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# Tail Probability Ratios of Normal and Student's $t$ Distributions

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## Abstract

The ratio of normal tail probabilities and the ratio of Student's  $t$  tail probabilities have gained an increased attention in statistics and related areas. However, they are not well studied in the literature. In this paper, we systematically study the functional behaviors of these two ratios. Meanwhile, we explore their difference as well as their relationship. It is surprising that the two ratios behave very different to each other. Finally, we conclude the paper by conducting some lower and upper bounds for the two ratios.

KEY WORDS: Lower bound; Normal distribution; Student's  $t$  distribution; Tail probability ratio; Upper bound

# 1 Introduction

Let  $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$  be the probability density function, and  $\bar{\Phi}(x) = \int_x^\infty \phi(t)dt$  be the tail probability of the standard normal distribution. The ratio  $\bar{\Phi}(x)/\phi(x)$  is known as Mills' ratio, which was well studied in the literature. See, for example, Birnbaum (1942), Sampford (1953), Boyd (1959), Ruben (1962), Ruben (1964), Rabinowitz (1969), Kerridge & Cook (1976), Bryc (2002), and Yun (2009). It is also known that Mills' ratio was extended to explore the Student's  $t$  distribution (Soms 1976, Soms 1980, Shao 1999, Finner, Dickhaus & Roters 2008).

Without loss of generality, let  $a > 0$  be a given constant. We define

$$R(x; a) = \bar{\Phi}(x + a)/\bar{\Phi}(x), \quad -\infty < x < \infty,$$

to be the ratio of two normal tail probabilities. The ratio  $R(x; a)$  has gained an increased attention in statistics and related areas, with typical examples including: (1) Assuming that  $X \sim N(0, 1)$ , the ratio  $R(x; a)$  is the conditional probability of  $X > x + a$  given that  $X > x$ . That is,  $R(x; a)$  can be referred to the hazard rate of the standard normal distribution; (2) Tusnády's lemma plays an important role in the KMT/Hungarian coupling of the empirical distribution function with a Brownian bridge. In Carter & Pollard (2004), the authors proposed some novel bounds for  $R(x; a)$  and then successfully applied them to sharpen the Tusnády's inequality; (3)  $R(x; a)$  was referred to as the *expected adverse impact* in Aguinis & Smith (2007) for the purpose of comparing the minority to the majority group; And (4)  $R(x; a)$  was also employed by Yuan, Tong & Ng (2011) to achieve the conditional type I error rate for superiority test conditioned on establishment of non-inferiority in clinical trials. In clinical trial studies, it is often desirable to test superiority conditioned on establishment of non-inferiority based on the same primary endpoint. Ng (2003) pointed out that switching between non-inferiority and superiority

without any adjustment has a logic flaw similar to a post-hoc specification for the null hypothesis, even though the family-wise type I error rate is not inflated. To overcome this problem, Yuan et al. (2011) proposed to control the conditional type I error rate of the second-step superiority test at the nominal significance level, which leads to a much lower significance level for the second-step superiority test. Nevertheless, to derive the conditional type I error rate of the second-step superiority test, we require  $R(x; a)$  to be a monotonically decreasing function of  $x$  on the entire line.

In contrast to Mills' ratio, to the best of our knowledge, little has been studied in the literature for the ratio  $R(x; a)$ . This motivates us to systematically study the functional behaviors of  $R(x; a)$  in this paper, including the curve behavior as well as the corresponding lower and upper bounds. Meanwhile, we will study another closely related ratio, namely, the ratio of two Student's  $t$  tail probabilities. Specifically, for any  $a > 0$  and  $\nu > 1$ , we define

$$r_\nu(x; a) = \overline{F}_\nu(x + a)/\overline{F}_\nu(x), \quad -\infty < x < \infty,$$

where

$$f_\nu(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} (1 + t^2/\nu)^{-\frac{\nu+1}{2}}, \quad \text{and} \quad \overline{F}_\nu(x) = \int_x^\infty f_\nu(t)dt,$$

are the probability density function and the tail probability of the Student's  $t$  distribution with  $\nu$  degrees of freedom, respectively.

The remainder of the paper is organized as follows. In Section 2, we explore the curve behaviors of  $R(x; a)$  and  $r_\nu(x; a)$  as a function of  $x$ . In Section 3, we discuss the relationship between the two ratios from an asymptotic point of view. We then conduct inequalities in Sections 4 and 5 that provide the lower and upper bounds for  $R(x; a)$  and  $r_\nu(x; a)$ , respectively, on the entire line of  $x$ .

## 2 Functional Behaviors of $R(x; a)$ and $r_\nu(x; a)$

**Theorem 1.** For the ratio  $R(x; a)$  with  $a > 0$ , we have

- (1)  $\lim_{x \rightarrow \infty} R(x; a) = 0$ ;
- (2)  $\lim_{x \rightarrow -\infty} R(x; a) = 1$ ;
- (3)  $R(x; a)$  is a monotonically decreasing function of  $x$  on  $x \in \mathbb{R}$ .

**Proof.** (1) Noting that for any  $a > 0$ , both  $\lim_{x \rightarrow \infty} \bar{\Phi}(x + a) = 0$  and  $\lim_{x \rightarrow \infty} \bar{\Phi}(x) = 0$ .

Then by L'Hôpital's rule (Abramowitz & Stegun 1972) we have

$$\lim_{x \rightarrow \infty} R(x; a) = \lim_{x \rightarrow \infty} \frac{\bar{\Phi}'(x + a)}{\bar{\Phi}'(x)} = \lim_{x \rightarrow \infty} \frac{-e^{-(x+a)^2/2}}{-e^{-x^2/2}} = 0,$$

where  $\bar{\Phi}'(x)$  denotes the first derivative of  $\bar{\Phi}(x)$  on  $x$ .

(2) Note that  $\lim_{x \rightarrow -\infty} \bar{\Phi}(x) = 1 > 0$  and  $\lim_{x \rightarrow -\infty} \bar{\Phi}(x + a) = 1$ . We have

$$\lim_{x \rightarrow -\infty} R(x; a) = \frac{\lim_{x \rightarrow -\infty} \bar{\Phi}(x + a)}{\lim_{x \rightarrow -\infty} \bar{\Phi}(x)} = 1.$$

(3) The first derivative of  $R(x; a)$  is

$$\begin{aligned} R'(x; a) &= \frac{\left(\int_{x+a}^{\infty} e^{-t^2/2} dt\right)' \left(\int_x^{\infty} e^{-t^2/2} dt\right) - \left(\int_{x+a}^{\infty} e^{-t^2/2} dt\right) \left(\int_x^{\infty} e^{-t^2/2} dt\right)'}{\left(\int_x^{\infty} e^{-t^2/2} dt\right)^2} \\ &= \frac{-e^{-(x+a)^2/2} \int_x^{\infty} e^{-t^2/2} dt + e^{-x^2/2} \int_{x+a}^{\infty} e^{-t^2/2} dt}{\left(\int_x^{\infty} e^{-t^2/2} dt\right)^2}. \end{aligned} \quad (1)$$

