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Polarized angular distributions in the decays of the $^3D_2$ state of charmonium produced in unpolarized proton-antiproton annihilation

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Abstract Using the helicity formalism, we calculate the combined angular distribution functions of the polarized gamma photons and electron in the triple cascade process $\bar{p}p \to {}^3D_2 \to \chi_J + \gamma_1 \to (\psi + \gamma_2) + \gamma_1 \to (e^+ + e^-) + \gamma_1 + \gamma_2$ ($J = 0, 1, 2$), when $\bar{p}$ and p are unpolarized. We also present the partially integrated angular distribution functions in different cases. Our results show that by measuring the two-particle angular distribution of γ_1 and γ_2 and that of γ_2 and e^- with the polarization of either one of the two particles, one can determine the relative magnitudes as well as the relative phases of all the helicity amplitudes in the two radiative decay processes ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$.

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1 Introduction

Charmonium spectroscopy above the open charm $D\bar{D}$ threshold has aroused intense interest recently [1, 2]. In particular, the search for the D_2 charmonium states has captured much attention as the widths of these states are expected to be narrow. The decays of the D_2 states into $D\bar{D}$ are forbidden by parity conservation and the decays of the D_2 states into $D\bar{D}^*$ or $\bar{D}D^*$ are also forbidden by energy conservation. In fact, the observation of the radiative decays of these D charmonium states is an important component of the planned PANDA experiments at FAIR [3], which study charmonium spectroscopy in $\bar{p}p$ annihilation.

In a previous paper [4], it is shown that by measuring the combined angular distribution of the two photons and of the electron, regardless of their polarizations, in the sequential decay process originating from unpolarized $\bar{p}p$ collisions, namely, $\bar{p}p \to {}^3D_2 \to \chi_J + \gamma_1 \to (\psi + \gamma_2) + \gamma_1 \to (e^+e^-) + \gamma_1 + \gamma_2$ ($J = 0, 1, 2$), one can extract the relative magnitudes as well as the cosines of the relative phases of all the angular-momentum helicity amplitudes in the radiative decay processes ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$. The sines of the relative phases can only be determined uniquely for $J = 2$. By including the measurement of the polarization of one of the decay particles, one may also obtain unambiguously the sines of these relative phases for the $J = 1$ case. Moreover, one may be able to get this complete information on all the helicity amplitudes by measuring the simultaneous angular distributions of only two decay particles at a time. So in this paper we calculate the angular distributions of different combinations of the final stable decay products, γ_1, γ_2 and e^-, with the determination of the polarization of one decay particle in the above cascade process when $\bar{p}$ and p are unpolarized. Our final model-independent expressions for the angular distribution functions are valid in the $\bar{p}p$ center-of-mass frame and they are written as sums of terms involving products of the Wigner D^J functions whose arguments are the angles representing the directions of the final electron and of the two photons. The coefficients in these expansions are functions of the angular-momentum helicity amplitudes, which contain all the dynamics of the individual decay processes. We stress that our expressions are independent of any dynamical models and are based only on the general principles of quantum mechanics and symmetry. This is important because we can then learn about the true dynamics of the charmonium system from the decays of the charmonium states.

Once the combined angular distribution of γ_1, γ_2 and e^- and the polarization of any one of the particles in unpolarized $\bar{p}p$ collisions are experimentally measured, our expressions will enable one to calculate the relative magnitudes as well as the relative phases of all the angular-momentum helicity amplitudes in the two radiative decay processes mentioned above for all values of J. In addition, one can also determine the relative magnitudes of the angular-momentum helicity amplitudes in the process $\psi \to e^+e^-$. Our results on the partially integrated angular distributions where the combined angular distribution function of γ_1, γ_2 and e^- is integrated over the direction of one of the three particles are quite interesting. They show that by measuring the two-particle angular distribution of γ_1 and γ_2 and that of γ_2 and e^- with the polarization of either one of the two particles, one can also get complete information on the helicity amplitudes. This is of the same advantage as in the case of the decays of the ${}^3D_2$ charmonium state produced in polarized proton-antiproton collisions [5].

The format of the rest of the paper is as follows: in Sect. 2, we give the calculation for the combined angular distribution with polarization determination of the electron and of the two photons in the cascade process $\bar{p}p \to {}^3D_2 \to \chi_J + \gamma_1 \to \psi + \gamma_2 + \gamma_1 \to e^+ + e^- + \gamma_1 + \gamma_2$ ($J = 0, 1, 2$), when $\bar{p}$ and p are unpolarized. We then show how the measurement of this combined angular distribution of polarized γ_1, γ_2 and e^- enables us to obtain complete information on the helicity amplitudes in the two radiative decay processes ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$. We also present three different results for the combined angular distribution, in which the polarization of only one of the three particles, γ_1, γ_2 and e^-, is measured. In Sect 3, we present the results for the partially integrated angular distributions in different cases where the combined angular distribution function of the three particles is integrated over the direction of one particle. These results can all be expressed in terms of the orthogonal spherical harmonic functions. We point out how the measurement of these two-particle angular distributions will again give us complete information on all the helicity amplitudes in the two radiative decay processes. Finally, in Sect. 4, we make some concluding remarks.

2 Calculations for the combined angular distribution of polarized γ_1, γ_2 and e^-

We consider the cascade process, $\bar{p}(\lambda_1) + p(\lambda_2) \to {}^3D_2(\nu) \to \chi_J(\sigma) + \gamma_1(\mu) \to \psi(\rho)$ $+\gamma_2(\kappa) + \gamma_1(\mu) \to e^-(\alpha_1) + e^+(\alpha_2) + \gamma_2(\kappa) + \gamma_1(\mu)$ ($J = 0, 1, 2$), in the $\bar{p}p$ center-of-mass frame or the ${}^3D_2$ rest frame, where J is the angular momentum of the χ resonance and the Greek symbols after the particle symbols represent their helicities except for the stationary ${}^3D_2$ resonance, in which case the symbol ν represents the z component of the angular momentum. We choose the z axis to be in the direction of motion of χ_J in the $^3D_2$ rest frame. The x and y axes are arbitrary in our discussions. The experimentalists can choose them according to their convenience. The probability amplitude for the above cascade process can be written as a product of the matrix elements for the individual sequential processes. Since only the helicities of the initial and the final particles, namely, λ_1, λ_2, μ, κ, α_1 and α_2, are observed, we write the probability amplitude for the cascade process in the $^3D_2$ rest frame as

$$T^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} = \sum_{\nu}^{-2\to+2}\sum_{\rho}^{-1\to+1}\sum_{\sigma}^{-J\to+J} {}_D\langle {}^3D_2(\nu)|B|\bar{p}(\lambda_1), p(\lambda_2)\rangle_D \times {}_D\langle \chi_J(\sigma), \gamma_1(\mu)|A|{}^3D_2(\nu)\rangle_D$$

$$\times\, {}_D\langle \psi(\rho), \gamma_2(\kappa)|E|\chi_J(\sigma)\rangle_D \times {}_D\langle e^-(\alpha_1), e^+(\alpha_2)|C|\psi(\rho)\rangle_D . \qquad (1)$$

We sum over the helicities and the spin indices of the unobserved intermediate particles in (1). The symbols B, A, E and C represent the appropriate transition operators. The subscript D attached to the bra or the ket vector indicates that each individual matrix element is evaluated in the $^3D_2$ rest frame. In the first two matrix elements the $^3D_2$ rest frame is the same as the c.m. frame of the two particles. In the last two matrix elements $\langle \psi\gamma_2|E|\chi_J\rangle$ and $\langle e^-e^+|C|\psi\rangle$ this is not the case. To avoid confusion, we should clarify what we mean by the two-particle helicity states when they are not in their c.m. frame. For example, the two-particle state $|\psi(\rho), \gamma_2(\kappa)\rangle_D$ defined in the $^3D_2$ rest frame, which is not the c.m. frame of ψ and γ_2, has the following meaning. First construct the two-particle helicity state $|\psi(\rho), \gamma_2(\kappa)\rangle_{\chi_J}$ in the χ_J rest frame (which is the same as the c.m. frame of ψ and γ_2) according to the usual conventions [6] with ψ and γ_2 having equal and opposite momenta and helicities ρ and κ, respectively. Then

$$|\psi(\rho), \gamma_2(\kappa)\rangle_D = U_\Lambda({}^3D_2, \chi_J)\,|\psi(\rho), \gamma_2(\kappa)\rangle_{\chi_J}, \qquad (2)$$

where $U_\Lambda(A,B)$ is the unitary operator corresponding to the Lorentz transformation $\Lambda(A,B)$ which takes the system from the Lorentz frame where B is at rest to the Lorentz frame where A is at rest. It is important to clarify this point since in general ψ and γ_2 do not have definite helicities in the $^3D_2$ rest frame. A similar meaning also holds for the two-particle state $|e^-(\alpha_1), e^+(\alpha_2)\rangle_D$.

Let us now consider the matrix elements in (1) one by one. First,

$$ {}_{D}\langle {}^{3}D_2(\nu)|B|\overline{p}(\lambda_1), p(\lambda_2)\rangle_{D} = \langle 2\nu|B|\, p(\theta,\phi);\lambda_1\lambda_2\rangle , \tag{3} $$

where $\langle 2\nu|$ is the one-particle helicity state, or the angular-momentum state, of ${}^{3}D_2$ in its own rest frame and $p(\theta,\phi)$ is the magnitude of the c.m. momentum of $\overline{p}$, which is taken to be in the direction (θ,ϕ) in the coordinate system we have chosen. Using the usual expansion [6] of the two-particle helicity state in the c.m. frame in terms of the angular-momentum states we find [7]

$$ {}_{D}\langle {}^{3}D_2(\nu)|B|\overline{p}(\lambda_1), p(\lambda_2)\rangle_{D} = \sqrt{\frac{5}{4\pi}}\, B_{\lambda_1\lambda_2} D^{2}_{\nu\lambda}(\phi,\theta,-\phi) , \tag{4} $$

where

$$ \lambda = \lambda_1 - \lambda_2 , \tag{5} $$

and $B_{\lambda_1\lambda_2}$ are the angular-momentum helicity amplitudes.

Similarly, the matrix element for the process ${}^{3}D_2 \to \chi_J + \gamma_1$ with χ_J and γ_1 moving along the +z and –z directions, respectively, can be written as

$$ {}_{D}\langle \chi_J(\sigma), \gamma_1(\mu)|A|{}^{3}D_2(\nu)\rangle_{D} = \langle p_{\chi_J}(0,0);\sigma\mu|A|2\nu\rangle $$

$$ = \sqrt{\frac{5}{4\pi}}\, A^{J}_{\sigma\mu} D^{2*}_{\nu,\sigma-\mu}(0,0,0) = \sqrt{\frac{5}{4\pi}}\, A^{J}_{\sigma\mu}\, \delta_{\nu,\sigma-\mu} , \tag{6} $$

where $p_{\chi_J}(0,0)$ is the magnitude of the momentum of χ_J along the z axis in the ${}^{3}D_2$ rest frame and the $A^{J}_{\nu\mu}$ are the angular-momentum helicity amplitudes for this process.

Next we notice that the matrix elements for the process $\chi_J \to \psi + \gamma_2$ in the ${}^{3}D_2$ and the χ_J rest frames are equal. That is,

$$ {}_{D}\langle \psi(\rho), \gamma_2(\kappa)|E|\chi_J(\sigma)\rangle_{D} = {}_{\chi_J}\langle \psi(\rho), \gamma_2(\kappa)|U^{\dagger}_{\Lambda}({}^{3}D_2,\chi_J)\, E\, U_{\Lambda}({}^{3}D_2,\chi_J)|\chi_J(\sigma)\rangle_{\chi_J} $$

$$ = {}_{\chi_J}\langle \psi(\rho), \gamma_2(\kappa)|E|\chi_J(\sigma)\rangle_{\chi_J} . \tag{7} $$

In (7) we have used the fact that the transition operator E is invariant under Lorentz transformations:

$$ U^{\dagger}_{\Lambda} E\, U_{\Lambda} = E . \tag{8} $$

Using (7) we can now write

$$ {}_{D}\langle \psi(\rho), \gamma_2(\kappa) | E | \chi_J(\sigma) \rangle_D = {}_{\chi_J}\langle p'(\theta',\phi'); \rho\kappa | E | J\sigma \rangle_{\chi_J} , \tag{9} $$

where $p'(\theta',\phi')$ is the magnitude of the ψ three-momentum in the χ_J rest frame or the ψ-γ_2 c.m. frame. Moreover, in this frame, the index σ is the z-component of the total angular momentum of χ_J. Again using the expansion of the two-particle helicity state in the c.m. frame in terms of the angular-momentum states we obtain

$$ {}_{D}\langle \psi(\rho), \gamma_2(\kappa) | E | \chi_J(\sigma) \rangle_D = \sqrt{\frac{2J+1}{4\pi}}\, E^J_{\rho\kappa} D^{J*}_{\sigma,\rho-\kappa}(\phi',\theta',-\phi') , \tag{10} $$

where $E^J_{\rho\kappa}$ are the angular-momentum helicity amplitudes for the process.

