any $i \ne j$ and $k \ge 1$,

$$\beta_{ij}^{(k)} = \sum_{q_1=1}^{k} \bar{a}_{q_1;ij}^{(k)} \beta_{ij;ij}^{(q_1)},$$

where

$$\begin{aligned}
\bar{a}_{1;ij}^{(1)} &= \frac{1}{(1+\bar{\beta}_{ii;i}^{(1)})(1+\bar{\beta}_{jj;ij}^{(1)})} \\
\bar{a}_{1;ij}^{(2)} &= \frac{\bar{\beta}_{ii;i}^{(2)}}{(1+\bar{\beta}_{ii;i}^{(1)})^2(1+\bar{\beta}_{jj;ij}^{(1)})} + \frac{\bar{\beta}_{jj;ij}^{(2)}}{(1+\bar{\beta}_{ii;i}^{(1)})(1+\bar{\beta}_{jj;ij}^{(1)})^2}, \\
\bar{a}_{2;ij}^{(2)} &= \bar{a}_{1;ij}^{(1)} \\
\bar{a}_{1;ij}^{(3)} &= \frac{1}{1+\bar{\beta}_{jj;ij}^{(1)}} \left(\frac{(\bar{\beta}_{ii;i}^{(2)})^2}{(1+\bar{\beta}_{ii;i}^{(1)})^3} - \frac{\bar{\beta}_{ii;i}^{(3)}}{(1+\bar{\beta}_{ii;i}^{(1)})^2} \right) \\
&\quad + \frac{1}{1+\bar{\beta}_{ii;i}^{(1)}} \left(\frac{(\bar{\beta}_{jj;ij}^{(2)})^2}{(1+\bar{\beta}_{jj;ij}^{(1)})^3} - \frac{\bar{\beta}_{jj;ij}^{(3)}}{(1+\bar{\beta}_{jj;ij}^{(1)})^2} \right) \\
&\quad + \frac{\bar{\beta}_{ii;i}^{(2)} \bar{\beta}_{jj;ij}^{(2)}}{(1+\bar{\beta}_{ii;i}^{(1)})^2(1+\bar{\beta}_{jj;ij}^{(1)})^2} \\
\bar{a}_{2;ij}^{(3)} &= \bar{a}_{1;ij}^{(2)}, \quad \bar{a}_{3;ij}^{(3)} = \bar{a}_{1;ij}^{(1)}.
\end{aligned}$$

By recalling that $\bar{\beta}_{ii;C} = \tau_p \gamma \bar{n}^{-1} \beta_{ii;C}$ for any $C \subset \{1,\ldots,p\}$ and using (S.12), (54) and (55), it follows that

$$\max_{1\le i\ne j\le p} \max_{1\le l\le 3} \max_{1\le q_1\le l} |\bar{a}_{q_1;ij}^{(l)} - \bar{a}_{q_1}^{(l)}(\gamma,\tau)| = O_{\mathrm{P}}\left(\sqrt{\frac{\log p}{n}}\right).$$

As a result, it is easily seen that, for $1 \leq k, l \leq 3$,

$$\begin{aligned} & \frac{1}{m(m-1)} \sum_{1 \leq i \neq j \leq m} \frac{1}{\bar{n}} \beta_{ij}^{(k)} \beta_{ij}^{(l)} \\ = & \sum_{q_1=1}^{k} \sum_{q_2=1}^{l} \bar{a}_{q_1}^{(k)}(\gamma,\tau) \bar{a}_{q_2}^{(l)}(\gamma,\tau) \left[\frac{1}{m(m-1)} \sum_{1 \leq i \neq j \leq m} \frac{1}{\bar{n}} \beta_{ij;ij}^{(q_1)} \beta_{ij;ij}^{(q_2)} \right] + o_{\mathrm{P}}(1). \end{aligned}$$

Also, $\bar{n}^{-1}\mathrm{tr}(\Sigma^{-l}) \to h_{l,0}(\gamma)$ for all $l \geq 1$. Moreover, since $\|\Sigma_{-i}^{-1}\| \leq 1$, and $\Sigma^{-1} - \Sigma_{-i}^{-1}$ is a rank one matrix with norm ≤ 1, it follows that $|\mathrm{tr}(\Sigma^{-l}) - \mathrm{tr}(\Sigma_{-i}^{-l})| \leq 2^l - 1$. Applying it again, with Σ replaced by Σ_{-i} and Σ_{-i} replaced by Σ_{-ij}, and combining with the previous bound, we obtain $\bar{n}^{-1}|\mathrm{tr}(\Sigma^{-l}) - \mathrm{tr}(\Sigma_{-ij}^{-l})| \leq \bar{n}^{-1}2^{l+1} \to 0$ uniformly in i, j. So the task of proving (S.22) reduces to showing that for any $1 \leq q_1, q_2 \leq 3$,

$$\text{(S.24)} \qquad \frac{1}{m(m-1)} \sum_{1 \leq i \neq j \leq m} \left(\frac{1}{\bar{n}} \beta_{ij;ij}^{(q_1)} \beta_{ij;ij}^{(q_2)} - \mu_{ij}^{(q_1+q_2)} \right) \xrightarrow{\mathrm{P}} 0,$$

where, it may be recalled, $\mu_C^{(l)} = \bar{n}^{-1}\mathrm{tr}(\Sigma_{-C}^{-l})$.