To show that  $R(x; a)$  is a monotonically decreasing function of  $x$ , it suffices to show that  $R'(x; a) \leq 0$  for any  $x \in \mathbb{R}$ . Or equivalently, by (1) it suffices to show that

$$e^{(x+a)^2/2} \int_{x+a}^{\infty} e^{-t^2/2} dt \leq e^{x^2/2} \int_x^{\infty} e^{-t^2/2} dt, \quad -\infty < x < \infty. \quad (2)$$

For ease of notation, we define  $g(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2} dt$ . By the Newton-Leibnitz formula,

$$\begin{aligned} g'(x) &= xe^{x^2/2} \int_x^\infty e^{-t^2/2} dt + e^{x^2/2} \left(-e^{-x^2/2}\right) \\ &= xe^{x^2/2} \int_x^\infty e^{-t^2/2} dt - 1. \end{aligned}$$

When  $x \leq 0$ , we have  $g'(x) < 0$ . When  $x > 0$ , by the result of Gordon (1941),

$$\int_x^\infty e^{-t^2/2} dt \leq \frac{1}{x} e^{-x^2/2},$$

we still have  $g'(x) \leq 0$ . This implies that  $g(x)$  is a monotonically decrease function of  $x$  on  $x \in \mathbb{R}$ . Therefore,  $g(x+a) \leq g(x)$ , i.e., (2) holds. This completes the proof.

**Theorem 2.** For the ratio  $r_\nu(x; a)$  with  $a > 0$  and  $\nu > 0$ , we have

$$(1) \lim_{x \rightarrow \infty} r_\nu(x; a) = \lim_{x \rightarrow -\infty} r_\nu(x; a) = 1;$$

(2)  $r_\nu(x; a)$  is a monotonically increasing function of  $x$  on  $[\sqrt{\nu}, \infty)$ .

**Proof.** (1) For ease of notation, we define  $k_\nu = \Gamma\left(\frac{\nu+1}{2}\right)/(\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right))$ . Given that  $\lim_{x \rightarrow \infty} \overline{F}_\nu(x+a) = \lim_{x \rightarrow \infty} \overline{F}_\nu(x) = 0$  for any  $a > 0$ , by the L'Hôpital's rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} r_\nu(x; a) &= \lim_{x \rightarrow \infty} \frac{\overline{F}'_\nu(x+a)}{\overline{F}'_\nu(x)} \\ &= \lim_{x \rightarrow \infty} \frac{-k_\nu \cdot (1+(x+a)^2/\nu)^{-\frac{\nu+1}{2}}}{-k_\nu \cdot (1+x^2/\nu)^{-\frac{\nu+1}{2}}} \\ &= \lim_{x \rightarrow \infty} \left( \frac{x^2 + 2ax + a^2 + \nu}{x^2 + \nu} \right)^{-\frac{\nu+1}{2}} \\ &= 1. \end{aligned}$$

On the other side, given that  $\lim_{x \rightarrow -\infty} \overline{F}_\nu(x+a) = \lim_{x \rightarrow -\infty} \overline{F}_\nu(x) = 1$ , we have

$$\lim_{x \rightarrow -\infty} r_\nu(x; a) = \frac{\lim_{x \rightarrow -\infty} \overline{F}_\nu(x+a)}{\lim_{x \rightarrow -\infty} \overline{F}_\nu(x)} = 1.$$

(2) The first derivative of  $r_\nu(x; a)$  is given as

$$r'_\nu(x) = \frac{g(x)}{\left[ \int_x^\infty (1 + t^2/\nu)^{-\frac{\nu+1}{2}} dt \right]^2},$$

where

$$g(x) = \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \int_{x+a}^\infty \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} dt - \left(1 + \frac{(x+a)^2}{\nu}\right)^{-\frac{\nu+1}{2}} \int_x^\infty \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} dt.$$

Let  $C(x) = (1 + x^2/\nu)^{-\frac{\nu+1}{2}} (1 + (x+a)^2/\nu)^{-\frac{\nu+1}{2}}$ . With some algebra, we have

$$\begin{aligned} g(x) &= C(x) \int_x^\infty \left\{ \left[ \frac{1 + (x+a)^2/\nu}{1 + (t+a)^2/\nu} \right]^{\frac{\nu+1}{2}} - \left[ \frac{1 + x^2/\nu}{1 + t^2/\nu} \right]^{\frac{\nu+1}{2}} \right\} dt \\ &= C(x) \int_x^\infty \left\{ [h(a)]^{\frac{\nu+1}{2}} - [h(0)]^{\frac{\nu+1}{2}} \right\} dt, \end{aligned}$$

where  $h(\cdot)$  is defined in Lemma 1. When  $x \geq \sqrt{\nu}$ , by Lemma 1 we know that  $h(y)$  is a monotonically increasing function of  $y$  on  $[0, \infty)$ . This implies that  $h(a) > h(0)$  and thus  $g(x) > 0$ . Therefore,  $r_\nu(x; a)$  is a monotonically increasing function of  $x$  on  $[\sqrt{\nu}, \infty)$ .

**Lemma 1.** Assume that  $\nu > 0$  and  $t \geq x$ . Define

$$h(y) = \frac{1 + (x+y)^2/\nu}{1 + (t+y)^2/\nu}. \quad (3)$$

Then  $h(y)$  is a monotonically increasing function of  $y$  on  $[\sqrt{\nu} - x, \infty)$ .

**Proof.** The first derivative of  $h(y)$  is

$$\begin{aligned} h'(y) &= \frac{2(x+y)[\nu + (t+y)^2] - 2(t+y)[\nu + (x+y)^2]}{[\nu + (t+y)^2]^2} \\ &= \frac{2(t-x)[(x+y)(t+y) - \nu]}{[\nu + (t+y)^2]^2}. \end{aligned}$$

When  $y > \sqrt{\nu} - x$ , we have  $(x+y)(t+y) \geq (x+y)^2 \geq \nu$  by the fact that  $t \geq x$ . Further,  $h'(y) \geq 0$ . This shows that  $h(y)$  is a monotonically increasing function of  $y$  on  $[\sqrt{\nu} - x, \infty)$ .

From Theorems 1 and 2, we observe that the two ratios behave very different to each other. For visualization, we set  $a = 1$  and plot the curve of  $R(x; 1)$  in Figure 1, as well as the curves of  $r_\nu(x; 1)$  with various  $\nu$  values.

### 3 Relationship between $R(x; a)$ and $r_\nu(x; a)$

In this section we investigate the relationship between the two ratios from the asymptotic point of view. Specifically, we have the following asymptotic result for  $r_\nu(x; a)$  with respect to the degrees of freedom  $\nu$ .