For the matrix element of the final process $\psi(\rho) \to e^-(\alpha_1) + e^+(\alpha_2)$ the situation is more involved. We have

$$ {}_{D}\langle e^-(\alpha_1), e^+(\alpha_2) | C | \psi(\rho) \rangle_D = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2) | U^\dagger_\Lambda({}^3D_2,\psi)\, C U_\Lambda({}^3D_2,\chi_J)\, U_\Lambda(\chi_J,\psi) | \psi(\rho) \rangle_\psi $$

$$ = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2) | U^\dagger_\Lambda({}^3D_2,\psi)\, C U_\Lambda({}^3D_2,\psi)\, U^{\square}_\Lambda(\;D_2,\psi)\, U_\Lambda(\;D_2,\chi_J)\, U_\Lambda(\chi_J,\psi) | \psi(\rho) \rangle_\psi $$

$$ = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2) | C U^\dagger_\Lambda({}^3D_2,\psi)\, U_\Lambda({}^3D_2,\chi_J)\, U_\Lambda(\chi_J,\psi) | \psi(\rho) \rangle_\psi . \tag{11} $$

In the first equality of (11) we have made use of the fact that the single-particle state $|\psi(\rho)\rangle_D$ was also part of the two-particle helicity state in (10). It was obtained by successively performing two unitary operations corresponding to two Lorentz transformations, the first taking the ψ state from its rest frame to the χ_J rest frame and the second taking it from the χ_J rest frame to the ${}^3D_2$ rest frame. In the last equality of (11) we now make use of the fact that

$$ U_\Lambda({}^3D_2,\chi_J)\, U_\Lambda(\chi_J,\psi) = U_\Lambda({}^3D_2,\psi)\, U_{R_W} , \tag{12} $$

where U_{R_W} is a unitary operator corresponding to a pure rotation, usually called "Wigner rotation". Using (12) and the unitarity of U_Λ, (11) now leads to

$$ {}_{D}\langle e^-(\alpha_1), e^+(\alpha_2) | C | \psi(\rho) \rangle_D = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2) | C U_{R_W} | \psi(\rho) \rangle_\psi $$

$$ = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2) | U_{R_W} U^\dagger_{R_W} C U_{R_W} | \psi(\rho) \rangle_\psi = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2) | U_{R_W} C | \psi(\rho) \rangle_\psi , \tag{13} $$

since

$$ U^\dagger_{R_W} C U_{R_W} = C . \tag{14} $$

Using the expansion of the two-particle helicity state in terms of the angular-momentum states, we can write the right-hand side of (13) as

$$ {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2) | U_{R_W} C | \psi(\rho) \rangle_{\psi} = \sqrt{\frac{3}{4\pi}} D^{1*}_{\rho\alpha}(R_W^{-1}\hat{e}_{\psi}) C_{\alpha_1\alpha_2} = \sqrt{\frac{3}{4\pi}} C_{\alpha_1\alpha_2} D^{1*}_{\rho\alpha}(\phi'', \theta'', -\phi''), \qquad (15) $$

where

$$ \alpha = \alpha_1 - \alpha_2, \qquad (16) $$

$\hat{e}_{\psi}$ is a unit vector in the direction of the e^- three-momentum in the ψ rest frame, R_W is the (3×3) rotation matrix and $C_{\alpha_1\alpha_2}$ are the angular-momentum helicity amplitudes for the process. We should mention here that if the electron-positron pair is created by a virtual photon via $\bar{q}q \to \gamma \to e^+e^-$, the helicity zero amplitude C_{++} or C_{--} is of the order m/E when compared to the helicity 1 amplitude C_{+-} or C_{-+}. Since $E \cong M_{\psi}/2$ where M_{ψ} is the rest mass of ψ, $m/E \cong 3.3\times10^{-4}$ and the helicity zero amplitude is relatively negligible.

The Wigner-rotated unit vector $R_W^{-1}\hat{e}_{\psi}$ can be obtained in the following way. If R represents the (4×4) matrix whose spatial part gives the (3×3) matrix R_W mentioned above, then, from the definition of U_{R_W} in (12),

$$ R = \Lambda^{-1}({}^3D_2, \psi)\Lambda({}^3D_2, \chi_J)\Lambda(\chi_J, \psi), \qquad (17) $$

where Λ are the (4×4) Lorentz transformation matrices. Now we note that the electron is highly relativistic in the ψ rest frame and its four-momentum vector p_{e_ψ} can be represented to a very good approximation by

$$ p_{e_\psi} = \frac{M_{\psi}}{2}(1, \hat{e}_{\psi}), \qquad (18) $$

and therefore

$$ R^{-1}p_{e_\psi} = \Lambda^{-1}(\chi_J, \psi)\Lambda^{-1}({}^3D_2, \chi_J)\Lambda({}^3D_2, \psi)p_{e_\psi} $$

$$ = \Lambda^{-1}(\chi_J, \psi)\Lambda^{-1}({}^3D_2, \chi_J)\Lambda({}^3D_2, \psi)\Lambda^{-1}({}^3D_2, \psi)p_{e_D} = \Lambda^{-1}(\chi_J, \psi)\Lambda^{-1}({}^3D_2, \chi_J)p_{e_D}. \qquad (19) $$

In (19) the four-momentum of e^- in the ${}^3D_2$ rest frame is given by

$$ p_{e_D} = E_{e_D}(1, \hat{e}_D), \qquad (20) $$

where $E_{e_{\rm D}}$ is the relativistic energy of $\rm e^-$, and $\hat{e}_{\rm D}$ is the unit vector in the direction of the three-momentum of $\rm e^-$ in the ${}^3\rm D_2$ rest frame. From (18) we also have

$$R^{-1} p_{e_\psi} = \frac{M_\psi}{2}(1, R_W^{-1}\hat{e}_\psi)\,. \tag{21}$$

Combining (19)-(21) we get

$$\frac{M_\psi}{2}(1, R_W^{-1}\hat{e}_\psi) = \Lambda^{-1}(\chi_{\rm J}, \psi)\Lambda^{-1}({}^3\rm D_2, \chi_{\rm J}) E_{e_{\rm D}}(1, \hat{e}_{\rm D})\,. \tag{22}$$

The spatial part of the right-hand side of (22) gives, within a normalization factor, the Wigner-rotated unit vector $\hat{\hat{e}} = (R_W^{-1}\hat{e}_\psi)$ in terms of the angles $(\tilde{\theta}'', \tilde{\phi}'')$, which give the direction of $\rm e^-$ in the ${}^3\rm D_2$ rest frame.

We emphasize that the angles (θ', ϕ') of ψ and (θ'', ϕ'') of $\rm e^-$ are directions in the $\chi_{\rm J}$ and the ψ rest frames, respectively. They are not the same as the corresponding angles measured in the ${}^3\rm D_2$ rest frame or the lab frame. However, the different reference frames are related to each other through the Lorentz transformation. The equations relating these angles are given in [8].

Using (4), (6), (10) and (15) we can now write the amplitude in (1) as

$$\begin{aligned} \mathrm{T}^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} &= \frac{5\sqrt{3(2\mathrm{J}+1)}}{(4\pi)^2} \mathrm{C}_{\alpha_1\alpha_2}\, \mathrm{B}_{\lambda_1\lambda_2} \sum_{\rho}^{-1,0,+1} \mathrm{E}^{\mathrm{J}}_{\rho\kappa}\, \mathrm{D}^{1*}_{\rho\alpha}(\phi'', \theta'', -\phi'') \\ &\times \sum_{\sigma}^{-\mathrm{J}\to+\mathrm{J}} \mathrm{A}^{\mathrm{J}}_{\sigma\mu}\, \mathrm{D}^{\mathrm{J}*}_{\sigma,\rho\text{-}\kappa}(\phi', \theta', -\phi')\, \mathrm{D}^2_{\sigma-\mu,\lambda}(\phi, \theta, -\phi)\,. \end{aligned} \tag{23}$$

Because of the C and the P invariances [6], the angular-momentum helicity amplitudes in (23) are not all independent. We have

$$\mathrm{B}_{\lambda_1\lambda_2} \overset{\mathrm{P}}{=} -\mathrm{B}_{-\lambda_1,-\lambda_2} \overset{\mathrm{C}}{=} -\mathrm{B}_{\lambda_2\lambda_1}\,,$$

$$\mathrm{A}^{\mathrm{J}}_{\sigma\mu} \overset{\mathrm{P}}{=} (-1)^{\mathrm{J}+1}\,\mathrm{A}^{\mathrm{J}}_{-\sigma,-\mu}\,,$$

$$\mathrm{E}^{\mathrm{J}}_{\rho\kappa} \overset{\mathrm{P}}{=} (-1)^{\mathrm{J}}\,\mathrm{E}^{\mathrm{J}}_{-\rho,-\kappa}\,,$$

and

$$\mathrm{C}_{\alpha_1\alpha_2} \overset{\mathrm{P}}{=} \mathrm{C}_{-\alpha_1,-\alpha_2} \overset{\mathrm{C}}{=} \mathrm{C}_{\alpha_2\alpha_1}\,. \tag{24}$$

From the first equation in (24), we get

$$B_{\frac{1}{2},\frac{1}{2}} = B_{-\frac{1}{2},-\frac{1}{2}} = 0 \tag{25}$$

Making use of the symmetry relations of (24) we now re-label the independent angular-momentum helicity amplitudes as follows:

$$B_1 = \sqrt{2}\, B_{\frac{1}{2},-\frac{1}{2}} = -\sqrt{2}\, B_{-\frac{1}{2},\frac{1}{2}} ,$$

$$A_\sigma = A^J_{\sigma,1} = (-1)^{J+1} A^J_{-\sigma,-1} \qquad (\sigma = -1, 0, \ldots, +J) ,$$

$$E_\rho = E^J_{\rho-1,-1} = (-1)^J E^J_{-\rho+1,1} \qquad (\rho = 0, 1, \ldots, +J) ,$$

$$C_\alpha = C_{|\alpha_1-\alpha_2|} = C_{\alpha_1\alpha_2} \qquad (\alpha = 0, 1) . \tag{26}$$