For convenience, define $\delta_{ij}^{(q_1,q_2)} := \bar{n}^{-1}\beta_{ij;ij}^{(q_1)}\beta_{ij;ij}^{(q_2)} - \mu_{ij}^{(q_1+q_2)}$. Then, observe that by (S.14), for some constant $K_1 > 0$,

$$\text{(S.25)} \qquad \max_{1 \leq i \neq j \leq p} \mathbb{E}\left[(\delta_{ij}^{(q_1+q_2)})^2 \right] \leq K_1.$$

Next, for any $l \geq 1$, and $i \neq j \neq i'$ we can make use of (S.10) to write $\bar{n}^{-1}\beta_{ij;ij}^{(l)} = \bar{n}^{-1}\beta_{ij;iji'}^{(l)} + e_{ij;iji'}^{(l)}$ and $\mu_{ij}^{(l)} = \mu_{iji'}^{(l)} + \tilde{e}_{ij;iji'}^{(l)}$, for appropriate $e_{ij;iji'}^{(l)}$ and $\tilde{e}_{ij;iji'}^{(l)}$. Similarly, noticing that for $i', j' \notin C \subset \{1, \ldots, p\}$,

$$\begin{aligned} \Sigma_{-C}^{-1} = \; & \Sigma_{-i'j'C}^{-1} - \frac{1}{d_{i'j';C}} \Big[(1 + \bar{\beta}_{j'j';i'j'C}^{(1)}) \Sigma_{-i'j'C}^{-1} \bar{\zeta}_{i'} \bar{\zeta}_{i'}' \Sigma_{-i'j'C}^{-1} \\ & + (1 + \bar{\beta}_{i'i';i'j'C}^{(1)}) \Sigma_{-i'j'C}^{-1} \bar{\zeta}_{j'} \bar{\zeta}_{j'}' \Sigma_{-i'j'C}^{-1} \\ & - \bar{\beta}_{i'j';i'j'C}^{(1)} (\Sigma_{-i'j'C}^{-1} \bar{\zeta}_{i'} \bar{\zeta}_{j'}' \Sigma_{-i'j'C}^{-1} + \Sigma_{-i'j'C}^{-1} \bar{\zeta}_{j'} \bar{\zeta}_{i'}') \Sigma_{-i'j'C}^{-1} \Big], \end{aligned}$$

where $d_{i'j';C} = (1 + \bar{\beta}_{i'i';i'j'C}^{(1)})(1 + \bar{\beta}_{j'j';i'j'C}^{(1)}) - (\bar{\beta}_{i'j';i'j'C}^{(1)})^2 \geq 1$.

Using this, for $i \neq j \neq i' \neq j'$, and $l \geq 1$, we can write $\beta_{ij;ij}^{(l)} = \beta_{ij;iji'j'}^{(l)} + e_{ij;iji'j'}^{(l)}$ and $\mu_{ij}^{(l)} = \mu_{iji'j'}^{(l)} + \tilde{e}_{ij;iji'j'}^{(l)}$. It can be seen that $e_{ij;iji'}^{(l)}$, $\tilde{e}_{ij;iji'}^{(l)}$, $e_{ij;iji'j'}^{(l)}$ and $\tilde{e}_{ij;iji'j'}^{(l)}$ are lower order term in the sense that contributions of any term involving one of these to the expectations $\mathbb{E}(\bar{\delta}_{ij}^{(q_1,q_2)}\bar{\delta}_{i'j'}^{(q_1,q_2)})$

are asymptotically negligible when $i \neq j$, $i' \neq j'$ and either $i \neq i', j'$ or $j \neq i', j'$. The latter statement can be formally verified by using Hölder's inequality together with (S.13) and (S.14), with a sufficiently large value of r.

Furthermore, by independence of the columns of Z, we have, for $i \neq j \neq i'$,

$$\begin{aligned} &\mathbb{E}\left[\left(\frac{1}{\bar{n}}\beta^{(q_1)}_{ij;iji'}\beta^{(q_2)}_{ij;iji'} - \mu^{(q_1+q_2)}_{iji'}\right)\left(\frac{1}{\bar{n}}\beta^{(q_1)}_{i'j;iji'}\beta^{(q_2)}_{i'j;iji'} - \mu^{(q_1+q_2)}_{iji'}\right)\right] \\ = \;& \mathbb{E}\left[\left(\frac{1}{\bar{n}}\beta^{(q_1+q_2)}_{jj;iji'} - \mu^{(q_1+q_2)}_{iji'}\right)^2\right] \;\leq\; K_2 n^{-1} \end{aligned}$$

for some constant $K_2 > 0$, while for $i \neq j \neq i' \neq j'$,

$$\mathbb{E}\left[\left(\frac{1}{\bar{n}}\beta^{(q_1)}_{ij;iji'j'}\beta^{(q_2)}_{ij;iji'j'} - \mu^{(q_1+q_2)}_{iji'j'}\right)\left(\frac{1}{\bar{n}}\beta^{(q_1)}_{i'j';iji'j'}\beta^{(q_2)}_{i'j';iji'j'} - \mu^{(q_1+q_2)}_{iji'j'}\right)\right] \;=\; 0.$$

Combining these facts, we conclude that

$$\frac{1}{m^2(m-1)^2}\sum_{1\leq i\neq j\neq i'} \mathbb{E}\left(\delta^{(q_1,q_2)}_{ij}\delta^{(q_1,q_2)}_{i'j}\right) \to 0, \tag{S.26}$$

and

$$\frac{1}{m^2(m-1)^2}\sum_{1\leq i\neq j\neq i'\neq j'} \mathbb{E}\left(\delta^{(q_1,q_2)}_{ij}\delta^{(q_1,q_2)}_{i'j'}\right) \to 0. \tag{S.27}$$

Now, (S.24) follows by combining (S.25), (S.26) and (S.27), and thus the proof of (S.22) is complete.