**Theorem 3.** *For any fixed  $x \in (-\infty, \infty)$ , we have*

$$\lim_{\nu \rightarrow \infty} r_\nu(x; a) = R(x; a).$$

**Proof.** Let  $m_\nu(t) = (1 + t^2/\nu)^{-\frac{\nu+1}{2}}$ . For any fixed  $\nu \geq 1$ , by Lemma 2 we have

$$|m_\nu(t)| \leq \left(1 + \frac{t^2}{\nu+1}\right)^{-\frac{\nu+1}{2}} \leq \left(1 + \frac{t^2}{2}\right)^{-1} = n(t), \quad -\infty < t < \infty,$$

where  $n(t) = (1 + t^2/2)^{-1}$ . Note that  $n(t)$  is integrable on the interval  $[x, \infty)$ , i.e., for any  $x$  value,  $\int_x^\infty |n(t)| dt < \infty$ . Thus by the dominated convergence theorem, the integration is interchangeable with the limit operation for the sequence  $m_\nu(t)$ . That is,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \int_x^\infty (1 + t^2/\nu)^{-\frac{\nu+1}{2}} dt &= \int_x^\infty \lim_{\nu \rightarrow \infty} (1 + t^2/\nu)^{-\frac{\nu+1}{2}} dt \\ &= \int_x^\infty e^{-t^2/2} dt. \end{aligned}$$



Further, since the limit of the denominator,  $\int_x^\infty e^{-t^2/2} dt$ , is non-zero for any fixed  $x \in (-\infty, \infty)$ , we have

$$\begin{aligned} \lim_{\nu \rightarrow \infty} r_\nu(x; a) &= \frac{\lim_{\nu \rightarrow \infty} \int_{x+a}^\infty (1+t^2/\nu)^{-\frac{\nu+1}{2}} dt}{\lim_{\nu \rightarrow \infty} \int_x^\infty (1+t^2/\nu)^{-\frac{\nu+1}{2}} dt} \\ &= \frac{\int_{x+a}^\infty e^{-t^2/2} dt}{\int_x^\infty e^{-t^2/2} dt} \\ &= R(x; a). \end{aligned}$$

This completes the proof.

**Lemma 2.** *Let  $s(y) = (1 + t^2/y)^{-\frac{y}{2}}$ , where  $t$  is a constant. Then,  $s(y)$  is a strictly decreasing function of  $y$  on  $(0, \infty)$ .*

**Proof.** The first derivative of  $s(y)$  is

$$s'(y) = -\frac{1}{2} \left(1 + \frac{t^2}{y}\right)^{-\frac{y}{2}} \left[ \log \left(1 + \frac{t^2}{y}\right) - \frac{t^2/y}{1 + t^2/y} \right].$$

Let

$$r(y) = \log \left(1 + \frac{t^2}{y}\right) - \frac{t^2/y}{1 + t^2/y}.$$

Note that  $\lim_{y \rightarrow \infty} r(y) = 0$ , and

$$r'(y) = \frac{-t^2/y^2}{1 + t^2/y} + \frac{t^2/y^2}{(1 + t^2/y)^2} = \frac{-t^4/y^3}{(1 + t^2/y)^2} \leq 0, \quad \forall y > 0.$$

We have  $r(y) \geq 0$  for any  $y > 0$ . This implies that  $s'(y) \leq 0$  for any  $y > 0$ . Therefore,  $s(y)$  is a strictly decreasing function of  $y$  on  $(0, \infty)$ .

## 4 Bounds of $R(x; a)$

For the bounds of  $\bar{\Phi}(x)$ , a seminal inequality was given by Mills (1926),

$$\left(\frac{1}{x} - \frac{1}{x^3}\right) \phi(x) < \bar{\Phi}(x) < \frac{1}{x} \phi(x), \quad x > 1. \quad (4)$$

It is known that (4) has been improved in the literature, e.g., in Birnbaum (1942) and Sampford (1953). Let  $p(x, \gamma) = (\gamma + 1)/\{x^2 + (2/\pi)(\gamma + 1)^2\}^{1/2} + \gamma x\}$ . Boyd (1959) proposed the following inequality for  $\bar{\Phi}(x)$ ,

$$p(x, \gamma_{\min})\phi(x) < \bar{\Phi}(x) < p(x, \gamma_{\max})\phi(x), \quad x > 0, \quad (5)$$

where  $\gamma_{\max} = 2/(\pi - 2)$  and  $\gamma_{\min} = \pi - 1$ . It can be shown that  $p(x, \gamma_{\min}) \geq 1/x - 1/x^3$  and  $p(x, \gamma_{\max}) \leq 1/x$ . Therefore, (5) gives more accurate bounds for  $\bar{\Phi}(x)$  compared to (4). By (5), we have the following bounds for  $R(x; a)$ ,

$$\frac{p(x+a, \gamma_{\min})\phi(x+a)}{p(x, \gamma_{\max})\phi(x)} < R(x; a) < \frac{p(x+a, \gamma_{\max})\phi(x+a)}{p(x, \gamma_{\min})\phi(x)}, \quad x > 0. \quad (6)$$

Recently, Carter & Pollard (2004) proposed another two important inequalities for  $R(x; a)$  that were used to sharpen the Tusnády inequality:

$$\exp[-a\rho(x+a)] \leq R(x; a) \leq \exp[-a\rho(x)], \quad (7)$$

$$\exp[-a\rho(x) - a^2/2] \leq R(x; a) \leq \exp[-a\rho(x+a) + a^2/2], \quad (8)$$

where  $\rho(x) = \phi(x)/\bar{\Phi}(x)$ .

Note that both the lower and upper bounds in (7) and (8) depend on the term  $\bar{\Phi}(\cdot)$  itself through  $\rho(x)$ . To have inequality bounds of  $R(x; a)$  independent of  $\bar{\Phi}(\cdot)$ , by the fact that

$$\frac{1}{p(x, \gamma_{\max})} < \rho(x) < \frac{1}{p(x, \gamma_{\min})}, \quad x > 0,$$

we propose the following two new inequalities,

$$\exp\left[-\frac{a}{p(x+a, \gamma_{\min})}\right] \leq R(x; a) \leq \exp\left[-\frac{a}{p(x, \gamma_{\max})}\right], \quad x > 0, \quad (9)$$

$$\exp\left[-\frac{a}{p(x, \gamma_{\min})} - \frac{a^2}{2}\right] \leq R(x; a) \leq \exp\left[-\frac{a}{p(x+a, \gamma_{\max})} + \frac{a^2}{2}\right], \quad x > 0. \quad (10)$$

In Figure 2, we compare the performance of the proposed three inequalities, (6), (9) and (10), under various settings. We observe that the bounds in (6) performs better when  $a$  is large. In other cases, the bounds in (9) or (10) performs better. We also observe that the bounds in (9) and (10) are comparable in general.

Combining (6), (9) and (10), we propose the following compounded bounds for  $R(x; a)$ ,

$$L_1(x, a) \leq R(x; a) \leq U_1(x, a), \quad x > 0, \quad (11)$$

where

$$L_1(x, a) = \max \left\{ \frac{p(x+a, \gamma_{\min})\phi(x+a)}{p(x, \gamma_{\max})\phi(x)}, \exp \left[ \frac{-a}{p(x+a, \gamma_{\min})} \right], \exp \left[ \frac{-a}{p(x, \gamma_{\min})} - \frac{a^2}{2} \right] \right\},$$

$$U_1(x, a) = \min \left\{ \frac{p(x+a, \gamma_{\max})\phi(x+a)}{p(x, \gamma_{\min})\phi(x)}, \exp \left[ \frac{-a}{p(x, \gamma_{\max})} \right], \exp \left[ \frac{-a}{p(x+a, \gamma_{\max})} + \frac{a^2}{2} \right] \right\}.$$