When $\bar{p}$ and p are unpolarized, the normalized function describing the combined angular distribution of the electron and the two photons whose polarizations are also observed can be written as

$$W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') = N_J \sum_{\lambda_1,\lambda_2}^{\pm\frac{1}{2}} \sum_{\alpha_2}^{\pm\frac{1}{2}} T^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} T^{\alpha_1\alpha_2\mu\kappa *}_{\lambda_1\lambda_2} , \tag{27}$$

where the subscripts $\mu\kappa\alpha_1$ of W represent the polarizations that are measured in the angular distribution. The normalization constant, N_J, in (27) is determined by requiring that the integral of the distribution function $W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$ over all the directions of γ_1, γ_2 and e^- or over all the angles, $(\theta,\phi;\theta',\phi';\theta'',\phi'')$, is 1. In (27) we sum over the helicities α_2 since e^+ is not observed. Substituting (23) into (27) and performing the various sums will then give us an expression for the angular distribution function $W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$ in terms of the Wigner D^J functions. After very long algebra, we get

$$\begin{aligned} W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') = \frac{1}{4(4\pi)^3} \sum_{L_3}^{0,1,2} \sum_{L_1}^{0,2,4} \gamma_{L_3} \beta_{L_1} \sum_{L_2}^{0\to 2J} (-1)^{\frac{1}{2}(1+\mu)L_2} (-1)^{\frac{1}{2}(1-\kappa)(L_3+L_2)} \\ \times \sum_{d}^{0\to d_m} \sum_{d'}^{0\to d'_m} \Big\{ \alpha^{L_1L_2}_{d+} \Big[\varepsilon^{L_3L_2}_{d'+} \big(D_1 + D_1^* + D_2 + D_2^* \big) + \varepsilon^{L_3L_2}_{d'-} \big(D_1 - D_1^* + D_2 - D_2^* \big) \Big] \\ + \alpha^{L_1L_2}_{d-} \Big[\varepsilon^{L_3L_2}_{d'+} \big(D_1 - D_1^* - D_2 + D_2^* \big) + \varepsilon^{L_3L_2}_{d'-} \big(D_1 + D_1^* - D_2 - D_2^* \big) \Big] \Big\} , \end{aligned} \tag{28}$$

where

$$d_m = \mathrm{Min}(L_1, L_2, 3) ,$$
$$d'_m = \mathrm{Min}(L_3, L_2, J) , \tag{29}$$

and we have used the following normalizations for the angular-momentum helicity amplitudes B_1, A_σ, E_ρ and C_α defined in (26):

$$\left|B_1\right|^2 = \left|C_0\right|^2 + \left|C_1\right|^2 = \sum_{\sigma}^{-1\to J} \left|A_\sigma\right|^2 = \sum_{\rho}^{0\to J} \left|E_\rho\right|^2 = 1\,. \tag{30}$$

In (28) the angle-dependent terms are given by

$$D_1 = D^{L_1 *}_{-\mu d,0}(\phi,\theta,-\phi)\, D^{L_3}_{\kappa d',0}(\phi'',\theta'',-\phi'')\, D^{L_2}_{-\mu d,\kappa d'}(\phi',\theta',-\phi') \tag{31}$$

and

$$D_2 = D^{L_1 *}_{\mu d,0}(\phi,\theta,-\phi)\, D^{L_3}_{\kappa d',0}(\phi'',\theta'',-\phi'')\, D^{L_2}_{\mu d,\kappa d'}(\phi',\theta',-\phi')\,. \tag{32}$$

The arguments of the Wigner functions D^{L_1}, D^{L_2} and D^{L_3} in (31) and (32) are $(\phi,\theta,-\phi)$ the direction of $\overline{p}$ with respect to χ_J (or the angles between γ_1 and the proton) in the ${}^3D_2$ rest frame, $(\phi',\theta',-\phi')$ the direction of ψ (or the angles between γ_2 and ${}^3D_2$) in the χ_J rest frame, and $(\phi'',\theta'',-\phi'')$ the direction of the e^- momentum in the ψ rest frame, respectively. The coefficients γ_{L_3}, β_{L_1}, $\alpha^{L_1L_2}_{d\pm}$ and $\varepsilon^{L_3L_2}_{d'\pm}$, which are independent of the angles in (28), are defined as follows:

$$\gamma_{L_3} = -\sqrt{3}\,(-1)^{(\alpha_1+\frac{1}{2})L_3}\left[\,\langle 11;00|L_3\,0\rangle\,|C_0|^2 - \langle 11;1-1|L_3\,0\rangle\,|C_1|^2\,\right], \tag{33}$$

$$\beta_{L_1} = -\sqrt{5}\,\langle 22;1-1|L_1\,0\rangle\,|B_1|^2, \tag{34}$$

$$\alpha^{L_1L_2}_{d\pm} = (-1)^{J+1}\sqrt{5(2J+1)}\left(1-\frac{\delta_{d0}}{2}\right)\sum_{s(d)}\left[A_{\frac{s+d}{2}}\,A^*_{\frac{s-d}{2}} \pm A^*_{\frac{s+d}{2}}\,A_{\frac{s-d}{2}}\right]$$
$$\times\left\langle JJ;\frac{s+d}{2},-\frac{s-d}{2}\middle|L_2\,d\right\rangle\left\langle 22;\frac{s+d-2}{2},-\frac{s-d-2}{2}\middle|L_1\,d\right\rangle,$$
$$s(d) = -(2J-d),\,-(2J-d)+2,\,\ldots,\,+(2J-d)\,, \tag{35}$$

$$\varepsilon^{L_3L_2}_{d'\pm} = (-1)^{J}\sqrt{3(2J+1)}\left(1-\frac{\delta_{d'0}}{2}\right)\sum_{s'(d')}\left[E_{\frac{s'+d'}{2}}\,E^*_{\frac{s'-d'}{2}} \pm E^*_{\frac{s'+d'}{2}}\,E_{\frac{s'-d'}{2}}\right]$$
$$\times\left\langle JJ;\frac{s'+d'}{2},-\frac{s'-d'}{2}\middle|L_2\,d'\right\rangle\left\langle 11;\frac{s'+d'-2}{2},-\frac{s'-d'-2}{2}\middle|L_3\,d'\right\rangle,$$
$$s'(d') = d',\,d'+2,\,\ldots,\,2J-d'\,. \tag{36}$$

Since the combined angular distribution in (28) is expressed as a sum of products of the orthogonal Wigner D^J functions, we can obtain the coefficients of the D^J angular functions as

$$\gamma_{L_3}\,\beta_{L_1}\,(-1)^{\frac{1}{2}(1+\mu)L_2}\,(-1)^{\frac{1}{2}(1-\kappa)(L_3+L_2)}\left\{\alpha^{L_1L_2}_{d+}\left[\,\varepsilon^{L_3L_2}_{d'+}\left(1+\delta_{d0}\right)\left(1+\delta_{d'0}\right)+\varepsilon^{L_3L_2}_{d'-}\left(1+\delta_{d0}\right)\left(1-\delta_{d'0}\right)\,\right]\right.$$
$$\left.+\,\alpha^{L_1L_2}_{d-}\left[\varepsilon^{L_3L_2}_{d'+}\left(1-\delta_{d0}\right)\left(1+\delta_{d'0}\right)+\varepsilon^{L_3L_2}_{d'-}\left(1-\delta_{d0}\right)\left(1-\delta_{d'0}\right)\right]\right\}$$

$$= 4(2L_1+1)(2L_2+1)(2L_3+1)\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\, D_1^*\, d\Omega\, d\Omega'\, d\Omega''. \qquad (37)$$

In calculating (37), we made use of the orthogonality relation,

$$\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} D_{mm'}^{j*}(\alpha,\beta,\gamma)\, D_{\mu\mu'}^{j'}(\alpha,\beta,\gamma)\sin\beta\, d\beta = \frac{8\pi^2}{(2j+1)}\,\delta_{m\mu}\,\delta_{m'\mu'}\,\delta_{jj'}. \qquad (38)$$

When we have sufficient experimental data for the angular distribution function $W_{\mu\kappa\alpha_1}$ where the final polarizations, μ, κ and α_1, of all the three decay particles are measured, the integral on the right side of (37) can be determined numerically for all possible allowed values of L_1, L_2, L_3, d and d'. Thus we can obtain the different coefficients γ_{L_3}, β_{L_1}, $\alpha_{d\pm}^{L_1L_2}$ and $\varepsilon_{d'\pm}^{L_3L_2}$ on the left side of (37). From these coefficients we can determine the relative magnitudes as well as the relative phases of the A and the E helicity amplitudes in the radiative decay processes $^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$, respectively, for the $J=1$ and the $J=2$ cases. For the $J=0$ case, there is only one independent helicity amplitude for each radiative decay process and that is fixed by our normalization. We can also obtain the relative magnitudes of the C helicity amplitudes in the final decay process $\psi \to e^+e^-$ for all values of J. For example, in the $J=1$ case, the measurements of the $(L_1L_2L_3dd')$ coefficients will give us the following. First the measurement of the (20000) and the (40000) coefficients yields α_{0+}^{20} and α_{0+}^{40}, and with the normalization $|A_{-1}|^2+|A_0|^2+|A_1|^2=1$, the relative magnitudes of A_{-1}, A_0 and A_1 are determined. Next measuring the (01000) coefficient gives ε_{0+}^{01} and hence $|E_1|^2$. With the normalization $|E_0|^2+|E_1|^2=1$ we can determine the relative magnitudes of E_0 and E_1. The relative magnitudes of C_0 and C_1 can then be obtained from the measurement of the (00100) coefficient and the normalization $|C_0|^2+|C_1|^2=1$. After having obtained all the relative magnitudes, now measuring the (02101) coefficient gives both $\mathrm{Re}(E_1E_0^*)$ and $\mathrm{Im}(E_1E_0^*)$. Thus the relative phase between E_0 and E_1 is determined. Finally the measurement of the (22010) and the (42010) coefficients yields $\mathrm{Re}(A_0A_{-1}^*)$, $\mathrm{Re}(A_1A_0^*)$, $\mathrm{Im}(A_0A_{-1}^*)$ and

$\mathrm{Im}\left(A_1 A_0^*\right)$. Hence the relative phases among A_{-1}, A_0 and A_1 are also obtained.

It is interesting to note that using (28) we can easily obtain different combined angular distribution functions where the polarizations of only one or two of the decay products γ_1, γ_2 and e^- are measured. Suppose we are interested in only measuring the polarization μ of γ_1, the normalized combined angular distribution of γ_1, γ_2 and e^- will then become

$$W_\mu(\theta,\phi;\theta',\phi';\theta'',\phi'') = \frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}} \sum_{\kappa}^{\pm 1} W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$$

$$= \frac{1}{8(4\pi)^3} \sum_{L_3}^{0,2} \sum_{L_1}^{0,2,4} \gamma_{L_3}\, \beta_{L_1} \sum_{L_2}^{0\to 2J} (-1)^{\frac{1}{2}(1+\mu)L_2}$$

$$\times \sum_{d}^{0\to d_m} \sum_{d'}^{0\to d'_m} \left\{ \alpha_{d+}^{L_1L_2} \left[\varepsilon_{d'+}^{L_3L_2} \left(1+(-1)^{L_2}\right)\left(\tilde{D}_1+\tilde{D}_1^*+\tilde{D}_2+\tilde{D}_2^*\right) + \varepsilon_{d'-}^{L_3L_2} \left(1-(-1)^{L_2}\right)\left(\tilde{D}_1-\tilde{D}_1^*+\tilde{D}_2-\tilde{D}_2^*\right) \right] \right.$$

$$\left. + \alpha_{d-}^{L_1L_2} \left[\varepsilon_{d'+}^{L_3L_2} \left(1+(-1)^{L_2}\right)\left(\tilde{D}_1-\tilde{D}_1^*-\tilde{D}_2+\tilde{D}_2^*\right) + \varepsilon_{d'-}^{L_3L_2} \left(1-(-1)^{L_2}\right)\left(\tilde{D}_1+\tilde{D}_1^*-\tilde{D}_2-\tilde{D}_2^*\right) \right] \right\} , \qquad (39)$$

where

$$\tilde{D}_1 = D_1(\kappa=1) = D_{-\mu d,0}^{L_1 *}(\phi,\theta,-\phi)\, D_{d',0}^{L_3}(\phi'',\theta'',-\phi'')\, D_{-\mu d,d'}^{L_2}(\phi',\theta',-\phi') \qquad (40)$$

and

$$\tilde{D}_2 = D_2(\kappa=1) = D_{\mu d,0}^{L_1 *}(\phi,\theta,-\phi)\, D_{d',0}^{L_3}(\phi'',\theta'',-\phi'')\, D_{\mu d,d'}^{L_2}(\phi',\theta',-\phi') . \qquad (41)$$

The coefficients of the D^J angular functions in (39) can be obtained from

$$\gamma_{L_3}\, \beta_{L_1} (-1)^{\frac{1}{2}(1+\mu)L_2}$$

$$\times \left\{ \alpha_{d+}^{L_1L_2} \left[\varepsilon_{d'+}^{L_3L_2} \left(1+(-1)^{L_2}\right)\left(1+\delta_{d0}\right)\left(1+\delta_{d'0}\right) + \varepsilon_{d'-}^{L_3L_2} \left(1-(-1)^{L_2}\right)\left(1+\delta_{d0}\right)\left(1-\delta_{d'0}\right) \right] \right.$$

$$\left. + \alpha_{d-}^{L_1L_2} \left[\varepsilon_{d'+}^{L_3L_2} \left(1+(-1)^{L_2}\right)\left(1-\delta_{d0}\right)\left(1+\delta_{d'0}\right) + \varepsilon_{d'-}^{L_3L_2} \left(1-(-1)^{L_2}\right)\left(1-\delta_{d0}\right)\left(1-\delta_{d'0}\right) \right] \right\}$$

$$= 8(2L_1+1)(2L_2+1)(2L_3+1) \int W_\mu(\theta,\phi;\theta',\phi';\theta'',\phi'')\, \tilde{D}_1^*\, d\Omega\, d\Omega'\, d\Omega'' , \qquad (42)$$

where L_3 can only take the values 0 and 2.