Unlike most of the results on Mills' ratio that hold only for  $x > 0$ , for the ratio  $R(x; a)$  we can derive the lower and upper bounds on the entire line of  $x$ . Note that  $\bar{\Phi}(x) = 1 - \bar{\Phi}(-x)$  and  $\phi(-x) = \phi(x)$ . For  $x \leq 0$ , we have  $\rho(x) = \phi(x)/(1 - \bar{\Phi}(-x))$  and by (5),

$$p(-x, \gamma_{\min})\phi(x) < \bar{\Phi}(-x) < p(-x, \gamma_{\max})\phi(x), \quad x \leq 0. \quad (12)$$

This leads to

$$\frac{\phi(x)}{1 - p(-x, \gamma_{\min})\phi(x)} \leq \rho(x) \leq \frac{\phi(x)}{1 - p(-x, \gamma_{\max})\phi(x)}, \quad x \leq 0. \quad (13)$$

With (12) and (13), in what follows we present the bounds of  $R(x; a)$  for  $x \leq 0$ . (i) When  $-a < x \leq 0$ , noting that  $R(x; a) = \bar{\Phi}(x+a)/[1 - \bar{\Phi}(-x)]$ , we have

$$L_2(x, a) \leq R(x; a) \leq U_2(x, a), \quad -a < x \leq 0,$$

where

$$L_2(x, a) = \max \left\{ \frac{p(x+a, \gamma_{\min})\phi(x+a)}{1-p(-x, \gamma_{\min})\phi(x)}, \exp \left[ \frac{-a}{p(x+a, \gamma_{\min})} \right], \right. \\ \left. \exp \left[ \frac{-a\phi(x)}{1-p(-x, \gamma_{\max})\phi(x)} - \frac{a^2}{2} \right] \right\},$$

$$U_2(x, a) = \min \left\{ \frac{p(x+a, \gamma_{\max})\phi(x+a)}{1-p(-x, \gamma_{\max})\phi(x)}, \exp \left[ \frac{-a\phi(x)}{1-p(-x, \gamma_{\min})\phi(x)} \right], \right. \\ \left. \exp \left[ \frac{-a}{p(x+a, \gamma_{\max})} + \frac{a^2}{2} \right] \right\}.$$

(ii) When  $x \leq -a$ , noting that  $R(x; a) = [1 - \bar{\Phi}(-x - a)]/[1 - \bar{\Phi}(-x)]$ , we have

$$L_3(x, a) \leq R(x; a) \leq U_3(x, a), \quad x \leq -a,$$

where

$$L_3(x, a) = \max \left\{ \frac{1-p(-x-a, \gamma_{\max})\phi(x+a)}{1-p(-x, \gamma_{\min})\phi(x)}, \exp \left[ \frac{-a\phi(x+a)}{1-p(-x-a, \gamma_{\max})\phi(x+a)} \right], \right. \\ \left. \exp \left[ \frac{-a\phi(x)}{1-p(-x, \gamma_{\max})\phi(x)} - \frac{a^2}{2} \right] \right\},$$

$$U_3(x, a) = \min \left\{ \frac{1-p(-x-a, \gamma_{\min})\phi(x+a)}{1-p(-x, \gamma_{\max})\phi(x)}, \exp \left[ \frac{-a\phi(x)}{1-p(-x, \gamma_{\min})\phi(x)} \right], \right. \\ \left. \exp \left[ \frac{-a\phi(x+a)}{1-p(-x-a, \gamma_{\min})\phi(x+a)} + \frac{a^2}{2} \right] \right\}.$$

As an illustration, we present in Figure 3 the true ratio  $R(x; a)$  and its lower and upper bounds on the entire line of  $x$  for  $a = 0.5$ . The visualized plot shows that the proposed bounds are relatively accurate on the whole range.

## 5 Bounds of $r_\nu(x; a)$

For the bounds of  $\bar{F}_\nu(x)$ , an inequality analogous to (4) was first given by Soms (1976),

$$\left( \frac{1}{x} - \frac{\nu}{(\nu+2)x^3} \right) \left( 1 + \frac{x^2}{\nu} \right) f_\nu(x) < \bar{F}_\nu(x) < \frac{1}{x} \left( 1 + \frac{x^2}{\nu} \right) f_\nu(x), \quad x > 0. \quad (14)$$

Denote  $k_\nu = \Gamma(\frac{\nu+1}{2})/(\sqrt{\nu\pi}\Gamma(\frac{\nu}{2}))$  as in Section 2. Let

$$\gamma_{\max} = \frac{4k_\nu^2}{1 - 4k_\nu^2}, \quad \text{and} \quad \gamma_{\min} = \frac{\nu}{2(\nu + 2)k_\nu^2} - 1.$$

Soms (1980) showed that for any  $\nu \geq 2$ ,

$$(1 + x^2/\nu) f_\nu(x) p_\nu(x, \gamma_{\min}) \leq \overline{F}_\nu(x) \leq (1 + x^2/\nu) f_\nu(x) p_\nu(x, \gamma_{\max}), \quad x > 0, \quad (15)$$

where  $p_\nu(x, \gamma) = (1 + \gamma)/\{[x^2 + 4k_\nu^2(1 + \gamma)^2]^{1/2} + \gamma x\}$ . For the special case when  $\nu = 2$ ,  $\gamma_{\max} = \gamma_{\min} = 1$  so that (15) is actually an equality. When  $\nu = 1$ , the inequality (15) still holds with the definition of  $\gamma_{\max}$  and  $\gamma_{\min}$  interchanged. It can be shown that (15) gives more accurate bounds for  $\overline{F}_\nu(x)$  compared to (14).

By (15), we have the following bounds for  $r_\nu(x; a)$ ,

$$c_1(x, a) \leq r_\nu(x; a) \leq d_1(x, a), \quad x > 0, \quad (16)$$

where

$$c_1(x, a) = \frac{[1 + (x + a)^2/\nu] f_\nu(x + a) p_\nu(x + a, \gamma_{\min})}{(1 + x^2/\nu) f_\nu(x) p_\nu(x, \gamma_{\max})},$$

$$d_1(x, a) = \frac{[1 + (x + a)^2/\nu] f_\nu(x + a) p_\nu(x + a, \gamma_{\max})}{(1 + x^2/\nu) f_\nu(x) p_\nu(x, \gamma_{\min})}.$$

In what follows we propose a new inequality for  $r_\nu(x; a)$ . Let  $\Psi_\nu(x) = -\log \overline{F}_\nu(x)$  and its derivative  $\psi_\nu(x) = (d/dx)\Psi_\nu(x) = f_\nu(x)/\overline{F}_\nu(x)$ . By Lagrange mean value theorem, we have  $\Psi_\nu(x + a) - \Psi_\nu(x) = a\Psi'_\nu(\xi) = a\psi_\nu(\xi)$ , where  $x < \xi < x + a$ . This equivalent to  $r_\nu(x; a) = \exp\{-a\psi_\nu(\xi)\}$ . By (15), we have

$$\frac{1}{(1 + x^2/\nu) p_\nu(x, \gamma_{\max})} \leq \psi_\nu(x) \leq \frac{1}{(1 + x^2/\nu) p_\nu(x, \gamma_{\min})}.$$

By Lemma A2 from Soms (1980), we have  $\gamma_{\max} > 0$  for any  $\nu \geq 1$ . By Lemmas A3 and A5 from Soms (1980), we have  $\gamma_{\min} > 0$  for any  $\nu \geq 1$ . Now given that both  $\gamma_{\max}$  and

$\gamma_{\min}$  are positive, it is easy to verify that  $p_\nu(x, \gamma_{\min})$  and  $p_\nu(x, \gamma_{\max})$  are both decreasing functions of  $x$  on  $(0, \infty)$ . This gives

$$\frac{1}{[1 + (x + a)^2/\nu] p_\nu(x, \gamma_{\max})} \leq \psi_\nu(\xi) \leq \frac{1}{(1 + x^2/\nu) p_\nu(x + a, \gamma_{\min})}.$$

Further, we have

$$c_2(x, a) \leq r_\nu(x; a) \leq d_2(x, a), \quad x > 0, \quad (17)$$

where

$$c_2(x, a) = \exp \left\{ \frac{-a}{(1 + x^2/\nu) p_\nu(x + a, \gamma_{\min})} \right\},$$

$$d_2(x, a) = \exp \left\{ \frac{-a}{[1 + (x + a)^2/\nu] p_\nu(x, \gamma_{\max})} \right\}.$$

To compare the performance of the proposed inequalities (16) and (17), we present the bounds, under various settings, in Figure 4 for  $\nu = 1$  and in Figure 5 for  $\nu = 10$ . For both  $\nu$  values, we observe that the bounds in (16) performs better when  $a$  is large, whereas the bounds in (17) performs better when  $a$  is small. By combining (16) and (17) we have

$$l_1(x, a) \leq r_\nu(x; a) \leq u_1(x, a), \quad x > 0, \quad (18)$$

where  $l_1(x, a) = \max \{c_1(x, a), c_2(x, a)\}$  and  $u_1(x, a) = \min \{d_1(x, a), d_2(x, a)\}$ .

The same as in Section 4, we now derive the lower and upper bounds of  $r_\nu(x; a)$  for  $x \leq 0$ . Note that  $\overline{F}_\nu(x) = 1 - \overline{F}_\nu(-x)$  and  $f_\nu(-x) = f_\nu(x)$ . Also by (15), we have

$$(1 + x^2/\nu) f_\nu(x) p_\nu(-x, \gamma_{\min}) \leq \overline{F}_\nu(-x) \leq (1 + x^2/\nu) f_\nu(x) p_\nu(-x, \gamma_{\max}), \quad x \leq 0.$$

(i) When  $-a < x \leq 0$ , noting that  $r_\nu(x; a) = \overline{F}_\nu(x + a)/[1 - \overline{F}_\nu(-x)]$ , we have

$$l_2(x, a) \leq r_\nu(x; a) \leq u_2(x, a), \quad -a < x \leq 0,$$

where

$$l_2(x, a) = \frac{[1 + (x + a)^2/\nu] f_\nu(x + a)p_\nu(x + a, \gamma_{\min})}{1 - (1 + x^2/\nu) f_\nu(x)p_\nu(-x, \gamma_{\min})},$$

$$u_2(x, a) = \frac{[1 + (x + a)^2/\nu] f_\nu(x + a)p_\nu(x + a, \gamma_{\max})}{1 - (1 + x^2/\nu) f_\nu(x)p_\nu(-x, \gamma_{\max})}.$$

(ii) When  $x \leq -a$ , noting that  $r_\nu(x; a) = [1 - \bar{F}_\nu(-x - a)]/[1 - \bar{F}_\nu(-x)]$ , we have

$$l_3(x, a) \leq r_\nu(x; a) \leq u_3(x, a), \quad x \leq -a,$$

where

$$l_3(x, a) = \frac{1 - [1 + (x + a)^2/\nu] f_\nu(x + a)p_\nu(-x - a, \gamma_{\max})}{1 - (1 + x^2/\nu) f_\nu(x)p_\nu(-x, \gamma_{\min})},$$

$$u_3(x, a) = \frac{1 - [1 + (x + a)^2/\nu] f_\nu(x + a)p_\nu(-x - a, \gamma_{\min})}{1 - (1 + x^2/\nu) f_\nu(x)p_\nu(-x, \gamma_{\max})}.$$

Finally, we present in Figure 6 the true ratio  $r_\nu(x; a)$  and its lower and upper bounds on the entire line of  $x$  for  $\nu = 5$  and  $a = 0.1$ . The visualized plot shows that the proposed bounds perform quite well, especially when  $x$  is negative or when  $x$  is large.

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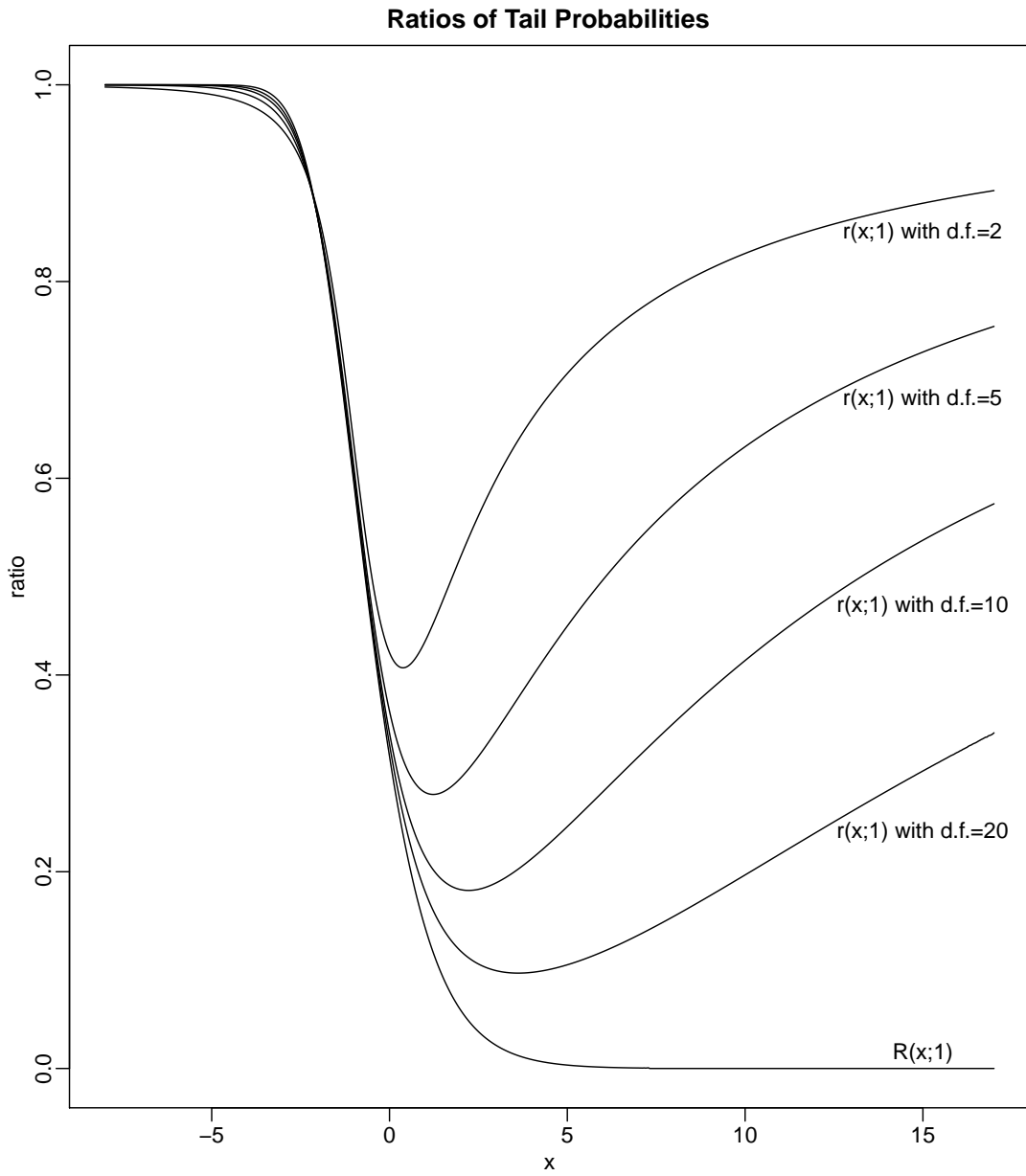