Similarly, the normalized combined angular distribution of γ_1, γ_2 and e^- where only the polarization κ of γ_2 is measured can be written as

$$W_\kappa(\theta,\phi;\theta',\phi';\theta'',\phi'') = \frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}} \sum_{\mu}^{\pm 1} W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$$

$$= \frac{1}{8(4\pi)^3} \sum_{L_3}^{0,2} \sum_{L_1}^{0,2,4} \gamma_{L_3} \beta_{L_1} \sum_{L_2}^{0\to 2J} (-1)^{\frac{1}{2}(1-\kappa)L_2} \sum_{d}^{0\to d_m} \sum_{d'}^{0\to d'_m}$$

$$\times\Big\{ \alpha_{d+}^{L_1L_2} \Big[\varepsilon_{d'+}^{L_3L_2} \big((-1)^{L_2}+1\big)\big(\hat{D}_1+\hat{D}_1^*+\hat{D}_2+\hat{D}_2^*\big) + \varepsilon_{d'-}^{L_3L_2} \big((-1)^{L_2}+1\big)\big(\hat{D}_1-\hat{D}_1^*+\hat{D}_2-\hat{D}_2^*\big)\Big]$$

$$+ \alpha_{d-}^{L_1L_2} \Big[\varepsilon_{d'+}^{L_3L_2} \big((-1)^{L_2}-1\big)\big(\hat{D}_1-\hat{D}_1^*-\hat{D}_2+\hat{D}_2^*\big) + \varepsilon_{d'-}^{L_3L_2} \big((-1)^{L_2}-1\big)\big(\hat{D}_1+\hat{D}_1^*-\hat{D}_2-\hat{D}_2^*\big)\Big]\Big\}, \qquad (43)$$

where

$$\hat{D}_1 = D_1(\mu=1) = D_{-d,0}^{L_1*}(\phi,\theta,-\phi)\, D_{\kappa d',0}^{L_3}(\phi'',\theta'',-\phi'')\, D_{-d,\kappa d'}^{L_2}(\phi',\theta',-\phi') \qquad (44)$$

and

$$\hat{D}_2 = D_2(\mu=1) = D_{d,0}^{L_1*}(\phi,\theta,-\phi)\, D_{\kappa d',0}^{L_3}(\phi'',\theta'',-\phi'')\, D_{d,\kappa d'}^{L_2}(\phi',\theta',-\phi'). \qquad (45)$$

The coefficients in (43) can be obtained from

$$\gamma_{L_3} \beta_{L_1} (-1)^{\frac{1}{2}(1-\kappa)L_2}$$

$$\times\Big\{ \alpha_{d+}^{L_1L_2} \Big[\varepsilon_{d'+}^{L_3L_2} \big((-1)^{L_2}+1\big)\big(1+\delta_{d0}\big)\big(1+\delta_{d'0}\big) + \varepsilon_{d'-}^{L_3L_2} \big((-1)^{L_2}+1\big)\big(1+\delta_{d0}\big)\big(1-\delta_{d'0}\big)\Big]$$

$$+ \alpha_{d-}^{L_1L_2} \Big[\varepsilon_{d'+}^{L_3L_2} \big((-1)^{L_2}-1\big)\big(1-\delta_{d0}\big)\big(1+\delta_{d'0}\big) + \varepsilon_{d'-}^{L_3L_2} \big((-1)^{L_2}-1\big)\big(1-\delta_{d0}\big)\big(1-\delta_{d'0}\big)\Big]\Big\}$$

$$= 8(2L_1+1)(2L_2+1)(2L_3+1) \int W_\kappa(\theta,\phi;\theta',\phi';\theta'',\phi'')\, \hat{D}_1^*\, d\Omega\, d\Omega'\, d\Omega'', \qquad (46)$$

where again L_3 can only take the values 0 and 2.

If we are only interested in measuring the polarization α_1 of e^-, the combined angular distribution of γ_1, γ_2 and e^- will become

$$W_{\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') = \frac{1}{4} \sum_{\kappa}^{\pm 1} \sum_{\mu}^{\pm 1} W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$$

$$= \frac{1}{16(4\pi)^3} \sum_{L_3}^{0,1,2} \sum_{L_1}^{0,2,4} \gamma_{L_3} \beta_{L_1} \sum_{L_2}^{0\to 2J} \sum_{d}^{0\to d_m} \sum_{d'}^{0\to d'_m} \Big\{ \alpha_{d+}^{L_1L_2} \big((-1)^{L_2}+1\big)\Big[\varepsilon_{d'+}^{L_3L_2} \big(1+(-1)^{L_3}\big)$$

$$\times\big(D'_1+D'^*_1+D'_2+D'^*_2\big) + \varepsilon_{d'-}^{L_3L_2} \big(1-(-1)^{L_3}\big)\big(D'_1-D'^*_1+D'_2-D'^*_2\big)\Big] + \alpha_{d-}^{L_1L_2} \big((-1)^{L_2}-1\big)$$

$$\times\Big[\varepsilon_{d'+}^{L_3L_2} \big(1-(-1)^{L_3}\big)\big(D'_1-D'^*_1-D'_2+D'^*_2\big) + \varepsilon_{d'-}^{L_3L_2} \big(1+(-1)^{L_3}\big)\big(D'_1+D'^*_1-D'_2-D'^*_2\big)\Big]\Big\}, \qquad (47)$$

where

$$D'_1 = D_1(\mu=\kappa=1) = D_{-d,0}^{L_1*}(\phi,\theta,-\phi)\, D_{d',0}^{L_3}(\phi'',\theta'',-\phi'')\, D_{-d,d'}^{L_2}(\phi',\theta',-\phi') \qquad (48)$$

and

$$D'_2 = D_2(\mu=\kappa=1) = D^{L_1 *}_{d,0}(\phi,\theta,-\phi)\, D^{L_3}_{d',0}(\phi'',\theta'',-\phi'')\, D^{L_2}_{d,d'}(\phi',\theta',-\phi')\,. \tag{49}$$

The coefficients in (47) can be obtained from

$$\begin{aligned}
&\gamma_{L_3}\,\beta_{L_1} \\
&\times\Big\{\alpha^{L_1L_2}_{d+}\left((-1)^{L_2}+1\right)\Big[\varepsilon^{L_3L_2}_{d'+}\left(1+(-1)^{L_3}\right)\left(1+\delta_{d0}\right)\left(1+\delta_{d'0}\right)+\varepsilon^{L_3L_2}_{d'-}\left(1-(-1)^{L_3}\right)\left(1+\delta_{d0}\right)\left(1-\delta_{d'0}\right)\Big] \\
&+\alpha^{L_1L_2}_{d-}\left((-1)^{L_2}-1\right)\Big[\varepsilon^{L_3L_2}_{d'+}\left(1-(-1)^{L_3}\right)\left(1-\delta_{d0}\right)\left(1+\delta_{d'0}\right)+\varepsilon^{L_3L_2}_{d'-}\left(1+(-1)^{L_3}\right)\left(1-\delta_{d0}\right)\left(1-\delta_{d'0}\right)\Big]\Big\} \\
&=16(2L_1+1)(2L_2+1)(2L_3+1)\int W_{\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\, D'^{*}_1\, d\Omega\, d\Omega'\, d\Omega''\,,
\end{aligned} \tag{50}$$

where this time L_3 can take the values 0, 1 and 2. If we now average over the polarizations α_1 of e^- in (47) as well, we get

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha_1}^{\pm\frac{1}{2}} W_{\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') = \frac{1}{8}\sum_{\kappa}^{\pm 1}\sum_{\mu}^{\pm 1}\sum_{\alpha_1}^{\pm\frac{1}{2}} W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') \\
&=\frac{1}{8(4\pi)^3}\sum_{L_3}^{0,2}\sum_{L_1}^{0,2,4}\gamma_{L_3}\,\beta_{L_1}\sum_{L_2}^{0\to 2J}\sum_{d}^{0\to d_m}\sum_{d'}^{0\to d'_m}\Big\{\Big[\alpha^{L_1L_2}_{d+}\varepsilon^{L_3L_2}_{d'+}+\alpha^{L_1L_2}_{d-}\varepsilon^{L_3L_2}_{d'-}\Big] \\
&+(-1)^{L_2}\Big[\alpha^{L_1L_2}_{d+}\varepsilon^{L_3L_2}_{d'+}-\alpha^{L_1L_2}_{d-}\varepsilon^{L_3L_2}_{d'-}\Big]\Big\}\Big[\left(D'_2+D'^{*}_2\right)+(-1)^{L_2}\left(D'_1+D'^{*}_1\right)\Big].
\end{aligned} \tag{51}$$

Using (35) and (36), the terms inside the braces of (51) can be simplified as

$$\Big[\alpha^{L_1L_2}_{d+}\,\varepsilon^{L_3L_2}_{d'+}+\alpha^{L_1L_2}_{d-}\,\varepsilon^{L_3L_2}_{d'-}\Big]+(-1)^{L_2}\Big[\alpha^{L_1L_2}_{d+}\,\varepsilon^{L_3L_2}_{d'+}-\alpha^{L_1L_2}_{d-}\,\varepsilon^{L_3L_2}_{d'-}\Big]=2\,\alpha^{L_1L_2}_{d}\,\varepsilon^{L_3L_2}_{d'}\,, \tag{52}$$

where

$$\begin{aligned}
&\alpha^{L_1L_2}_{d} = (-1)^{J+1}\sqrt{5(2J+1)}\left(1-\frac{\delta_{d0}}{2}\right)\sum_{s(d)}\Big[A_{\frac{s+d}{2}}\,A^{*}_{\frac{s-d}{2}}+(-1)^{L_2}A^{*}_{\frac{s+d}{2}}\,A_{\frac{s-d}{2}}\Big] \\
&\qquad\times\left\langle JJ;\frac{s+d}{2},-\frac{s-d}{2}\middle| L_2\, d\right\rangle\left\langle 22;\frac{s+d-2}{2},-\frac{s-d-2}{2}\middle| L_1\, d\right\rangle, \\
&s(d) = -(2J-d),\,-(2J-d)+2,\,\ldots,\,+(2J-d)\,,
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\varepsilon^{L_3L_2}_{d'} = (-1)^{J}\sqrt{3(2J+1)}\left(1-\frac{\delta_{d'0}}{2}\right)\sum_{s'(d')}\Big[E_{\frac{s'+d'}{2}}\,E^{*}_{\frac{s'-d'}{2}}+(-1)^{L_2}E^{*}_{\frac{s'+d'}{2}}\,E_{\frac{s'-d'}{2}}\Big] \\
&\qquad\times\left\langle JJ;\frac{s'+d'}{2},-\frac{s'-d'}{2}\middle| L_2\, d'\right\rangle\left\langle 11;\frac{s'+d'-2}{2},-\frac{s'-d'-2}{2}\middle| L_3\, d'\right\rangle, \\
&s'(d') = d',\, d'+2,\,\ldots,\, 2J-d'\,.
\end{aligned} \tag{54}$$

By combining (52) and (51), we now recover the results in [4], where the polarizations of the decay particles are not measured.