Figure 1: The curve of  $R(x; 1)$ , and the curves of  $r_\nu(x; 1)$  with various degrees of freedom  $\nu$ .

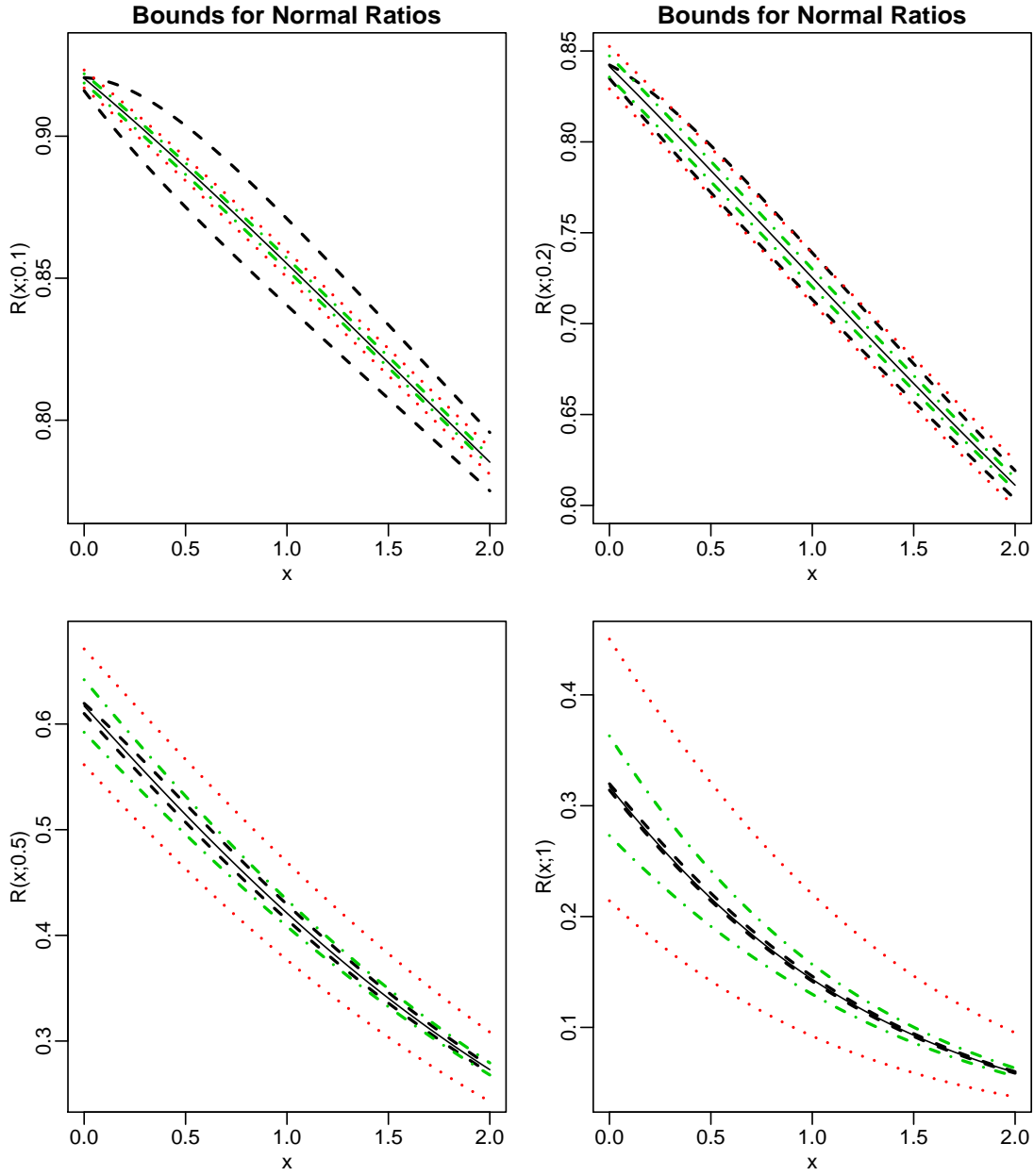


Figure 2: Bounds for the ratio  $R(x; a)$  with various  $a$  values, where the solid lines represent the true ratio  $R(x; a)$ , the dashed lines represent the bounds in (6), the dotted lines represent the bounds in (9), and the dash-dotted lines represent the bounds in (10).

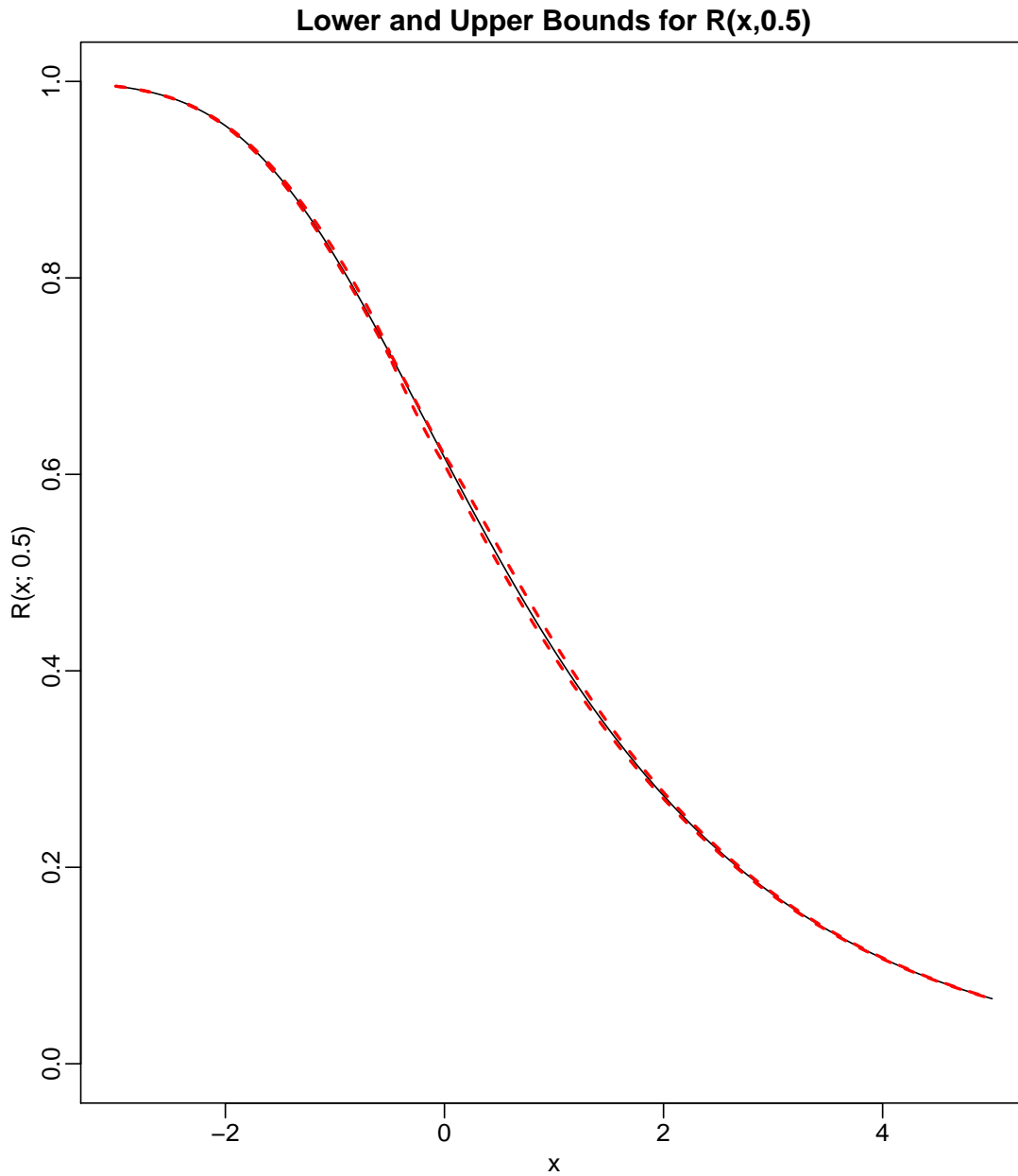


Figure 3: Lower and upper bounds for  $R(x;0.5)$  on the entire line of  $x$ , where the solid line represents the true ratio  $R(x;0.5)$ , and the dashed lines represent its lower and upper bounds.

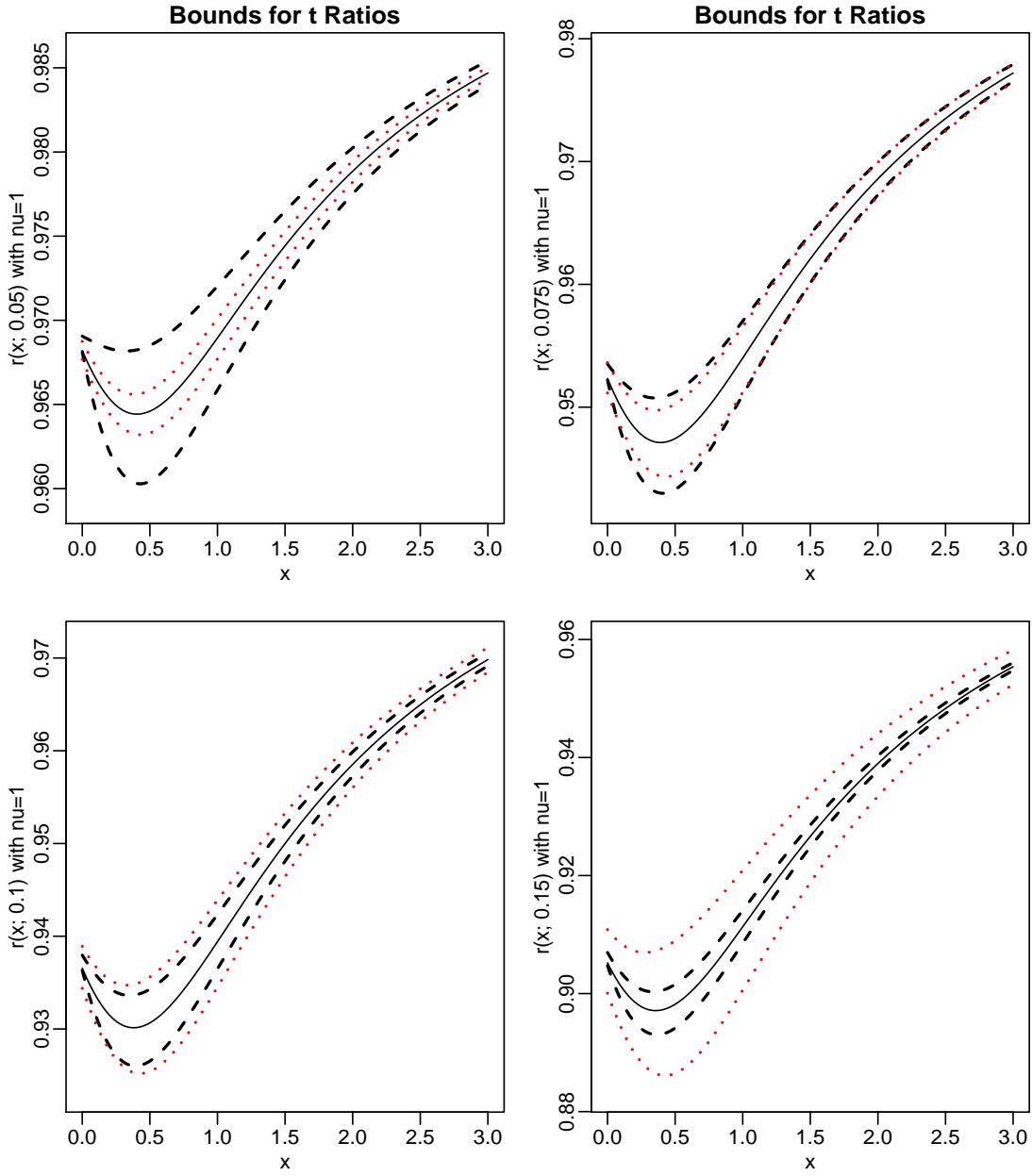


Figure 4: Bounds for the ratio  $r_1(x; a)$  with various  $a$  values, where the solid lines represent the true ratio  $r_1(x; a)$ , the dashed lines represent the bounds in (16), and the dotted lines represent the bounds in (17).