Using (42), (46) or (50) it can be seen that once the combined angular distribution

$W_{\mu}(\theta,\phi;\theta',\phi';\theta'',\phi'')$, $W_{\kappa}(\theta,\phi;\theta',\phi';\theta'',\phi'')$ or $W_{\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$ is measured, one can also get the same information on the helicity amplitudes as one obtained from measuring $W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$ where the polarizations of the three particles γ_1, γ_2 and e^- are observed. In other words, by measuring the combined angular distribution of the decay particles γ_1, γ_2 and e^- and the polarization of any one particle, we can get complete information on the helicity amplitudes in the radiative decay processes ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$. In addition, we can also get the relative magnitudes of the helicity amplitudes in the final decay process $\psi \to e^+e^-$.

3 Partially integrated angular distributions

The partially integrated angular distributions obtained from (28) will look a lot simpler and we will gain greater insight from them. There are three different cases in which the polarization and the angular distribution of only one particle are measured. We find that these results are just the same as the single-particle angular distribution functions given in [4], where the polarizations of the individual particles are not measured. So the measurement of the polarizations for the single-particle angular distributions does not give us any further information. Nevertheless, we will find that the measurement of the polarizations of the decay particles can provide us more information on the helicity amplitudes when we measure the simultaneous angular distributions of two particles. We now consider three different cases of two-particle angular distributions. In deriving these results, we will frequently make use of (38) and the following property of the D^J functions.

$$\int_0^{2\pi} d\phi \int_0^{\pi} D^{L*}_{MM'}(\phi,\theta,-\phi)\sin\theta\, d\theta = \int_0^{2\pi} d\phi \int_0^{\pi} D^{L}_{MM'}(\phi,\theta,-\phi)\sin\theta\, d\theta$$

$$= 2\pi\, \delta_{M-M',0} \int_0^{\pi} d^L_{MM'}(\theta)\sin\theta\, d\theta = 2\pi\, k_{LM}\,, \tag{55}$$

where

$$k_{LM} = \int_0^{\pi} d^L_{MM}(\theta)\sin\theta\, d\theta\,. \tag{56}$$

We will express the final results in terms of the orthogonal spherical harmonics by making use of the relation:

$$D^L_{M0} = \sqrt{\frac{4\pi}{2L+1}}\, Y^*_{LM} \tag{57}$$

Since there is only one independent helicity amplitude in each of the radiative decay processes ${}^3D_2 \to \chi_0 + \gamma_1$ and $\chi_0 \to \psi + \gamma_2$ and that is fixed by normalization, we will only concentrate on the $J=1$ and the $J=2$ cases.

Case 1: We will first integrate over the angles (θ,ϕ) or the direction of the first photon γ_1 and then average over the polarizations of γ_1. The combined angular distribution of γ_2 and e^- and the polarization of only one of the two particles are measured. The explicit expressions are given in the following.

$J=1$ (only κ is measured):

$$\begin{aligned}
\tilde{\tilde{W}}_\kappa(\theta',\phi';\theta'',\phi'') &= \frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}}\sum_{\mu}^{\pm 1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta\, d\theta\, d\phi \\
&= \frac{1}{4\pi}\left[Y_{00}(\theta'')Y_{00}(\theta') - \frac{1}{\sqrt{5}}\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2\right)\left(|E_0|^2 - \frac{1}{2}|E_1|^2\right)Y_{00}(\theta'')Y_{20}(\theta')\right] \\
&+ \frac{1}{4\pi}\gamma_2\left\{\frac{1}{\sqrt{10}}\left(|E_0|^2 - 2|E_1|^2\right)Y_{20}(\theta'')Y_{00}(\theta') - \frac{1}{5\sqrt{2}}\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2\right)Y_{20}(\theta'')Y_{20}(\theta')\right. \\
&- \frac{3}{5\sqrt{2}}\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2\right)\mathrm{Re}\left(E_1E_0^*\right)\mathrm{Re}\left[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')\right] \\
&\left.- \frac{3}{5\sqrt{2}}\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2\right)\mathrm{Im}\left(E_1E_0^*\right)\mathrm{Im}\left[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')\right]\right\}.
\end{aligned} \tag{58}$$

$J=1$ (only α_1 is measured):

$$\begin{aligned}
\tilde{\tilde{W}}_{\alpha_1}(\theta',\phi';\theta'',\phi'') &= \frac{1}{4}\sum_{\kappa}^{\pm 1}\sum_{\mu}^{\pm 1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta\, d\theta\, d\phi \\
&= \frac{1}{4\pi}\left[Y_{00}(\theta'')Y_{00}(\theta') - \frac{1}{\sqrt{5}}\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2\right)\left(|E_0|^2 - \frac{1}{2}|E_1|^2\right)Y_{00}(\theta'')Y_{20}(\theta')\right] \\
&+ \frac{1}{4\pi}\gamma_1\left\{\sqrt{\frac{3}{10}}\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2\right)\mathrm{Im}\left(E_1E_0^*\right)\mathrm{Im}\left[Y_{11}(\theta'',\phi'')Y_{21}(\theta',\phi')\right]\right\} \\
&+ \frac{1}{4\pi}\gamma_2\left\{\frac{1}{\sqrt{10}}\left(|E_0|^2 - 2|E_1|^2\right)Y_{20}(\theta'')Y_{00}(\theta') - \frac{1}{5\sqrt{2}}\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2\right)Y_{20}(\theta'')Y_{20}(\theta')\right. \\
&\left.- \frac{3}{5\sqrt{2}}\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2\right)\mathrm{Re}\left(E_1E_0^*\right)\mathrm{Re}\left[Y_{21}(\theta'',\phi'')Y_{21}(\theta',\phi')\right]\right\}.
\end{aligned} \tag{59}$$

An inspection of (58) and (59) shows that we can determine the relative magnitudes of the E helicity amplitudes as well as the relative phase between E_0 and E_1 when we measure the combined angular distribution of γ_2 and e^- and the polarization of either one of the particles for the $J=1$ case. In (58), for example, we can first get the combination

$\left(|E_0|^2 - 2|E_1|^2\right)$ from the coefficient of $Y_{20}(\theta'')Y_{00}(\theta')$. From this combination and the normalization $|E_0|^2 + |E_1|^2 = 1$, we can determine $|E_0|$ and $|E_1|$. Then we can obtain $\mathrm{Re}\left(E_1E_0^*\right)$ and $\mathrm{Im}\left(E_1E_0^*\right)$ from the coefficients of $\mathrm{Re}\left[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')\right]$ and $\mathrm{Im}\left[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')\right]$, respectively, since the factor $\left(|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2\right)$ in these coefficients can be obtained from the coefficient of $Y_{20}(\theta'')Y_{20}(\theta')$. Therefore the relative phase between E_0 and E_1 can also be known. It should be noted that the measurement of the polarization of one of the decay particles is essential for getting the sine of the relative phase between the two independent E helicity amplitudes.

$J = 2$ (only κ is measured):

$$\tilde{\tilde{W}}_\kappa(\theta',\phi';\theta'',\phi'')$$

$$
\begin{aligned}
&= \frac{1}{4\pi}\Bigg\{Y_{00}(\theta'')Y_{00}(\theta') + \frac{\sqrt{5}}{7}\left(|A_{-1}|^2 + 2|A_0|^2 + |A_1|^2 - 2|A_2|^2\right)\left(|E_0|^2 + \frac{1}{2}|E_1|^2 - |E_2|^2\right)Y_{00}(\theta'')Y_{20}(\theta') \\
&- \frac{4}{7}\left(|A_{-1}|^2 - \frac{3}{2}|A_0|^2 + |A_1|^2 - \frac{1}{4}|A_2|^2\right)\left(|E_0|^2 - \frac{2}{3}|E_1|^2 + \frac{1}{6}|E_2|^2\right)Y_{00}(\theta'')Y_{40}(\theta')\Bigg\} \\
&+ \frac{1}{4\pi}\gamma_2\Bigg\{\frac{1}{\sqrt{10}}\left(|E_0|^2 - 2|E_1|^2 + |E_2|^2\right)Y_{20}(\theta'')Y_{00}(\theta') \\
&+ \frac{1}{7\sqrt{2}}\left(|A_{-1}|^2 + 2|A_0|^2 + |A_1|^2 - 2|A_2|^2\right)\left(|E_0|^2 - |E_1|^2 - |E_2|^2\right)Y_{20}(\theta'')Y_{20}(\theta') \\
&+ \frac{\sqrt{3}}{7\sqrt{2}}\left(|A_{-1}|^2 + 2|A_0|^2 + |A_1|^2 - 2|A_2|^2\right)\left[\mathrm{Re}\left(E_1E_0^*\right) - \sqrt{6}\,\mathrm{Re}\left(E_2E_1^*\right)\right]\mathrm{Re}\left[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')\right] \\
&+ \frac{\sqrt{3}}{7\sqrt{2}}\left(|A_{-1}|^2 + 2|A_0|^2 + |A_1|^2 - 2|A_2|^2\right)\left[\mathrm{Im}\left(E_1E_0^*\right) - \sqrt{6}\,\mathrm{Im}\left(E_2E_1^*\right)\right]\mathrm{Im}\left[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')\right] \\
&- \frac{2\sqrt{3}}{7}\left(|A_{-1}|^2 + 2|A_0|^2 + |A_1|^2 - 2|A_2|^2\right)\mathrm{Re}\left(E_2E_0^*\right)\mathrm{Re}\left[Y_{2,2\kappa}(\theta'',\phi'')Y_{2,2\kappa}(\theta',\phi')\right] \\
&- \frac{2\sqrt{3}}{7}\left(|A_{-1}|^2 + 2|A_0|^2 + |A_1|^2 - 2|A_2|^2\right)\mathrm{Im}\left(E_2E_0^*\right)\mathrm{Im}\left[Y_{2,2\kappa}(\theta'',\phi'')Y_{2,2\kappa}(\theta',\phi')\right] \\
&- \frac{4}{7\sqrt{10}}\left(|A_{-1}|^2 - \frac{3}{2}|A_0|^2 + |A_1|^2 - \frac{1}{4}|A_2|^2\right)\left(|E_0|^2 + \frac{4}{3}|E_1|^2 + \frac{1}{6}|E_2|^2\right)Y_{20}(\theta'')Y_{40}(\theta') \\
&- \frac{4}{7\sqrt{6}}\left(|A_{-1}|^2 - \frac{3}{2}|A_0|^2 + |A_1|^2 - \frac{1}{4}|A_2|^2\right)\left[\sqrt{6}\,\mathrm{Re}\left(E_1E_0^*\right) + \mathrm{Re}\left(E_2E_1^*\right)\right]\mathrm{Re}\left[Y_{2\kappa}(\theta'',\phi'')Y_{4\kappa}(\theta',\phi')\right] \\
&- \frac{4}{7\sqrt{6}}\left(|A_{-1}|^2 - \frac{3}{2}|A_0|^2 + |A_1|^2 - \frac{1}{4}|A_2|^2\right)\left[\sqrt{6}\,\mathrm{Im}\left(E_1E_0^*\right) + \mathrm{Im}\left(E_2E_1^*\right)\right]\mathrm{Im}\left[Y_{2\kappa}(\theta'',\phi'')Y_{4\kappa}(\theta',\phi')\right]
\end{aligned}
$$

$$-\frac{4}{7}\left(|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right)\mathrm{Re}\left(E_2E_0^*\right)\mathrm{Re}\left[Y_{2,2\kappa}(\theta'',\phi'')Y_{4,2\kappa}(\theta',\phi')\right]$$
$$-\frac{4}{7}\left(|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right)\mathrm{Im}\left(E_2E_0^*\right)\mathrm{Im}\left[Y_{2,2\kappa}(\theta'',\phi'')Y_{4,2\kappa}(\theta',\phi')\right]\Bigg\}. \qquad (60)$$