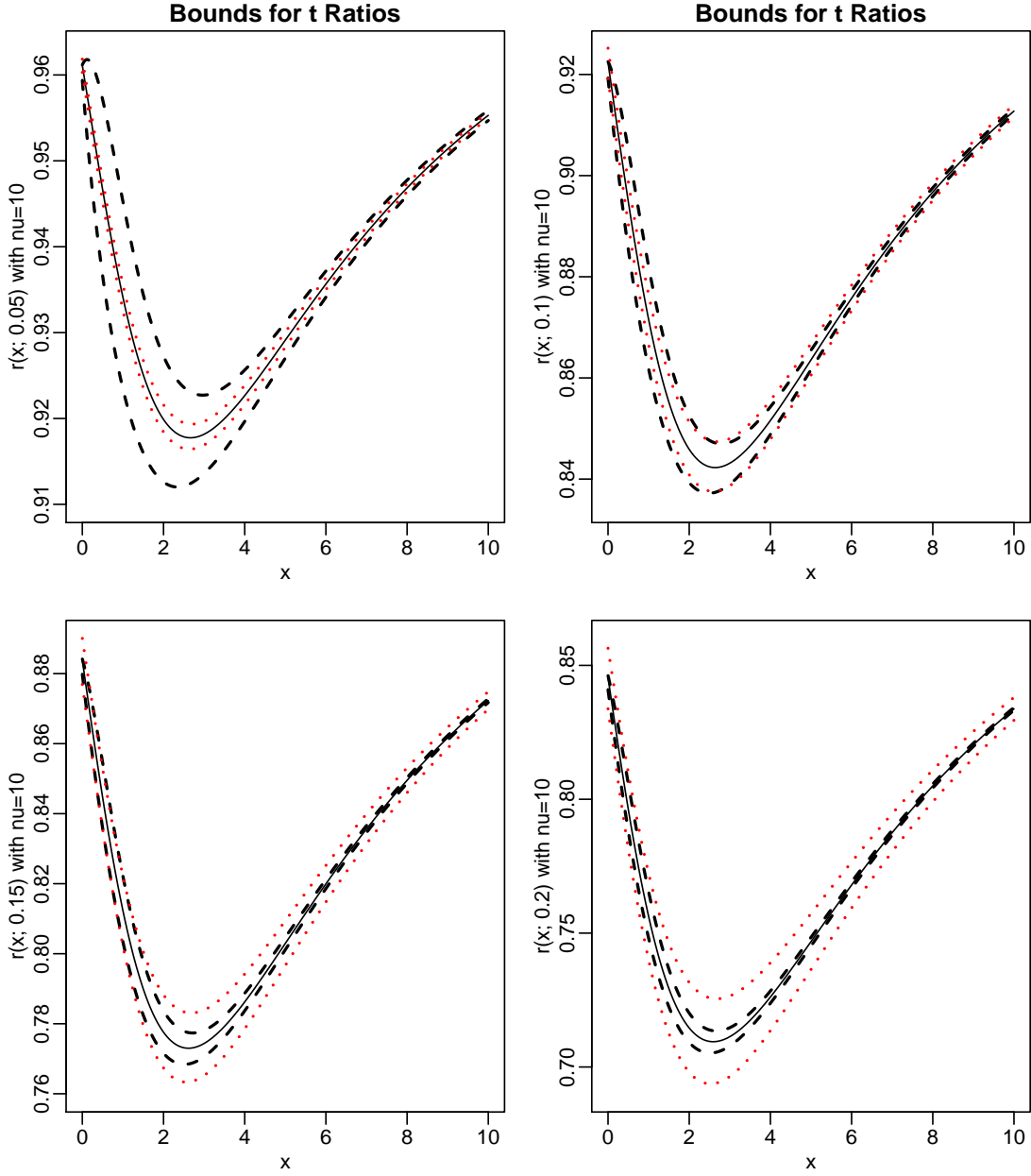


Figure 5: Bounds for the ratio  $r_{10}(x; a)$  with various  $a$  values, where the solid lines represent the true ratio  $r_{10}(x; a)$ , the dashed lines represent the bounds in (16), and the dotted lines represent the bounds in (17).

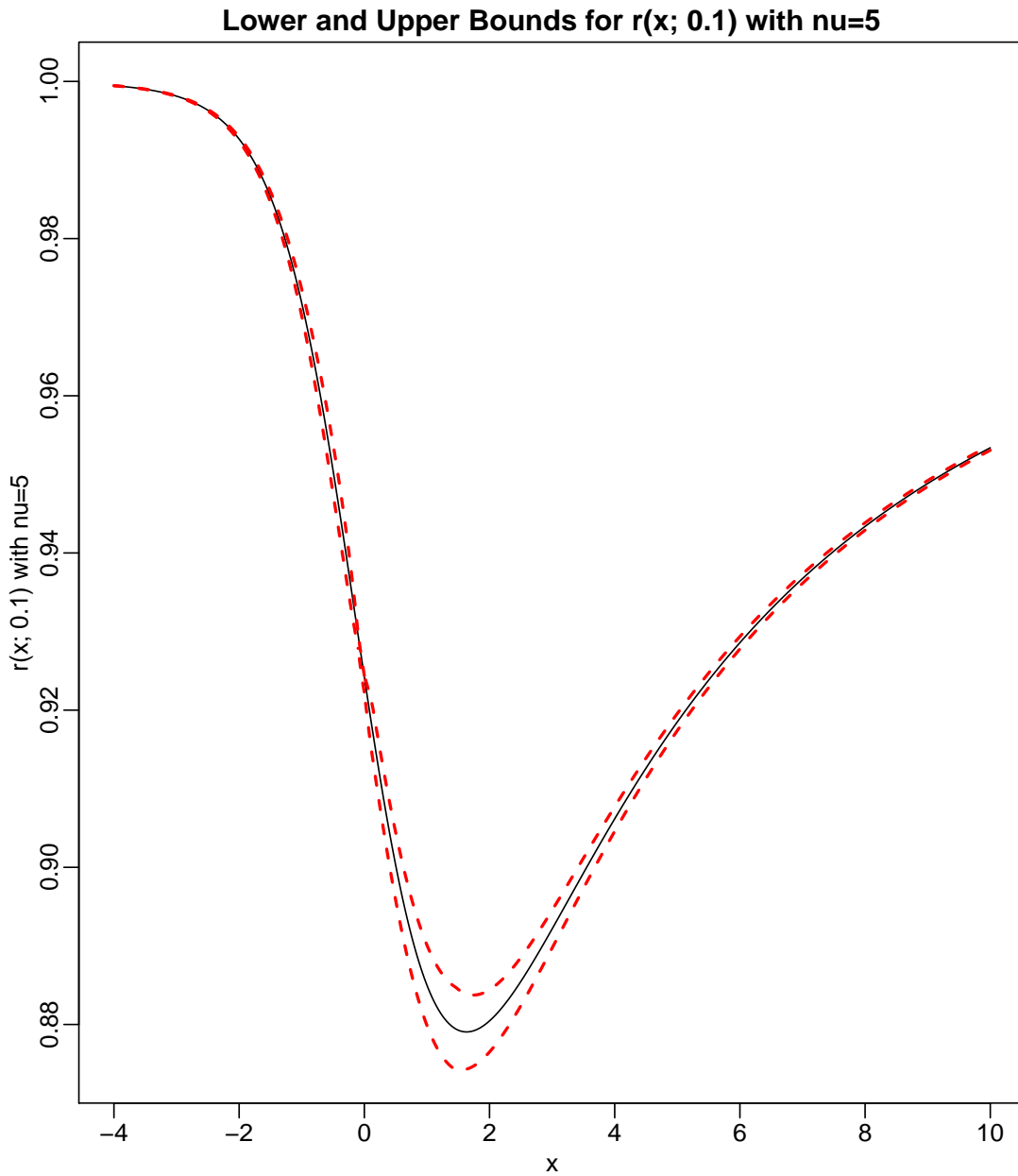


Figure 6: Lower and upper bounds for  $r_5(x; 0.1)$  on the entire line of  $x$ , where the solid line represents the true ratio  $r_5(x; 0.1)$ , and the dashed lines represent its lower and upper bounds.