$J=2$ (only α_1 is measured):

$$\tilde{\tilde{W}}_{\alpha_1}(\theta',\phi';\theta'',\phi'')$$

$$=\frac{1}{4\pi}\Bigg\{Y_{00}(\theta'')Y_{00}(\theta')+\frac{\sqrt{5}}{7}\left(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2\right)\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)Y_{00}(\theta'')Y_{20}(\theta')$$
$$-\frac{4}{7}\left(|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right)\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)Y_{00}(\theta'')Y_{40}(\theta')\Bigg\}$$
$$-\frac{\sqrt{10}}{56\pi}\gamma_1\Bigg\{\left(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2\right)\left[\mathrm{Im}\left(E_1E_0^*\right)+\sqrt{6}\,\mathrm{Im}\left(E_2E_1^*\right)\right]\mathrm{Im}\left[Y_{11}(\theta'',\phi'')Y_{21}(\theta',\phi')\right]$$
$$-\frac{4}{3}\left(|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right)\times\left[\sqrt{6}\,\mathrm{Im}\left(E_1E_0^*\right)-\mathrm{Im}\left(E_2E_1^*\right)\right]\mathrm{Im}\left[Y_{11}(\theta'',\phi'')Y_{41}(\theta',\phi')\right]\Bigg\}$$
$$+\frac{1}{4\pi}\gamma_2\Bigg\{\frac{1}{\sqrt{10}}\left(|E_0|^2-2|E_1|^2+|E_2|^2\right)Y_{20}(\theta'')Y_{00}(\theta')$$
$$+\frac{1}{7\sqrt{2}}\left(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2\right)\left(|E_0|^2-|E_1|^2-|E_2|^2\right)Y_{20}(\theta'')Y_{20}(\theta')$$
$$+\frac{\sqrt{3}}{7\sqrt{2}}\left(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2\right)\left[\mathrm{Re}\left(E_1E_0^*\right)-\sqrt{6}\,\mathrm{Re}\left(E_2E_1^*\right)\right]\mathrm{Re}\left[Y_{21}(\theta'',\phi'')Y_{21}(\theta',\phi')\right]$$
$$-\frac{2\sqrt{3}}{7}\left(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2\right)\mathrm{Re}\left(E_2E_0^*\right)\mathrm{Re}\left[Y_{22}(\theta'',\phi'')Y_{22}(\theta',\phi')\right]$$
$$-\frac{4}{7\sqrt{10}}\left(|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right)\left(|E_0|^2+\frac{4}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)Y_{20}(\theta'')Y_{40}(\theta')$$
$$-\frac{4}{7\sqrt{6}}\left(|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right)\left[\sqrt{6}\,\mathrm{Re}\left(E_1E_0^*\right)+\mathrm{Re}\left(E_2E_1^*\right)\right]\mathrm{Re}\left[Y_{21}(\theta'',\phi'')Y_{41}(\theta',\phi')\right]$$
$$-\frac{4}{7}\left(|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right)\mathrm{Re}\left(E_2E_0^*\right)\mathrm{Re}\left[Y_{22}(\theta'',\phi'')Y_{42}(\theta',\phi')\right]\Bigg\}. \qquad (61)$$

Using (60) or (61) we can then determine the relative magnitudes as well as the relative phases among the E helicity amplitudes for the $J=2$ case. Take (60) for example. First from the normalization of the E helicity amplitudes and the coefficient of $Y_{20}(\theta'')Y_{00}(\theta')$ we have $|E_0|^2+|E_1|^2+|E_2|^2=1$ and $|E_0|^2-2|E_1|^2+|E_2|^2=\eta$, where η is a known constant. A third relation among the magnitudes of the E helicity amplitudes can then be obtained from dividing the coefficient of $Y_{20}(\theta'')Y_{20}(\theta')$ by that of $Y_{00}(\theta'')Y_{20}(\theta')$. Thus we can get the relative magnitudes of the E helicity amplitudes by solving these three equations. Once these magnitudes are known, we can then determine the factors $\left(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2\right)$ and $\left(|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right)$ from the coefficients of $Y_{00}(\theta'')Y_{20}(\theta')$ and $Y_{00}(\theta'')Y_{40}(\theta')$. After getting these factors, we can obtain $\mathrm{Re}\left(E_1E_0^*\right)$ and $\mathrm{Re}\left(E_2E_1^*\right)$ from the coefficients of $\mathrm{Re}\left[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')\right]$ and $\mathrm{Re}\left[Y_{2\kappa}(\theta'',\phi'')Y_{4\kappa}(\theta',\phi')\right]$. Likewise, $\mathrm{Im}\left(E_1E_0^*\right)$ and $\mathrm{Im}\left(E_2E_1^*\right)$ can be obtained from the coefficients of $\mathrm{Im}\left[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')\right]$ and $\mathrm{Im}\left[Y_{2\kappa}(\theta'',\phi'')Y_{4\kappa}(\theta',\phi')\right]$. Thus the relative phases among the E helicity amplitudes can also be determined. As in the $J=1$ case, the measurement of the polarization of one of the two particles is essential for getting the sines of the relative phases among the E helicity amplitudes.

Case 2: We will integrate over (θ'',ϕ'') and average over the polarizations of the electron. The combined angular distribution of the two photons and the polarization of either one of them are measured. We obtain

$J=1$ (only μ is measured):

$$\tilde{\tilde{W}}_\mu(\theta,\phi;\theta',\phi')=\frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}}\sum_{\kappa}^{\pm1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta''\,d\theta''\,d\phi''$$

$$=\frac{1}{4\pi}\Bigg\{Y_{00}(\theta')Y_{00}(\theta)-\frac{1}{\sqrt{5}}\left(|A_{-1}|^2-2|A_0|^2+|A_1|^2\right)\left(|E_0|^2-\frac{1}{2}|E_1|^2\right)Y_{20}(\theta')Y_{00}(\theta)$$
$$-\frac{\sqrt{5}}{7}\left(|A_{-1}|^2-\frac{1}{2}|A_0|^2-|A_1|^2\right)Y_{00}(\theta')Y_{20}(\theta)$$
$$+\frac{1}{7}\left(|A_{-1}|^2+|A_0|^2-|A_1|^2\right)\left(|E_0|^2-\frac{1}{2}|E_1|^2\right)Y_{20}(\theta')Y_{20}(\theta)$$
$$+\frac{\sqrt{3}}{7}\left(|E_0|^2-\frac{1}{2}|E_1|^2\right)\left[\sqrt{6}\,\mathrm{Re}\left(A_0A_{-1}^*\right)-\mathrm{Re}\left(A_1A_0^*\right)\right]\mathrm{Re}\left[Y_{2\mu}(\theta',\phi')Y_{2\mu}^*(\theta,\phi)\right]$$
$$-\frac{\sqrt{3}}{7}\left(|E_0|^2-\frac{1}{2}|E_1|^2\right)\left[\sqrt{6}\,\mathrm{Im}\left(A_0A_{-1}^*\right)-\mathrm{Im}\left(A_1A_0^*\right)\right]\mathrm{Im}\left[Y_{2\mu}(\theta',\phi')Y_{2\mu}^*(\theta,\phi)\right]$$
$$+\frac{2\sqrt{6}}{7}\left(|E_0|^2-\frac{1}{2}|E_1|^2\right)\mathrm{Re}\left(A_1A_{-1}^*\right)\mathrm{Re}\left[Y_{2,2\mu}(\theta',\phi')Y_{2,2\mu}^*(\theta,\phi)\right]$$
$$-\frac{2\sqrt{6}}{7}\left(|E_0|^2-\frac{1}{2}|E_1|^2\right)\mathrm{Im}\left(A_1A_{-1}^*\right)\mathrm{Im}\left[Y_{2,2\mu}(\theta',\phi')Y_{2,2\mu}^*(\theta,\phi)\right]$$
$$-\frac{2}{21}\left(|A_{-1}|^2-4|A_0|^2+6|A_1|^2\right)Y_{00}(\theta')Y_{40}(\theta)$$

$$+\frac{2}{21\sqrt{5}}\left(\left|A_{-1}\right|^2+8\left|A_0\right|^2+6\left|A_1\right|^2\right)\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)Y_{20}(\theta')Y_{40}(\theta)$$

$$+\frac{4\sqrt{3}}{21}\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)\left[\mathrm{Re}\left(A_0A_{-1}^*\right)+\sqrt{6}\,\mathrm{Re}\left(A_1A_0^*\right)\right]\mathrm{Re}\left[Y_{2\mu}(\theta',\phi')Y_{4\mu}^*(\theta,\phi)\right]$$

$$-\frac{4\sqrt{3}}{21}\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)\left[\mathrm{Im}\left(A_0A_{-1}^*\right)+\sqrt{6}\,\mathrm{Im}\left(A_1A_0^*\right)\right]\mathrm{Im}\left[Y_{2\mu}(\theta',\phi')Y_{4\mu}^*(\theta,\phi)\right]$$

$$+\frac{4\sqrt{2}}{7}\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)\mathrm{Re}\left(A_1A_{-1}^*\right)\mathrm{Re}\left[Y_{2,2\mu}(\theta',\phi')Y_{4,2\mu}^*(\theta,\phi)\right]$$

$$-\frac{4\sqrt{2}}{7}\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)\mathrm{Im}\left(A_1A_{-1}^*\right)\mathrm{Im}\left[Y_{2,2\mu}(\theta',\phi')Y_{4,2\mu}^*(\theta,\phi)\right]\Bigg\}. \qquad (62)$$

$J=1$ (only κ is measured):

$$\tilde{\tilde{W}}_\kappa(\theta,\phi;\theta',\phi')=\frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}}\sum_{\mu}^{\pm1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta''\,d\theta''\,d\phi''$$

$$=\frac{1}{4\pi}\Bigg\{Y_{00}(\theta')Y_{00}(\theta)-\frac{1}{\sqrt{5}}\left(\left|A_{-1}\right|^2-2\left|A_0\right|^2+\left|A_1\right|^2\right)\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)Y_{20}(\theta')Y_{00}(\theta)$$

$$-\frac{\sqrt{5}}{7}\left(\left|A_{-1}\right|^2-\frac{1}{2}\left|A_0\right|^2-\left|A_1\right|^2\right)Y_{00}(\theta')Y_{20}(\theta)$$

$$+\frac{\sqrt{15}}{14}(-1)^{\frac{1}{2}(1-\kappa)}\left|E_1\right|^2\left[\sqrt{6}\,\mathrm{Im}\left(A_0A_{-1}^*\right)+\mathrm{Im}\left(A_1A_0^*\right)\right]\mathrm{Im}\left[Y_{11}(\theta',\phi')Y_{21}^*(\theta,\phi)\right]$$

$$+\frac{1}{7}\left(\left|A_{-1}\right|^2+\left|A_0\right|^2-\left|A_1\right|^2\right)\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)Y_{20}(\theta')Y_{20}(\theta)$$

$$+\frac{\sqrt{3}}{7}\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)\left[\sqrt{6}\,\mathrm{Re}\left(A_0A_{-1}^*\right)-\mathrm{Re}\left(A_1A_0^*\right)\right]\mathrm{Re}\left[Y_{21}(\theta',\phi')Y_{21}^*(\theta,\phi)\right]$$

$$+\frac{2\sqrt{6}}{7}\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)\mathrm{Re}\left(A_1A_{-1}^*\right)\mathrm{Re}\left[Y_{22}(\theta',\phi')Y_{22}^*(\theta,\phi)\right]$$

$$-\frac{2}{21}\left(\left|A_{-1}\right|^2-4\left|A_0\right|^2+6\left|A_1\right|^2\right)Y_{00}(\theta')Y_{40}(\theta)$$

$$+\frac{2\sqrt{15}}{21}(-1)^{\frac{1}{2}(1-\kappa)}\left|E_1\right|^2\left[\mathrm{Im}\left(A_0A_{-1}^*\right)-\sqrt{6}\,\mathrm{Im}\left(A_1A_0^*\right)\right]\mathrm{Im}\left[Y_{11}(\theta',\phi')Y_{41}^*(\theta,\phi)\right]$$

$$+\frac{2}{21\sqrt{5}}\left(\left|A_{-1}\right|^2+8\left|A_0\right|^2+6\left|A_1\right|^2\right)\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)Y_{20}(\theta')Y_{40}(\theta)$$

$$+\frac{4\sqrt{3}}{21}\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)\left[\mathrm{Re}\left(A_0A_{-1}^*\right)+\sqrt{6}\,\mathrm{Re}\left(A_1A_0^*\right)\right]\mathrm{Re}\left[Y_{21}(\theta',\phi')Y_{41}^*(\theta,\phi)\right]$$

$$+\frac{4\sqrt{2}}{7}\left(\left|E_0\right|^2-\frac{1}{2}\left|E_1\right|^2\right)\mathrm{Re}\left(A_1A_{-1}^*\right)\mathrm{Re}\left[Y_{22}(\theta',\phi')Y_{42}^*(\theta,\phi)\right]\Bigg\}. \qquad (63)$$

J = 2 (only μ is measured):

$$\tilde{\tilde{W}}_{\mu}(\theta,\phi;\theta',\phi')$$

$$=\frac{1}{4\pi}\left\{Y_{00}(\theta')Y_{00}(\theta)+\frac{\sqrt{5}}{7}\left(|A_1|^2+2|A_0|^2+|A_1|^2-2|A_2|^2\right)\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)Y_{20}(\theta')Y_{00}(\theta)\right.$$

$$-\frac{4}{7}\left(|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right)\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)Y_{40}(\theta')Y_{00}(\theta)$$

$$-\frac{\sqrt{5}}{7}\left(|A_{-1}|^2-\frac{1}{2}|A_0|^2-|A_1|^2-\frac{1}{2}|A_2|^2\right)Y_{00}(\theta')Y_{20}(\theta)$$

$$-\frac{5}{49}\left(|A_{-1}|^2-|A_0|^2-|A_1|^2+|A_2|^2\right)\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)Y_{20}(\theta')Y_{20}(\theta)$$

$$+\frac{5\sqrt{6}}{49}\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)$$

$$\times\left[\mathrm{Re}\left(A_0A_{-1}^*\right)-\frac{1}{\sqrt{6}}\mathrm{Re}\left(A_1A_0^*\right)+\mathrm{Re}\left(A_2A_1^*\right)\right]\mathrm{Re}\left[Y_{2\mu}(\theta',\phi')Y_{2\mu}^*(\theta,\phi)\right]$$

$$-\frac{5\sqrt{6}}{49}\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)$$

$$\times\left[\mathrm{Im}\left(A_0A_{-1}^*\right)-\frac{1}{\sqrt{6}}\mathrm{Im}\left(A_1A_0^*\right)+\mathrm{Im}\left(A_2A_1^*\right)\right]\mathrm{Im}\left[Y_{2\mu}(\theta',\phi')Y_{2\mu}^*(\theta,\phi)\right]$$

$$+\frac{10\sqrt{6}}{49}\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)\left[\mathrm{Re}\left(A_1A_{-1}^*\right)+\mathrm{Re}\left(A_2A_0^*\right)\right]\mathrm{Re}\left[Y_{2,2\mu}(\theta',\phi')Y_{2,2\mu}^*(\theta,\phi)\right]$$

$$-\frac{10\sqrt{6}}{49}\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)\left[\mathrm{Im}\left(A_1A_{-1}^*\right)+\mathrm{Im}\left(A_2A_0^*\right)\right]\mathrm{Im}\left[Y_{2,2\mu}(\theta',\phi')Y_{2,2\mu}^*(\theta,\phi)\right]$$

$$+\frac{4\sqrt{5}}{49}\left(|A_{-1}|^2+\frac{3}{4}|A_0|^2-|A_1|^2+\frac{1}{8}|A_2|^2\right)\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)Y_{40}(\theta')Y_{20}(\theta)$$

$$+\frac{30}{49}\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)$$

$$\times\left[\mathrm{Re}\left(A_0A_{-1}^*\right)-\frac{1}{\sqrt{6}}\mathrm{Re}\left(A_1A_0^*\right)-\frac{1}{6}\mathrm{Re}\left(A_2A_1^*\right)\right]\mathrm{Re}\left[Y_{4\mu}(\theta',\phi')Y_{2\mu}^*(\theta,\phi)\right]$$

$$-\frac{30}{49}\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)$$

$$\times\left[\mathrm{Im}\left(A_0A_{-1}^*\right)-\frac{1}{\sqrt{6}}\mathrm{Im}\left(A_1A_0^*\right)-\frac{1}{6}\mathrm{Im}\left(A_2A_1^*\right)\right]\mathrm{Im}\left[Y_{4\mu}(\theta',\phi')Y_{2\mu}^*(\theta,\phi)\right]$$

$$+\frac{20\sqrt{2}}{49}\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)\left[\mathrm{Re}\left(A_1A_{-1}^*\right)-\frac{3}{4}\mathrm{Re}\left(A_2A_0^*\right)\right]\mathrm{Re}\left[Y_{4,2\mu}(\theta',\phi')Y_{2,2\mu}^*(\theta,\phi)\right]$$

$$-\frac{20\sqrt{2}}{49}\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)\left[\mathrm{Im}\left(A_1A_{-1}^*\right)-\frac{3}{4}\mathrm{Im}\left(A_2A_0^*\right)\right]\mathrm{Im}\left[Y_{4,2\mu}(\theta',\phi')Y_{2,2\mu}^*(\theta,\phi)\right]$$

$$-\frac{4}{21}\left(\left|A_{-1}\right|^2-4\left|A_0\right|^2+6\left|A_1\right|^2-4\left|A_2\right|^2\right)Y_{00}(\theta')Y_{40}(\theta)$$

$$-\frac{2\sqrt{5}}{147}\left(\left|A_{-1}\right|^2-8\left|A_0\right|^2+6\left|A_1\right|^2+8\left|A_2\right|^2\right)\left(\left|E_0\right|^2+\frac{1}{2}\left|E_1\right|^2-\left|E_2\right|^2\right)Y_{20}(\theta')Y_{40}(\theta)$$

$$+\frac{20}{147}\left(\left|E_0\right|^2+\frac{1}{2}\left|E_1\right|^2-\left|E_2\right|^2\right)$$

$$\times\left[\mathrm{Re}\left(A_0A_{-1}^*\right)+\sqrt{6}\,\mathrm{Re}\left(A_1A_0^*\right)-6\,\mathrm{Re}\left(A_2A_1^*\right)\right]\mathrm{Re}\left[Y_{2\mu}(\theta',\phi')Y_{4\mu}^*(\theta,\phi)\right]$$

$$-\frac{20}{147}\left(\left|E_0\right|^2+\frac{1}{2}\left|E_1\right|^2-\left|E_2\right|^2\right)$$

$$\times\left[\mathrm{Im}\left(A_0A_{-1}^*\right)+\sqrt{6}\,\mathrm{Im}\left(A_1A_0^*\right)-6\,\mathrm{Im}\left(A_2A_1^*\right)\right]\mathrm{Im}\left[Y_{2\mu}(\theta',\phi')Y_{4\mu}^*(\theta,\phi)\right]$$

$$+\frac{20\sqrt{2}}{49}\left(\left|E_0\right|^2+\frac{1}{2}\left|E_1\right|^2-\left|E_2\right|^2\right)\left[\mathrm{Re}\left(A_1A_{-1}^*\right)-\frac{4}{3}\mathrm{Re}\left(A_2A_0^*\right)\right]\mathrm{Re}\left[Y_{2,2\mu}(\theta',\phi')Y_{4,2\mu}^*(\theta,\phi)\right]$$

$$-\frac{20\sqrt{2}}{49}\left(\left|E_0\right|^2+\frac{1}{2}\left|E_1\right|^2-\left|E_2\right|^2\right)\left[\mathrm{Im}\left(A_1A_{-1}^*\right)-\frac{4}{3}\mathrm{Im}\left(A_2A_0^*\right)\right]\mathrm{Im}\left[Y_{2,2\mu}(\theta',\phi')Y_{4,2\mu}^*(\theta,\phi)\right]$$

$$+\frac{8}{147}\left(\left|A_{-1}\right|^2+6\left|A_0\right|^2+6\left|A_1\right|^2+\left|A_2\right|^2\right)\left(\left|E_0\right|^2-\frac{2}{3}\left|E_1\right|^2+\frac{1}{6}\left|E_2\right|^2\right)Y_{40}(\theta')Y_{40}(\theta)$$

$$+\frac{20\sqrt{6}}{147}\left(\left|E_0\right|^2-\frac{2}{3}\left|E_1\right|^2+\frac{1}{6}\left|E_2\right|^2\right)$$

$$\times\left[\mathrm{Re}\left(A_0A_{-1}^*\right)+\sqrt{6}\,\mathrm{Re}\left(A_1A_0^*\right)+\mathrm{Re}\left(A_2A_1^*\right)\right]\mathrm{Re}\left[Y_{4\mu}(\theta',\phi')Y_{4\mu}^*(\theta,\phi)\right]$$

$$-\frac{20\sqrt{6}}{147}\left(\left|E_0\right|^2-\frac{2}{3}\left|E_1\right|^2+\frac{1}{6}\left|E_2\right|^2\right)$$

$$\times\left[\mathrm{Im}\left(A_0A_{-1}^*\right)+\sqrt{6}\,\mathrm{Im}\left(A_1A_0^*\right)+\mathrm{Im}\left(A_2A_1^*\right)\right]\mathrm{Im}\left[Y_{4\mu}(\theta',\phi')Y_{4\mu}^*(\theta,\phi)\right]$$

$$+\frac{40\sqrt{6}}{147}\left(\left|E_0\right|^2-\frac{2}{3}\left|E_1\right|^2+\frac{1}{6}\left|E_2\right|^2\right)\left[\mathrm{Re}\left(A_1A_{-1}^*\right)+\mathrm{Re}\left(A_2A_0^*\right)\right]\mathrm{Re}\left[Y_{4,2\mu}(\theta',\phi')Y_{4,2\mu}^*(\theta,\phi)\right]$$

$$-\frac{40\sqrt{6}}{147}\left(\left|E_0\right|^2-\frac{2}{3}\left|E_1\right|^2+\frac{1}{6}\left|E_2\right|^2\right)\left[\mathrm{Im}\left(A_1A_{-1}^*\right)+\mathrm{Im}\left(A_2A_0^*\right)\right]\mathrm{Im}\left[Y_{4,2\mu}(\theta',\phi')Y_{4,2\mu}^*(\theta,\phi)\right]$$

$$+\frac{20}{21}\left(\left|E_0\right|^2-\frac{2}{3}\left|E_1\right|^2+\frac{1}{6}\left|E_2\right|^2\right)\mathrm{Re}\left(A_2A_{-1}^*\right)\mathrm{Re}\left[Y_{4,3\mu}(\theta',\phi')Y_{4,3\mu}^*(\theta,\phi)\right]$$

$$-\frac{20}{21}\left(\left|E_0\right|^2-\frac{2}{3}\left|E_1\right|^2+\frac{1}{6}\left|E_2\right|^2\right)\mathrm{Im}\left(A_2A_{-1}^*\right)\mathrm{Im}\left[Y_{4,3\mu}(\theta',\phi')Y_{4,3\mu}^*(\theta,\phi)\right]\Bigg\}. \qquad (64)$$

$J=2$ (only κ is measured):

$$\tilde{\tilde{W}}_{\kappa}(\theta,\phi;\theta',\phi')$$

$$=\frac{1}{4\pi}\left\{Y_{00}(\theta')Y_{00}(\theta)+\frac{\sqrt{5}}{7}\left(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2\right)\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)Y_{20}(\theta')Y_{00}(\theta)\right.$$

$$-\frac{4}{7}\left(|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right)\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)Y_{40}(\theta')Y_{00}(\theta)$$

$$-\frac{\sqrt{5}}{7}\left(|A_{-1}|^2-\frac{1}{2}|A_0|^2-|A_1|^2-\frac{1}{2}|A_2|^2\right)Y_{00}(\theta')Y_{20}(\theta)$$

$$+\frac{\sqrt{30}}{14}(-1)^{\frac{1}{2}(1-\kappa)}\left(|E_1|^2+2|E_2|^2\right)$$

$$\times\left[\mathrm{Im}\left(A_0A_{-1}^*\right)+\frac{1}{\sqrt{6}}\mathrm{Im}\left(A_1A_0^*\right)-\frac{1}{3}\mathrm{Im}\left(A_2A_1^*\right)\right]\mathrm{Im}\left[Y_{11}(\theta',\phi')Y_{21}^*(\theta,\phi)\right]$$

$$-\frac{5}{49}\left(|A_{-1}|^2-|A_0|^2-|A_1|^2+|A_2|^2\right)\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)Y_{20}(\theta')Y_{20}(\theta)$$

$$+\frac{5\sqrt{6}}{49}\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)$$

$$\times\left[\mathrm{Re}\left(A_0A_{-1}^*\right)-\frac{1}{\sqrt{6}}\mathrm{Re}\left(A_1A_0^*\right)+\mathrm{Re}\left(A_2A_1^*\right)\right]\mathrm{Re}\left[Y_{21}(\theta',\phi')Y_{21}^*(\theta,\phi)\right]$$

$$+\frac{10\sqrt{6}}{49}\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)\left[\mathrm{Re}\left(A_1A_{-1}^*\right)+\mathrm{Re}\left(A_2A_0^*\right)\right]\mathrm{Re}\left[Y_{22}(\theta',\phi')Y_{22}^*(\theta,\phi)\right]$$

$$+\frac{2\sqrt{15}}{7\sqrt{7}}(-1)^{\frac{1}{2}(1-\kappa)}\left(|E_1|^2-\frac{1}{2}|E_2|^2\right)$$

$$\times\left[\mathrm{Im}\left(A_0A_{-1}^*\right)+\frac{1}{\sqrt{6}}\mathrm{Im}\left(A_1A_0^*\right)+\frac{1}{2}\mathrm{Im}\left(A_2A_1^*\right)\right]\mathrm{Im}\left[Y_{31}(\theta',\phi')Y_{21}^*(\theta,\phi)\right]$$

$$+\frac{5\sqrt{6}}{7\sqrt{7}}(-1)^{\frac{1}{2}(1-\kappa)}\left(|E_1|^2-\frac{1}{2}|E_2|^2\right)\mathrm{Im}\left(A_2A_0^*\right)\mathrm{Im}\left[Y_{32}(\theta',\phi')Y_{22}^*(\theta,\phi)\right]$$

$$+\frac{4\sqrt{5}}{49}\left(|A_{-1}|^2+\frac{3}{4}|A_0|^2-|A_1|^2+\frac{1}{8}|A_2|^2\right)\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)Y_{40}(\theta')Y_{20}(\theta)$$

$$+\frac{30}{49}\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)$$

$$\times\left[\mathrm{Re}\left(A_0A_{-1}^*\right)-\frac{1}{\sqrt{6}}\mathrm{Re}\left(A_1A_0^*\right)-\frac{1}{6}\mathrm{Re}\left(A_2A_1^*\right)\right]\mathrm{Re}\left[Y_{41}(\theta',\phi')Y_{21}^*(\theta,\phi)\right]$$

$$+\frac{20\sqrt{2}}{49}\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)\left[\mathrm{Re}\left(A_1A_{-1}^*\right)-\frac{3}{4}\mathrm{Re}\left(A_2A_0^*\right)\right]\mathrm{Re}\left[Y_{42}(\theta',\phi')Y_{22}^*(\theta,\phi)\right]$$

$$-\frac{4}{21}\left(|A_{-1}|^2-4|A_0|^2+6|A_1|^2-4|A_2|^2\right)Y_{00}(\theta')Y_{40}(\theta)$$

$$+\frac{2\sqrt{5}}{21}(-1)^{\frac{1}{2}(1-\kappa)}\left(|E_1|^2+2|E_2|^2\right)$$

$$\times\left[\mathrm{Im}\left(A_0A_{-1}^*\right)-\sqrt{6}\,\mathrm{Im}\left(A_1A_0^*\right)+2\,\mathrm{Im}\left(A_2A_1^*\right)\right]\mathrm{Im}\left[Y_{11}(\theta',\phi')Y_{41}^*(\theta,\phi)\right]$$

$$
-\frac{2\sqrt{5}}{147}\left(|A_{-1}|^2-8|A_0|^2+6|A_1|^2+8|A_2|^2\right)\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)Y_{20}(\theta')Y_{40}(\theta)
$$
$$
+\frac{20}{147}\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)
$$
$$
\times\left[\mathrm{Re}\left(A_0A_{-1}^*\right)+\sqrt{6}\,\mathrm{Re}\left(A_1A_0^*\right)-6\,\mathrm{Re}\left(A_2A_1^*\right)\right]\mathrm{Re}\left[Y_{21}(\theta',\phi')Y_{41}^*(\theta,\phi)\right]
$$
$$
+\frac{20\sqrt{2}}{49}\left(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right)\left[\mathrm{Re}\left(A_1A_{-1}^*\right)-\frac{4}{3}\mathrm{Re}\left(A_2A_0^*\right)\right]\mathrm{Re}\left[Y_{22}(\theta',\phi')Y_{42}^*(\theta,\phi)\right]
$$
$$
+\frac{4\sqrt{10}}{21\sqrt{7}}(-1)^{\frac{1}{2}(1-\kappa)}\left(|E_1|^2-\frac{1}{2}|E_2|^2\right)
$$
$$
\times\left[\mathrm{Im}\left(A_0A_{-1}^*\right)-\sqrt{6}\,\mathrm{Im}\left(A_1A_0^*\right)-3\,\mathrm{Im}\left(A_2A_1^*\right)\right]\mathrm{Im}\left[Y_{31}(\theta',\phi')Y_{41}^*(\theta,\phi)\right]
$$
$$
-\frac{40\sqrt{2}}{21\sqrt{7}}(-1)^{\frac{1}{2}(1-\kappa)}\left(|E_1|^2-\frac{1}{2}|E_2|^2\right)\mathrm{Im}\left(A_2A_0^*\right)\mathrm{Im}\left[Y_{32}(\theta',\phi')Y_{42}^*(\theta,\phi)\right]
$$
$$
-\frac{20}{21}(-1)^{\frac{1}{2}(1-\kappa)}\left(|E_1|^2-\frac{1}{2}|E_2|^2\right)\mathrm{Im}\left(A_2A_{-1}^*\right)\mathrm{Im}\left[Y_{33}(\theta',\phi')Y_{43}^*(\theta,\phi)\right]
$$
$$
+\frac{8}{147}\left(|A_{-1}|^2+6|A_0|^2+6|A_1|^2+|A_2|^2\right)\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)Y_{40}(\theta')Y_{40}(\theta)
$$
$$
+\frac{20\sqrt{6}}{147}\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)
$$
$$
\times\left[\mathrm{Re}\left(A_0A_{-1}^*\right)+\sqrt{6}\,\mathrm{Re}\left(A_1A_0^*\right)+\mathrm{Re}\left(A_2A_1^*\right)\right]\mathrm{Re}\left[Y_{41}(\theta',\phi')Y_{41}^*(\theta,\phi)\right]
$$
$$
+\frac{40\sqrt{6}}{147}\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)\left[\mathrm{Re}\left(A_1A_{-1}^*\right)+\mathrm{Re}\left(A_2A_0^*\right)\right]\mathrm{Re}\left[Y_{42}(\theta',\phi')Y_{42}^*(\theta,\phi)\right]
$$
$$
+\frac{20}{21}\left(|E_0|^2-\frac{2}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right)\mathrm{Re}\left(A_2A_{-1}^*\right)\mathrm{Re}\left[Y_{43}(\theta',\phi')Y_{43}^*(\theta,\phi)\right]\Bigg\}. \qquad (65)
$$

An examination of (62)-(65) shows that we can determine the relative magnitudes as well as the relative phases among the A helicity amplitudes for both the $J=1$ and $J=2$ cases when the simultaneous angular distribution of γ_1 and γ_2 is measured with the polarization of either one of the two photons. As in case 1, the measurement of the polarization is essential for getting the sines of the relative phases of these helicity amplitudes uniquely.

Case 3: We first integrate over (θ',ϕ') or the direction of the second photon γ_2 and then average over the polarizations of γ_2. The polarizations and the combined angular distribution of γ_1 and e^- are measured. We find that all the phases of the A and the E helicity amplitudes can then be determined for both the $J=1$ and $J=2$ cases when the combined angular distribution of γ_1 and e^- is measured together with the polarizations of both particles. In addition, all the relative magnitudes of A and E are determined for the $J=1$ case (but not for the $J=2$ case). Since all of this information can be obtained from the previous two cases where the polarization of only one particle is measured, we do not provide the long expressions here.

4 Concluding remarks

We have derived three model-independent expressions for the combined angular distribution of the final electron and the two gamma photons in the cascade process, $\overline{p}p \to {}^3D_2 \to \chi_J + \gamma_1 \to \psi + \gamma_2 + \gamma_1 \to e^+ + e^- + \gamma_2 + \gamma_1$ ($J = 0, 1, 2$), when $\overline{p}$ and p are unpolarized and the polarization of any one of the three decay particles is measured. Our expressions are based only on the general principles of quantum mechanics and the symmetry of the problem. We have also derived the partially integrated angular distribution functions which give the angular distributions of (γ_1,γ_2) and (γ_2,e^-) with the measurement of the polarization of one particle in each case. Once these angular distributions are experimentally measured, our expressions can be used to extract all the independent helicity amplitudes in the radiative decays ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$ for all values of J. In fact, the analysis of the angular correlations in the final decay products will serve to verify the value of J for the intermediate χ state in the cascade process. These experimentally determined values of the helicity amplitudes can then be compared with the predictions of various dynamical models. The great advantage of measuring the angular distributions with the polarization of one particle is that one can get not only the relative magnitudes of the helicity amplitudes but also both the cosines and sines of the relative phases of the helicity amplitudes in the decay processes ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$. This is important because the helicity amplitudes are in general complex. Therefore by measuring the combined angular distribution of γ_1, γ_2 and e^- with the polarization of any one of the three particles, we can obtain complete information on the helicity amplitudes in the two radiative decay processes. Alternatively, we can get the same information by measuring the two-particle angular distribution of γ_2 and e^- and that of γ_1 and γ_2 with the polarization of either one of the two particles.

It is of great advantage that we express all the angular distribution functions in terms of the orthogonal functions such as the Wigner D^J functions and the spherical harmonics. Because of this feature of our results, we can get the coefficients of these functions, which are functions of the angular-momentum helicity amplitudes, by just doing a numerical integration of the measured angular distributions.

Both the theorists and the experimentalists would like to express their results in terms of the multipole amplitudes in the radiative transitions ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$ ($J = 0, 1, 2$). The relationship between the helicity and the multipole amplitudes are given by the orthogonal transformations [9, 10]

$$A_\sigma = \sum_{\mathrm{Max}(k=|2-J|;1)}^{J+2} a_k \left(\frac{2k+1}{5}\right)^{1/2} \langle k-1; J, \sigma | 2, \sigma-1 \rangle,$$

$$E_\rho = \sum_{k=1}^{J+1} e_k \left(\frac{2k+1}{2J+1}\right)^{1/2} \langle k1; 1, \rho-1 | J\rho \rangle, \quad (J = 0, 1, 2), \tag{66}$$

where a_k and e_k are the radiative multipole amplitudes in ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$, respectively. Since the transformations of (66) are orthogonal,

$$\sum_\sigma |A_\sigma|^2 = \sum_k |a_k|^2 = 1,$$

$$\sum_\rho |E_\rho|^2 = \sum_k |e_k|^2 = 1 \tag{67}$$

It is noteworthy that the decay process ${}^3D_2 \to \chi_2 + \gamma_1$ has four independent helicity amplitudes corresponding to four multipole amplitudes $E1$, $M2$, $E3$ and $M4$. In any potential model for heavy quarkonia, the $M4$ multipole amplitude is zero to order v^2/c^2 because in this approximation there is no fourth rank tensor component in the transition operator [11]. Moreover, in non-relativistic potential models, the $E1$ multipole amplitude should be the dominant contribution. Using (66), we can then obtain simple relationships among the A and the E helicity amplitudes for $J = 1$ and $J = 2$:

$$3|A_{-1}|^2 \simeq 2|A_0|^2 \simeq 2|A_1|^2 \simeq 3|A_2|^2 \quad (J = 2)$$

$$|A_{-1}|^2 \simeq 2|A_0|^2 \simeq 6|A_1|^2 \quad (J = 1)$$

$$6|E_0|^2 \simeq 2|E_1|^2 \simeq |E_2|^2 \quad (J = 2)$$

$$|E_0|^2 \simeq |E_1|^2 \quad (J = 1) \tag{68}$$

So by measuring the angular distributions, one can further test the validity of the non-relativistic potential models.

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